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# Practical consensus in bounded confidence opinion dynamics ☆

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ABSTRACT

Opinion dynamics expressed by the bounded confidence discrete-time heterogeneous Hegselmann–Krause model is considered. A policy for the adaptation of the agents confidence thresholds based on heterophily, maximum number of neighbors and non-influencing similarity interval is proposed. The policy leads to the introduction of the concepts of practical clustering and practical consensus. Several properties of the agents dynamic behaviors are proved by exploiting the roles of the agents having at each time-step the maximum and the minimum opinions. The convergence in finite time to (a maximum number of) practical clusters and, for sufficiently large threshold bounds, the convergence to a practical consensus are proved. Sufficient conditions for reaching a practical consensus around a stubborn are derived too. Numerical simulations verify the theoretical results.



## 1. Introduction

Opinion dynamics has been widely used for representing the time evolution of agents indicators in social networks (Proskurnikov & Tempo, 2018). The agents are the nodes of a corresponding graph and the *opinion* is the state of each agent which is interpreted as a measure of the intensity of his will toward a particular action or goal. In the Hegselmann–Krause (HK) model, in particular, the nodes interacting with each agent are selected as his neighbors, i.e. those agents who have an opinion close to his own one up to some confidence thresholds (Hegselmann & Krause, 2002; Tangredi et al., 2017).

Different types of HK models can be defined according to the characteristics of the confidence thresholds. When each agent selects the neighbors with lower and upper opinions by using the same confidence threshold, the model is said *symmetric*, and *asymmetric* otherwise. When all agents have the same interval of confidence, possibly asymmetric, the HK model is called *homogeneous*, see among others Blondel et al. (2009), Etesami and Başar (2015), and *heterogeneous* otherwise. The heterogeneity of the confidence thresholds among the nodes allows one to represent more general scenarios such as the presence of agents who are open-minded or closed-minded (Chazelle & Wang, 2017). On the other hand, the heterogeneous HK model exhibits behaviors much more complex than the homogeneous case and the analysis of the former class becomes far from trivial both in continuous-time (Altafini & Ceragioli, 2018; Frasca et al., 2019; Yang et al., 2014) and in discrete-time (Proskurnikov & Tempo, 2018).

In this paper we consider discrete-time asymmetric heterogeneous HK models. Numerical studies have shown some interesting phenomena induced by the heterogeneity, see among others Han et al. (2019) and Lorenz (2010). Some theoretical results for this class of HK models have been proposed in the literature by introducing specific model structures. A modified symmetric heterogeneous model is considered in Cheng and Yu (2019) where it is shown that by adding in the opinion dynamics the presence of group pressure, i.e. the average of all opinions influences the opinion of all agents, the convergence to the consensus in finite time can be easily proved. In Chazelle and Wang (2017) the confidence thresholds are fixed for each pair of nodes, thus corresponding to an undirected graph representation of the network which allows one to prove the convergence of the opinions to static agents. The heterogeneity feature of the model analyzed in Parasnis et al. (2018) comes from a physical connectivity graph which underlines a symmetric homogeneous confidence bound HK model. The type of models considered in the papers (Chazelle & Wang, 2017; Cheng & Yu, 2019; Parasnis et al., 2018) do not apply for our framework where the agents connected with a node are not uniquely identified and the possible interaction of an agent with others is determined only by his confidence intervals.

A one-sided asymmetry model, i.e. the lower (or the upper) confidence threshold is assumed to be the same for all agents,

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is analyzed in Coulson et al. (2015) while a not side-dependent confidence version of the heterogeneous HK model has been considered in Mirtabatabaei and Bullo (2012). Our results on the dynamic properties of the maximum and minimum opinions are valid also for the models analyzed in Coulson et al. (2015) and Mirtabatabaei and Bullo (2012), while the arguments used in these papers for proving the finite time convergence of the opinions to a limiting opinion vector cannot be used in our framework which refers to a model with adaptive confidence bounds.

The asymmetric heterogeneous HK model considered in this paper is characterized by an adaptation policy of the agents confidence thresholds. The policy takes origin from the preliminary idea presented by the authors in Iervolino et al. (2018) where each agent increases his thresholds looking for someone different to him. Indeed, heterophily, i.e. the tendency to interact with those who have different opinions, has been shown to play a key role on emergent behaviors in opinion dynamics (Motsch & Tadmor, 2014). In this line, the lower (upper) threshold of each agent is varied such that he would connect with at least one neighbor with a lower (upper) opinion; moreover it is assumed that each agent has a maximum number of interacting agents. On the other hand, we introduce a (small) similarity interval for all agents which reflects the fact that if two agents have similar, i.e. very close, opinions they do not interact anymore. This feature is coherent with the classical HK model dynamics where the intensity of the contribution of each agent's opinion on the opinion variation of another agent is *proportional* to their distance.

A first contribution of the paper is the definition of the *practical consensus* concept which occurs if all opinions belong to the same similarity interval. Our idea of practical consensus has relevant differences with the apparently similar concept of *quasi-consensus* introduced in Su et al. (2017). The latter corresponds to the situation when the graph is complete, i.e. all agents' opinions belong to the same symmetric confidence interval, instead the practical consensus refers to the case where all the agents do not interact anymore because their opinions belong to the same similarity interval. A further major contribution of the paper is the derivation of sufficient conditions for the convergence in finite time of the opinions to a maximum number of practical clusters and, for sufficiently large confidence thresholds, to practical consensus. The effects due to the presence of a stubborn are analyzed too, both formally and numerically.

The rest of the paper is organized as follows. In Section 2, by considering the discrete-time heterogeneous HK model, the definitions of the similarity interval, the practical clustering and the practical consensus are introduced. In Section 3 the proposed thresholds adaptation policy is described and structured via algorithms which exploit the idea of the active neighbors. A sociological interpretation of the model and the policies adopted in the paper is presented at the end of that section. Some dynamic properties of the extreme opinions of the network are proved in Section 4. The possible steady state solutions are analyzed in Section 5 while the proof of finite time convergence to a practical consensus is presented in Section 6. The impact of the presence of a stubborn on the validity of the former results is analyzed in Section 7. Numerical experiments verifying the theoretical results are discussed in Section 8. Section 9 concludes the paper by tracing some directions for future research.

## 2. Opinion dynamics and practical consensus

The opinion dynamics model considered in this paper is relative to $N$ agents whose opinions are represented through scalar state variables $x_i \in [0, 1]$, $i \in \mathcal{I} = \{1, \ldots, N\}$. In order to deal with significant cases, we will consider scenarios with $N \geq 3$ hereinafter. Similarly to the bounded confidence HK model, we define the scalar *influence function* $\phi_{ij}(x_i, x_j) : [0, 1]^2 \to \{0, 1\}$, $i, j \in \mathcal{I}$ which is equal to 1 when $x_j$ influences the opinion evolution of the agent $i$ and 0 otherwise. If $\phi_{ij}(x_i, x_j) = 1$ we say that the agent $j$ is an *active neighbor* of the agent $i$. In the following we assume $\phi_{ii}(x_i) = 1$ for all $i \in \mathcal{I}$. For the sake of notation, we use $x_i^+ := x_i(k + 1)$ and $x_i := x_i(k)$ for all $i \in \mathcal{I}$, where $k \in \mathbb{N}_0$ is the discrete time variable.

The opinion dynamics model is described by

$$x_i^+ = x_i + \frac{1}{\sum_{j=1}^N \phi_{ij}(x_i, x_j)} \sum_{j=1}^N \phi_{ij}(x_i, x_j)(x_j - x_i) \tag{1}$$

for all $i \in \mathcal{I}$, or equivalently

$$x_i^+ = \frac{1}{\sum_{j=1}^N \phi_{ij}(x_i, x_j)} \sum_{j=1}^N \phi_{ij}(x_i, x_j)\, x_j \tag{2}$$

for all $i \in \mathcal{I}$. The model (2) has an interesting interpretation: the agent opinion at the next time-step, i.e. $x_i^+$, is equal to the average of the neighbors opinions, including his own one. By considering the choice $x_i(0) \in [0, 1]$, from (2) it follows straightforwardly that $x_i \in [0, 1]$ for all time-steps.

A typical steady state behavior of interest for the dynamic system (2) is when all opinions become equal in finite time. This situation can be formally defined as follows.

**Definition 1** (*Consensus*)**.** The system (2) is said to reach a consensus if there exist a finite time-step $\hat{k}$ and a constant $\bar{c}$ such that for all $i \in \mathcal{I}$ and for all $h \in \mathbb{N}_0$ it is

$$\hat{x}_i^{+h} = \bar{c}, \tag{3}$$

with $\hat{x}_i^{+h} := x_i(\hat{k} + h)$ for all $i \in \mathcal{I}$.

Another typical steady state behavior of (2) is the clustering where the agents reach different (constant) values of opinions, each one corresponding to a subgroup of agents with the same opinion.

**Definition 2** (*Clustering*)**.** The system (2) is said to reach a clustering if there exist a finite time-step $\hat{k}$, different constants $\bar{c}_\mu \in [0, 1]$ and constant subsets of indices $\Sigma_\mu \subseteq \mathcal{I}$, $\mu = 1, \ldots, M$, $M \leq N$, with $\bigcup_{\mu=1}^M \Sigma_\mu = \mathcal{I}$, $\Sigma_{\mu_1} \bigcap \Sigma_{\mu_2} = \emptyset$ for any $\mu_1 \neq \mu_2$, such that for all $h \in \mathbb{N}_0$ it is

$$\hat{x}_i^{+h} = \bar{c}_\mu \tag{4}$$

for all $i \in \Sigma_\mu$, $\mu = 1, \ldots, M$, with $\hat{x}_i^{+h} := x_i(\hat{k} + h)$ for all $i \in \mathcal{I}$.

