Remarks on two integral operators and numerical methods for CSIE[☆]

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ABSTRACT

In this paper the author extends the mapping properties of some singular integral operators in Zygmund spaces equipped with uniform norm. As a by-product quadrature methods for solving CSIE having indices 0 and 1 are proposed. Their stability and convergence are proved and error estimates in Zygmund norm are given. Some numerical tests are also shown.

1. Introduction and preliminaries

It is well-known (see, for instance, [1]) that the operator

$$(D^{\alpha,-\alpha}f)(x) = (\cos \pi \alpha) v^{\alpha,-\alpha}(x) f(x) - \frac{\sin \pi \alpha}{\pi} \int_{-1}^{1} f(y) \frac{v^{\alpha,-\alpha}(y)}{y-x} dy,$$

with $v^{\rho,\sigma}(x) = (1-x)^{\rho}(1+x)^{\sigma}$, $\rho, \sigma > -1$, a Jacobi weight, and its inverse $D^{-\alpha,\alpha}$ are isometric maps in the couples of spaces $(L_w^2, L_{1/w}^2)$ and $(L_{1/w}^2, L_w^2)$, $w = v^{\frac{\alpha}{2},-\frac{\alpha}{2}}$, respectively. Analogous properties, but under the assumption $\int_{-1}^{1} f(x) v^{-\alpha,\alpha-1}(x) dx = 0$, hold true for the operator

$$(D^{-\alpha,\alpha-1}f)(x) = (\cos \pi \alpha) v^{-\alpha,\alpha-1}(x) f(x) + \frac{\sin \pi \alpha}{\pi} \int_{-1}^{1} f(y) \frac{v^{-\alpha,\alpha-1}(y)}{y-x} dy,$$

and its inverse $D^{\alpha,1-\alpha}$ in the couples of spaces $(L_u^2, L_{1/u}^2)$ and $(L_{1/u}^2, L_u^2)$, $u = v^{-\frac{\alpha}{2},\frac{\alpha-1}{2}}$, respectively.

By contrast, in the space of continuous functions in $[-1, 1]$, $C^0 := C^0([-1, 1])$, equipped with the uniform norm the previous operators are unbounded.

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Now, we introduce the space

$$C_{v^{\rho,\sigma}} = \left\{ f \in C^0((-1,1)) : \lim_{|x| \to 1} (f v^{\rho,\sigma})(x) = 0 \right\}, \quad \rho, \sigma \geq 0,$$

equipped with the norm

$$\|f\|_{C_{v^{\rho,\sigma}}} := \max_{|x| \leq 1} |(f v^{\rho,\sigma})(x)| = \|f v^{\rho,\sigma}\|_\infty,$$

with the obvious modifications if ρ e/o σ are zeros (letting $C_{v^{0,0}} = C^0$).

A subspace of $C_{v^{\rho,\sigma}}$ is the Sobolev space

$$W_r(v^{\rho,\sigma}) := \{ f \in C_{v^{\rho,\sigma}} : f^{(r-1)} \in AC(-1,1) \text{ and } \|f^{(r)} \varphi^r v^{\rho,\sigma}\|_\infty < \infty \},$$

where $AC(-1,1)$ is the set of all functions that are absolutely continuous in every compact set of $(-1,1)$, equipped with the norm

$$\|f\|_{W_r(v^{\rho,\sigma})} = \|f v^{\rho,\sigma}\|_\infty + \|f^{(r)} \varphi^r v^{\rho,\sigma}\|_\infty.$$

Another subspace is the Zygmund space

$$Z_r(v^{\rho,\sigma}) := Z_{r,k}(v^{\rho,\sigma}) = \left\{ f \in C_{v^{\rho,\sigma}} : \sup_{t>0} \frac{\Omega_\varphi^k(f,t)_{v^{\rho,\sigma}}}{t^r} < +\infty \right\}$$

equipped with the norm

$$\|f\|_{Z_r(v^{\rho,\sigma})} = \|f v^{\rho,\sigma}\|_\infty + \sup_{t>0} \frac{\Omega_\varphi^k(f,t)_{v^{\rho,\sigma}}}{t^r},$$

where $r > 0$ is an arbitrary real number, $k > r$ is integer,

$$\Omega_\varphi^k(f,t)_{v^{\rho,\sigma}} := \sup_{0 < h \leq t} \|(\Delta_{h\varphi}^k f) v^{\rho,\sigma}\|_{C(I_{h,k})}, \tag{1}$$

$$\Delta_{h\varphi}^k f(x) := \sum_{i=0}^{k} (-1)^i \binom{k}{i} f\left(x + \frac{kh}{2}\varphi(x) - ih\varphi(x)\right),$$

$I_{h,k} := [-1 + 4k^2h^2, 1 - 4k^2h^2]$, $0 < t < 1$ and $\varphi(x) = \sqrt{1-x^2}$. We remark that $W_r(v^{\rho,\sigma}) \subseteq Z_r(v^{\rho,\sigma})$ when $r = k$.

With the above notations, in [2,3] the authors showed that $D^{\alpha,-\alpha}$ is bounded and invertible in the couple $(Z_r(v^{\alpha,0}), Z_r(v^{0,\alpha}))$ and, moreover, $D^{-\alpha,\alpha-1}$, under the assumption $\int_{-1}^{1} f(x) v^{-\alpha,\alpha-1}(x)dx = 0$, is bounded and invertible in the couple $(Z_r(v^{0,0}), Z_r(v^{\alpha,1-\alpha}))$.

As a first contribution of this paper, we prove that $D^{\alpha,-\alpha}$ is bounded and invertible in the couple $(Z_r(v^{\alpha+\gamma,\delta}), Z_r(v^{\gamma,\alpha+\delta}))$ and, analogously, $D^{-\alpha,\alpha-1}$, under the assumption $\int_{-1}^{1} f(x) v^{-\alpha,\alpha-1}(x)dx = 0$, is bounded and invertible in the couple $(Z_r(v^{\gamma,\delta}), Z_r(v^{\alpha+\gamma,1-\alpha+\delta}))$. Obviously, the parameters α, γ, δ have to satisfy suitable conditions that we will assign explicitly.

Then, the results in [2,3] have been extended to larger functional spaces. This fact suggested us to propose two numerical methods to approximate the solutions of the equations

$$(D^{\alpha,-\alpha} + K^{\alpha,-\alpha})f(x) = g(x) \tag{2}$$

and

$$(D^{-\alpha,\alpha-1} + K^{-\alpha,\alpha-1})f(x) = g(x), \qquad \int_{-1}^{1} f(x) v^{-\alpha,\alpha-1}(x)dx = 0, \tag{3}$$

where $K^{\alpha,-\alpha}$ and $K^{\alpha,1-\alpha}$ are compact perturbations. Eqs. (2) and (3) are well-known Cauchy equations of indices $\chi = 0$ and $\chi = 1$, respectively.

The proposed numerical methods are stable and convergent for any choice of the parameter $0 < \alpha < 1$. We would like to emphasize that the range of α is $(0, 1)$, taking into account that the numerical methods proposed in [4], using the properties proved in [2,3], lead to strong restrictions: $\frac{1}{2} \leq \alpha < 1$ for Eq. (2) and $\alpha = \frac{1}{2}$ for Eq. (3).

The paper is organized as follows. In Section 2 we give the main results. Section 3 contains the description of the quadrature methods we propose for Eqs. (2) and (3). We show that both of them are stable and convergent and lead to solve well-conditioned linear systems. The proofs of the main results are given in Section 4, while Appendix contains some proofs that are more technical. Finally, in Section 5, we show the efficiency of our procedures by some numerical tests.

2. Main results

We first give some preliminary definition and result. In the following $\mathcal{C}$ denotes a positive constant which may have different values in different formulas. We will write $\mathcal{C} \neq \mathcal{C}(a, b, \ldots)$ to say that $\mathcal{C}$ is independent of the parameters $a, b, \ldots$. If $A, B > 0$ are quantities depending on some parameters, we write $A \sim B$, if there exists a positive constant $\mathcal{C}$ independent of the parameters of A and B, such that

$$\frac{B}{\mathcal{C}} \leq A \leq \mathcal{C}B.$$

We denote by

$$\omega_\varphi^k(f, t)_{v^{\rho,\sigma}} = \Omega_\varphi^k(f, t)_{v^{\rho,\sigma}} + \inf_{P \in \mathbb{P}_{k-1}} \|(f - P)v^{\rho,\sigma}\|_{C[-1, -1+4k^2 t^2]} + \inf_{P \in \mathbb{P}_{k-1}} \|(f - P)v^{\rho,\sigma}\|_{C[1-4k^2 t^2, 1]}$$

the kth φ-modulus of continuity, where Ω_φ^k is defined in (1) and $\mathbb{P}_m$ is the set of all polynomials of degree at most m, and by

$$E_m(f)_{v^{\rho,\sigma}} = \inf_{P_m \in \mathbb{P}_m} \|(f - P_m)v^{\rho,\sigma}\|_\infty$$

the error of best approximation of a function $f \in C_{v^{\rho,\sigma}}$. The following inequality is well-known for $k < m$ [5]:

$$E_m(f)_{v^{\rho,\sigma}} \leq \mathcal{C}\omega_\varphi^k\left(f, \frac{1}{m}\right)_{v^{\rho,\sigma}}, \quad \mathcal{C} \neq \mathcal{C}(m, f). \tag{4}$$

2.1. Mapping properties of the operators $D^{\alpha, -\alpha}$ and $D^{\alpha, 1-\alpha}$

The following theorems, that extend Theorem 3.1 in [3] and Theorem 3.3 in [2], allow to characterize the smoothness of $D^{\alpha, -\alpha}f$, $D^{-\alpha, \alpha}f$, $D^{\alpha, 1-\alpha}f$ and $D^{-\alpha, 1-\alpha}f$.

