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Well-posedness result for the Kuramoto–Velarde equation

Giuseppe Maria Coclite[1] · **Lorenzo di Ruvo**[2]

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Abstract
The Kuramoto–Velarde equation describes slow space-time variations of disturbances at interfaces, diffusion–reaction fronts and plasma instability fronts. It also describes Benard–Marangoni cells that occur when there is large surface tension on the interface in a microgravity environment. Under appropriate assumption on the initial data, of the time T, and the coefficients of such equation, we prove the well-posedness of the classical solutions for the Cauchy problem, associated with this equation.

Keywords Existence · Uniqueness · Stability · Kuramoto–Velarde equation · Cauchy problem

Mathematics Subject Classification 35G25 · 35K55

1 Introduction

In this paper, we investigate the well-posedness of the following Cauchy problem:

$$\begin{cases} \partial_t u + \kappa u \partial_x u + \nu \partial_x^2 u + \delta \partial_x^3 u \\ \qquad +\beta^2 \partial_x^4 u + \gamma (\partial_x u)^2 + \alpha u \partial_x^2 u = 0, & 0 < t < T, \quad x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

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✉ Giuseppe Maria Coclite
giuseppemaria.coclite@poliba.it

Lorenzo di Ruvo
lorenzo.diruvo77@gmail.com

1 Dipartimento di Meccanica, Matematica e Management, Politecnico di Bari, Via E. Orabona 4, Bari 70125, Italy

2 Dipartimento di Matematica, Università di Bari, Via E. Orabona 4, Bari 70125, Italy

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with $\kappa,\ \nu,\ \delta,\ \beta,\ \gamma,\ \alpha \in \mathbb{R}$.

On the initial datum, we assume

$$u_0 \in H^2(\mathbb{R}), \quad u_0 \neq 0 \tag{1.2}$$

and one of the following

$$\beta \neq 0, \quad \beta^2 > \frac{2\tau^2 T}{\log(A_0)}, \tag{1.3}$$

$$\beta \neq 0, \quad 2\|u_0\|^2_{H^1(\mathbb{R})} + 1 < \sup_{\lambda \in \left(0, \frac{1}{4}\right)} \frac{\left(1 + e^{\frac{4\tau^2\sqrt{1-4\lambda}T}{\beta^2}}\right)\sqrt{1-4\lambda}}{e^{\frac{4\tau^2\sqrt{1-4\lambda}T}{\beta^2}} - 1}, \tag{1.4}$$

$$\beta \neq 0, \quad \beta^2 > \left(2\|u_0\|^2_{H^1(\mathbb{R})} + 1\right)\tau^2 T, \tag{1.5}$$

$$\beta \neq 0, \quad \beta^2 > 4\tau^2 T, \quad \|u_0\|^2_{H^1(\mathbb{R})} \leq \frac{-\tau^2 T + \sqrt{\tau^2 T\beta^2 - 3\tau^4 T^2}}{2\tau^2 T}, \tag{1.6}$$

where

$$\tau^2 = \max\left\{\kappa^2 + \alpha^2,\ 2\nu^2,\ \frac{3(\gamma - 2\alpha)^2}{4},\ 1\right\} \neq 0, \quad A_0 = \frac{\|u_0\|^2_{H^1(\mathbb{R})} + 1}{\|u_0\|^2_{H^1(\mathbb{R})}}. \tag{1.7}$$

From a physical point of view, Eq. (1.1), known as the Kuramoto–Velarde equation, describes slow space-time variations of disturbances at interfaces, diffusion–reaction fronts and plasma instability fronts [1–3]. It also describes Benard–Marangoni cells that occur when there is large surface tension on the interface [4–6] in a microgravity environment. This situation arises in crystal growth experiments aboard an orbiting space station, although the free interface is metastable with respect to small perturbations. In particular, the nonlinearities, $\gamma(\partial_x u)^2$ and $\alpha u \partial_x^2 u$, model pressure destabilization effects striving to rupture the interface. Moreover, in [7], (1.1) is deduced to describe the long waves on a viscous fluid flowing down an inclined plane, while, in [8], (1.1) is deduced to model the drift waves in a plasma.

In [9–14] (1.1) is used to model the spinodal decomposition of phase separating systems in an external field, while, in [15–17], (1.1) is used to describe the spatiotemporal evolution of the morphology of steps on crystal surfaces. Finally, in [18–21], (1.1) is deduced to describe the growth of thermodynamically unstable crystal surfaces with strongly anisotropic surface tension.

From a mathematical point of view, in [22], the exact solutions for (1.1) are studied, while in [23], the initial boundary problem is analyzed. In [1,24], the existence of the solitons is proven, while in [25], the existence of traveling wave solutions for (1.1) is analyzed. In [26], the author analyzes the existence of the periodic solution for (1.1), under appropriate assumptions on $\kappa,\ \nu,\ \delta,\ \beta,\ \gamma,\ \alpha$. The well-posedness of the Cauchy problem for (1.1) is proven in [27], using the energy space technique and assuming $\kappa = 0$, and in [28], through a priori estimates together with an application of the Cauchy-Kovalevskaya and choosing

$$\gamma = 2\alpha. \tag{1.8}$$

In particular, in [27], the author gives some suitable conditions on $\nu,\ \delta,\ \beta,\ \gamma,\ \alpha$, and prove the local well-posedness of (1.1), with $\kappa = 0$. Instead, in [28], under Assumptions (1.2) and (1.8), the authors prove well-posedness of (1.1), for each choose of β and T.

Observe that (1.1) generalizes the following equation:

$$\partial_t u + \kappa u \partial_x u + \nu \partial_x^2 u + \delta \partial_x^3 u + \beta^2 \partial_x^4 u = 0, \tag{1.9}$$

that (1.9) was also independently deduced by Kuramoto [29–31] to describe the phase turbulence in reaction-diffusion systems, and by Sivashinsky [32] to describe plane flame propagation, taking into account the combined influence of diffusion and thermal conduction of the gas on the stability of a plane flame front.

Equation (1.9) can be used to study incipient instabilities in several physical and chemical systems [33–35]. Moreover, (1.9), which is also known as the Benney–Lin equation [36,37], was derived by Kuramoto in the study of phase turbulence in Belousov–Zhabotinsky reactions [38].