We are interested to analyze scenarios which approximate in some sense the above concepts of consensus and clustering. To this aim we generalize the notions above by introducing the definitions of *practical consensus* and *practical clustering*, respectively.

**Definition 3** (*Practical Consensus*)**.** The system (2) is said to reach a practical consensus if there exist a finite time-step $\hat{k}$ and a small $\epsilon_c \geq 0$ such that for all $i, j \in \mathcal{I}$ and for all $h \in \mathbb{N}_0$ it is

$$|\hat{x}_i^{+h} - \hat{x}_j^{+h}| \leq \epsilon_c, \tag{5}$$

with $\hat{x}_i^{+h} := x_i(\hat{k} + h)$ for all $i \in \mathcal{I}$ and for all $h \in \mathbb{N}_0$. In particular, a constant practical consensus is a practical consensus where

$$\hat{x}_i^{+h} = \hat{x}_i \tag{6}$$

holds for all $i \in \mathcal{I}$.

Note that the practical consensus definition, which corresponds to the convergence of the sequences of distances between all pairs of opinions to the set defined by (5), does not require the system to be at a regime with constant opinions which is the further condition required for the constant practical consensus.

By generalizing Definition 2, the practical clustering is defined as the situation when groups of agents have a relative distance smaller than $\epsilon_c$ inside each group but larger than $\epsilon_c$ for all pair of agents belonging to two different and not interacting groups.

**Definition 4** (*Practical Clustering*)**.** The system (2) is said to reach a practical clustering if there exist a finite time-step $\hat{k}$, a small $\epsilon_c \geq 0$ and constant largest subsets of indices $\Sigma_\mu \subseteq \mathcal{I}$, $\mu = 1, \ldots, M$, $M \leq N$, with $\bigcup_{\mu=1}^{M} \Sigma_\mu = \mathcal{I}$, $\Sigma_{\mu_1} \bigcap \Sigma_{\mu_2} = \emptyset$ for any $\mu_1 \neq \mu_2$, such that the inequalities (5) are satisfied only for all pairs $i, j \in \Sigma_\mu$, $\mu = 1, \ldots, M$. In particular, a constant practical clustering is a practical clustering where (6) holds for all $i \in \mathcal{I}$ and for all $h \in \mathbb{N}_0$.

It is easy to verify that the practical clustering definition reduces to that of practical consensus for $M = 1$. Moreover for $\epsilon_c = 0$ the definitions of practical consensus and practical clustering reduce to the classical definitions of consensus and clustering, respectively.

## 3. Influence function

The possible convergence of the opinions to (practical) consensus or clustering depends on the definition of the influence function $\phi_{ij}(x_i, x_j)$ in (2). For instance, in the case of symmetric homogeneous HK models the preserving average condition is satisfied and there exist conditions for which the system (2) converges to the consensus which is the average of the initial conditions, i.e. $\bar{c} = \frac{1}{N}\sum_{i=1}^{N} x_i(0)$, see Blondel et al. (2009).

In this section we present the definition of the influence function $\phi_{ij}(x_i, x_j)$ for all $i, j \in \mathcal{I}$, for our asymmetric heterogeneous HK model in the form (2), which includes the proposed thresholds variation policy.

### 3.1. Similarity interval

Let us introduce what we call the nominal form of the influence function, say $\bar{\phi}_{ij}(x_i, x_j)$ which depends on the difference $x_j - x_i$ through the following conditions

$$\bar{\phi}_{ij}(x_i, x_j) = \begin{cases} 1, & \text{if } -\ell_i < x_j - x_i < -\epsilon \\ 1, & \text{if } \epsilon < x_j - x_i < u_i \\ 0, & \text{otherwise} \end{cases} \tag{7}$$

for all $i, j \in \mathcal{I}$ with $i \neq j$. In the case $i = j$ we assume $\bar{\phi}_{ii}(x_i) = 1$ for all $i \in \mathcal{I}$ and for all $x_i$. Fig. 1 shows a picture of the nominal function. The small parameter $\epsilon \geq 0$ determines the *similarity interval*, i.e. $x_j - x_i \in [-\epsilon, \epsilon]$ for all $i, j \in \mathcal{I}$, for which two agents with sufficiently close opinions do not influence each others. For simplicity we assume $\epsilon$ to be the same for all agents. We call $\ell_i$ and $u_i$ the lower and the upper (confidence) thresholds of the agent $i$, respectively, assumed to be bounded for all agents, i.e. $\ell_i \in [\epsilon, \ell_{\max}]$, $u_i \in [\epsilon, u_{\max}]$, $i \in \mathcal{I}$ with $\ell_{\max} \in [\epsilon, 1]$ and $u_{\max} \in [\epsilon, 1]$. With some abuse of notation, the first (second) condition in (7) is intended to be excluded in the case $\ell_i = \epsilon$ ($u_i = \epsilon$), $i \in \mathcal{I}$.

The confidence set of the $i$-th agent is divided in two intervals: the lower confidence interval $L_i = (-\ell_i, -\epsilon)$ and the upper confidence interval $U_i = (\epsilon, u_i)$. The opinion dynamics model (2) is said symmetric if for each $i \in \mathcal{I}$ it is $\ell_i = u_i$, asymmetric otherwise. The agent $j$ is said a *potentially active lower neighbor* of $i$ if $x_j - x_i \in L_i$. Therefore the set of potentially active lower neighbors of $i$, say $\mathcal{P}_{L_i}(x) \subseteq \mathcal{I}$, is defined as $\mathcal{P}_{L_i}(x) = \{j \in \mathcal{I} : x_j - x_i \in L_i\}$ where $x$ is the vector of the opinions. Analogously, the agent $j$ is said a *potentially active upper neighbor* of $i$ if $x_j - x_i \in U_i$. Therefore, the set of potentially active upper neighbors of $i$, say $\mathcal{P}_{U_i}(x) \subseteq \mathcal{I}$, is defined as $\mathcal{P}_{U_i}(x) = \{j \in \mathcal{I} : x_j - x_i \in U_i\}$. Not all potential neighbors of the agent $i$ will contribute to $x_i^+$, as it will be detailed below.

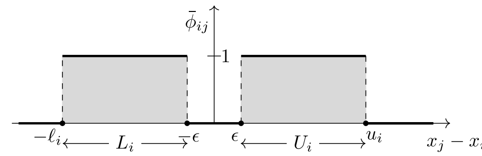


**Fig. 1.** The nominal influence function $\bar{\phi}_{ij}(x_i, x_j)$. The thresholds adaptation policy determines possible variations of $\ell_i$ and/or $u_i$ at each time-step.

### 3.2. Maximum number of active neighbors

In the following we assume that the agents fix a maximum number of lower and upper agents, say $\nu_L \in \{1, \ldots, N-1\}$ and $\nu_U \in \{1, \ldots, N-1\}$ respectively, which can influence their opinions. The potentially lower (upper) agents who influence the agent $i$ are called *active* lower (upper) neighbors of the agent $i$. The maximum number of active lower (upper) neighbors is assumed to be the same for all agents.

The determination of the active neighbors and the construction of the corresponding influence function $\phi_{ij}(x_i, x_j)$ for $i, j \in \mathcal{I}$ in (2) is synthesized by Algorithm 1. The parameters are $\epsilon$, $\nu_L$ and $\nu_U$. The inputs of the algorithm are the opinions $x_j$, $j \in \mathcal{I}$, and the bounds $\ell_i$ and $u_i$ of the agent $i$. The outputs of the algorithm are $\phi_{ij}(x_i, x_j)$, $j \in \mathcal{I}$, the number of active lower neighbors $N_{L_i}(x) \in \{0, \ldots, \nu_L\}$ and the number of active upper neighbors $N_{U_i}(x) \in \{0, \ldots, \nu_U\}$ at the current time-step.

For each agent $i$ the algorithm selects the potential neighbors of the agent $i$, i.e. the sets $\mathcal{P}_{L_i}(x)$ and $\mathcal{P}_{U_i}(x)$ defined above. The algorithm then determines the agents who are active neighbors of $i$ by taking into account the limitation of the maximum number of neighbors. The numbers of such lower and upper neighbors are $N_{L_i}(x) = \min\{\nu_L, \mathrm{card}(\mathcal{P}_{L_i}(x))\}$ and $N_{U_i}(x) = \min\{\nu_U, \mathrm{card}(\mathcal{P}_{U_i}(x))\}$, respectively. Clearly it will be $\sum_{j=1}^{N} \phi_{ij}(x_i, x_j) = N_{L_i}(x) + N_{U_i}(x) + 1$.

The indices in $\mathcal{P}_{L_i}(x)$ ($\mathcal{P}_{U_i}(x)$) are then ordered by decreasing (increasing) values of the corresponding opinions through a sorting operation on the indices. The ordering for potential neighbors with the same opinion is arbitrary. Say $\widehat{\mathcal{P}}_{L_i}(x)$ and $\widehat{\mathcal{P}}_{U_i}(x)$ the corresponding ordered sets of indices. Then, the set of the indices of the active lower (upper) neighbors of the agent $i$, say $\mathcal{N}_{L_i}(x)$ ($\mathcal{N}_{U_i}(x)$), is given by the first $N_{L_i}(x)$ ($N_{U_i}(x)$) elements of $\widehat{\mathcal{P}}_{L_i}(x)$ ($\widehat{\mathcal{P}}_{U_i}(x)$). The $j$-th element of the set $\widehat{\mathcal{P}}_{L_i}$ ($\widehat{\mathcal{P}}_{U_i}$) is indicated with $\widehat{\mathcal{P}}_{L_i}[j]$ ($\widehat{\mathcal{P}}_{U_i}[j]$).