Theorem 2.1. *Let* $0 < \alpha < 1$. *If the parameters* γ, δ *satisfy*

$$\begin{cases} \max\left\{0, -\dfrac{\alpha}{2} + \dfrac{1}{4}\right\} \leq \gamma < \min\left\{-\dfrac{\alpha}{2} + \dfrac{3}{4}, \dfrac{1-\alpha}{2}\right\} \\ \max\left\{0, -\dfrac{\alpha}{2} + \dfrac{1}{4}\right\} \leq \delta < \min\left\{-\dfrac{\alpha}{2} + \dfrac{3}{4}, \dfrac{1-\alpha}{2}\right\}, \end{cases} \tag{5}$$

then $\forall f \in C_{v^{\gamma, \alpha+\delta}}$ *such that* $\int_0^1 \frac{\omega_\varphi^k(f,t)_{v^{\gamma,\alpha+\delta}}}{t} dt < +\infty$ *we have*

$$E_m(D^{-\alpha, \alpha}f)_{v^{\alpha+\gamma, \delta}} \leq \mathcal{C} \int_0^{\frac{1}{m}} \frac{\omega_\varphi^k(f, t)_{v^{\gamma, \alpha+\delta}}}{t} dt \tag{6}$$

and $\forall f \in C_{v^{\alpha+\gamma, \delta}}$ *such that* $\int_0^1 \frac{\omega_\varphi^k(f,t)_{v^{\alpha+\gamma,\delta}}}{t} dt < +\infty$ *we get*

$$E_m(D^{\alpha, -\alpha}f)_{v^{\gamma, \alpha+\delta}} \leq \mathcal{C} \int_0^{\frac{1}{m}} \frac{\omega_\varphi^k(f, t)_{v^{\alpha+\gamma, \delta}}}{t} dt, \tag{7}$$

where $\mathcal{C} \neq \mathcal{C}(m, f)$.

Theorem 2.2. *Let* $0 < \alpha < 1$ *and assume that the parameters* γ, δ *satisfy*

$$\begin{cases} \max\left\{0, -\dfrac{\alpha}{2} + \dfrac{1}{4}\right\} \leq \gamma < -\dfrac{\alpha}{2} + \dfrac{1}{2} \\ \max\left\{0, \dfrac{\alpha}{2} - \dfrac{1}{4}\right\} \leq \delta < \dfrac{\alpha}{2}. \end{cases} \tag{8}$$

Then $\forall f \in C_{v^{\alpha+\gamma, 1-\alpha+\delta}}$ *such that* $\int_0^{\frac{1}{m}} \frac{\omega_\varphi^k(f,t)_{v^{\alpha+\gamma,1-\alpha+\delta}}}{t} dt < +\infty$ *we have*

$$E_m(D^{\alpha, 1-\alpha}f)_{v^{\gamma, \delta}} \leq \mathcal{C} \int_0^{\frac{1}{m}} \frac{\omega_\varphi^k(f, t)_{v^{\alpha+\gamma, 1-\alpha+\delta}}}{t} dt$$

and $\forall f \in C_{v^{\gamma,\delta}}$ such that $\int_0^{\frac{1}{m}} \frac{\omega_\varphi^k(f,t)_{v^{\gamma,\delta}}}{t} dt < +\infty$ we obtain

$$E_m(D^{-\alpha,\alpha-1}f)_{v^{\alpha+\gamma,1-\alpha+\delta}} \leq \mathcal{C} \int_0^{\frac{1}{m}} \frac{\omega_\varphi^k(f,t)_{v^{\gamma,\delta}}}{t} dt,$$

where $\mathcal{C} \neq \mathcal{C}(m,f)$.

As a consequence of the above two theorems, we get the following two corollaries, respectively.

Corollary 2.1. *Let* $0 < \alpha < 1$ *and* γ, δ *satisfying* (5). *Then, for* $r > 0$, $D^{\alpha,-\alpha} : Z_r(v^{\alpha+\gamma,\delta}) \to Z_r(v^{\gamma,\alpha+\delta})$ *is a continuous and invertible map and* $D^{-\alpha,\alpha} : Z_r(v^{\gamma,\alpha+\delta}) \to Z_r(v^{\alpha+\gamma,\delta})$ *is its inverse. Moreover, the equivalences*

$$\|D^{\alpha,-\alpha}f\|_{Z_r(v^{\gamma,\alpha+\delta})} \sim \|f\|_{Z_r(v^{\alpha+\gamma,\delta})} \tag{9}$$

and

$$\|D^{-\alpha,\alpha}f\|_{Z_r(v^{\alpha+\gamma,\delta})} \sim \|f\|_{Z_r(v^{\gamma,\alpha+\delta})} \tag{10}$$

hold. The constants in "$\sim$" are independent of f.

Corollary 2.2. *Let* $0 < \alpha < 1$ *and* γ, δ *satisfying* (8). *Then, for* $r > 0$, *the operators* $D^{-\alpha,\alpha-1} : Z_r(v^{\gamma,\delta}) \to Z_r(v^{\alpha+\gamma,1-\alpha+\delta})$ *and* $D^{\alpha,1-\alpha} : Z_r(v^{\alpha+\gamma,1-\alpha+\delta}) \to Z_r(v^{\gamma,\delta})$ *are continuous linear maps. If* $\int_{-1}^1 f(x)v^{-\alpha,\alpha-1}(x)dx = 0$, $D^{\alpha,1-\alpha}$ *is the inverse of* $D^{-\alpha,\alpha-1}$ *and the following equivalences*

$$\|D^{-\alpha,\alpha-1}f\|_{Z_r(v^{\alpha+\gamma,1-\alpha+\delta})} \sim \|f\|_{Z_r(v^{\gamma,\delta})}$$

and

$$\|D^{\alpha,1-\alpha}f\|_{Z_r(v^{\gamma,\delta})} \sim \|f\|_{Z_r(v^{\alpha+\gamma,1-\alpha+\delta})}$$

hold true. The constants in "$\sim$" are independent of f.

2.2. Mapping properties of the operators $K^{\alpha,-\alpha}$ and $K^{\alpha,1-\alpha}$

In Eqs. (2) and (3) we consider compact perturbations of the following forms

$$(K^{\alpha,-\alpha}f)(x) = \int_{-1}^1 k(x,y)f(y)v^{\alpha,-\alpha}(y)dy$$

and

$$(K^{-\alpha,\alpha-1}f)(x) = \int_{-1}^1 k(x,y)f(y)v^{-\alpha,\alpha-1}(y)dy,$$

respectively. We assume that the kernel $k(x,y)$ is smooth or can be written as

$$k(x,y) = \frac{h(x,y) - h(x,x)}{x-y}.$$

The latter case, where $k(x,y)$ can be weakly singular, has been considered in [4,6,7]. With the notations $k(x,y) = k_x(y) = k_y(x)$ and $h(x,y) = h_x(y)$, we state the following lemmas.

Lemma 2.1. *Let* $0 < \alpha < 1$. *If, for* $s > 0$ *and* γ, δ *satisfying* (5)*, we have*

$$\sup_{|x|\leq 1} v^{\gamma,\alpha+\delta}(x)\|k_x\|_{Z_s} < +\infty \quad \text{and} \quad \sup_{|y|\leq 1} \|k_y\|_{Z_s(v^{\gamma,\alpha+\delta})} < +\infty \tag{11}$$

or

$$\sup_{|x|\leq 1} \|h_x\|_{Z_s} < +\infty \quad \text{when } k(x,y) = \frac{h(x,y) - h(x,x)}{x-y}, \tag{12}$$

then the operator $K^{\alpha,-\alpha} : Z_r(v^{\alpha+\gamma,\delta}) \to Z_r(v^{\gamma,\alpha+\delta})$ *is compact for all* $0 < r < s$.

Lemma 2.2. *Let* $0 < \alpha < 1$. *If, for* $s > 0$ *and* γ, δ *satisfying* (8)*, the kernel* k *satisfies*

$$\sup_{|x|\leq 1} v^{\alpha+\gamma,1-\alpha+\delta}(x)\|k_x\|_{Z_s} < +\infty \quad \text{and} \quad \sup_{|y|\leq 1} \|k_y\|_{Z_s(v^{\alpha+\gamma,1-\alpha+\delta})} < +\infty \tag{13}$$

or (12) *then the operator* $K^{-\alpha,\alpha-1} : Z_r(v^{\gamma,\delta}) \to Z_r(v^{\alpha+\gamma,1-\alpha+\delta})$ *is compact for all* $0 < r < s$.

The above lemmas, where no restriction on the parameter α appears, provide the assumptions on $k(x, y)$ and $g(x)$ under which Eqs. (2) and (3) are unisolvent in the spaces $Z_r(v^{\alpha+\gamma,\delta})$ and $Z_r(v^{\gamma,\delta})$, respectively. In fact, using the Fredholm alternative theorem (see, for instance, [8]), by Corollaries 2.1 and 2.2 and Lemmas 2.1 and 2.2, we easily deduce the following propositions.

Proposition 2.1. *Assuming that the kernel k satisfies* (11) *or* (12) *with $s > 0$ and that* $\mathrm{Ker}(D^{\alpha,-\alpha} + K^{\alpha,-\alpha}) = \{0\}$ *in $Z_r(v^{\alpha+\gamma,\delta})$ with $r < s$, Eq.* (2) *admits a unique solution in $Z_r(v^{\alpha+\gamma,\delta})$ for every choice of the right-hand side $g \in Z_r(v^{\gamma,\alpha+\delta})$.*

Proposition 2.2. *If the kernel k satisfies* (13) *or* (12) *with $s > 0$ and* $\mathrm{Ker}(D^{-\alpha,\alpha-1} + K^{-\alpha,\alpha-1}) = \{0\}$ *in $Z_r(v^{\gamma,\delta})$ with $\int_{-1}^{1} f(x)v^{-\alpha,\alpha-1}(x)dx = 0$ and $r < s$, then the problem* (3) *is uniquely solvable in $Z_r(v^{\gamma,\delta})$ for every choice of the right-hand side $g \in Z_r(v^{\alpha+\gamma,1-\alpha+\delta})$.*

For example, the kernel $k(x, y) = |x - y|^{\mu}, \mu > 0$, fulfills the assumptions (11) and (13) with $s = \mu$ and the kernel $k(x, y) = d(x)\frac{|y-a|^{\mu}-|x-a|^{\mu}}{(y-x)}$, where d is an analytic function, a is fixed in $(1, 1)$ and $\mu > 0$, satisfies (12) with $h(x, y) = d(x)|y - a|^{\mu}$ and $s = \mu$.

Finally, we remark that in the assumptions (11)–(13), if r is an integer, the norms in the Zygmund spaces can be replaced by the corresponding simpler norms in the Sobolev spaces.

3. Numerical methods

Bearing in mind the space $C_{v^{\alpha+\gamma,\delta}}$, for all $F \in C_{v^{\alpha+\gamma,\delta}}$ we denote by $L_m^{\alpha,-\alpha}F$ the Lagrange polynomial interpolating the function F at the zeros $t_1 < \cdots < t_m$ of $p_m^{\alpha,-\alpha}$ that is the m-th Jacobi polynomial of parameters α and $-\alpha$. Moreover, we denote by $L_m^{-\alpha,\alpha}F, F \in C_{v^{\gamma,\alpha+\delta}}$, the Lagrange polynomial based on the zeros $x_1, \ldots, x_m$ of $p_m^{-\alpha,\alpha}$.