The dynamical properties and the existence of exact solutions for (1.9) have been investigated in [39–44]. In [45–47], the control problem for (1.9) with periodic boundary conditions, and on a bounded interval are studied, respectively. In [48], the problem of global exponential stabilization of (1.9) with periodic boundary conditions is analyzed. In [49], it is proposed a generalization of optimal control theory for (1.9), while in [50] the problem of global boundary control of (1.9) is considered. In [51], the existence of solitonic solutions for (1.9) is proven. In [28,52–54], the well-posedness of the Cauchy problem for (1.9) is proven, using the energy space technique, the fixed point method, a priori estimates together with an application of the Cauchy–Kovalevskaya Theorem and a priori estimates together with an application of the Aubin–Lions Lemma, respectively. Instead, in [55–57], the initial-boundary value problem for (1.1) is studied, using a priori estimates together with an application of the Cauchy–Kovalevskaya Theorem, and the energy space technique, respectively. Finally, following [58–60], in [61], the convergence of the solution of (1.9) to the unique entropy one of the Burgers equation is proven.

The main result of this paper is the following theorem.

Theorem 1.1 *Assuming that* (1.2) *and one within* (1.3), (1.4), (1.5), (1.6) *hold, there exists a unique solution u of* (1.1), *such that*

$$u \in H^1((0,T)\times\mathbb{R}) \cap L^\infty(0,T;H^2(\mathbb{R})). \tag{1.10}$$

Moreover, if u_1 and u_2 are two solutions of (1.1) *in correspondence of the initial data $u_{1,0}$ and $u_{2,0}$, we have that*

$$\|u_1(t,\cdot) - u_2(t,\cdot)\|_{L^2(\mathbb{R})} \le e^{Ct}\left\|u_{1,0} - u_{2,0}\right\|_{L^2(\mathbb{R})}, \tag{1.11}$$

for some suitable $C > 0$, and every $0 \le t \le T$.

Compared to [27], Theorem 1.1 gives some conditions on u_0, β and T to have classical solutions for (1.1), under Assumption (1.2). Moreover, the argument of Theorem 1.1 relies on deriving suitable a priori estimates together with the existence result in [28].

The paper is organized as follows. In Sect. 2, we prove some a priori estimates of (1.1), under Assumptions (1.3), (1.4), (1.5) and (1.6), respectively. Those play a key role in the proof of our main result, which is given in Sect. 3.

2 A priori estimates

In this section, we prove some a priori estimates on u.

We prove the following result.

Lemma 2.1 *We have that*

$$\frac{\|u(t,\cdot)\|^2_{H^1(\mathbb{R})}}{\|u(t,\cdot)\|^2_{H^1(\mathbb{R})}+1} \le \frac{\|u_0\|^2_{H^1(\mathbb{R})}}{\|u_0\|^2_{H^1(\mathbb{R})}+1} e^{\frac{2\tau^2 t}{\beta^2}}, \tag{2.1}$$

for every $0 \le t \le T$*, where* τ^2 *is defined in* (1.7)*. In particular, if* (1.3) *holds, there exists a constant* $C > 0$*, such that*

$$\|u(t,\cdot)\|^2_{H^1(\mathbb{R})} \le C, \tag{2.2}$$

for every $0 \le t \le T$*. Moreover,*

$$\|u\|_{L^\infty((0,T)\times\mathbb{R})} \le C, \tag{2.3}$$

$$\int_0^t \left\|\partial_x^2 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \le C, \tag{2.4}$$

$$\int_0^t \left\|\partial_x^3 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \le C, \tag{2.5}$$

$$\int_0^t \|\partial_x u(s,\cdot)\|^4_{L^4(\mathbb{R})} ds \le C, \tag{2.6}$$

for every $0 \le t \le T$*.*

Proof Let $0 \le t \le T$ be given. Multiplying (1.1) by $2u - 2\partial_x^2 u$, an integration on $\mathbb{R}$ gives

$$\begin{aligned}
&\frac{d}{dt}\left(\|u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}\right)\\
&\quad = 2\int_{\mathbb{R}} u\partial_x u dx - 2\int_{\mathbb{R}} \partial_x^2 u \partial_t u dx\\
&\quad = -2\kappa\int_{\mathbb{R}} u^2\partial_x u dx + 2\kappa\int_{\mathbb{R}} u\partial_x u\partial_x^2 u dx - 2\nu\int_{\mathbb{R}} u\partial_x^2 u dx\\
&\qquad + 2\nu\int_{\mathbb{R}} (\partial_x^2 u)^2 dx - 2\delta\int_{\mathbb{R}} u\partial_x^3 u dx + 2\delta\int_{\mathbb{R}} \partial_x^2 u\partial_x^3 u dx\\
&\qquad - 2\beta^2\int_{\mathbb{R}} u\partial_x^4 u dx + 2\beta\int_{\mathbb{R}} \partial_x^2 u\partial_x^3 u dx - 2\gamma\int_{\mathbb{R}} u(\partial_x u)^2 dx\\
&\qquad + 2\gamma\int_{\mathbb{R}} (\partial_x u)^2\partial_x^2 u dx - 2\alpha\int_{\mathbb{R}} u^2\partial_x^2 u dx + 2\alpha\int_{\mathbb{R}} u(\partial_x^2 u)^2 dx\\
&\quad = 2\kappa\int_{\mathbb{R}} u\partial_x u\partial_x^2 u dx - 2\nu\int_{\mathbb{R}} u\partial_x^2 u dx + 2\nu\int_{\mathbb{R}} (\partial_x^2 u)^2 dx\\
&\qquad + 2\delta\int_{\mathbb{R}} \partial_x u\partial_x^2 u dx + 2\beta^2\int_{\mathbb{R}} \partial_x u\partial_x^3 u dx - 2\beta^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\qquad - 2\gamma\int_{\mathbb{R}} u(\partial_x u)^2 dx - 2\alpha\int_{\mathbb{R}} u^2\partial_x^2 u dx + 2\alpha\int_{\mathbb{R}} u(\partial_x^2 u)^2 dx\\
&\quad = 2\kappa\int_{\mathbb{R}} u\partial_x u\partial_x^2 u dx - 2\nu\int_{\mathbb{R}} u\partial_x^2 u dx + 2\nu\int_{\mathbb{R}} (\partial_x^2 u)^2 dx\\
&\qquad - 2\beta^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - 2\beta^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - 2\gamma\int_{\mathbb{R}} u(\partial_x u)^2 dx\\
&\qquad - 2\alpha\int_{\mathbb{R}} u^2\partial_x^2 u dx + 2\alpha\int_{\mathbb{R}} u(\partial_x^2 u)^2 dx.
\end{aligned}$$