### 3.3. Thresholds variation

Once the connections of the agent $i$ have been determined through Algorithm 1, we can now define a strategy for updating the upper and lower thresholds at the next time-step, which will depend on the current opinions. For the sake of simplicity in the following we omit in $\ell_i$ and $u_i$ the dependence on the opinions and therefore on time too.

The proposed policy consists in increasing the lower (upper) threshold of the agent $i$ by a constant amount $\gamma \geq 0$, if the agent has no lower (upper) active neighbors; otherwise the corresponding threshold is kept equal to its previous value. The

**Algorithm 1:** Values of the influence function of the agent $i$ for all agents $j \in \mathcal{I}$

```
Parameter: ε, ν_L, ν_U
Input    : {x_j}_{j∈I}, ℓ_i, u_i
Output   : {φ_ij}_{j∈I}, N_{L_i}, N_{U_i}
begin
    {φ_ij}_{j∈I} ← 0;
    φ_ii ← 1;
    P_{L_i} ← find{j ∈ I : −ℓ_i < x_j − x_i < −ε};
    N_{L_i} ← min{ν_L, card(P_{L_i})};
    if N_{L_i} ≠ 0 then
        P̂_{L_i} ← indexsort{x_j, j ∈ P_{L_i}, 'descend'};
        for j ← 1 to N_{L_i} do
            φ_{i P̂_{L_i}[j]} ← 1;
        end
    end
    P_{U_i} ← find{j ∈ I : ε < x_j − x_i < u_i};
    N_{U_i} ← min{ν_U, card(P_{U_i})};
    if N_{U_i} ≠ 0 then
        P̂_{U_i} ← indexsort{x_j, j ∈ P_{U_i}, 'ascend'};
        for j ← 1 to N_{U_i} do
            φ_{i P̂_{U_i}[j]} ← 1;
        end
    end
end
```

choice $\gamma = 0$ is left to account for the case of no thresholds variation strategy. The idea is applied through Algorithm 2. The parameters of the algorithm are $\gamma$, $\ell_{\max}$ and $u_{\max}$. The inputs are the current thresholds $\ell_i$ and $u_i$ and the numbers of active lower and upper neighbors $N_{L_i}(x)$ and $N_{U_i}(x)$, respectively. The outputs of the algorithm are the lower and the upper thresholds at the next time-step, i.e. $\ell_i^+$ and $u_i^+$. The algorithm proceeds as follows. If the agent $i$ has no lower neighbors then the lower threshold at the next time-step is chosen as $\ell_i^+ = \min\{\ell_i+\gamma, \ell_{\max}\}$. The upper threshold increases in a similar way by considering the active upper neighbors: if the agent $i$ has no upper neighbors then his upper threshold is updated according to $u_i^+ = \min\{u_i+\gamma, u_{\max}\}$.

**Algorithm 2:** Thresholds updating for the agent $i$

```
Parameter: γ, ℓ_max, u_max
Input    : ℓ_i, u_i, N_{L_i}, N_{U_i}
Output   : ℓ_i^+, u_i^+
begin
    if N_{L_i} == 0 ∧ ℓ_i < ℓ_max then
        ℓ_i^+ ← min{ℓ_i + γ, ℓ_max};
    end
    if N_{U_i} == 0 ∧ u_i < u_max then
        u_i^+ ← min{u_i + γ, u_max};
    end
end
```

From Definition 3 (Definition 4) it is easy to verify that by using the influence function defined by Algorithm 1 and Algorithm 2, any practical consensus (clustering) with $\epsilon_c = \epsilon$ will be a constant practical consensus (clustering). Indeed, from (5) all the absolute values of the differences between opinion pairs in the same practical cluster cannot be larger than $\epsilon$, therefore the agents belonging to the same cluster do not influence each other according to Algorithm 1 and from (2) the equality (6) directly follows.

The bounded confidence models analyzed in Coulson et al. (2015) and Mirtabatabaei and Bullo (2012) can be obtained as particular cases of our proposed policy, with the slight difference that in our model the agents with the same opinion are not considered as neighbors. In particular, by considering the policy defined by Algorithm 1 and Algorithm 2 with $\epsilon = 0$, $\gamma = 0$ and $\nu_L = \nu_U = N-1$, the one-sided asymmetric HK model considered in Coulson et al. (2015) can be obtained by selecting $u_i(0) = u_{\max}$ for all $i \in \mathcal{I}$ and $\ell_{\max} \in [0, u_{\max}]$, while the heterogeneous symmetric HK model in Mirtabatabaei and Bullo (2012) can be obtained by choosing $u_i(0) = \ell_i(0)$ for all $i \in \mathcal{I}$.

The theoretical results presented below are subjected to the following assumptions.

**Assumptions 5.** *The initial conditions of the system (2) are such that $x_i(0) \in [0, 1]$, $i \in \mathcal{I}$; the initial lower and upper thresholds are such that $\ell_i(0) \in [\epsilon, \ell_{\max}]$, $u_i(0) \in [\epsilon, u_{\max}]$, $i \in \mathcal{I}$, with $\ell_{\max} \in [\epsilon, 1]$, $u_{\max} \in [\epsilon, 1]$; the dynamics of the system (2) are subject to the policy defined by Algorithm 1 and Algorithm 2; the maximum number of lower and upper neighbors are chosen such that $\nu_L \in \{1, \dots, N-1\}$, $\nu_U \in \{1, \dots, N-1\}$.*

### 3.4. A sociological interpretation

The *opinion* of an agent has been interpreted in opinion dynamics as a cognitive orientation of some intensity toward a particular object (Friedkin, 2015). Alternatively, one could consider the state of each agent as a measure of his *skill* in a particular field, so as in collaboration networks (Xie et al., 2016). Let us consider the model (1). If $x_j - x_i \in L_i$, one could say that the skill level $x_i$ is larger than $x_j$. For $\ell_i = \epsilon$ it is always $\mathcal{P}_{L_i} = \emptyset$ and the agent $i$ can be considered as a stubborn towards less-skilled agents. Analogously, for $u_i = \epsilon$ it is always $\mathcal{P}_{U_i} = \emptyset$ and the agent $i$ acts like a stubborn towards more-skilled agents. In spite of the interpretation of the state in (1) as the skill of an agent, we prefer to use the term opinion according to the terminology typically adopted for the model (2) in the control systems literature.

An interpretation of our model derives from the sociological meanings of homophily and heterophily. It is a common experience that the creation of a new edge between two nodes is facilitated by their similarity, which is the classical homophily principle (McPherson et al., 2001). At the same time, the importance of weak ties, i.e. edges between nodes with non similar behaviors, for the connectivity of a social network has been widely recognized in the sociological literature (Granovetter, 1973). The idea of weak ties has stimulated a deeper analysis on how homophilous and heterophilous behaviors influence the dynamics of collaboration networks, see among others Rivera et al. (2010) and Yokomatsu and Kotani (2020). The measure of the similarity between two agents adopted herein is the Euclidean distance between their states. As a result, looking at Fig. 1, our analysis can be framed within the context of heterophilous opinion dynamics. More specifically, the heterophilous behavior justifies the use of the similarity interval introduced in (7). Indeed, the motivation for assuming $\phi_{ij} = 0$ for all $i, j \in \mathcal{I}$ such that $|x_j - x_i| \le \epsilon$, i.e. no interactions between agents in the same similarity interval, is that in the heterophilous framework two similar agents do not provide contribution in changing their opinions (Yokomatsu & Kotani, 2020). In the classical Hegselmann–Krause model dynamics, the intensity of the contribution of each agent's opinion on the opinion variation of another agent is proportional to their difference. Therefore the similarity interval $[-\epsilon, \epsilon]$ can be interpreted as the interval of distance values between pairs of (heterophilous) agents such that their mutual interaction is neglected.

The meaning of practical consensus introduced in Definition 3 follows from that of the similarity interval. In Motsch and Tadmor

(2014) it is shown that the consensus in opinion dynamics is enhanced by heterophily, provided that sufficiently strong heterophilous interactions are present and nonzero interaction levels are considered for the agents with similar opinions. In our case, the absence of any interaction for all agents being in the same similarity interval leads to the practical consensus scenario with $\epsilon_c = \epsilon$, where the agents share a common interval of opinion values, i.e. $|x_j - x_i| \leq \epsilon$ for all $i, j \in \mathcal{I}$, rather than a unique consensus value.

Finally, the heterophilous behavior provides a motivation for the policies implemented in Algorithm 1 and Algorithm 2. The former allows the agent to compute the values of his influence function by taking into account the limitations on the maximum number of his neighbors (Dunbar, 2010). Algorithm 2 implements the agent strategy of increasing his connectivity thresholds thus searching for agents with more distant opinions. This is an instinctive behavior for agents who are isolated (Lobel & Sadler, 2016). In our model this corresponds to $\mathcal{P}_{L_i}$, $\mathcal{P}_{U_i}$ and $\mathcal{S}_i = \{j \in \mathcal{I} : |x_j - x_i| \leq \varepsilon\}$ being empty sets. However, Algorithm 2 increases the connectivity thresholds also if $\mathcal{S}_i$ is not empty. This situation corresponds to the consideration that agents with many similar neighbors also tend to be heterophilous (Lobel & Sadler, 2016).