One naturally uses the following representations:

$$L_m^{\alpha,-\alpha}(F, x) = \sum_{i=1}^{m} \varphi_i^{\alpha,-\alpha}(x)(Fv^{\alpha+\gamma,\delta})(t_i)$$

and

$$L_m^{-\alpha,\alpha}(F, x) = \sum_{i=1}^{m} \varphi_i^{-\alpha,\alpha}(x)(Fv^{\gamma,\alpha+\delta})(x_i),$$

where

$$\varphi_i^{\alpha,-\alpha}(x) = \frac{l_i^{\alpha,-\alpha}(x)}{v^{\alpha+\gamma,\delta}(t_i)}, \qquad \varphi_i^{-\alpha,\alpha}(x) = \frac{l_i^{-\alpha,\alpha}(x)}{v^{\gamma,\alpha+\delta}(x_i)}, \quad i = 1, \ldots, m,$$

being $l_i^{\alpha,-\alpha}$ and $l_i^{-\alpha,\alpha}(x), i = 1, \ldots, m$, the fundamental Lagrange polynomials.

That being stated, with

$$f_m(x) = \sum_{j=1}^{m} \varphi_j^{\alpha,-\alpha}(x)a_j, \quad a_j = (v^{\alpha+\gamma,\delta}f_m)(t_j),$$

the unknown polynomial, we proceed to solve the equations

$$(D^{\alpha,-\alpha} + K_m^{\alpha,-\alpha})f_m = g_m, \quad m \geq 1, \tag{14}$$

where

$$g_m(x) := L_m^{-\alpha,\alpha}(g, x) = \sum_{i=1}^{m} \varphi_i^{-\alpha,\alpha}(x)b_i, \quad b_i = (v^{\gamma,\alpha+\delta}g)(x_i),$$

and

$$(K_m^{\alpha,-\alpha}f_m)(x) := L_m^{-\alpha,\alpha}(\tilde{K}_m^{\alpha,-\alpha}f_m, x) = \sum_{i=1}^{m} \varphi_i^{-\alpha,\alpha}(x)(v^{\gamma,\alpha+\delta}\tilde{K}_m^{\alpha,-\alpha}f_m)(x_i), \tag{15}$$

with

$$(\tilde{K}_m^{\alpha,-\alpha}f_m)(x) = \int_{-1}^{1} L_m^{\alpha,-\alpha}(k_x, y)f_m(y)v^{\alpha,-\alpha}(y)dy.$$

Using well-known properties of the operator $D^{\alpha,-\alpha}$ and standard procedures (see, for instance, [9, Chapter 9] and [10, Lemma 1.15]), the finite dimensional equations (14) can be written in the form of linear systems as follows

$$v^{\gamma,\alpha+\delta}(x_i) \sum_{j=1}^{m} \frac{\lambda_j^{\alpha,-\alpha}}{v^{\alpha+\gamma,\delta}(t_j)} \left[\frac{\sin(\alpha\pi)}{\pi} \frac{1}{(x_i - t_j)} + k(x_i, t_j) \right] a_j = b_i, \quad i = 1, \dots, m \tag{16}$$

or

$$v^{\gamma,\alpha+\delta}(x_i) \sum_{j=1}^{m} \frac{\lambda_j^{\alpha,-\alpha}}{v^{\alpha+\gamma,\delta}(t_j)} \left[\frac{\sin(\alpha\pi)}{\pi} \frac{1}{(x_i - t_j)} + \frac{h(x_i, t_j) - h(x_i, x_i)}{x_i - t_j} \right] a_j = b_i, \quad i = 1, \dots, m, \tag{17}$$

when $k(x, y) = \frac{h(x,y) - h(x,x)}{y - x}$.

Note that the systems (16) and (17) are well-defined, in fact, letting $x_{m,i} = \cos\tau_{m,i}$ and $t_{m,j} = \cos\theta_{m,j}$, $i, j = 1, \dots, m$, in [7] the authors proved that

$$\min_{i,j=1,\dots,m} |\tau_{m,i} - \theta_{m,j}| \geq \frac{\mathcal{C}}{m}, \tag{18}$$

from which we can easily deduce that there exists a positive constant $\mathcal{C} \neq \mathcal{C}(m, j, i)$ s.t.

$$\frac{v^{\gamma,\alpha+\delta}(x_i)\lambda_j^{\alpha,-\alpha}}{v^{\alpha+\gamma,\delta}(t_j)|t_j - x_i|} > \mathcal{C}.$$

Now, we will show that, under suitable assumptions on the kernel $k(x, y)$, the right-hand side g and the parameters α, β, γ, δ, the system (16) admits a unique solution, say $\mathbf{a}^* = (a_1^*, \dots, a_m^*)^T$, and that the sequence

$$\{f_m^*\}_m, \quad f_m^*(x) = \sum_{j=1}^{m} \varphi_j^{\alpha,-\alpha}(x) a_j^*,$$

for $m \to +\infty$, converges in $Z_r(v^{\alpha+\gamma,\delta})$ to the solution f^* of (2). Denoting by A_m the matrix of the system (16) and by $\text{cond}(A_m)$ its condition number in uniform norm, we establish the following theorem.

Theorem 3.1. *Let $0 < \alpha < 1$ and assume that γ, δ fulfill the conditions (5). If $\text{Ker}(D^{\alpha,-\alpha} + K^{\alpha,-\alpha}) = \{0\}$ in $Z_r(v^{\alpha+\gamma,\delta})$, $k(x, y)$ satisfies (11) and*

$$g \in Z_s(v^{\gamma,\alpha+\delta}), \tag{19}$$

with $0 < r < s$, then the system (16), for a sufficiently large m (say $m > m_0$), is unisolvent and

$$\sup_m \frac{\text{cond}(A_m)}{\log^2 m} < +\infty. \tag{20}$$

Moreover the following error estimate

$$\|f^* - f_m^*\|_{Z_r(v^{\alpha+\gamma,\delta})} = \mathcal{O}\left(\frac{\log m}{m^{s-r}}\right) \tag{21}$$

holds true, where the constants in "$\mathcal{O}$" are independent of m.

Analogous considerations can be done for system (17). In this case we state the following theorem.

Theorem 3.2. *Let us assume that $0 < \alpha < 1$ and γ, δ satisfy (5). If $\text{Ker}(D^{\alpha,-\alpha} + K^{\alpha,-\alpha}) = \{0\}$ in $Z_r(v^{\alpha+\gamma,\delta})$ for $r > 0$ and, with $s > r$, k and g fulfill (12) and (19), respectively, then the system (16), for a sufficiently large m (say $m > m_0$), admits a unique solution and the condition number of its matrix in uniform norm satisfies (20). Moreover the following error estimate*

$$\|f^* - f_m^*\|_{Z_r(v^{\alpha+\gamma,\delta})} = \mathcal{O}\left(\frac{\log^2 m}{m^{s-r}}\right) \tag{22}$$

holds true, where the constants in "$\mathcal{O}$" are independent of m.

Now, we will briefly describe the numerical method we propose to solve the problem (3).

Recalling the spaces $C_{v^{\gamma,\delta}}$ and $C_{v^{\alpha+\gamma,1-\alpha+\delta}}$, with the above introduced notations, let

$$L_m^{-\alpha,\alpha-1}(F, x) = \sum_{i=1}^{m} \varphi_i^{-\alpha,\alpha-1}(x)(Fv^{\gamma,\delta})(\bar{t}_i)$$

be the Lagrange polynomial of the function $F \in C_{v^{\gamma,\delta}}$ based on the zeros $\bar{t}_1 < \cdots < \bar{t}_m$ of $p_m^{-\alpha,\alpha-1}$, where $\varphi_i^{-\alpha,\alpha-1}(x) = \frac{l_i^{-\alpha,\alpha-1}(x)}{v^{\gamma,\delta}(\bar{t}_i)}$, $i = 1, \ldots, m$.

For functions F belonging to $C_{v^{\alpha+\gamma,1-\alpha+\delta}}$ we introduce the polynomial

$$L_{m-1}^{\alpha,1-\alpha}(F, x) = \sum_{i=1}^{m-1} \varphi_i^{\alpha,1-\alpha}(x)(F v^{\alpha+\gamma,1-\alpha+\delta})(\bar{x}_i),$$

where $\bar{x}_1 < \cdots < \bar{x}_{m-1}$ are the zeros of $p_{m-1}^{\alpha,1-\alpha}$ and $\varphi_i^{\alpha,1-\alpha}(x) = \frac{l_i^{\alpha,1-\alpha}(x)}{v^{\alpha+\gamma,1-\alpha+\delta}(\bar{x}_i)}$, $i = 1, \ldots, m-1$.

Thus, with

$$f_m(x) = \sum_{j=1}^{m} \varphi_j^{-\alpha,\alpha-1}(x)a_j, \quad a_j = (v^{\gamma,\delta}f_m)(t_j),$$

we solve the finite dimensional equations

$$\begin{cases} (D^{-\alpha,\alpha-1}f_m)(x) + (K_{m-1}^{-\alpha,\alpha-1}f_m)(x) = g_{m-1}(x), \\ \int_{-1}^{1} f_m(x)v^{-\alpha,\alpha-1}(x)dx = 0, \end{cases} \quad m \geq 1, \tag{23}$$

where

$$g_{m-1}(x) := L_{m-1}^{\alpha,1-\alpha}(g, x) = \sum_{i=1}^{m-1} \varphi_i^{\alpha,1-\alpha}(x)b_i, \quad b_i = (v^{\alpha+\gamma,1-\alpha+\delta}g)(\bar{x}_i),$$

and

$$\begin{aligned} (K_{m-1}^{-\alpha,\alpha-1}f_m)(x) &:= L_{m-1}^{\alpha,1-\alpha}(\tilde{K}_m^{-\alpha,\alpha-1}f_m, x) \\ &= \sum_{i=1}^{m-1} \varphi_i^{\alpha,1-\alpha}(x)(v^{\alpha+\gamma,1-\alpha+\delta}\tilde{K}_m^{-\alpha,\alpha-1}f_m)(\bar{x}_i), \end{aligned}$$

with

$$(\tilde{K}_m^{-\alpha,\alpha-1}f_m)(x) = \int_{-1}^{1} L_m^{-\alpha,\alpha-1}(k_x, y)f_m(y)v^{-\alpha,\alpha-1}(y)dy.$$

The finite dimensional equations (23) are equivalent to the linear systems

$$\begin{cases} v^{\alpha+\gamma,1-\alpha+\delta}(\bar{x}_i) \sum_{j=1}^{m} \frac{\lambda_j^{-\alpha,\alpha-1}}{v^{\gamma,\delta}(\bar{t}_j)} \left[\frac{\sin(\alpha\pi)}{\pi} \frac{1}{(\bar{t}_j - \bar{x}_i)} + k(\bar{x}_i, \bar{t}_j) \right] a_j = b_i, \quad i = 1, \ldots, m-1, \\ \sum_{j=1}^{m} \frac{\lambda_j^{-\alpha,\alpha-1}}{v^{\gamma,\delta}(\bar{t}_j)} a_j = 0 \end{cases} \tag{24}$$

or

$$\begin{cases} v^{\alpha+\gamma,1-\alpha+\delta}(\bar{x}_i) \sum_{j=1}^{m} \frac{\lambda_j^{-\alpha,\alpha-1}}{v^{\gamma,\delta}(\bar{t}_j)} \left[\frac{\sin(\alpha\pi)}{\pi} \frac{1}{(\bar{t}_j - \bar{x}_i)} + \frac{h(\bar{x}_i, \bar{t}_j) - h(\bar{x}_i, \bar{x}_i)}{\bar{x}_i - \bar{t}_j} \right] a_j = b_i, \quad i = 1, \ldots, m-1, \\ \sum_{j=1}^{m} \frac{\lambda_j^{-\alpha,\alpha-1}}{v^{\gamma,\delta}(\bar{t}_j)} a_j = 0, \end{cases} \tag{25}$$

when $k(x, y) = \frac{h(x,y) - h(x,x)}{y-x}$. Using the same argument that we used for the systems (16) and (17), we deduce that (24) and (25) are well-defined, too.