Therefore,

$$
\begin{aligned}
&\frac{d}{dt}\left(\|u(t,\cdot)\|^2_{L^2(\mathbb{R})}+\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}\right)\\
&\quad+2\beta^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+2\beta^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad=2\kappa\int_{\mathbb{R}}u\partial_x u\partial_x^2 u dx-2\nu\int_{\mathbb{R}}u\partial_x^2 u dx+2\nu\int_{\mathbb{R}}(\partial_x^2 u)^2 dx\\
&\quad-2\gamma\int_{\mathbb{R}}u(\partial_x u)^2 dx-2\alpha\int_{\mathbb{R}}u^2\partial_x^2 u dx+2\alpha\int_{\mathbb{R}}u(\partial_x^2 u)^2 dx.
\end{aligned}
\tag{2.7}
$$

Observe that

$$
\begin{aligned}
2\nu\int_{\mathbb{R}}(\partial_x^2 u)^2 dx&=2\nu\int_{\mathbb{R}}\partial_x^2 u\partial_x^2 u dx=-2\nu\int_{\mathbb{R}}\partial_x u\partial_x^3 u dx,\\
2\alpha\int_{\mathbb{R}}u(\partial_x^2 u)^2 dx&=2\alpha\int_{\mathbb{R}}u\partial_x^2 u\partial_x^2 u dx=-2\alpha\int_{\mathbb{R}}u\partial_x u\partial_x^3 u dx.
\end{aligned}
\tag{2.8}
$$

Moreover,

$$
\begin{aligned}
\int_{\mathbb{R}}u(\partial_x u)^2 dx&=\int_{\mathbb{R}}u\partial_x u\partial_x u dx=-\int_{\mathbb{R}}u\partial_x(u\partial_x u)dx\\
&=-\int_{\mathbb{R}}u(\partial_x u)^2 dx-\int_{\mathbb{R}}u^2\partial_x^2 u dx.
\end{aligned}
$$

Therefore,

$$
2\int_{\mathbb{R}}u(\partial_x u)^2 dx=-\int_{\mathbb{R}}u^2\partial_x^2 u dx.
\tag{2.9}
$$

Consequently, by (2.8) and (2.9), we have that

$$
\begin{aligned}
&\frac{d}{dt}\left(\|u(t,\cdot)\|^2_{L^2(\mathbb{R})}+\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}\right)\\
&\quad+2\beta^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+2\beta^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad=2\kappa\int_{\mathbb{R}}u\partial_x u\partial_x^2 u dx-2\nu\int_{\mathbb{R}}u\partial_x^2 u dx-2\nu\int_{\mathbb{R}}\partial_x u\partial_x^3 u dx\\
&\quad+(\gamma-2\alpha)\int_{\mathbb{R}}u^2\partial_x^2 u dx-2\alpha\int_{\mathbb{R}}u\partial_x u\partial_x^3 u dx.
\end{aligned}
\tag{2.10}
$$

Due to the Young inequality,

$$
\begin{aligned}
2|\kappa|\int_{\mathbb{R}}|u||\partial_x u||\partial_x^2 u|dx&=2\int_{\mathbb{R}}\left|\frac{\kappa u\partial_x u}{\beta}\right|\left|\beta\partial_x^2 u\right|dx\\
&\le\frac{\kappa^2}{\beta^2}\int_{\mathbb{R}}u^2(\partial_x u)^2 dx+\beta^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\le\frac{\kappa^2}{\beta^2}\|u(t,\cdot)\|^2_{L^\infty(\mathbb{R})}\|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}+\beta^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
2|\nu|\int_{\mathbb{R}}|u||\partial_x^2 u|dx&=2\int_{\mathbb{R}}\left|\frac{\nu u}{\beta}\right|\left|\beta\partial_x^2 u\right|dx\\
&\le\frac{2\nu^2}{\beta^2}\|u(t,\cdot)\|^2_{L^2(\mathbb{R})}+\frac{\beta^2}{2}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},
\end{aligned}
$$

$$
\begin{aligned}
2|\nu|\int_{\mathbb{R}}|\partial_x u||\partial_x^3 u|dx &= \int_{\mathbb{R}}\left|\frac{2\nu\partial_x u}{\beta}\right|\left|\beta\partial_x^3 u\right|dx\\
&\leq \frac{2\nu^2}{\beta^2}\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{\beta^2}{2}\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
|\gamma-2\alpha|\int_{\mathbb{R}}u^2|\partial_x^2 u|dx &= 2\int_{\mathbb{R}}\left|\frac{\sqrt{3}(\gamma-2\alpha)u^2}{2\beta}\right|\left|\frac{\beta\partial_x^2 u}{\sqrt{3}}\right|dx\\
&\leq \frac{3(\gamma-2\alpha)^2}{4\beta^2}\int_{\mathbb{R}}u^4dx+\frac{\beta^2}{3}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\leq \frac{3(\gamma-\alpha)^2}{4\beta^2}\left\|u(t,\cdot)\right\|^2_{L^\infty(\mathbb{R})}\left\|u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{\beta^2}{3}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
2|\alpha|\int_{\mathbb{R}}|u\partial_x u||\partial_x^3 u|dx &= 2\int_{\mathbb{R}}\left|\frac{\alpha u\partial_x u}{\beta}\right|\left|\beta\partial_x^3 u\right|dx\\
&\leq \frac{\alpha^2}{\beta^2}\int_{\mathbb{R}}u^2(\partial_x u)^2dx+\beta^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\leq \frac{\alpha^2}{\beta^2}\left\|u(t,\cdot)\right\|^2_{L^\infty(\mathbb{R})}\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\beta^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

It follows from (1.3) and (2.7) that

$$
\begin{aligned}
&\frac{d}{dt}\left(\left\|u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right)\\
&\quad+\frac{\beta^2}{6}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{\beta^2}{2}\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\leq \frac{\kappa^2+\alpha^2}{\beta^2}\left\|u(t,\cdot)\right\|^2_{L^\infty(\mathbb{R})}\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad+\frac{2\nu^2}{\beta^2}\left(\left\|u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right)\\
&\quad+\frac{3(\gamma-2\alpha)^2}{4\beta^2}\left\|u(t,\cdot)\right\|^2_{L^\infty(\mathbb{R})}\left\|u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\leq \frac{\kappa^2+\alpha^2}{\beta^2}\left\|u(t,\cdot)\right\|^2_{L^\infty(\mathbb{R})}\left(\left\|u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right)\\
&\quad+\frac{2\nu^2}{\beta^2}\left(\left\|u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right)\\
&\quad+\frac{3(\gamma-2\alpha)^2}{4\beta^2}\left\|u(t,\cdot)\right\|^2_{L^\infty(\mathbb{R})}\left(\left\|u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right)\\
&\quad+\frac{1}{\beta^2}\left(\left\|u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\right).
\end{aligned}
\tag{2.11}
$$