## 4. Properties of the extreme opinions

In this section we prove some properties of the opinion dynamics (2) with the policy defined by Algorithm 1 and Algorithm 2, by focusing on those we call maximum and minimum agents. At each time-step, the maximum (minimum) agent is selected among the agents with maximum (minimum) opinion at that time. More specifically, say $\mathcal{I}_M \subseteq \mathcal{I}$ the set of indices defined by

$$\mathcal{I}_M = \arg\max_{i \in \mathcal{I}} x_i. \tag{8}$$

Then the maximum agent at the time-step $k$, say $i_M$, is one of the agents belonging to the set $\mathcal{I}_M$, selected among those who have the minimum lower confidence threshold at that time-step, i.e. $i_M \in \widehat{\mathcal{I}}_M \subseteq \mathcal{I}_M$ with

$$\widehat{\mathcal{I}}_M = \arg\min_{i \in \mathcal{I}_M} \ell_i. \tag{9}$$

In particular, if the previous maximum $i_M^-$ is still an element of $\widehat{\mathcal{I}}_M$ then he is selected again as the current maximum agent, i.e. $i_M = i_M^-$. Otherwise $i_M$ is arbitrarily chosen among the elements of the set defined by (9).

Analogously, the minimum agent $i_m$ is chosen belonging to the set $\widehat{\mathcal{I}}_m \subseteq \mathcal{I}_m$ defined by

$$\widehat{\mathcal{I}}_m = \arg\min_{i \in \mathcal{I}_m} u_i \tag{10}$$

with the set of indices $\mathcal{I}_m \subseteq \mathcal{I}$ given by

$$\mathcal{I}_m = \arg\min_{i \in \mathcal{I}} x_i. \tag{11}$$

In particular, if $i_m^- \in \widehat{\mathcal{I}}_m$ then it is still chosen $i_m = i_m^-$, where $i_m^-$ is the minimum agent at the previous time-step. Otherwise $i_m$ is arbitrarily chosen as an element of the set defined by (10).

We now prove that the maximum (minimum) opinion cannot increase (decrease) in time. This is a known result for the classical asymmetric heterogeneous HK model, see Motsch and Tadmor (2014), but it must be proved to be valid in our case where a thresholds variation policy has been introduced. A direct consequence of this fact is that the evolution of the measure of the convex hull of the opinions is non-increasing over time.

**Lemma 6.** *Consider the system (2) with Assumptions 5. Then for any $\epsilon \geq 0$ and $\gamma \geq 0$ the inequalities*

$$x^+_{i^+_M} \leq x_{i_M} \tag{12a}$$

$$x^+_{i^+_m} \geq x_{i_m} \tag{12b}$$

*and*

$$x^+_{i^+_M} - x^+_{i^+_m} \leq x_{i_M} - x_{i_m} \tag{13}$$

*hold, where $x_\bullet := x_\bullet(k)$, $x^+_\bullet := x_\bullet(k+1)$, $i_\bullet := i_\bullet(k)$, $i^+_\bullet := i_\bullet(k+1)$, the symbol $\bullet$ is used for any subscript, for all $k \in \mathbb{N}_0$.*

**Proof.** First assume that the maximum agent does not change for two consecutive time-steps, i.e. $i^+_M = i_M$ which implies $x^+_{i^+_M} = x^+_{i_M}$. Since the maximum agent can only have neighbors with lower opinions, the inequality (12a) directly follows from (2).

Consider the case when at the next time-step the maximum agent changes, i.e. $i^+_M \neq i_M$. In the case that $x_{i_M} - x_{i^+_M} > \epsilon$, an upper bound of the opinion of the agent $i^+_M$ at the next time-step can be obtained by assuming that the opinion of $i^+_M$ at the next time-step, i.e. $x^+_{i^+_M}$, is the effect of his connection with $\nu_U$ agents with maximum opinion $x_{i_M}$. From (2) one obtains

$$\begin{aligned} x^+_{i^+_M} &\leq \frac{1}{\nu_U + 1}(\nu_U x_{i_M} + x_{i^+_M}) \\ &= x_{i_M} - \frac{1}{\nu_U + 1}(x_{i_M} - x_{i^+_M}) \leq x_{i_M} \end{aligned} \tag{14}$$

where we used the condition $x_{i_M} \geq x_{i^+_M}$ which is by definition of the maximum agent at each time-step. In the case that $x_{i_M} - x_{i^+_M} \leq \epsilon$ it is $N_{U_{i^+_M}}(x) = 0$ and $N_{L_{i^+_M}}(x) \geq 0$. If $N_{L_{i^+_M}}(x) = 0$ it is $x^+_{i^+_M} = x_{i^+_M} \leq x_{i_M}$. Otherwise $x^+_{i^+_M} < x_{i^+_M} \leq x_{i_M}$. Then (12a) is verified.

In order to prove (12b), let us assume that the minimum agent does not change for two consecutive time-steps, i.e. $i^+_m = i_m$ which implies $x^+_{i^+_m} = x^+_{i_m}$. Since the minimum agent can only have neighbors with larger opinions, the inequality (12b) directly follows from (2). With an analogous procedure to that presented for the maximum agent one can prove that (12b) holds also when the minimum changes over time, i.e. $i^+_m \neq i_m$. Indeed, in the case that $x_{i^+_m} - x_{i_m} > \epsilon$ a lower bound of the opinion of the agent $i^+_m$ at the next time-step can be obtained by considering $x^+_{i^+_m}$ to be determined by $\nu_L$ agents with minimum opinion $x_{i_m}$. From (2) one obtains

$$\begin{aligned} x^+_{i^+_m} &\geq \frac{1}{\nu_L + 1}(\nu_L x_{i_m} + x_{i^+_m}) \\ &= x_{i_m} - \frac{1}{\nu_L + 1}(x_{i_m} - x_{i^+_m}) \geq x_{i_m} \end{aligned} \tag{15}$$

where we used the condition $x_{i_m} \leq x_{i^+_m}$ which is by definition of the minimum agent at each time-step. In the case that $x_{i^+_m} - x_{i_m} \leq \epsilon$ it is $N_{L_{i^+_m}}(x) = 0$ and $N_{U_{i^+_m}}(x) \geq 0$. If $N_{U_{i^+_m}}(x) = 0$ it is $x^+_{i^+_m} = x_{i^+_m} \geq x_{i_m}$. Otherwise $x^+_{i^+_m} > x_{i^+_m} \geq x_{i_m}$. Then (12b) is verified.

From (12b) one can write $-x^+_{i^+_m} \leq -x_{i_m}$ and by adding (12a) the validity of (13) at any time-step directly follows. □

In the following we show that if the maximum (minimum) agent changes from one time-step to the next, then the maximum (minimum) opinion is strictly decreasing (increasing).

**Theorem 7.** *Consider the system (2) with Assumptions 5. Then, for any $\epsilon \geq 0$ and $\gamma \geq 0$ the following implications*

$$i^+_M \neq i_M \implies x^+_{i^+_M} < x_{i_M} \tag{16a}$$

$$i_m^+ \neq i_m \implies x_{i_m^+}^+ > x_{i_m} \tag{16b}$$

*hold, where* $x_\bullet := x_\bullet(k)$, $x_\bullet^+ := x_\bullet(k+1)$, $i_\bullet := i_\bullet(k)$, $i_\bullet^+ := i_\bullet(k+1)$, *the symbol* $\bullet$ *is used for any subscript, for all* $k \in \mathbb{N}_0$.

**Proof.** We first verify the implication (16a). Consider the maximum agents at two consecutive steps, i.e. $i_M$ and $i_M^+$, and suppose that $i_M^+ \neq i_M$.

Firstly, let us consider the case $x_{i_M} > x_{i_M^+}$. If $x_{i_M} - x_{i_M^+} > \epsilon$, by repeating the considerations used for (14), the inequality (16a) directly follows because $x_{i_M} - x_{i_M^+} > 0$. If $x_{i_M} - x_{i_M^+} \leq \epsilon$, it has been shown in the proof of Lemma 6 that $x_{i_M^+}^+ \leq x_{i_M^+}$, which implies (16a).

We now show that each agent $i \in \mathcal{I}_M \setminus \{i_M\}$, where $\mathcal{I}_M$ is given by (8), cannot be the maximum agent at the next time-step. According to (9) it is $\ell_i \geq \ell_{i_M}$, for all agents $i \in \mathcal{I}_M$. Therefore, since $x_i = x_{i_M}$ for all $i \in \mathcal{I}_M$, it is $\mathcal{N}_{U_i}(x) = \mathcal{N}_{U_{i_M}}(x) = \emptyset$ and $\mathcal{N}_{L_i}(x) \supseteq \mathcal{N}_{L_{i_M}}(x)$. Then it is $N_{L_i}(x) \geq N_{L_{i_M}}(x)$ for all $i \in \mathcal{I}_M$, i.e. the maximum agent has the minimum number of lower neighbors within the set $\mathcal{I}_M$ and all lower neighbors of $i_M$ are also neighbors of any agent $i$ with $i \in \mathcal{I}_M \setminus \{i_M\}$. For all $i \in \mathcal{I}_M$ such that $N_{L_i}(x) > N_{L_{i_M}}(x)$, from (2) it will be $x_i^+ < x_{i_M}^+$ and then any of such agents cannot be the maximum at the next step. For all $i \in \mathcal{I}_M \setminus \{i_M\}$ such that $N_{L_i}(x) = N_{L_{i_M}}(x) > 0$, it will be $\mathcal{N}_{L_i}(x) = \mathcal{N}_{L_{i_M}}(x)$ and from (2) it is $x_i^+ = x_{i_M}^+$, however any of such agents cannot be the maximum at next step because the maximum selection rule with (9) would choose $i_M^+ = i_M$. Finally for all $i \in \mathcal{I}_M \setminus \{i_M\}$ such that $N_{L_i}(x) = N_{L_{i_M}}(x) = 0$ it will be $\ell_i^+ \geq \ell_{i_M}^+$ because it was $\ell_i \geq \ell_{i_M}$, all such lower bounds will be increased by the same amount $\gamma$, and the maximum selection rule with (9) would choose $i_M^+ = i_M$.