We state the following theorem.

Theorem 3.3. *Let* $0 < \alpha < 1$ *and assume that* γ, δ *fulfill the conditions* (8). *If* $\text{Ker}(D^{-\alpha,\alpha-1} + K^{-\alpha,\alpha-1}) = \{0\}$ *in* $Z_r(v^{\gamma,\delta})$, *with* $\int_{-1}^{1} f(x)v^{-\alpha,\alpha-1}(x)dx = 0$ *and* $r > 0$, *and, for* $0 < r < s$, $k(x, y)$ *satisfies* (13) *or* (12) *and*

$$g \in Z_s(v^{\alpha+\gamma,1-\alpha+\delta}),$$

then, for a sufficiently large m (say m > m₀), the system (24) is unisolvent and for the condition number in uniform norm of its matrix A_m, we have

$$\sup_m \frac{\text{cond}(A_m)}{\log^2 m} < +\infty. \tag{26}$$

Moreover, the solution f_m^* of (23) converges to the solution f^* of (3) and if $k(x, y)$ satisfies (13) we have

$$\|f^* - f_m^*\|_{Z_r(v^{\gamma,\delta})} = \mathcal{O}\left(\frac{\log m}{m^{s-r}}\right), \tag{27}$$

while if $k(x, y)$ satisfies (12) we get

$$\|f^* - f_m^*\|_{Z_r(v^{\gamma,\delta})} = \mathcal{O}\left(\frac{\log^2 m}{m^{s-r}}\right). \tag{28}$$

Here the constants in "$\mathcal{O}$" are independent of m.

4. Proofs

The complete proofs of Theorems 2.1 and 2.2 and Corollaries 2.1 and 2.2 can be found in [3,2], apart from some little technical changes. Here we limit ourselves to give the main steps of the proofs of Theorem 2.1 and Corollary 2.1, omitting the details.

We first introduce the de la Vallée Poussin sum

$$V_m^{\rho,\sigma}(f, x) = \sum_{j=0}^{2m} \mu_j c_j^{\rho,\sigma}(f) p_j^{\rho,\sigma}(x), \tag{29}$$

where

$$\mu_j := \begin{cases} 1 & \text{if } j \leq m \\ \dfrac{2m + 1 - j}{m + 1} & \text{if } j > m \end{cases} \tag{30}$$

and

$$c_j^{\rho,\sigma}(f) = \int_{-1}^{1} f(x) p_j^{\rho,\sigma}(x) v^{\rho,\sigma}(x) dx \tag{31}$$

are the coefficients of the mth partial Fourier sum of a function f. Obviously $V_m^{\rho,\sigma} f \in \mathbb{P}_{2m-1}$ and if $f \in \mathbb{P}_m$ then $V_m^{\rho,\sigma} f = f$.

Now, assuming that $0 < \alpha < 1$ and the parameters γ and δ satisfy (5), using [11, Theorem 2.2] (see also [2, Theorem 2.1]), we deduce

$$\|v^{\gamma,\alpha+\delta} V_m^{-\alpha,\alpha}(f)\|_\infty \leq \mathcal{C} \|v^{\gamma,\alpha+\delta} f\|_\infty, \quad \forall f \in C_{v^{\gamma,\alpha+\delta}}, \tag{32}$$

and

$$\|v^{\alpha+\gamma,\delta} V_m^{\alpha,-\alpha}(f)\|_\infty \leq \mathcal{C} \|v^{\alpha+\gamma,\delta} f\|_\infty, \quad \forall f \in C_{v^{\alpha+\gamma,\delta}},$$

where $\mathcal{C} \neq \mathcal{C}(m, f)$. Moreover, under the previous assumptions on the parameters α, γ, δ and for $r > 0$, we get

$$c_j^{\alpha,-\alpha}(D^{-\alpha,\alpha} f) = c_j^{-\alpha,\alpha}(f), \quad \forall f \in Z_r(v^{\gamma,\alpha+\delta}), \tag{33}$$

and

$$c_j^{-\alpha,\alpha}(D^{\alpha,-\alpha} f) = c_j^{\alpha,-\alpha}(f), \quad \forall f \in Z_r(v^{\alpha+\gamma,\delta}). \tag{34}$$

In order to prove (7), since (see [3, Proposition 3.1])

$$\lim_m E_m(D^{-\alpha,\alpha} f)_{v^{\gamma,\alpha+\delta}} = 0,$$

by (32), we get

$$E_{2m}(D^{\alpha,-\alpha} f)_{v^{\gamma,\alpha+\delta}} \leq \|[D^{\alpha,-\alpha} f - V_m^{-\alpha,\alpha}(D^{\alpha,-\alpha} f)] v^{\gamma,\alpha+\delta}\|_\infty$$

$$\leq \sum_{i=0}^{\infty} \|[V_{2^{i+1}m}^{-\alpha,\alpha}(D^{\alpha,-\alpha} f) - V_{2^i m}^{-\alpha,\alpha}(D^{\alpha,-\alpha} f)] v^{\gamma,\alpha+\delta}\|_\infty.$$

Now, setting $N = 2^i m$, by (29), (31) and (34), we deduce the following integral representation of $V_N^{-\alpha,\alpha}(D^{\alpha,-\alpha}f)$:

$$V_N^{-\alpha,\alpha}(D^{\alpha,-\alpha}f, x) = \int_{-1}^{1} H_N^*(x, y)f(y)v^{\alpha,-\alpha}(y)dy,$$

where

$$H_N^*(x, y) = \sum_{j=0}^{2N} \mu_j p_j^{\alpha,-\alpha}(y)p_j^{-\alpha,\alpha}(x),$$

with $\mu_j, j = 0, \ldots, 2N$, defined in (30). Then, denoting by $P_N \in \mathbb{P}_N$ the polynomial of best approximation of $f \in C_{v^{\alpha+\gamma,\delta}}$ and applying the Remez-type inequality (see [5])

$$\|Qv^{\rho,\sigma}\|_p \leq \mathcal{C}\|Qv^{\rho,\sigma}\|_{L^p\left(\left[-1+\frac{c}{m^2}, 1-\frac{c}{m^2}\right]\right)}, \quad \forall Q \in \mathbb{P}_m, \ 1 \leq p \leq +\infty, \ m > \sqrt{C},$$

with $p = \infty$, for the generic term of the above series we have

$$\|[V_{2N}^{-\alpha,\alpha}(D^{\alpha,-\alpha}f) - V_N^{-\alpha,\alpha}(D^{\alpha,-\alpha}f)]v^{\gamma,\alpha+\delta}\|_\infty$$
$$= \|[V_{2N}^{-\alpha,\alpha}(D^{\alpha,-\alpha}(f - P_N)) - V_N^{-\alpha,\alpha}(D^{\alpha,-\alpha}(f - P_N))]v^{\gamma,\alpha+\delta}\|_\infty$$
$$\leq E_N(f)_{v^{\alpha+\gamma,\delta}} \max_{x \in \left[-1+\frac{c}{N^2}, 1-\frac{c}{N^2}\right]} v^{\gamma,\alpha+\delta}(x) \int_{-1+\frac{c}{N^2}}^{1-\frac{c}{N^2}} |H_{2N}^*(x, y) - H_N^*(x, y)|v^{-\gamma,-\alpha-\delta}(y)dy.$$

Finally, since (see [3, Lemma 4.4])

$$\max_{x \in \left[-1+\frac{c}{N^2}, 1-\frac{c}{N^2}\right]} v^{\gamma,\alpha+\delta}(x) \int_{-1+\frac{c}{N^2}}^{1-\frac{c}{N^2}} |H_{2N}^*(x, y) - H_N^*(x, y)|v^{-\gamma,-\alpha-\delta}(y)dy \leq \mathcal{C},$$

where $\mathcal{C} \neq \mathcal{C}(N)$, by (4) we get

$$E_{2m}(D^{\alpha,-\alpha}f)_{v^{\gamma,\alpha+\delta}} \leq \mathcal{C} \sum_{i=0}^{\infty} E_{2^i m}(f)_{v^{\alpha+\gamma,\delta}} \leq \mathcal{C} \sum_{i=0}^{\infty} \omega_\varphi^k\left(f, \frac{1}{2^i m}\right)_{v^{\alpha+\gamma,\delta}}$$
$$\sim \mathcal{C} \int_0^{\frac{1}{2m}} \frac{\omega_\varphi^k(f, t)_{v^{\alpha+\gamma,\delta}}}{t} dt,$$

i.e., (7). The proof of (6) is similar.

Concerning Corollary 2.1, we first prove the invertibility of $D^{\alpha,-\alpha}$ and that its two-sided inverse is $D^{-\alpha,\alpha}$. By (32), (29), (34) and (33), we have

$$v^{\gamma,\alpha+\delta}(x)(D^{\alpha,-\alpha}D^{-\alpha,\alpha}f)(x) = \lim_m v^{\gamma,\alpha+\delta}(x)V_m^{-\alpha,\alpha}(D^{\alpha,-\alpha}D^{-\alpha,\alpha}f, x)$$
$$= \lim_m v^{\gamma,\alpha+\delta}(x) \sum_{j=0}^{2m} \mu_j c_j^{-\alpha,\alpha}(D^{\alpha,-\alpha}D^{-\alpha,\alpha}f)p_j^{-\alpha,\alpha}(x)$$
$$= \lim_m v^{\gamma,\alpha+\delta}(x) \sum_{j=0}^{2m} \mu_j c_j^{\alpha,-\alpha}(D^{-\alpha,\alpha}f)p_j^{-\alpha,\alpha}(x)$$
$$= \lim_m v^{\gamma,\alpha+\delta}(x) \sum_{j=0}^{2m} \mu_j c_j^{-\alpha,\alpha}(f)p_j^{-\alpha,\alpha}(x)$$
$$= \lim_m v^{\gamma,\alpha+\delta}(x)V_m^{-\alpha,\alpha}(f, x) = v^{\gamma,\alpha+\delta}(x)f(x),$$

then $D^{-\alpha,\alpha}$ is the right-inverse of $D^{\alpha,-\alpha}$. Analogously it is possible to prove that $D^{-\alpha,\alpha}$ is also the left-inverse of $D^{\alpha,-\alpha}$.