Thanks to the Hölder inequality,

$$
u^2(t,x)=2\int_{-\infty}^{x}u\partial_x u dx\leq 2\int_{\mathbb{R}}|u||\partial_x u|dx\leq 2\left\|u(t,\cdot)\right\|_{L^2(\mathbb{R})}\left\|\partial_x u(t,\cdot)\right\|_{L^2(\mathbb{R})}.
$$

Hence,

$$
\left\|u(t,\cdot)\right\|^2_{L^\infty(\mathbb{R})}\leq 2\left\|u(t,\cdot)\right\|_{L^2(\mathbb{R})}\left\|\partial_x u(t,\cdot)\right\|_{L^2(\mathbb{R})}. \tag{2.12}
$$

Therefore, by the Young inequality,

$$\|u(t,\cdot)\|^2_{L^\infty(\mathbb{R})} \le \|u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}. \tag{2.13}$$

Consequently, by (2.11),

$$\begin{aligned}
&\frac{d}{dt}\left(\|u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}\right)\\
&\quad + \frac{\beta^2}{6}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\beta^2}{2}\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\le \frac{\kappa^2+\alpha^2}{\beta^2}\left(\|u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}\right)^2\\
&\quad + \frac{2\nu^2}{\beta^2}\left(\|u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}\right)\\
&\quad + \frac{3(\gamma-2\alpha)^2}{\beta^2}\left(\|u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}\right)^2\\
&\quad + \frac{1}{\beta^2}\left(\|u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}\right).
\end{aligned} \tag{2.14}$$

We define the following function

$$X(t) := \|u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})} = \|u(t,\cdot)\|^2_{H^1(\mathbb{R})} > 0. \tag{2.15}$$

It follows from (1.7), (2.14) and (2.15) that

$$\frac{dX(t)}{dt} + \frac{\beta^2}{6}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\beta^2}{2}\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \le \frac{2\tau^2}{\beta^2}X(t)\left(X(t)+1\right). \tag{2.16}$$

Therefore, we have that

$$\frac{dX(t)}{dt} \le \frac{2\tau^2}{\beta^2}X(t)\left(X(t)+1\right). \tag{2.17}$$

Consequently, by (2.15),

$$\frac{1}{X(t)\left(X(t)+1\right)}\frac{dX(t)}{dt} \le \frac{2\tau^2}{\beta^2}.$$

Integrating on $(0,t)$, we have that

$$\log\left(\frac{X(t)}{X(t)+1}\right) - \log\left(\frac{X_0}{X_0+1}\right) \le \frac{2\tau^2 t}{\beta^2}.$$

Hence, by (1.2) and (2.15), we get

$$\log\left(\frac{\|u(t,\cdot)\|^2_{H^1(\mathbb{R})}}{\|u(t,\cdot)\|^2_{H^1(\mathbb{R})}+1}\right) \le \log\left(\frac{\|u_0\|^2_{H^1(\mathbb{R})}}{\|u_0\|^2_{H^1(\mathbb{R})}+1}\right) + \frac{2\tau^2 t}{\beta^2},$$

which gives (2.1).

Assume (1.3) and we prove (2.2). We begin by observing that, by (2.1),

$$\frac{\|u(t,\cdot)\|^2_{H^1(\mathbb{R})}}{\|u(t,\cdot)\|^2_{H^1(\mathbb{R})}+1} \le \frac{\|u_0\|^2_{H^1(\mathbb{R})}}{\|u_0\|^2_{H^1(\mathbb{R})}+1}e^{\frac{2\tau^2 T}{\beta^2}}. \tag{2.18}$$

We assume by contradiction that (2.2) does not hold, i.e.,

$$\|u(t,\cdot)\|^2_{H^1(\mathbb{R})} = \infty. \tag{2.19}$$

Therefore, by (2.18) and (2.19),

$$\lim_{\|u(t,\cdot)\|^2_{H^1(\mathbb{R})}\to\infty} \frac{\|u(t,\cdot)\|^2_{H^1(\mathbb{R})}}{\|u(t,\cdot)\|^2_{H^1(\mathbb{R})}+1} = 1 \le \frac{\|u_0\|^2_{H^1(\mathbb{R})}}{\|u_0\|^2_{H^1(\mathbb{R})}+1} e^{\frac{2\tau^2 T}{\beta^2}}. \tag{2.20}$$

It follows from (1.7) and (2.20) that

$$\beta^2 \le \frac{2\tau^2 T}{\log(A_0)},$$

which contradicts (1.3).

We prove (2.3). Due to (2.2) and (2.13), we have that

$$\|u(t,\cdot)\|^2_{L^\infty(\mathbb{R})} \le C.$$

Hence,

$$\|u\|^2_{L^\infty((0,T)\times\mathbb{R})} \le C,$$

which gives (2.3).

We prove (2.4) and (2.5). Thanks to (2.2) and (2.14),

$$\begin{aligned}\frac{d}{dt}&\left(\|u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}\right)\\ &+ \frac{\beta^2}{6}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\beta^2}{2}\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \le C.\end{aligned}$$

Integrating on $(0,t)$, by (1.2), we get

$$\begin{aligned}\|u(t,\cdot)\|^2_{L^2(\mathbb{R})} &+ \|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}\\ &+ \frac{\beta^2}{6}\int_0^t \left\|\partial_x^2 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds + \frac{\beta^2}{2}\int_0^t \left\|\partial_x^3 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds\\ &\le C_0 + Ct \le C,\end{aligned}$$

which gives (2.4) and (2.5), respectively.