The proof of (16b) can be easily obtained by applying similar arguments to the minimum agent and the corresponding sets. □

An interesting consequence of Theorem 7 is that if $i_M^+ \neq i_M$ ($i_m^+ \neq i_m$) and the agent $i_M$ ($i_m$) returns to be the maximum (minimum) agent in any future time-step he will have an opinion lower (larger) than $x_{i_M}$ ($x_{i_m}$).

## 5. Practical clustering conditions

The system (2) with the policy defined by Algorithm 1 and Algorithm 2 is well posed. Indeed, it can be easily verified that the system has a unique solution for each set of initial conditions $x_i(0) \in [0, 1]$, $\ell_i(0) \in [\epsilon, \ell_{\max}]$, $u_i(0) \in [\epsilon, u_{\max}]$, $i \in \mathcal{I}$, and for any small $\epsilon \geq 0$, $\gamma \geq 0$, $\nu_L \in \{1, \dots, N-1\}$, $\nu_U \in \{1, \dots, N-1\}$, $\ell_{\max} \in [\epsilon, 1]$, $u_{\max} \in [\epsilon, 1]$. In the following we exclude the trivial case $\ell_{\max} = u_{\max} = \epsilon$ which corresponds to all opinions being constant for all $k \in \mathbb{N}_0$.

Agents opinions which satisfy the conditions of constant practical consensus or those of constant practical clustering are steady state solutions of the system (2) by definition. We now show that constant practical clusters in the sense of Definition 4 with $\epsilon_c = \epsilon$ are the only possible steady state solutions with constant opinions.

**Lemma 8.** *Consider the system (2) with Assumptions 5. Then for any* $\epsilon \geq 0$ *and* $\gamma > 0$*, any steady state solution with constant opinions is a constant practical clustering as in Definition 4 with* $\epsilon_c = \epsilon$*. Moreover, the distance between two practical clusters is such that*

$$|\hat{x}_i - \hat{x}_j| \geq b_{\max} \tag{17}$$

*with*

$$b_{\max} = \max\{\ell_{\max}, u_{\max}\}, \tag{18}$$

*for any* $i \in \Sigma_{\mu_1}$, $j \in \Sigma_{\mu_2}$, $\mu_1 \neq \mu_2$, *and the number of practical clusters M satisfies the inequality*

$$M \leq \left\lfloor \frac{x_{i_M}(0) - x_{i_m}(0) + b_{\max}}{b_{\max}} \right\rfloor. \tag{19}$$

**Proof.** Assume by contradiction that there exists a constant solution of the system (2) satisfying (6) and not being a practical clustering. Suppose $b_{\max} = \ell_{\max}$, i.e. $\ell_{\max} \geq u_{\max}$. Let us consider the maximum opinion among all agents, say $\hat{x}_{\hat{i}_{M1}}$, and say $\Sigma_1 \subseteq \mathcal{I}$ the set of indices such that $\hat{x}_{\hat{i}_{M1}} - \hat{x}_i \leq \epsilon$ with $i \in \mathcal{I}$. By definition it is $\phi_{\hat{i}_{M1}i}(\hat{x}_{\hat{i}_{M1}}, \hat{x}_i) = 0$ for all $i \in \Sigma_1 \setminus \{\hat{i}_{M1}\}$. Any agent in $\Sigma_1$ does not have upper neighbors and he cannot interact with any lower neighbor otherwise he would decrease his opinion by contradicting the constant steady state assumption. Therefore according to Algorithm 2 the thresholds of all agents in $\Sigma_1$ will increase until they will be equal to $\ell_{\max}$ at the finite time-step

$$\bar{k}_1 = \hat{k} + \left\lceil \frac{\ell_{\max} - \min\{\hat{\ell}_i\}_{i\in\Sigma_1}}{\gamma} \right\rceil. \tag{20}$$

As a consequence it must be

$$\hat{x}_j \notin (\min\{\hat{x}_i\}_{i\in\Sigma_1} - \ell_{\max}, \hat{x}_{\hat{i}_{M1}} - \epsilon) \tag{21}$$

for all $j \in \mathcal{I}$, i.e. there are no agents outside $\Sigma_1$ with opinions closer than $\ell_{\max}$ to the opinions of the agents in $\Sigma_1$.

Let us indicate with $\hat{i}_{M2}$ the agent having the constant steady state opinion given by $\hat{x}_{\hat{i}_{M2}} = \max\{\hat{x}_i\}_{i\in\mathcal{I}\setminus\Sigma_1}$, and say $\Sigma_2 \subset \mathcal{I}$ the set of indices of the agents similar to $\hat{i}_{M2}$, i.e. $\hat{x}_{\hat{i}_{M2}} - \hat{x}_i \leq \epsilon$ for all $i \in \mathcal{I} \setminus \Sigma_1$. Since $\ell_{\max} \geq u_{\max}$ all agents in $\Sigma_2$ do not have upper neighbors. Therefore it must be $\hat{x}_j \notin (\hat{x}_{\hat{i}_{M2}}, \hat{x}_{\hat{i}_{M2}} + u_{\max})$ for all $j \in \mathcal{I}$. By combining these expressions with (21), it follows that $\hat{x}_j \notin (\hat{x}_{\hat{i}_{M2}}, \hat{x}_{\hat{i}_{M1}} - \epsilon)$ for all $j \in \mathcal{I}$, which implies

$$\hat{x}_j \notin (\min\{\hat{x}_i\}_{i\in\Sigma_1} - b_{\max}, \hat{x}_{\hat{i}_{M1}} - \epsilon) \tag{22}$$

for all $j \in \mathcal{I}$. By iterating the arguments above one obtains

$$\hat{x}_j \notin (\min\{\hat{x}_i\}_{i\in\Sigma_\mu} - b_{\max}, \hat{x}_{\hat{i}_{M\mu}} - \epsilon) \tag{23}$$

for all $j \in \mathcal{I}$, with

$$\hat{x}_{\hat{i}_{M\mu}} = \max\{\hat{x}_i\}_{i\in\mathcal{I}\setminus\bigcup_{m=1}^{\mu-1}\Sigma_m} \tag{24}$$

i.e. any steady state solution with constant opinions must be a practical clustering according to Definition 4.

The inequality (17) directly follows from (23). For the case $b_{\max} = u_{\max}$ analogous arguments can be applied by starting from the minimum agent.

The inequality (19) comes from Definition 4 and the property (17) which includes (5). Since from Lemma 6 it is $x_i \in [x_{i_m}(0), x_{i_M}(0)]$ for all $i \in \mathcal{I}$ and for all $k \in \mathbb{N}_0$, being $M$ the number of clusters, there must be at most $M-1$ intervals between any two practical clusters of minimum amplitude equal to $b_{\max}$. On the other hand, each practical cluster can also correspond to all agents of the cluster having the same opinion. Therefore, it must be $(M-1)b_{\max} \leq x_{i_M}(0) - x_{i_m}(0)$, from which (19) directly follows. □

It should be noticed that in Lemma 8 it is assumed that $\gamma$ is strictly positive. In the case $\gamma = 0$ by using similar arguments of Lemma 8 it can be easily proved that the minimum distance between pairs of clusters will be $\min\{\ell_i(0), u_i(0)\}_{i\in\mathcal{I}}$.

## 6. Large thresholds and practical consensus

In this section we analyze the opinion dynamics when the upper bound of the thresholds is sufficiently large. In particular, we consider the case that either $\ell_{\max}$ or $u_{\max}$ are such that $b_{\max}$ is larger or equal to the difference between the maximum and minimum initial opinions. If the initial conditions are not known, one can choose $\ell_{\max}$ or $u_{\max}$ equal to 1, i.e. $b_{\max} = 1$.

In the following we consider the not restrictive case that among all agents' initial opinions at least one is strictly larger than $x_{i_m}(0)$ and strictly smaller than $x_{i_M}(0)$. A direct consequence of this assumption is that the system cannot exhibit practical clustering with more than one cluster, i.e. any steady state solution with constant opinions must be a practical consensus. The result below proves the convergence to such a solution for almost all initial conditions.

**Theorem 9.** *Consider the system (2) with Assumptions 5, $x_{\bar{\imath}}(0) \in (x_{i_m}(0), x_{i_M}(0))$ for some $\bar{\imath} \in \mathcal{I}$ and*

$$b_{\max} \geq x_{i_M}(0) - x_{i_m}(0) \tag{25}$$

*with $b_{\max}$ given by (18). Then, for any $\epsilon > 0$ and $\gamma > 0$ the system converges to a constant practical consensus with $\epsilon_c = \epsilon$.*

**Proof.** From (25) and (19) in Lemma 8 it follows that any practical clustering must have $M \leq 2$. By using (17) in Lemma 8 and (12) in Lemma 6, the only possibility of having $M = 2$ is that $b_{\max} = x_{i_M}(0) - x_{i_m}(0)$ and

$$x_i = x_i(0) \in \{x_{i_m}(0), x_{i_M}(0)\} \tag{26}$$

for all $i \in \mathcal{I}$ and for all $k \in \mathbb{N}_0$. This situation cannot occur if there exists at least one initial opinion $x_{\bar{\imath}}(0) \in (x_{i_m}(0), x_{i_M}(0))$ for some $\bar{\imath} \in \mathcal{I}$ which contradicts (26). Therefore, the practical clustering, if any, must be a practical consensus, i.e. $M = 1$.