Moreover, we note that by (4) and [5]

$$\Omega_\varphi^k\left(f, \frac{1}{m}\right)_{v^{\rho,\sigma}} \leq \frac{\mathcal{C}}{m^k} \sum_{i=0}^{m} (1+i)^{k-1} E_i(f)_{v^{\rho,\sigma}},$$

where $\mathcal{C} \neq \mathcal{C}(f, m)$, it is possible to deduce that

$$\|f\|_{Z_r(v^{\rho,\sigma})} \sim \|fv^{\rho,\sigma}\|_\infty + \sup_{i \geq 0}(1+i)^r E_i(f)_{v^{\rho,\sigma}}, \tag{35}$$

where the constants in "$\sim$" are independent of f.

Therefore, since, applying [12, Theorem 3.1] (with $w_1 = v^{\alpha+\gamma,\delta}$ and $w_2 = v^{\gamma,\alpha+\delta}$), we get

$$
\begin{aligned}
\|v^{\gamma,\alpha+\delta}D^{\alpha,-\alpha}f\|_\infty &\leq \mathcal{C}\|fv^{\alpha+\gamma,\delta}\|_\infty + \int_0^1 \frac{\omega_\varphi^k(f,t)_{v^{\alpha+\gamma,\delta}}}{t}\,dt \\
&\leq \mathcal{C}\|f\|_{Z_r(v^{\alpha+\gamma,\delta})}
\end{aligned}
$$

and, using (7), we obtain

$$
\begin{aligned}
\sup_{i\geq 0}(1+i)^r E_i(D^{\alpha,-\alpha}f)_{v^{\gamma,\alpha+\delta}} &\leq \mathcal{C}\|f\|_{Z_r(v^{\alpha+\gamma,\delta})}\sup_{i\geq 0}\left(\frac{1+i}{i}\right)^r \\
&\leq \mathcal{C}\|f\|_{Z_r(v^{\alpha+\gamma,\delta})},
\end{aligned}
$$

by (35), we deduce

$$
\|D^{\alpha,-\alpha}f\|_{Z_r(v^{\gamma,\alpha+\delta})} \leq \mathcal{C}\|f\|_{Z_r(v^{\alpha+\gamma,\delta})}, \tag{36}
$$

where $\mathcal{C} \neq \mathcal{C}(f)$. In an analogous way, it is also possible to deduce that

$$
\|D^{-\alpha,\alpha}f\|_{Z_r(v^{\alpha+\gamma,\delta})} \leq \mathcal{C}\|f\|_{Z_r(v^{\gamma,\alpha+\delta})}, \quad \mathcal{C} \neq \mathcal{C}(f). \tag{37}
$$

Now, as a consequence of the invertibility and of the bounds (36)–(37), we get

$$
\|f\|_{Z_r(v^{\alpha+\gamma,\delta})} = \|D^{-\alpha,\alpha}D^{\alpha,-\alpha}f\|_{Z_r(v^{\alpha+\gamma,\delta})} \leq \mathcal{C}\|D^{\alpha,-\alpha}f\|_{Z_r(v^{\gamma,\alpha+\delta})}
$$

and

$$
\|f\|_{Z_r(v^{\gamma,\alpha+\delta})} = \|D^{\alpha,-\alpha}D^{-\alpha,\alpha}f\|_{Z_r(v^{\gamma,\alpha+\delta})} \leq \mathcal{C}\|D^{-\alpha,\alpha}f\|_{Z_r(v^{\alpha+\gamma,\delta})}
$$

and then the equivalences (9)–(10) follow.

Proof of Lemma 2.1. Assume that the kernel k satisfies (11).

It is easy to verify that the operator $K^{\alpha,-\alpha}$ is continuous in the couple of spaces $(Z_r(v^{\alpha+\gamma,\delta}), Z_r(v^{\gamma,\alpha+\delta}))$. Moreover, if we consider the sequence of the finite dimensional operators $\{K_m^{\alpha,-\alpha}\}_m$ defined in (15), we can complete the proof by proving that

$$
\|K^{\alpha,-\alpha} - K_m^{\alpha,-\alpha}\|_{Z_r(v^{\alpha+\gamma,\delta})\to Z_r(v^{\gamma,\alpha+\delta})} \leq \mathcal{C}\,\frac{\log m}{m^{s-r}}, \quad \mathcal{C} \neq \mathcal{C}(m). \tag{38}
$$

The proof of (38) can be found in the Appendix.

In the case where the kernel k satisfies (12), if we prove that

$$
\|v^{\gamma,\alpha+\delta}K^{\alpha,-\alpha}f\|_\infty \leq \mathcal{C}\|f\|_{Z_r(v^{\alpha+\gamma,\delta})} \tag{39}
$$

and

$$
E_m(K^{\alpha,-\alpha}f)_{v^{\gamma,\alpha+\delta}} \leq \mathcal{C}\|f\|_{Z_r(v^{\alpha+\gamma,\delta})}\frac{\log m}{m^s}, \tag{40}
$$

the boundedness of $K^{\alpha,-\alpha} : Z_r(v^{\gamma,\alpha+\delta}) \to Z_r(v^{\alpha+\gamma,\delta})$ follows by (35). The proofs of the above bounds imply some technical details that we will give in the Appendix.

Finally, using (35) together with (40), we obtain

$$
E_m(K^{\alpha,-\alpha}f)_{Z_r(v^{\gamma,\alpha+\delta})} \leq \mathcal{C}\sup_m m^r E_m(K^{\alpha,-\alpha}f)_{v^{\gamma,\alpha+\delta}} \leq \mathcal{C}\|f\|_{Z_r(v^{\alpha+\gamma,\delta})}\frac{\log m}{m^{s-r}}
$$

and, taking into account [13, p. 44], we deduce the compactness of the operator $K^{\alpha,-\alpha} : Z_r(v^{\alpha+\gamma,\delta}) \to Z_r(v^{\gamma,\alpha+\delta})$ for $r < s$ also when the kernel k satisfies (12). $\square$

Proof of Lemma 2.2. The proof is similar to the one of Lemma 2.1. $\square$

Proof of Theorem 3.1. Taking into account that

$$
(D^{\alpha,-\alpha} + K^{\alpha,-\alpha})f_m = (K^{\alpha,-\alpha} - K_m^{\alpha,-\alpha})f_m + (D^{\alpha,-\alpha} + K_m^{\alpha,-\alpha})f_m,
$$

we obtain

$$
\|f_m\|_{Z_r(v^{\alpha+\gamma,\delta})} \leq \|(D^{\alpha,-\alpha} + K^{\alpha,-\alpha})^{-1}\|_{Z_r(v^{\gamma,\alpha+\delta}) \to Z_r(v^{\gamma,\alpha+\delta})}
$$
$$
\times \left[\|(K^{\alpha,-\alpha} - K_m^{\alpha,-\alpha})f_m\|_{Z_r(v^{\gamma,\alpha+\delta})} + \|(D^{\alpha,-\alpha} + K_m^{\alpha,-\alpha})f_m\|_{Z_r(v^{\gamma,\alpha+\delta})} \right].
$$

Afterwards, using (38), we get

$$
\mathcal{C}\|f_m\|_{Z_r(v^{\alpha+\gamma,\delta})} \leq \|(D^{\alpha,-\alpha} + K_m^{\alpha,-\alpha})f_m\|_{Z_r(v^{\gamma,\alpha+\delta})},
$$

where, for $s > r$,

$$
\mathcal{C} \leq \|(D^{\alpha,-\alpha} + K^{\alpha,-\alpha})^{-1}\|_{Z_r(v^{\gamma,\alpha+\delta}) \to Z_r(v^{\alpha+\gamma,\delta})}^{-1} - \mathcal{O}\left(\frac{\log m}{m^{s-r}}\right).
$$

Consequently, denoting by $im L_m^{\alpha,\beta}$ the range of the operator $L_m^{\alpha,\beta}$, we deduce that $D^{\alpha,-\alpha} + K_m^{\alpha,-\alpha} : im L_m^{\alpha,-\alpha} \to im L_m^{-\alpha,\alpha}$ is invertible and, for a sufficiently large m (say $m > m_0$), $(D^{\alpha,-\alpha} + K_m^{\alpha,-\alpha})f = g_m$ has a unique solution $f_m \in im L_m^{\alpha,-\alpha}$.

Now we prove (20). By (16) and recalling that [14, p. 353, Eq. 15.3.10]

$$
\lambda_j^{\alpha,-\alpha} \sim \Delta t_j v^{\alpha,-\alpha}(t_j), \quad \Delta t_j = t_{j+1} - t_j, \tag{41}
$$

we get

$$
\|A_m\|_\infty \leq \mathcal{C} \max_{i=1,\dots,m} v^{\gamma,\alpha+\delta}(x_i) \left[\sum_{j=1}^m \frac{\lambda_j^{\alpha,-\alpha}}{v^{\alpha+\gamma,\delta}(t_j)|x_i - t_j|} + \sum_{j=1}^m \frac{\lambda_j^{\alpha,-\alpha}}{v^{\alpha+\gamma,\delta}(t_j)} |k(x_i, t_j)| \right]
$$
$$
\leq \mathcal{C} \max_{i=1,\dots,m} v^{\gamma,\alpha+\delta}(x_i) \left[\sum_{j=1}^m \frac{\Delta t_j}{|x_i - t_j|} v^{-\gamma,-\alpha-\delta}(t_j) + \sum_{j=1}^m \Delta t_j v^{-\gamma,-\alpha-\delta}(t_j)|k(x_i, t_j)| \right].
$$