Finally, we prove (2.6). We begin by observing that [62] [Lemma 2.3] says that

$$\|\partial_x u(t,\cdot)\|^4_{L^4(\mathbb{R})} \le 6\left(\|u(t,\cdot)\|^2_{L^2(\mathbb{R})} + \|\partial_x u(t,\cdot)\|^2_{L^2(\mathbb{R})}\right)\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

Consequently, by (2.2),

$$\|\partial_x u(t,\cdot)\|^4_{L^4(\mathbb{R})} \le C\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

Integrating on $(0,t)$, by (2.4), we have (2.6). □

Lemma 2.2 *We have that*

$$\frac{2\|u(t,\cdot)\|^2_{H^1(\mathbb{R})} + 1 - \sqrt{1-4\lambda}}{2\|u(t,\cdot)\|^2_{H^1(\mathbb{R})} + 1 + \sqrt{1-4\lambda}} \le \frac{2\|u_0\|^2_{H^1(\mathbb{R})} + 1 - \sqrt{1-4\lambda}}{2\|u_0\|^2_{H^1(\mathbb{R})} + 1 + \sqrt{1-4\lambda}} e^{\frac{4\tau^2\sqrt{1-4\lambda}t}{\beta^2}}, \tag{2.21}$$

for every $0 \le t \le T$ *and some* $\lambda \in (0, 1/4)$. *In particular, under Assumption* (1.4)*, we have* (2.2)*,* (2.3)*,* (2.4)*,* (2.5) *and* (2.6)*.*

Proof Arguing as in Lemma 2.1, we have (2.17). Therefore, by (1.4), (1.7) and (2.17), we have that

$$\frac{dX(t)}{dt} \le \frac{2\tau^2}{\beta^2} X(t)\,(X(t)+1) \le \frac{2\tau^2}{\beta^2}\left(X^2(t)+X(t)+\lambda\right).$$

Hence,

$$\frac{1}{2\left(X^2(t)+X(t)+\lambda\right)}\frac{dX(t)}{dt} \le \frac{\tau^2}{\beta^2}.$$

Thanks to (1.4) and (1.7), an integration on $(0,t)$ gives

$$\log\left(\frac{2X(t)+1-\sqrt{1-\lambda}}{2X(t)+1+\sqrt{1-4\lambda}}\right) - \log\left(\frac{2X_0+1-\sqrt{1-4\lambda}}{2X_0+1+\sqrt{1-4\lambda}}\right) \le \frac{4\tau^2\sqrt{1-4\lambda}t}{\beta^2}.$$

Consequently,

$$\log\left(\frac{2X(t)+1-\sqrt{1-4\lambda}}{2X(t)+1+\sqrt{1-4\lambda}}\right) \le \log\left(\frac{2X_0+1-\sqrt{1-4\lambda}}{2X_0+1+\sqrt{1-4\lambda}}\right) + \frac{4\tau^2\sqrt{1-4\lambda}t}{\beta^2}. \tag{2.22}$$

(1.2), (2.15) and (2.22) give (2.21).

We assume (1.4) and we prove (2.2). We begin by observing that, by (2.21), we have that

$$\frac{2\left\|u(t,\cdot)\right\|^2_{H^1(\mathbb{R})}+1-\sqrt{1-4\lambda}}{2\left\|u(t,\cdot)\right\|^2_{H^1(\mathbb{R})}+1+\sqrt{1-4\lambda}} \le \frac{2\left\|u_0\right\|^2_{H^1(\mathbb{R})}+1-\sqrt{1-4\lambda}}{2\left\|u_0\right\|^2_{H^1(\mathbb{R})}+1+\sqrt{1-4\lambda}} e^{\frac{4\tau^2\sqrt{1-4\lambda}T}{\beta^2}}. \tag{2.23}$$

We assume by contradiction that (2.2) does not hold, i.e., we have (2.19). Consequently, by (2.19) and (2.23),

$$\begin{aligned}\lim_{\|u(t,\cdot)\|^2_{H^1(\mathbb{R})}\to\infty} &\frac{2\left\|u(t,\cdot)\right\|^2_{H^1(\mathbb{R})}+1-\sqrt{1-4\lambda}}{2\left\|u(t,\cdot)\right\|^2_{H^1(\mathbb{R})}+1+\sqrt{1-4\lambda}}\\ =1 &\le \frac{2\left\|u_0\right\|^2_{H^1(\mathbb{R})}+1-\sqrt{1-4\lambda}}{2\left\|u_0\right\|^2_{H^1(\mathbb{R})}+1+\sqrt{1-4\lambda}} e^{\frac{4\tau^2\sqrt{1-4\lambda}T}{\beta^2}}.\end{aligned} \tag{2.24}$$

Therefore, to (2.24), we have that

$$2\left\|u_0\right\|^2_{H^1(\mathbb{R})}+1+\sqrt{1-4\lambda} \le \left(2\left\|u_0\right\|^2_{H^1(\mathbb{R})}+1-\sqrt{1-4\lambda}\right) e^{\frac{4\tau^2\sqrt{1-4\lambda}T}{\beta^2}},$$

that is

$$\left(1-e^{\frac{4\tau^2\sqrt{1-4\lambda}T}{\beta^2}}\right)\left(2\left\|u_0\right\|^2_{H^1(\mathbb{R})}+1\right) \le -\left(1+e^{\frac{4\tau^2\sqrt{1-4\lambda}T}{\beta^2}}\right)\sqrt{1-4\lambda}.$$

Hence,

$$2\left\|u_0\right\|^2_{H^1(\mathbb{R})}+1 \ge \frac{\left(1+e^{\frac{4\tau^2\sqrt{1-4\lambda}T}{\beta^2}}\right)\sqrt{1-4\lambda}}{e^{\frac{4\tau^2\sqrt{1-4\lambda}T}{\beta^2}}-1},$$

which contradicts (1.4).

Finally, arguing as in Lemma 2.1, we have (2.3), (2.4), (2.5) and (2.6). □

Lemma 2.3 *We have that*

$$\frac{-1}{2\left\|u(t,\cdot)\right\|^2_{H^1(\mathbb{R})}+1} \le \frac{-1}{2\left\|u_0\right\|^2_{H^1(\mathbb{R})}+1} + \frac{\tau^2 t}{\beta^2} \tag{2.25}$$

for every $0 \le t \le T$. *In particular, if* (1.5) *holds, we have* (2.2), (2.3), (2.4), (2.5) *and* (2.6).