From Lemma 6 it follows that the difference $x_{i_M} - x_{i_m}$, which is bounded in $[0, 1]$, is also non-increasing. From the theorem of convergence of bounded monotone sequences, the sequence $\{x_{i_M} - x_{i_m}\}_{k=0,1,\ldots}$ is convergent to its infimum, say $\bar{\sigma}$, which depends on the initial conditions $x_i(0)$, $\ell_i(0)$, $u_i(0)$, $i \in \mathcal{I}$. Clearly if $\bar{\sigma} \in [0, \epsilon]$ the proof is complete. We now show that it is not possible to have $\bar{\sigma} > \epsilon$.

The convergence property implies that it must be

$$x_{i_M} - x_{i_m} \geq \bar{\sigma} \tag{27}$$

for any time-step. The non-increasing property of the sequence implies that for any $\delta > 0$ there exists a finite time-step $\hat{k}$ such that $\hat{x}_{\hat{i}_M} - \hat{x}_{\hat{i}_m} \leq \bar{\sigma} + \delta$. Without loss of generality one can choose $\delta < \frac{\epsilon}{2}$. We now show that if $\bar{\sigma} > \epsilon$ there would exist a finite $h \in \mathbb{N}_0$ such that

$$\hat{x}^{+h}_{\hat{i}^{+h}_M} - \hat{x}^{+h}_{\hat{i}^{+h}_m} < \bar{\sigma} \tag{28}$$

thus contradicting (27). Clearly if $\bar{\sigma} \in [0, \epsilon]$, the condition (28) can never be satisfied because as soon as the measure of the convex hull of the opinions becomes less than or equal to $\epsilon$ all opinions will remain constant for any future time-step.

The following three cases related to the maximum agent $\hat{i}_M$ at $\hat{k}$ are possible: (i) he is interacting with some lower neighbor and he remains the maximum at next time-step, (ii) he is not interacting with lower neighbors and (as a consequence) he remains the maximum agent for some future time interval, (iii) he is interacting with some lower neighbor and at next step the maximum agent changes.

In the case (i) the agent $\hat{i}_M$ is interacting with some lower neighbor and it will be

$$\hat{x}^{+}_{\hat{i}_M} < \frac{1}{2}\left(\hat{x}_{\hat{i}_M} + \hat{x}_{\hat{i}_M} - \epsilon\right) = \hat{x}_{\hat{i}_M} - \frac{\epsilon}{2}. \tag{29}$$

If $\hat{i}^{+}_M = \hat{i}_M$, by using (29) and (12b) it will be

$$\hat{x}^{+}_{\hat{i}^{+}_M} - \hat{x}^{+}_{\hat{i}^{+}_m} = \hat{x}^{+}_{\hat{i}_M} - \hat{x}^{+}_{\hat{i}^{+}_m} < \hat{x}_{\hat{i}_M} - \frac{\epsilon}{2} - \hat{x}^{+}_{\hat{i}^{+}_m}$$

$$\leq \hat{x}_{\hat{i}_M} - \frac{\epsilon}{2} - \hat{x}_{\hat{i}_m} \leq \bar{\sigma} + \delta - \frac{\epsilon}{2} < \bar{\sigma} \tag{30}$$

which corresponds to (28) with $h = 1$.

In the case (ii) the agent $\hat{i}_M$ is not interacting with any lower neighbor and he remains the maximum agent for some finite time interval, say $\Delta$, by increasing his lower threshold until he will eventually interact with a lower neighbor, i.e. $\hat{i}^{+h}_M = \hat{i}_M$ for some $h = 1, \ldots, \Delta$ with

$$\Delta \leq \left\lceil \frac{\bar{\sigma} + \delta - \hat{\ell}_{\hat{i}_M}}{\gamma} \right\rceil. \tag{31}$$

If $\hat{i}^{+(\Delta+1)}_M = \hat{i}_M$ one can repeat the argument above and (28) holds with $h = \Delta + 1$. Otherwise, if $\hat{i}^{+(\Delta+1)}_M \neq \hat{i}_M$ the scenario (iii) must be considered.

In the case (iii) the maximum agent $\hat{i}_M$ is interacting with some lower neighbor but does not remain the maximum at the next time-step. Then, by using Theorem 7 it is $\hat{x}^{+}_{\hat{i}^{+}_M} < \hat{x}_{\hat{i}_M}$. Moreover it is

$$x^{+h}_{\hat{i}_M} < \frac{1}{\nu_U + 1}\left(x_{\hat{i}_M} - \frac{\varepsilon}{2} + \nu_U x_{\hat{i}_M}\right) = x_{\hat{i}_M} - \frac{\varepsilon}{2(\nu_U + 1)} \tag{32}$$

for any $h \geq 1$. The number of the time-steps (not necessarily consecutive), at which the agent $\hat{i}_M$ is the maximum and his opinion has not yet decreased by at least $\epsilon/2$, is upper bounded by $\Delta_{\max} + \nu_U + 1$ where

$$\Delta_{\max} = \left\lceil \frac{\bar{\sigma} + \delta - \min\{\hat{\ell}_i\}_{i\in\mathcal{I}}}{\gamma} \right\rceil. \tag{33}$$

Now we can apply to $\hat{i}^{+}_M$ the arguments presented above for $\hat{i}_M$. Since the number of agents is finite, by using (32) the inequality (28) will be satisfied for some

$$h \leq N(\Delta_{\max} + \nu_U + 1). \tag{34}$$

We have shown that in all cases (i), (ii) and (iii) for $\bar{\sigma} > \epsilon$ the condition (28) will be eventually satisfied for some finite $h \geq 1$ which contradicts (27). This means that the system eventually reaches the practical consensus and the proof is complete. □

Note that in Theorem 9 $\epsilon$ is strictly positive. The arguments used in the proof above cannot be easily extended to the case $\epsilon = 0$, which represents a further motivation for using the practical consensus concept introduced in this paper. Moreover, the hypothesis $\gamma > 0$ implies that there exists a finite time-step such that the heterogeneity of the model is lost, i.e. $\ell_i = \ell_{\max}$ and $u_i = u_{\max}$ for all $i \in \mathcal{I}$. On the other hand, the practical consensus may be reached before (asymmetric) homogeneity is achieved.

By using Theorem 9, an upper bound on the convergence time to the practical consensus can be determined, so as formalized by the following corollary.

**Corollary 10.** *Consider the system (2) with Assumptions 5, $x_{\bar{\imath}}(0) \in (x_{i_m}(0), x_{i_M}(0))$ for some $\bar{\imath} \in \mathcal{I}$, and (25). Then, for any $\epsilon > 0$ and $\gamma > 0$ the convergence time, say $k_c \in \mathbb{N}_0$, to a constant practical consensus is such that*

$$k_c \leq N(\min\{\Delta_M, \Delta_m\} + \Delta_\epsilon) \tag{35}$$

*where*

$$\Delta_M = \left\lceil \frac{x_{i_M}(0) - x_{i_m}(0) - \min\{\ell_i(0)\}_{i\in\mathcal{I}}}{\gamma} \right\rceil, \tag{36}$$

$$\Delta_m = \left\lceil \frac{x_{i_M}(0) - x_{i_m}(0) - \min\{u_i(0)\}_{i\in\mathcal{I}}}{\gamma} \right\rceil, \tag{37}$$

$$\Delta_\epsilon = \left\lceil \frac{x_{i_M}(0) - x_{i_m}(0) - \epsilon}{\frac{\epsilon}{2(\max\{\nu_L, \nu_U\}+1)}} \right\rceil. \tag{38}$$

**Proof.** By applying the considerations used in the proof of Theorem 9 for (34) and by replacing $\bar{\sigma}+\delta$ with $x_{i_M}(0)-x_{i_m}(0)$ one can straightforwardly obtain an upper bound $\Delta_M$ for the sum of the time intervals in which the maximum agent of each interval does not change his opinion, which can be expressed by (36). Analogously, an upper bound $\Delta_m$ for the sum of the time intervals in which the minimum agent of each interval does not change his opinion can be expressed by (37). The measure of the convex hull of the opinions is constant when both the maximum and the minimum agents do not change their opinions. Therefore, an upper bound on the sum of the time intervals in which the measure of the convex hull of the opinions remains constant is given by $\min\{\Delta_M, \Delta_m\}$.

If the maximum agent or the minimum agent change their opinions, by using (32) in the proof of Theorem 9 the measure of the convex hull reduces at least by an amount equal to $\epsilon/(2\max\{\nu_L, \nu_U\}+2)$. Therefore an upper bound for the total time-steps in which the measure of the convex hull of the opinions decreases up to $\epsilon$ is given by (38).

By combining (36)–(38), the convergence time to the practical consensus must satisfy the inequality (35). □

The way how the upper bound expressed by (35) has been obtained provides also its interpretation. In particular, the number of future time-steps (not necessarily consecutive), at which the agent $i_M$ ($i_m$) is the maximum (minimum) and his opinion is equal to $x_{i_M}$ ($x_{i_m}$), is upper bounded by $\Delta_M$ ($\Delta_m$). Moreover, the number of future time-steps (not necessarily consecutive) for the opinion of $i_M$ to decrease until the similarity interval of $i_m$ is reached is upper bounded by $\Delta_\epsilon$. By repeating the arguments above for the $N$ agents, an upper bound of the number of time-steps at which the convex hull of the opinions remains constant is given by $N\min\{\Delta_M, \Delta_m\}$, while an upper bound of the number of time-steps at which the convex hull of the opinions is decreasing and such that all opinions reach the same similarity interval is given by $N\Delta_\epsilon$.