Moreover, taking into account that, by virtue of (18), we have (see, for instance, [15, (5.16)])

$$
\sum_{j=1}^m \frac{\Delta t_j}{|x_i - t_j|} v^{-\gamma,-\alpha-\delta}(t_j) \leq \mathcal{C} v^{-\gamma,-\alpha-\delta}(x_i) \log m, \tag{42}
$$

we obtain

$$
\|A_m\|_\infty \leq \mathcal{C} \log m + \mathcal{C} \left(\int_{-1}^1 v^{-\gamma,-\alpha-\delta}(x)dx \right) \sup_{|x| \leq 1} v^{\gamma,\alpha+\delta}(x)\|k_x\|_\infty,
$$

from which we deduce

$$
\|A_m\|_\infty \leq \mathcal{C} \log m, \tag{43}
$$

using the assumptions on k and the parameters γ, δ. Now we estimate $\|A_m^{-1}\|_\infty$. By virtue of the equivalence of the system (16) with the equation $(D^{\alpha,-\alpha} + K_m^{\alpha,-\alpha})f_m = g_m$, for every $\theta = (\theta_1, \dots, \theta_m)$ there exists a unique $\xi = (\xi_1, \dots, \xi_m)$ such that $A_m^{-1}\theta = \xi$ if and only if $(D^{\alpha,-\alpha} + K_m^{\alpha,-\alpha})^{-1}\tilde{\theta}(x) = \tilde{\xi}(x)$, where

$$
\tilde{\theta}(x) = \sum_{j=1}^m \varphi_j^{-\alpha,\alpha}(x)\theta_j, \quad \theta_j = (\tilde{\theta} v^{\gamma,\alpha+\delta})(x_j)
$$

and

$$
\tilde{\xi}(x) = \sum_{j=1}^m \varphi_j^{\alpha,-\alpha}(x)\xi_j, \quad \xi_j = (\tilde{\xi} v^{\alpha+\gamma,\delta})(t_j).
$$

Then, for all θ, we get

$$
\|A_m^{-1}\theta\|_{l^\infty} = \|\xi\|_{l^\infty} \leq \|\tilde{\xi} v^{\alpha+\gamma,\delta}\|_\infty = \|(D^{\alpha,-\alpha} + K_m^{\alpha,-\alpha})^{-1}\tilde{\theta} v^{\alpha+\gamma,\delta}\|_\infty
$$
$$
\leq \|(D^{\alpha,-\alpha} + K_m^{\alpha,-\alpha})_{|\mathbb{P}_{m-1}}^{-1}\|_{C_{v^{\gamma,\alpha+\delta}} \to C_{v^{\alpha+\gamma,\delta}}} \|\theta\|_{l^\infty} \|L_m^{-\alpha,\alpha}\|_{C_{v^{\gamma,\alpha+\delta}} \to C_{v^{\alpha+\gamma,\delta}}}.
$$

Then, recalling that, for $0 < \alpha < 1$ and γ, δ satisfying (5), we have

$$\|v^{\gamma,\alpha+\delta}(L_m^{-\alpha,\alpha}f)\|_\infty \leq \mathcal{C}(\log m)\|fv^{\gamma,\alpha+\delta}\|_\infty, \quad \forall f \in C_{v^{\gamma,\alpha+\delta}}, \tag{44}$$

where $\mathcal{C} \neq \mathcal{C}(m,f)$, we obtain

$$\|A_m^{-1}\|_\infty \leq \mathcal{C} \log m. \tag{45}$$

Now, combining (43) with (45), (20) follows.

It remains to prove (21). To this end we use the following identity

$$(f - f_m) = (D^{\alpha,-\alpha} + K^{\alpha,-\alpha})^{-1} \left[(g - g_m) - (K^{\alpha,-\alpha} - K_m^{\alpha,-\alpha})f_m \right].$$

Now, by (35), (44) and the assumptions on g, we get

$$\|g - g_m\|_{Z_r(v^{\gamma,\alpha+\delta})} \leq \mathcal{C}\frac{\log m}{m^{s-r}}\|g\|_{Z_s^\infty(v^{\gamma,\alpha+\delta})}.$$

Finally, using (38), (21) follows. □

Proof of Theorem 3.2. The proof is similar to the one of Theorem 3.1, taking into account that

$$\|(K^{\alpha,-\alpha} - K_m^{\alpha,-\alpha})f_m\|_{Z_r(v^{\gamma,\alpha+\delta})} \leq \mathcal{C}\|f_m\|_{Z_r(v^{\alpha+\gamma,\delta})}\frac{\log^2 m}{m^{s-r}}, \tag{46}$$

where $\mathcal{C} \neq \mathcal{C}(m,f_m)$. The proof of (46) is given in the Appendix. □

Proof of Theorem 3.3. The proof runs as the one of Theorem 3.1. □

5. Numerical examples

In this section we show by some examples that our theoretical results are confirmed by the numerical tests.

In particular, according to (20) and (26), the systems (16), (17), (24) and (24) are well-conditioned except for a logarithmic factor and, by virtue of the estimates (21)–(22) and (27)–(28), the smoother are the kernels and the right-hand sides and the higher is the convergence order of the weighted approximation errors.

We point out that the singular integral equations considered in the examples are of the form (2) or (3) with $0 < \alpha < 1$. We recall that, by contrast, the numerical methods proposed in [4] impose strong restrictions on the choice of α.

In all the tables that follow we show the values, in two different points, of the weighed approximate solutions uf_m, where $u = v^{\alpha+\gamma,\delta}$ if $\chi = 0$ and $u = v^{\gamma,\delta}$ if $\chi = 1$, for increasing values of m. When we do not know the exact solution, for each m we bolded the digits that are exact with respect to $uf_{\bar{m}}$, with $\bar{m} > m$.

Example 5.1. We first consider the equation

$$\left(D^{\frac{3}{4},-\frac{3}{4}}f\right)(x) + \frac{1}{2}\int_{-1}^1 \log(x+2)y^3f(y)\left(\frac{1-y}{1+y}\right)^{\frac{3}{4}}dy = 1 - \frac{123\pi\log(x+2)}{128\sqrt{2}}$$

of index 0 whose exact solution is the function $f(x) = 1$.

Since $\alpha = \frac{3}{4}$, taking into account (5), we choose $\gamma = \delta = 0$. In this case both the kernel and the right-hand side are very smooth and, according to the estimate (21), the convergence of the weighted approximate solutions $v^{\frac{3}{4},0}f_m$ is very fast: it is sufficient to take $m = 4$ to get approximations with 15 exact decimal digits. The condition numbers in uniform norm of the matrices A_m of the solved linear systems (16) are less than 8.

Example 5.2. Now we take the integral equation

$$\left(D^{\frac{1}{5},-\frac{1}{5}}f\right)(x) + \frac{1}{5}\int_{-1}^1 \cos(x)\frac{\left|y-\frac{1}{2}\right|^{\frac{7}{2}} - \left|x-\frac{1}{2}\right|^{\frac{7}{2}}}{(y-x)}f(y)\left(\frac{1-y}{1+y}\right)^{\frac{1}{5}}dy = e^x\sin(x).$$

It has index 0 and its exact solution is unknown.

By virtue of (5), we take $\gamma = \delta = \frac{3}{20}$, being $\alpha = \frac{1}{5}$. In this case the kernel $k(x,y) = \cos(x)\frac{\left|y-\frac{1}{2}\right|^{\frac{7}{2}} - \left|x-\frac{1}{2}\right|^{\frac{7}{2}}}{(y-x)}$ is not much smooth, in fact, as announced at the end of Section 2, it satisfies (12) with $h(x,y) = \cos(x)\left|y-\frac{1}{2}\right|^{\frac{7}{2}}$ and $s = \frac{7}{2}$. Then, according to the estimate (22), we need to increase m to get exact decimal digits. In particular, as shown in Table 1, at the points $x = 0.1$ and $x = 0.4$ with $m = 500$ we reach at least 11 significant digits. The matrices A_m of the solved linear systems (17) have condition numbers less than 30.

Table 1

Example 5.2. Weighted approximate solutions $\left(v^{\frac{7}{20},\frac{3}{20}}f_m\right)(x)$.

m	$\left(v^{\frac{7}{20},\frac{3}{20}}f_m\right)(0.1)$	$\left(v^{\frac{7}{20},\frac{3}{20}}f_m\right)(0.4)$
10	**0.89**09744734332579	**1.22394**2090159346
50	**0.8910113**412791550	**1.223946**987916056
100	**0.8910112522**979449	**1.223947380**327240
200	**0.89101125088**04216	**1.223947378**443911
400	**0.8910112509**049636	**1.22394737897**8747
500	**0.8910112508943**384	**1.223947378966**167
600	**0.8910112508957**724	**1.223947378966**294

Table 2

Example 5.3. Weighted approximate solutions $\left(v^{\frac{1}{12},0}f_m\right)(x)$.

m	$\left(v^{\frac{1}{12},0}f_m\right)(0.5)$	$\left(v^{\frac{1}{12},0}f_m\right)(-0.3)$
10	**0.9407668**784351797	**0.296018**9055794727
20	**0.940766895583648**9	**0.2960188852484**730
30	**0.9407668955836**422	**0.2960188852484**725

Table 3

Example 5.4. Weighted approximate solutions $\left(v^{0,\frac{1}{8}}f_m\right)(x)$.

m	$\left(v^{0,\frac{1}{8}}f_m\right)(0.2)$	$\left(v^{0,\frac{1}{8}}f_m\right)(-0.5)$
10	$-$**0.4644119**973638529	$-$**0.7513**486578874185
50	$-$**0.464411881**0678438	$-$**0.7513490891**169037
100	$-$**0.46441188809**846486	$-$**0.7513490894**543699
300	$-$**0.46441188809**806697	$-$**0.7513490894**696305
500	$-$**0.4644118809807**871	$-$**0.7513490894698**759
600	$-$**0.4644118809809**778	$-$**0.7513490894695**284

Example 5.3. Let us consider the following equation

$$\left(D^{-\frac{1}{3},-\frac{2}{3}}f\right)(x) + \frac{1}{3}\int_{-1}^{1}(x+y)\sin(x+y)f(y)v^{-\frac{1}{3},-\frac{2}{3}}(y)dy = \cos(x)$$

of index $\chi = 1$ whose exact solution is unknown.

Here $\alpha = \frac{1}{3}$ and, taking into account (8), we take $\gamma = \frac{1}{12}$ and $\delta = 0$. Since both the kernel and the right-hand side are very smooth, according to the estimate (27), we need to solve a linear system of order only 20 to get approximations of the solution with 14 exact decimal digit (see Table 2). The condition numbers of the matrices of the solved linear systems (24) are less than 13.

Example 5.4. Finally, we take the equation

$$\left(D^{-\frac{3}{4},-\frac{1}{4}}f\right)(x) + \int_{-1}^{1}\frac{\cos(x-y)|x-y|^{\frac{9}{2}}}{(y-x)(3+x+y^3)^4}f(y)v^{-\frac{3}{4},-\frac{1}{4}}(y)dy = \sin(1+x).$$

Its index is $\chi = 1$ and we do not know its exact solution.