Proof Arguing as in Lemma 2.1, we have (2.17). Therefore, by (2.17), we have that

$$\frac{dX(t)}{dt} \le \frac{2\tau^2}{\beta^2}\left(X^2(t)+X(t)+\frac{1}{4}\right) = \frac{\tau^2}{2\beta^2}\left(2X(t)+1\right)^2.$$

Therefore,

$$\frac{2}{(2X(t)+1)^2}\frac{dX(t)}{dt} \le \frac{\tau^2}{\beta^2}.$$

Integrating on $(0,t)$, we have that

$$\frac{-1}{2X(t)+1} + \frac{1}{2X_0+1} \le \frac{\tau^2 t}{\beta^2},$$

that is,

$$\frac{-1}{2X(t)+1} \le \frac{-1}{2X_0+1} + \frac{\tau^2 t}{\beta^2}. \tag{2.26}$$

(2.25) follows from (1.2) and (2.15).

Assume (1.5) and we prove (2.2). We begin by observing that, by (2.25), we have that

$$\frac{-1}{2\left\|u(t,\cdot)\right\|^2_{H^1(\mathbb{R})}+1} \le \frac{-1}{2\left\|u_0\right\|^2_{H^1(\mathbb{R})}+1} + \frac{\tau^2 T}{\beta^2}. \tag{2.27}$$

We assume by contradiction that (2.2) does not hold, i.e., we have (2.19). It follows from (2.19) and (2.27) that

$$\lim_{\left\|u(t,\cdot)\right\|^2_{H^1(\mathbb{R})}\to\infty} \frac{-1}{2\left\|u(t,\cdot)\right\|^2_{H^1(\mathbb{R})}+1} = 0 \le \frac{-1}{2\left\|u_0\right\|^2_{H^1(\mathbb{R})}+1} + \frac{\tau^2 T}{\beta^2}. \tag{2.28}$$

Consequently, by (2.28), we have that

$$\beta^2 \le \left(2\left\|u_0\right\|^2_{H^1(\mathbb{R})}+1\right)\tau^2 T,$$

which contradicts (1.5).

Finally, arguing as in Lemma 2.1, we have (2.3), (2.4), (2.5) and (2.6). □

Lemma 2.4 *We have that*

$$\arctan\left(\frac{2\left\|u(t,\cdot)\right\|^2_{H^1(\mathbb{R})}+1}{\sqrt{4\lambda-1}}\right) \le \arctan\left(\frac{2\left\|u_0\right\|^2_{H^1(\mathbb{R})}+1}{\sqrt{4\lambda-1}}\right) + \frac{\tau^2\sqrt{4\lambda-1}t}{\beta^2}, \tag{2.29}$$

for every $0 \le t \le T$, *where*

$$\lambda > \frac{1}{4}. \tag{2.30}$$

In particular, assuming (1.6) *and taking*

$$\lambda \in \left(\frac{1}{4}, \frac{\beta^2 - 4\tau^2 T\left\|u_0\right\|^4_{H^1(\mathbb{R})} - 4\tau^2 T\left\|u_0\right\|^2_{H^1(\mathbb{R})}}{4\tau^2 T}\right), \tag{2.31}$$

we obtain the following inequality

$$\frac{2\left\|u(t,\cdot)\right\|_{H^1(\mathbb{R})}^2+1}{\sqrt{4\lambda-1}} \le \tan\left(\arctan\left(\frac{2\left\|u_0\right\|_{H^1(\mathbb{R})}^2+1}{\sqrt{4\lambda-1}}\right)+\frac{2\tau^2\sqrt{4\lambda-1}T}{\beta^2}\right), \tag{2.32}$$

where

$$\arctan\left(\frac{2\left\|u_0\right\|_{H^1(\mathbb{R})}^2+1}{\sqrt{4\lambda-1}}\right)+\frac{2\tau^2\sqrt{4\lambda-1}T}{\beta^2} \in \left(0,\ \frac{\pi}{2}\right). \tag{2.33}$$

Moreover, (2.2)*,* (2.3)*,* (2.4)*,* (2.5) *and* (2.6)*, hold.*

Proof Arguing as in Lemma 2.1, we have (2.17). Therefore, by (2.17) and (2.30), we have that

$$\frac{dX(t)}{dt} \le \frac{2\tau^2}{\beta^2}\left(X^2(t)+X(t)\right) \le \frac{2\tau^2}{\beta^2}\left(X^2(t)+X(t)+\lambda\right).$$

Hence,

$$\frac{1}{X^2(t)+X(t)+\lambda}\frac{dX(t)}{dt} \le \frac{2\tau^2}{\beta^2}.$$

Integrating on $(0,t)$, we have that

$$\frac{2}{\sqrt{4\lambda-1}}\arctan\left(\frac{2X(t)+1}{\sqrt{4\lambda-1}}\right)-\frac{2}{\sqrt{4\lambda-1}}\arctan\left(\frac{2X_0+1}{\sqrt{4\lambda-1}}\right) \le \frac{2\tau^2 t}{\beta^2}$$

Hence,

$$\arctan\left(\frac{2X(t)+1}{\sqrt{4\lambda-1}}\right) \le \arctan\left(\frac{2X_0+1}{\sqrt{4\lambda-1}}\right)+\frac{\tau^2\sqrt{4\lambda-1}t}{\beta^2}. \tag{2.34}$$

(1.2), (2.15) and (2.34) give (2.29).

Assume (1.6) and we prove (2.32). We begin by observe that, by (2.29),

$$\arctan\left(\frac{2\left\|u(t,\cdot)\right\|_{H^1(\mathbb{R})}^2+1}{\sqrt{4\lambda-1}}\right) \le \arctan\left(\frac{2\left\|u_0\right\|_{H^1(\mathbb{R})}^2+1}{\sqrt{4\lambda-1}}\right)+\frac{\tau^2\sqrt{4\lambda-1}T}{\beta^2}. \tag{2.35}$$

We assume by contradiction that (2.2) does not hold, i.e., we have (2.19). It follows from (2.19) and (2.35) that

$$\begin{aligned}&\lim_{\left\|u(t,\cdot)\right\|_{H^1(\mathbb{R})}^2\to\infty}\arctan\left(\frac{2\left\|u(t,\cdot)\right\|_{H^1(\mathbb{R})}^2+1}{\sqrt{4\lambda-1}}\right)\\&=\frac{\pi}{2} \le \arctan\left(\frac{2\left\|u_0\right\|_{H^1(\mathbb{R})}^2+1}{\sqrt{4\lambda-1}}\right)+\frac{\tau^2\sqrt{4\lambda-1}T}{\beta^2}.\end{aligned} \tag{2.36}$$