## 7. Stubbornness

An agent identified with a generic index $s\in\mathcal{I}$ is said a *stubborn* if he is anchored to his initial opinion, i.e. $x_s = x_s(0)$ for all $k\in\mathbb{N}_0$ for some $s\in\mathcal{I}$. By definition, a stubborn is not influenced by other opinions, i.e. in (2) it is $\phi_{sj}(x_s, x_j)=0$ for all $j\in\mathcal{I}\setminus\{s\}$.

A stubborn agent is characterized by $\ell_s=\epsilon$ and $u_s=\epsilon$ for all $k\in\mathbb{N}_0$ which implies $\bar{\phi}_{sj}(x_s, x_j)=0$ for all $j\in\mathcal{I}\setminus\{s\}$, see (7). Algorithm 1 sets for the stubborn $\phi_{ss}=1$ and $\phi_{sj}=0$ for all $j\in\mathcal{I}\setminus\{s\}$ because $\mathcal{P}_{L_s}$ and $\mathcal{P}_{U_s}$ are empty, therefore $N_{L_s}$ and $N_{U_s}$ are equal to 0. The thresholds variation strategy described by Algorithm 2 cannot be applied to the stubborn if $\gamma>0$, otherwise his thresholds would increase and the conditions $\ell_s=\epsilon$ and $u_s=\epsilon$ for all $k\in\mathbb{N}_0$ would not be valid. In other words, the system (2) with the policy defined by Algorithm 1 and Algorithm 2 does not include the presence of stubborn agents. To do so, Algorithm 2 has to be applied for all agents $i\in\mathcal{I}\setminus\{s\}$. In this section we analyze the influence of the presence of a stubborn on the results proved in the previous sections, and verify their validity by introducing extra conditions if required.

For what concerns the properties of the maximum and minimum agents illustrated in Section 4, it is easy to verify that the results in Lemma 6 hold also in presence of a single stubborn among the agents. In order to show that, we can consider two cases. First, if $i_M\neq s$, where $s$ identifies the stubborn, for all $k\in\mathbb{N}_0$ the same considerations of the proof of Lemma 6 can be directly applied. Otherwise, if at some time-step $i_M=s$, since $x_s^+=x_s$, it follows $x_{i_M}^+=x_{i_M}$ and by combining this condition with the dynamics (2), then (12a) holds. By applying similar arguments to the minimum agent it follows that the inequality (12b) holds also in presence of a stubborn. As a consequence, the result on the non-increasing measure of the convex hull of the opinions proved in Lemma 6 is still valid also in the presence of a stubborn.

The results on the maximum and minimum opinions proved in Theorem 7 can be shown to hold also in the presence of a stubborn for $k>0$. In particular, if $i_M=s$ it is not possible that $i_M^+\neq i_M$ because $x_s^+=x_s$ by definition and the algorithm would select the agent $s$ as the maximum agent according to (9), since he is characterized by the minimum lower threshold, i.e. $\ell_s=\epsilon$. Therefore if $i_M=s$ at some time-step, the maximum agent will remain the same in any future time-step. The conditions $i_M^+=s$, $i_M^+\neq i_M$ and $x_{i_M}>x_s$, conversely, could occur when the agent $i_M$ is influenced by lower neighbors that decrease his opinion such that $x_{i_M}^+\leq x_s^+$. Since $x_s^+=x_s$, it follows directly $x_s^+<x_{i_M}$, corresponding to (16a). By applying similar arguments to the minimum agent it follows that the inequality (16b) holds also in presence of a stubborn.

As regards the steady state solutions, the practical clustering results in (17)–(19) of Lemma 8 hold, if the practical cluster including the stubborn contains at least another agent. Otherwise, when the stubborn is the unique agent in a cluster, the distance from another cluster is such that

$$|\hat{x}_s-\hat{x}_j|\geq\beta_{\max} \tag{39}$$

with

$$\beta_{\max}=\min\{\ell_{\max}, u_{\max}\} \tag{40}$$

for all $j\in\mathcal{I}\setminus\{s\}$. In particular, the cluster including the stubborn must have a distance not smaller than $\beta_{\max}$ from the closest cluster and not smaller than $b_{\max}$ from the others. On the other hand each practical cluster can also correspond to all agents of the cluster having the same opinion. Therefore it must be

$$(M-2)b_{\max}+\beta_{\max}\leq x_{i_M}(0)-x_{i_m}(0) \tag{41}$$

from which the number of practical clusters when the stubborn is the unique agent in a cluster satisfies the inequality

$$M\leq\left\lfloor\frac{x_{i_M}(0)-x_{i_m}(0)+2b_{\max}-\beta_{\max}}{b_{\max}}\right\rfloor. \tag{42}$$

We are now ready to prove the convergence of the opinions to the practical consensus around the stubborn provided that sufficiently large thresholds bounds are selected.

**Theorem 11.** *Consider the system (2) with Assumptions 5 for all $i\in\mathcal{I}\setminus\{s\}$ where $s$ is the index of the stubborn, $x_s\in(x_{i_m}(0), x_{i_M}(0))$ and*

$$\beta_{\max}>\max\{x_{i_M}(0)-x_s, x_s-x_{i_m}(0)\} \tag{43}$$

*with $\beta_{\max}$ given by (40). Then, for any $\epsilon\geq 0$ and $\gamma>0$ the system converges to a practical consensus with $\epsilon_c=\epsilon$ around the stubborn opinion in finite time.*

**Proof.** We first prove that if the inequality (43) holds, any steady state solution must be a practical cluster around the stubborn. Assume by contradiction that there exists a steady state solution of the system (2) at a time instant $\hat{k}\in\mathbb{N}_0$ such that

$$\hat{x}_i^{+h}-x_s>\epsilon \tag{44}$$

for some $i\in\mathcal{I}$, for all $h\in\mathbb{N}_0$. Since the non-increasing property of the measure of the convex hull of the opinions holds also in the presence of a stubborn, it must be $x_{i_m}(0)\leq\hat{x}_i^{+h}\leq x_{i_M}(0)$. According to Algorithm 2 at least the maximum agent which might satisfy (44) would increase his thresholds until reaching a value equal or larger to $\beta_{\max}$ in a finite number of steps. Therefore such agent would interact with the stubborn and change his opinion by contradicting the steady state assumption.

In order to prove the convergence to the practical consensus around the stubborn one can use similar arguments of Theorem 9. If $i_M \neq s$ and $i_m \neq s$ for all $k \in \mathbb{N}_0$ the same considerations used in the proof of Theorem 9 hold. Otherwise, if at some time-step it is $i_M = s$ ($i_m = s$) one can use the arguments of Theorem 9 with the consideration that the maximum (minimum) agent does not interact with other agents, but the minimum (maximum) agent is influenced at least by an upper (lower) neighbor and the measure of the convex hull decreases. □

**Remark 12.** By using arguments similar to Theorem 9, it is easy to prove that the results in Theorem 11 hold also by relaxing the condition (43) with a non-strict inequality, provided that there exists at least one agent whose initial condition belongs to the interval $(x_s, x_{i_M}(0))$ and another agent with initial conditions belonging to the interval $(x_{i_m}(0), x_s)$.

## 8. Numerical experiments

In this section we provide a numerical analysis of transient and steady state behaviors of the system (2) with the policy described by Algorithm 1 and Algorithm 2. For simplicity, we consider the same initial values for the thresholds of all agents, i.e. $\ell_i(0) = 0.10$ and $u_i(0) = 0.10$ for all $i \in \mathcal{I}$. This choice does not affect the heterogeneity of the network because during transient the policy will determine different variations of the thresholds among the agents. We consider $N = 100$ agents with uniformly distributed initial opinions $x_i(0) \in [0, 1]$, $i \in \mathcal{I}$. Other policy parameters common to all simulations are $\epsilon = 0.01$ and $\gamma = 0.03$.

Fig. 2 shows the results obtained with and without the proposed policy. In the case (a) the thresholds are constant. The structure of the example allows one to apply Lemma 8 by considering $\ell_{\max} = u_{\max} = 0.10$, which implies a distance value among the clusters not smaller than 0.10. The numerical simulation confirms the results, i.e. there are three practical clusters with a minimum distance equal to 0.20. The second and third plots show that the thresholds variation policy determines a reduction of the number of practical clusters. Moreover in (b) the distance between the two practical clusters is larger than $\ell_{\max}$ in agreement with the inequality (17) of Lemma 8. In the last case, since the conditions of Theorem 9 hold, the practical consensus is reached. The opinions of the agents in the same practical cluster and the practical consensus do not have a unique common value but their relative distance is less than $\varepsilon$. Moreover, as stated in Lemma 6 the opinions evolution shows a non-increasing measure of their convex hull. The distribution of the steady state constant opinions $\hat{x}_i$, $i \in \mathcal{I}$, for different numerical tests corresponding to different values of $\ell_{\max} = u_{\max}$ and uniform initial opinions is shown in Fig. 3. The color denotes the number of agents which are included within a range of opinions of amplitude equal to 0.05. By increasing $\ell_{\max}$ and $u_{\max}$ the number of practical clusters decreases. Moreover, the practical consensus is reached for $\ell_{\max} = u_{\max} \geq 0.18$. The inequality (19) in Lemma 8 leads to $M \leq 5$ which is verified for all simulations.