Since $\alpha = \frac{3}{4}$, by (8), we take $\gamma = 0$ and $\delta = \frac{1}{8}$. As you can see in Table 3, taking $m = 500$ we get approximations of the weighted solution $v^{0,\frac{1}{8}}f$ at the points $x = 0.2$ and $x = -0.5$ with 12 exact decimal digits. This agrees with the theoretical expectations, in fact, since $k(x,y) = \frac{\cos(x-y)|x-y|^{\frac{9}{2}}}{(y-x)(3+x+y^3)^4}$ fulfills (12) with $h(x,y) = \frac{\cos(x-y)|x-y|^{\frac{9}{2}}}{(3+x+y^3)^4}$ and $s = \frac{9}{2}$, by (28), the convergence order in $C_{v^{0,\frac{1}{8}}}$ is $\frac{\log^2 m}{m^{\frac{9}{2}}}$.

Note that $k(x,y)$ satisfies also (13) with $s = \frac{7}{2}$, but, the numerical results in Table 3 show that the estimate (28) is more accurate than (27).

Also in this case the matrices of the linear systems (25) that we solve have small condition numbers, in fact, they are less than 18.

Acknowledgment

Appendix

Proof of (38). With $P_m(x, y)$ a polynomial of degree m with respect to both the variables separately, we have

$$(K^{\alpha,-\alpha} - K_m^{\alpha,-\alpha})f(x) = \int_{-1}^{1} [k(x,y) - P_m(x,y)]f(y)v^{\alpha,-\alpha}(y)dy$$
$$- L_m^{-\alpha,\alpha} \left(\int_{-1}^{1} L_m^{\alpha,-\alpha}([k_x - P_m(x,\cdot)], y)\, (fv^{\alpha,-\alpha})(y)dy, x \right),$$

being

$$\int_{-1}^{1} P_m(x,y)f(y)v^{\alpha,-\alpha}(y)dy - L_m^{-\alpha,\alpha}\left(\int_{-1}^{1} L_m^{\alpha,-\alpha}(P_m(x,\cdot), y)\,(fv^{\alpha,-\alpha})(y)dy, x \right) = 0.$$

Then, with $R(x,y) = k(x,y) - P_m(x,y)$, $P_{m,x}(y) = P_m(x,y)$ and $R_x(y) = R(x,y)$, we get

$$|v^{\gamma,\alpha+\delta}(x)(K^{\alpha,-\alpha} - K_m^{\alpha,-\alpha})f(x)| \leq \left| v^{\gamma,\alpha+\delta}(x) \int_{-1}^{1} R(x,y)f(y)v^{\alpha,-\alpha}(y)dy \right|$$
$$+ \left| v^{\gamma,\alpha+\delta}(x) L_m^{-\alpha,\alpha}\left(\int_{-1}^{1} L_m^{\alpha,-\alpha}(R_x, y)\,(fv^{\alpha,-\alpha})(y)dy, x \right) \right|$$
$$=: A + B.$$

We have

$$A \leq \|fv^{\alpha+\gamma,\delta}\|_\infty v^{\gamma,\alpha+\delta}(x) \int_{-1}^{1} |R(x,y)| v^{-\gamma,-\alpha-\delta}(y)dy$$
$$\leq \|fv^{\alpha+\gamma,\delta}\|_\infty \sup_{|x|\leq 1} v^{\gamma,\alpha+\delta}(x) \|k_x - P_{m,x}\|_\infty \int_{-1}^{1} v^{-\gamma,-\alpha-\delta}(y)dy$$

and, under the assumptions (5), we get

$$A \leq \mathcal{C}\|fv^{\alpha+\gamma,\delta}\|_\infty \sup_{|x|\leq 1} v^{\gamma,\alpha+\delta}(x)\|k_x - P_{m,x}\|_\infty.$$

Assuming the infimum on $P_{m,x}$ and taking into account that by [5]

$$E_m(f)_{v^{\rho,\sigma}} \leq \mathcal{C} \int_0^{\frac{1}{m}} \frac{\Omega_\varphi^k(f,t)_{v^{\rho,\sigma}}}{t}dt, \quad \mathcal{C} \neq \mathcal{C}(m,f),$$

for all $f \in Z_s(v^{\rho,\sigma})$ with $s > 0$, we have

$$E_m(f)_{v^{\rho,\sigma}} \leq \frac{\mathcal{C}}{m^s} \|f\|_{Z_s(v^{\rho,\sigma})}, \quad \mathcal{C} \neq \mathcal{C}(f,m), \tag{A.1}$$

we obtain

$$A \leq \frac{\mathcal{C}}{m^s} \|fv^{\alpha+\gamma,\delta}\|_\infty \sup_{|x|\leq 1} v^{\gamma,\alpha+\delta}(x) \|k_x\|_{Z_s}.$$

Now we estimate B. Using (44) and [16, Theorem 1], we get

$$B \leq \mathcal{C}(\log m) \sup_{|x|\leq 1} v^{\gamma,\alpha+\delta}(x) \left| \int_{-1}^{1} L_m^{\alpha,-\alpha}(R_x, y)\,(fv^{\alpha,-\alpha})(y)dy \right|$$
$$\leq \mathcal{C}(\log m)\|fv^{\alpha+\gamma,\delta}\|_\infty \sup_{|x|\leq 1} v^{\gamma,\alpha+\delta}(x) \int_{-1}^{1} |L_m^{\alpha,-\alpha}(R_x, y)| v^{-\gamma,-\alpha-\delta}(y)dy$$
$$\leq \mathcal{C}(\log m)\|fv^{\alpha+\gamma,\delta}\|_\infty \sup_{|x|\leq 1} v^{\gamma,\alpha+\delta}(x)\|k_x - P_{m,x}\|_\infty.$$

Moreover, taking the infimum on $P_{m,x}$ and using (A.1), we have

$$B \le \mathcal{C}\,\frac{\log m}{m^s}\|fv^{\alpha+\gamma,\delta}\|_\infty \sup_{|x|\le 1} v^{\gamma,\alpha+\delta}(x)\|k_x\|_{Z_s}.$$

Summing up, under the assumption on k, we conclude that

$$\|v^{\gamma,\alpha+\delta}(K^{\alpha,-\alpha}-K_m^{\alpha,-\alpha})f\|_\infty \le \mathcal{C}\|f\|_{Z_r(v^{\alpha+\gamma,\delta})}\frac{\log m}{m^s}.$$

Finally, since taking into account the equivalence (35) we get

$$\|(K^{\alpha,-\alpha}-K_m^{\alpha,-\alpha})f\|_{Z_r(v^{\gamma,\alpha+\delta})} \le \mathcal{C}\sup_m m^r\|v^{\gamma,\alpha+\delta}(K^{\alpha,-\alpha}-K_m^{\alpha,-\alpha})f\|_\infty,$$

the thesis follows. $\square$

Proof of (39). We write

$$v^{\gamma,\alpha+\delta}(x)(K^{\alpha,-\alpha}f)(x) = v^{\gamma,\alpha+\delta}(x)\left\{\int_{|x-y|>\frac{1+x}{4}}+\int_{x-\frac{1+x}{4}}^{x+\frac{1+x}{4}}\right\}\frac{h(x,y)-h(x,x)}{y-x}(fv^{\alpha,-\alpha})(y)dy$$

$$:= I_1 + I_2. \tag{A.2}$$

We have

$$|I_1| \le \mathcal{C}\|fv^{\alpha+\gamma,\delta}\|_\infty \sup_{|x|\le 1}\|h_x\|_\infty v^{\gamma,\alpha+\delta}(x)\left\{\int_{-1}^{x-\frac{1+x}{4}}+\int_{x+\frac{1+x}{4}}^1\right\}\frac{v^{-\gamma,-\alpha-\delta}(y)}{|y-x|}dy$$

$$=: \mathcal{C}\|fv^{\alpha+\gamma,\delta}\|_\infty \sup_{|x|\le 1}\|h_x\|_\infty\{I_1'+I_1''\}. \tag{A.3}$$

Concerning I_1', since $1-x<1-y$ and $|y-x|>\frac{1+x}{4}$, we have

$$I_1' \le \mathcal{C}(1+x)^{\alpha+\delta-1}\int_{-1}^{x-\frac{1+x}{4}}(1+y)^{-\alpha-\delta} \le \mathcal{C}. \tag{A.4}$$

In order to estimate I_1'' we consider two different cases: $x \ge \frac{1}{2}$ and $x < \frac{1}{2}$. For $x \ge \frac{1}{2}$ and $x+\frac{1+x}{4} \le y \le 1$, we have $1+x<1+y$ and $y-x \sim 1-x$, then it is easy to deduce

$$I_1'' \le \mathcal{C}(1-x)^{\gamma-1}\int_{x+3\frac{1-x}{4}}^1(1-y)^{-\gamma}dy \le \mathcal{C}.$$

For $x < \frac{1}{2}$, we write

$$I_1'' = \mathcal{C}v^{\gamma,\alpha+\delta}(x)\left\{\int_{x+\frac{1+x}{4}}^{1-\frac{1-x}{4}}+\int_{1-\frac{1-x}{4}}^1\right\}\frac{v^{-\gamma,-\alpha-\delta}(y)}{y-x}dy$$

$$=: A + B.$$

Since for $x+\frac{1+x}{4} \le y \le 1-\frac{1-x}{4}$ one has $1-x \sim 1-y$ and $y-x>\frac{1+x}{4}$, under the assumption (5), we get

$$A \le \mathcal{C}(1+x)^{\alpha+\delta-1}\int_{x+\frac{1+x}{4}}^1(1-y)^{-\alpha-\delta}dy \le \mathcal{C},$$

while, since for $1-\frac{1-x}{4} \le y \le 1$ one has $y-x \sim 1-x$ and $1+x<1+y$, we obtain

$$B \le \mathcal{C}(1-x)^{\gamma-1}\int_{1-\frac{1-x}{4}}^1(1-y)^{-\gamma}dy \le \mathcal{C}.$$

Then

$$I_1'' \le \mathcal{C} \tag{A.5}$$

for $x < \frac{1}{2}$, too.