Therefore, we obtain that

$$\frac{\pi}{2}-\arctan\left(\frac{2\left\|u_0\right\|_{H^1(\mathbb{R})}^2+1}{\sqrt{4\lambda-1}}\right)-\frac{\tau^2\sqrt{4\lambda-1}T}{\beta^2} \le 0,$$

that is

$$\arctan\left(\frac{2\left\|u_0\right\|_{H^1(\mathbb{R})}^2+1}{\sqrt{4\lambda-1}}\right)+\frac{\tau^2\sqrt{4\lambda-1}T}{\beta^2}-\frac{\pi}{2} \ge 0. \tag{2.37}$$

We consider the following function

$$F(\lambda)=\arctan\left(\frac{2\left\|u_0\right\|^2_{H^1(\mathbb{R})}+1}{\sqrt{4\lambda-1}}\right)+\frac{\tau^2\sqrt{4\lambda-1}T}{\beta^2}-\frac{\pi}{2}, \tag{2.38}$$

with

$$\lambda\in\left(\frac{1}{4},\,\infty\right). \tag{2.39}$$

Observe that, by (2.38) and (2.39),

$$\begin{aligned}\lim_{\lambda\to\frac{1}{4}^+}F(\lambda)=&\lim_{\lambda\to\frac{1}{4}^+}\arctan\left(\frac{2\left\|u_0\right\|^2_{H^1(\mathbb{R})}+1}{\sqrt{4\lambda-1}}\right)+\frac{\tau^2\sqrt{4\lambda-1}T}{\beta^2}-\frac{\pi}{2}=0,\\ \lim_{\lambda\to\infty}F(\lambda)=&\lim_{\lambda\to\infty}\arctan\left(\frac{2\left\|u_0\right\|^2_{H^1(\mathbb{R})}+1}{\sqrt{4\lambda-1}}\right)+\frac{\tau^2\sqrt{4\lambda-1}T}{\beta^2}-\frac{\pi}{2}=\infty.\end{aligned} \tag{2.40}$$

Consequently, by (2.38) and (2.40), (2.37) is verified if

$$F'(\lambda)>0\quad\text{for}\quad\lambda>\frac{1}{4}. \tag{2.41}$$

Observe that, by (2.38),

$$\begin{aligned}F'(\lambda)=&\frac{-1}{\frac{\left(2\left\|u_0\right\|^2_{H^1(\mathbb{R})}+1\right)^2}{4\lambda-1}+1}\frac{1}{4\lambda-1}\frac{2}{\sqrt{4\lambda-1}}+\frac{\tau^2T}{\beta^2}\frac{2}{\sqrt{4\lambda-1}}\\ =&\frac{-1}{4\left\|u_0\right\|^2_{H^1(\mathbb{R})}\left(\left\|u_0\right\|^2_{H^1(\mathbb{R})}+1\right)+4\lambda}\frac{2}{\sqrt{4\lambda-1}}+\frac{\tau^2T}{\beta^2}\frac{2}{\sqrt{4\lambda-1}}.\end{aligned} \tag{2.42}$$

Hence, $F'(\lambda)>0$ if and only if

$$\frac{-1}{4\left\|u_0\right\|^2_{H^1(\mathbb{R})}\left(\left\|u_0\right\|^2_{H^1(\mathbb{R})}+1\right)+4\lambda}\frac{2}{\sqrt{4\lambda-1}}+\frac{\tau^2T}{\beta^2}\frac{2}{\sqrt{4\lambda-1}}\geq 0,$$

that is

$$4\tau^2T\lambda+4\tau^2T\left\|u_0\right\|^4_{H^1(\mathbb{R})}+4\tau^2T\left\|u_0\right\|^2_{H^1(\mathbb{R})}-\beta^2\geq 0. \tag{2.43}$$

Thanks to (2.39), (2.43) is verified when

$$4\tau^2T\left\|u_0\right\|^4_{H^1(\mathbb{R})}+4\tau^2T\left\|u_0\right\|^2_{H^1(\mathbb{R})}-\beta^2\geq 0,$$

that is

$$\left\|u_0\right\|^2_{H^1(\mathbb{R})}\geq\frac{-\tau^2T+\sqrt{\tau^4T^2+\tau^2T\beta^2}}{2\tau^2T},$$

which contradicts (1.6).

Therefore, if we assume (1.6), (2.37) cannot hold. Observe that, by (2.42), $F'(\lambda)\leq 0$ when $\lambda>\frac{1}{4}$, if and only if,

$$4\tau^2T\lambda+4\tau^2T\left\|u_0\right\|^4_{H^1(\mathbb{R})}+4\tau^2T\left\|u_0\right\|^2_{H^1(\mathbb{R})}-\beta^2\leq 0,$$

that is

$$\lambda\leq\frac{\beta^2-4\tau^2T\left\|u_0\right\|^4_{H^1(\mathbb{R})}-4\tau^2T\left\|u_0\right\|^2_{H^1(\mathbb{R})}}{4\tau^2T}. \tag{2.44}$$

It follows from (2.30) and (2.44) that

$$\frac{1}{4} < \lambda \leq \frac{\beta^2 - 4\tau^2 T \|u_0\|^4_{H^1(\mathbb{R})} - 4\tau^2 T \|u_0\|^2_{H^1(\mathbb{R})}}{4\tau^2 T}.$$

λ exist, if

$$\frac{1}{4} < \frac{\beta^2 - 4\tau^2 T \|u_0\|^4_{H^1(\mathbb{R})} - 4\tau^2 T \|u_0\|^2_{H^1(\mathbb{R})}}{4\tau^2 T}.$$

Hence,

$$4\tau^2 T < \beta^2 - 4\tau^2 T \|u_0\|^4_{H^1(\mathbb{R})} - 4\tau^2 T \|u_0\|^2_{H^1(\mathbb{R})},$$

that is

$$4\tau^2 T \|u_0\|^4_{H^1(\mathbb{R})} + 4\tau^2 T \|u_0\|^2_{H^1(\mathbb{R})} + 4\tau^2 T - \beta^2 \leq 0,$$

which is guaranteed by Assumption (1.6). Therefore, (2.32) holds.