The sensitivity to different numbers of maximum neighbors can be analyzed through the numerical results shown in Fig. 4. In all numerical tests, the sufficient conditions of Theorem 9 hold and the practical consensus is always reached. The average value of the steady state opinions is shown to be decreasing (increasing) with respect to $\nu_L$ ($\nu_U$); the same trend can be observed for fixed $\nu_L$ and $u_i(0)$ by choosing larger values of $\ell_i(0)$ (or, equivalently, by choosing smaller values of $u_i(0)$, with fixed $\nu_L$ and $\ell_i(0)$). For $\nu_L = 0$ ($\nu_U = 0$), the maximum (minimum) agent becomes a stubborn and the opinions of all agents converge to the similarity

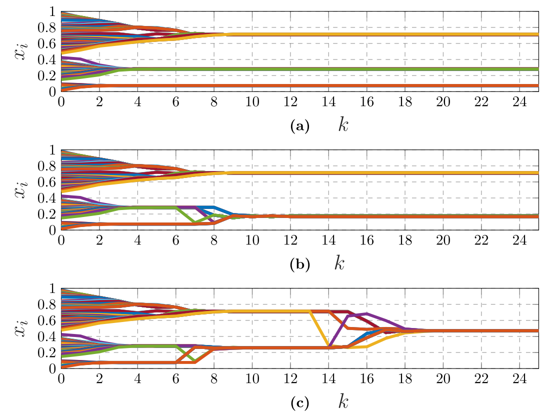


**Fig. 2.** Agents opinions time evolutions in the presence of the proposed policy: (a) only Algorithm 1 is applied, (b-c) complete policy implementation. The parameters are: $\nu_L = \nu_U = 15$ in all tests, $\ell_{\max} = 0.5$ and $u_{\max} = 0.2$ in (b), $\ell_{\max} = i_M(0) - i_m(0)$ and $u_{\max} = \min\{\ell_{\max}, 0.5\}$ in (c).

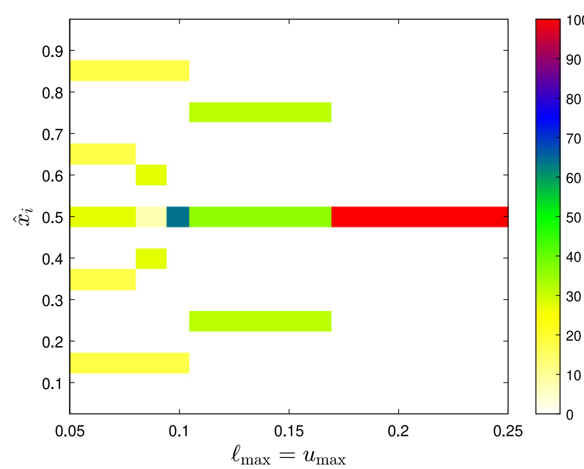

**Fig. 3.** Distribution of the constant steady state agents opinions $\hat{x}_i$, $i \in \mathcal{I}$, by varying $\ell_{\max} = u_{\max} \in [0.05, 0.25]$, $\nu_L = \nu_U = 20$ and a uniform distribution of initial opinions.

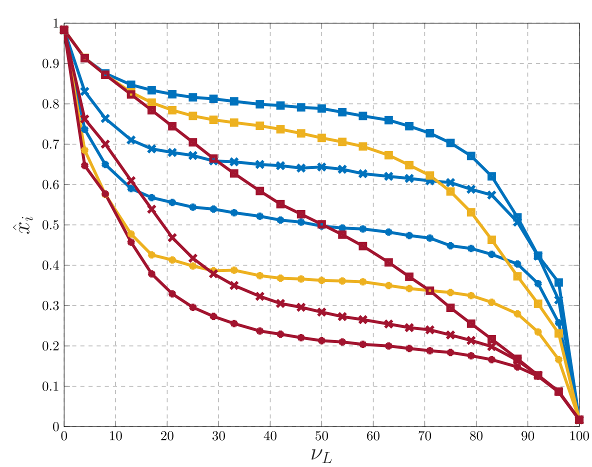

**Fig. 4.** Mean values of the averages of the steady state opinions corresponding to 100 runs with $\nu_L + \nu_U = 100$, $\ell_{\max} = u_{\max} = 1$ and for different values of the initial thresholds: $\ell_i(0) = 0.10$ (blue), $\ell_i(0) = 0.15$ (yellow), $\ell_i(0) = 0.50$ (red), $u_i(0) = 0.10$ (circle), $u_i(0) = 0.15$ (cross), $u_i(0) = 0.50$ (square). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

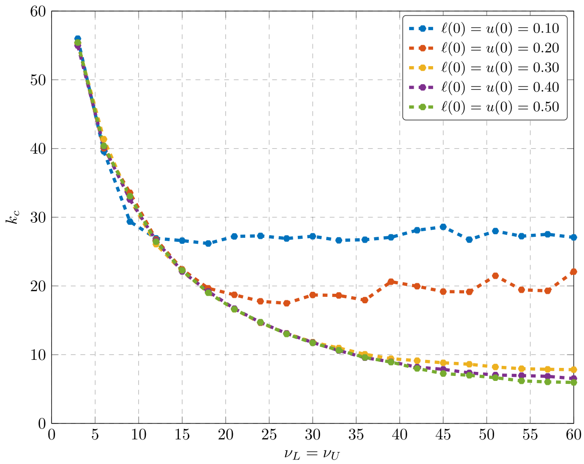


**Fig. 5.** Mean values of the convergence times $k_c$ corresponding to 100 runs for each value of $\nu_L = \nu_U$ and for different values of the initial thresholds, with $\ell_{\max} = u_{\max} = 1$ and uniform random distributions of initial opinions.

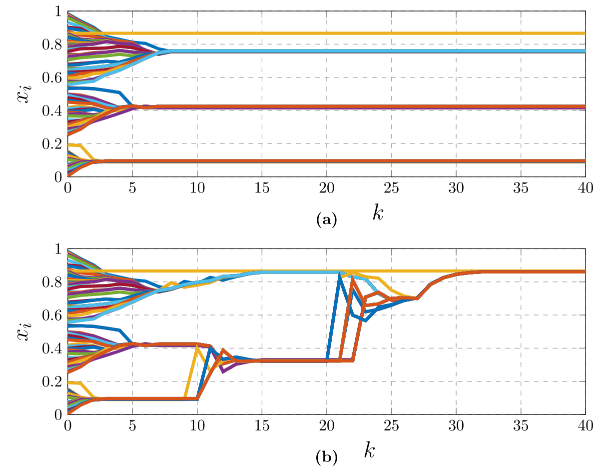


**Fig. 6.** Agents opinions time evolutions in the presence of a stubborn without (a) and with (b) the proposed policy, with $\nu_L = \nu_U = 15$ and $\ell_{\max} = u_{\max} = 1$.

interval of the maximum (minimum) agent. Some curves present, for $\nu_L > N\ell_i(0)$ and $\nu_U > Nu_i(0)$, i.e. for $\nu_L$ such that $\ell_i(0) < \nu_L/N < 1 - u_i(0)$, a sort of plateau corresponding to a low decreasing rate of the steady state opinion. A justification of this behavior can be provided by interpreting $N\ell_i(0)$ ($Nu_i(0)$) as the expected number of lower (upper) neighbors, due to the uniform initial conditions assumption. Finally, for $\ell_i(0) = u_i(0)$, $i \in \mathcal{I}$, and $\nu_L = \nu_U = N/2$ the average opinion is equal to 0.50, which is coherent with the well known preserving average behavior valid for homogeneous symmetric HK models.

In Fig. 5 it is shown the variation of the convergence time $k_c$ to the practical consensus for different values of the maximum number of neighbors. For $\ell_i(0) = u_i(0) \geq 0.30$ for all $i \in \mathcal{I}$, the convergence time is strictly decreasing. Up to $\nu_L = \nu_U = 15$ the convergence time is not much influenced by the initial thresholds. For small initial thresholds the convergence time does not change significantly after certain values of the maximum number of neighbors. The upper bound provided by (35) is about two orders of magnitude larger than the values obtained in the simulations. Although not directly comparable with our results because of the different policies adopted, the upper bound of the convergence time to the consensus proposed in Coulson et al. (2015) is of $\mathcal{O}(N^3)$ which is quite larger than our bound. The tendencies in Fig. 5 do not show the dependence on $\nu_L$ and $\nu_U$ expressed in the upper bound (38), which is obtained by considering the worst case scenario of $\nu_U$ upper ($\nu_L$ lower) neighbors pulling up (down) the opinion of $i_M$ ($i_m$) when he is no longer a maximum (minimum).

The theoretical results regarding the presence of a stubborn are confirmed by the results shown in Fig. 6. In the scenario (a) the confidence thresholds are constant and the stubborn eventually becomes an isolated practical cluster. The proposed policy is applied in the test (b) where the practical consensus around the stubborn opinion is reached, in accordance with Theorem 11.

## 9. Conclusion

The practical clustering and practical consensus concepts introduced in this paper represent a promising approach for the theoretical analysis of a quite general class of opinion dynamics represented by heterogeneous asymmetric HK models. The agents are characterized by a similarity interval which reproduces the fact that two agents with very similar opinions do not influence each other. We have proposed an adaptation policy for the confidence thresholds based on the interacting neighbors of each agent. The opinion dynamics under that policy has been proved to converge in finite time to a maximum number of practical clusters which corresponds to a practical consensus in the case of sufficiently large thresholds bounds. A theoretical analysis in the presence of a stubborn has been also presented. Numerical results have confirmed the theoretical findings. Moreover, the theoretical results presented in the paper can be applied to (or can recover the approach to) the classical (heterogeneous or homogeneous) HK model.

The analysis presented in this paper can inspire interesting future developments: to consider a more general heterogeneous framework with different thresholds bounds for the agents, to find sets of initial conditions such that a desired number of clusters is achieved, to consider weighted and asynchronous connections, to introduce a stubbornness for each agent. Future work will also focus on extending the model by allowing homophilous or heterophilous behaviors for each agent and by applying different behaviors along the components of vector state variables. All such scenarios, even though requiring major modifications of the proposed analytical approach, can take advantage from the similarity interval and practical clustering definitions thus representing directions for future research.

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