Now, combining (A.3) with (A.4)–(A.5), we deduce

$$|I_1| \le \mathcal{C}\|fv^{\alpha+\gamma,\delta}\|_\infty \sup_{|x|\le 1}\|h_x\|_\infty. \tag{A.6}$$

It remains to estimate I_2. We will distinguish between the two cases $x \leq \frac{1}{2}$ and $x > \frac{1}{2}$. For $x \leq \frac{1}{2}$, we have $1 \pm x \sim 1 \pm y$. Then

$$|I_2| \leq \|f v^{\alpha+\gamma,\delta}\|_\infty v^{\gamma,\alpha+\delta}(x) \int_{x-\frac{1+x}{4}}^{x+\frac{1+x}{4}} \left| \frac{h(x,y) - h(x,x)}{y-x} \right| v^{-\gamma,-\alpha-\delta}(y) dy$$

$$\leq C \|f v^{\alpha+\gamma,\delta}\|_\infty \int_{x-\frac{1+x}{4}}^{x+\frac{1+x}{4}} \left| \frac{h(x,y) - h(x,x)}{y-x} \right| dy.$$

For $x > \frac{1}{2}$ we write

$$|I_2| \leq \|f v^{\alpha+\gamma,\delta}\|_\infty v^{\gamma,\alpha+\delta}(x) \left\{ \int_{x-\frac{1+x}{4}}^{x-\frac{3}{4}(1-x)} + \int_{x-\frac{3}{4}(1-x)}^{x+\frac{3}{4}(1-x)} + \int_{\frac{3}{4}(1-x)}^{x-\frac{1+x}{4}} \right\} \left| \frac{h(x,y) - h(x,x)}{y-x} \right| v^{-\gamma,-\alpha-\delta}(y) dy$$

$$=: \|f v^{\alpha+\gamma,\delta}\|_\infty \{I_2' + I_2'' + I_2'''\}. \tag{A.7}$$

Concerning I_2' and I_2''', it is easy to verify that $1 + x \sim 1 + y$ and $|x - y| > \frac{3}{4}(1-x)$, then

$$I_2' + I_2''' \leq C \|f v^{\alpha+\gamma,\delta}\|_\infty \sup_{|x| \leq 1} \|h_x\|_\infty (1-x)^{\gamma-1} \left\{ \int_{-1}^{x-\frac{3}{4}(1-x)} + \int_{\frac{3}{4}(1-x)}^{1} \right\} (1-y)^{-\gamma} dy$$

$$\leq C \|f v^{\alpha+\gamma,\delta}\|_\infty \sup_{|x| \leq 1} \|h_x\|_\infty.$$

With regard to I_2'', we have $1 \pm x \sim 1 \pm y$ and then

$$I_2'' \leq C \|f v^{\alpha+\gamma,\delta}\|_\infty \int_{x-\frac{3}{4}(1-x)}^{x+\frac{3}{4}(1-x)} \left| \frac{h(x,y) - h(x,x)}{y-x} \right| dy. \tag{A.8}$$

Combining (A.7)–(A.8), we deduce

$$|I_2| \leq C \|f v^{\alpha+\gamma,\delta}\|_\infty \left\{ \sup_{|x| \leq 1} \|h_x\|_\infty + \int_{x-\frac{1+x}{4}}^{x+\frac{1+x}{4}} \left| \frac{h(x,y) - h(x,x)}{y-x} \right| dy \right\}$$

and, proceeding as in the proof of Lemma 5.2 in [2], we get

$$|I_2| \leq C \|f v^{\alpha+\gamma,\delta}\|_\infty \left\{ \sup_{|x| \leq 1} \|h_x\|_\infty + \sup_{|x| \leq 1} \int_0^1 \frac{\omega_\varphi(h_x, u)_\infty}{u} du \right\}. \tag{A.9}$$

Thus, substituting (A.6) and (A.9) into (A.2) and using the assumption (11), we get (39). $\square$

Proof of (40). With $p_N(x,y)$ a polynomial of degree $N = \lfloor \frac{m+1}{2} \rfloor$ with respect to both the variables x and y separately, we set $q(x,y) = \frac{p_N(x,y) - p_N(x,x)}{y-x}$. The latter is a polynomial of degree $N-1$ with respect to the variable y and of degree m with respect to the variable x and

$$P_m(x) = \int_{-1}^1 q(x,y) f(y) v^{\alpha,-\alpha}(y) dy$$

is a polynomial of degree m. Then, we have

$$E_m(K^{\alpha,-\alpha}f)_{v^{\gamma,\alpha+\delta}} \leq \sup_{|x| \leq 1} \left| v^{\gamma,\alpha+\delta}(x) \int_{-1}^1 [k(x,y) - q(x,y)](f v^{\alpha,-\alpha})(y) dy \right|$$

$$= \sup_{|x| \leq 1} \left| v^{\gamma,\alpha+\delta}(x) \int_{-1}^1 \frac{[h_x(y) - p_{N,x}(y)] - [h_x(x) - p_{N,x}(x)]}{y-x} (f v^{\alpha,-\alpha})(y) dy \right| \tag{A.10}$$

and, proceeding as done for (A.2), we deduce

$$E_m(K^{\alpha,-\alpha}f)_{v^{\gamma,\alpha+\delta}} \leq C \|f v^{\alpha+\gamma,\delta}\|_\infty \left\{ \sup_{|x| \leq 1} \|h_x - p_{N,x}\|_\infty + \sup_{|x| \leq 1} \int_0^1 \frac{\omega_\varphi(h_x - p_{N,x}, t)_\infty}{t} dt \right\}.$$

Taking $p_{N,x}$ as the polynomial of best approximation of h_x for every fixed x and recalling [17, Lemma 2.1]

$$\int_0^{\frac{1}{m}} \frac{\omega_\varphi(f-P,u)_\infty}{u} du \leq \mathcal{C} \int_0^{\frac{1}{m}} \frac{\omega_\varphi^k(f,u)_\infty}{u} du$$

and (A.1), we get (40) by virtue of the assumption (12). $\square$

Proof of (46). We set $q(x,y) = \frac{p_N(x,y)-p_N(x,x)}{y-x}$, where $p_N(x,y)$ is a polynomial of degree $N = \lfloor \frac{m}{2} \rfloor$ with respect to both the variables x and y separately. So, $q(x,y)$ is a polynomial of degree $N-1$ with respect to the variable y and of degree $m-1$ with respect to the variable x. With $R(x,y) = h(x,y) - p_N(x,y)$, we have

$$(K^{\alpha,-\alpha} - K_m^{\alpha,-\alpha})f_m(x) = \int_{-1}^1 \frac{R(x,y)-R(x,x)}{y-x}(f_m v^{\alpha,-\alpha})(y)dy$$
$$- L_m^{-\alpha,\alpha}\left(\int_{-1}^1 L_m^{\alpha,-\alpha}\left(\frac{R(x,\cdot)-R(x,x)}{\cdot-x}, y\right)(f_m v^{\alpha,-\alpha})(y)dy, x\right)$$

being

$$\int_{-1}^1 \frac{p_N(x,y)-p_N(x,x)}{y-x}(f_m v^{\alpha,-\alpha})(y)dy = L_m^{-\alpha,\alpha}\left(\int_{-1}^1 L_m^{\alpha,-\alpha}\left(\frac{p_N(x,\cdot)-p_N(x,x)}{\cdot-x}, y\right)(f_m v^{\alpha,-\alpha})(y)dy, x\right).$$

Then

$$v^{\gamma,\alpha+\delta}(x)|(K^{\alpha,-\alpha} - K_m^{\alpha,-\alpha})f_m(x)| \leq \left| v^{\gamma,\alpha+\delta}(x)\int_{-1}^1 \frac{R(x,y)-R(x,x)}{y-x}(f_m v^{\alpha,-\alpha})(y)dy \right|$$
$$+ \left| v^{\gamma,\alpha+\delta}(x)L_m^{-\alpha,\alpha}\left(\int_{-1}^1 L_m^{\alpha,-\alpha}\left(\frac{R(x,\cdot)-R(x,x)}{\cdot-x}, y\right)(f_m v^{\alpha,-\alpha})(y)dy, x\right) \right|$$
$$=: A+B. \tag{A.11}$$

Proceeding as done for the proof of (A.10), we obtain

$$A \leq \mathcal{C}\|f_m v^{\alpha+\gamma,\delta}\|_\infty \sup_{|x|\leq 1} \|h_x\|_{Z_s} \frac{\log m}{m^s}. \tag{A.12}$$

While, using the definition of $L_m^{-\alpha,\alpha}$, we have

$$B \leq \mathcal{C}(\log m)\max_{k=1,\ldots,m} v^{\gamma,\alpha+\delta}(x_k)\left|\int_{-1}^1 L_m^{\alpha,-\alpha}\left(\frac{R(x_k,\cdot)-R(x_k,x_k)}{\cdot-x_k}, y\right)(f_m v^{\alpha,-\alpha})(y)dy \right|$$

being, by (44),

$$\max_{|x|\leq 1} v^{\gamma,\alpha+\delta}(x)\sum_{k=1}^m \frac{|l_k^{-\alpha,\alpha}(x)|}{v^{\gamma,\alpha+\delta}(x_k)} = \|L_m^{-\alpha,\alpha}\|_{C_{v^{\gamma,\alpha+\delta}}\to C_{v^{\gamma,\alpha+\delta}}} \leq \log m.$$

Therefore, applying the Gaussian rule and recalling (41) and (42), we get

$$B \leq \mathcal{C}(\log m)\max_{k=1,\ldots,m} v^{\gamma,\alpha+\delta}(x_k)\sum_{i=1}^m \frac{|R(x_k,t_i)-R(x_k,x_k)|}{|t_i-x_k|}|f_m(t_i)|\lambda_i^{\alpha,-\alpha}$$
$$\leq \mathcal{C}(\log m)\|f_m v^{\alpha+\gamma,\delta}\|_\infty \max_{k=1,\ldots,m} v^{\gamma,\alpha+\delta}(x_k)\|R_x\|_\infty \sum_{i=1}^m \frac{\Delta t_i}{|t_i-x_k|}v^{-\gamma,-\alpha-\delta}(t_i)$$
$$\leq \mathcal{C}(\log^2 m)\|f_m v^{\alpha+\gamma,\delta}\|_\infty \sup_{|x|\leq 1}\|h_x - p_{N,x}\|_\infty.$$

Now, taking $p_{N,x}$ as the polynomial of best approximation of the function h_x for every fixed x and using the assumption (12) and (A.1), we obtain

$$B \leq \mathcal{C}\|f_m v^{\alpha+\gamma,\delta}\|_\infty \sup_{|x|\leq 1}\|h_x\|_{Z_s}\frac{\log^2 m}{m^s}. \tag{A.13}$$

Summing up, substituting (A.12) and (A.13) into (A.11), we get

$$\|v^{\gamma,\alpha+\delta}(K^{\alpha,-\alpha} - K_m^{\alpha,-\alpha})f_m\|_\infty \leq \mathcal{C}\|f_m v^{\alpha+\gamma,\delta}\|_\infty \sup_{|x|\leq 1}\|h_x\|_{Z_s}\frac{\log^2 m}{m^s}.$$

Finally, using the equivalence (35), the thesis follows. $\square$

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