Finally, thanks to (2.32) and (2.33), we have (2.2), while arguing as in Lemma 2.1, we have (2.3), (2.4), (2.5) and (2.6). □

Lemma 2.5 *Assume that one within* (1.3)*,* (1.4)*,* (1.5)*,* (1.6) *holds. There exist a constant* $C > 0$*, such that*

$$\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \beta^2 \int_0^t \left\|\partial_x^4 u(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \leq C, \tag{2.45}$$

for every $0 \leq t \leq T$*. In particular,*

$$\|\partial_x u\|_{L^\infty((0,T)\times\mathbb{R})} \leq C. \tag{2.46}$$

Proof Let $0 \leq t \leq T$. Multiplying (1.1) by $2\partial_x^4 u$, an integration on $\mathbb{R}$ gives

$$\begin{aligned}
\frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} =& 2\int_{\mathbb{R}} \partial_x^4 u \partial_t u dx \\
=& -2\kappa \int_{\mathbb{R}} u\partial_x u \partial_x^4 u dx - 2\nu \int_{\mathbb{R}} \partial_x^2 u \partial_x^4 u dx - 2\beta^2 \left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&+ 2\delta \int_{\mathbb{R}} \partial_x^3 u \partial_x^4 u dx - 2\gamma \int_{\mathbb{R}} (\partial_x u)^2 \partial_x^4 u dx - 2\alpha \int_{\mathbb{R}} u\partial_x^2 u \partial_x^4 u dx \\
=& 2\kappa \int_{\mathbb{R}} (\partial_x u)^2 \partial_x^3 u dx + 2\kappa \int_{\mathbb{R}} u\partial_x^2 u \partial_x^3 u dx + 2\nu \left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&- 2\beta^2 \left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - 2\gamma \int_{\mathbb{R}} (\partial_x u)^2 \partial_x^4 u dx - 2\alpha \int_{\mathbb{R}} u\partial_x^2 u \partial_x^4 u dx.
\end{aligned}$$

Therefore, we have that

$$\begin{aligned}
\frac{d}{dt}&\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \beta^2 \left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&= 2\kappa \int_{\mathbb{R}} (\partial_x u)^2 \partial_x^3 u dx + 2\kappa \int_{\mathbb{R}} u\partial_x^2 u \partial_x^3 u dx - 2\nu \left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&\quad - 2\gamma \int_{\mathbb{R}} (\partial_x u)^2 \partial_x^4 u dx - 2\alpha \int_{\mathbb{R}} u\partial_x^2 u \partial_x^4 u dx.
\end{aligned} \tag{2.47}$$

Due to (2.3) and the Young inequality,

$$2|\kappa \int_{\mathbb{R}} (\partial_x u)^2 |\partial_x^3 u| dx \leq \kappa^2 \|\partial_x u(t,\cdot)\|^4_{L^4(\mathbb{R})} + \left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},$$

$$
\begin{aligned}
2|\kappa| \int_{\mathbb{R}} |u||\partial_x^2 u||\partial_x^3 u| dx &\le 2|\kappa| \, \|u\|_{L^\infty((0,T)\times\mathbb{R})} \int_{\mathbb{R}} |\partial_x^2 u||\partial_x^3 u| dx \\
&\le 2C \int_{\mathbb{R}} |\partial_x^2 u||\partial_x^3 u| dx \le C \left\| \partial_x^2 u(t,\cdot) \right\|^2_{L^2(\mathbb{R})} + C \left\| \partial_x^3 u(t,\cdot) \right\|^2_{L^2(\mathbb{R})}, \\
2|\gamma| \int_{\mathbb{R}} (\partial_x u)^2 |\partial_x^4 u| dx &= 2 \int_{\mathbb{R}} \left| \frac{\gamma(\partial_x u)^2}{\beta} \right| \left| \beta \partial_x^4 u \right| dx \\
&\le \frac{\gamma^2}{\beta^2} \|\partial_x u(t,\cdot)\|^4_{L^4(\mathbb{R})} + \beta^2 \left\| \partial_x^4 u(t,\cdot) \right\|^2_{L^2(\mathbb{R})}, \\
2|\alpha| \int_{\mathbb{R}} |u||\partial_x^2 u||\partial_x^4 u| dx &\le 2|\alpha| \, \|u\|_{L^\infty((0,T)\times\mathbb{R})} \int_{\mathbb{R}} |\partial_x^2 u||\partial_x^4 u| dx \\
&\le C \int_{\mathbb{R}} |\partial_x^2 u||\partial_x^4 u| dx = \int_{\mathbb{R}} \left| \frac{C\partial_x^2 u}{\beta} \right| \left| \beta \partial_x^4 u \right| dx \\
&\le C \left\| \partial_x^2 u(t,\cdot) \right\|^2_{L^2(\mathbb{R})} + \frac{\beta^2}{2} \left\| \partial_x^4 u(t,\cdot) \right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

It follows from (2.47) that

$$
\begin{aligned}
&\frac{d}{dt} \left\| \partial_x^2 u(t,\cdot) \right\|^2_{L^2(\mathbb{R})} + \frac{\beta^2}{2} \left\| \partial_x^4 u(t,\cdot) \right\|^2_{L^2(\mathbb{R})} \\
&\quad \le \left(\kappa^2 + \frac{\gamma^2}{\beta^2} \right) \|\partial_x u(t,\cdot)\|^4_{L^4(\mathbb{R})} + C \left\| \partial_x^3 u(t,\cdot) \right\|^2_{L^2(\mathbb{R})} + C \left\| \partial_x^2 u(t,\cdot) \right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

Integrating on $(0, t)$, by (1.2), (2.4), (2.5) and (2.6), we get

$$
\begin{aligned}
\left\| \partial_x^2 u(t,\cdot) \right\|^2_{L^2(\mathbb{R})} + \frac{\beta^2}{2} \int_0^t \left\| \partial_x^4 u(s,\cdot) \right\|^2_{L^2(\mathbb{R})} ds
&\le C_0 + \left(\kappa^2 + \frac{\gamma^2}{\beta^2} \right) \int_0^t \|\partial_x u(s,\cdot)\|^4_{L^4(\mathbb{R})} ds \\
&\quad + C \int_0^t \left\| \partial_x^3 u(s,\cdot) \right\|^2_{L^2(\mathbb{R})} ds + C \int_0^t \left\| \partial_x^2 u(s,\cdot) \right\|^2_{L^2(\mathbb{R})} ds \\
&\le C_0 + C \le C,
\end{aligned}
$$

which gives (2.45).

Finally, we prove (2.46). Thanks to (2.2), (2.45) and the Hölder inequality,

$$
\begin{aligned}
(\partial_x u(t,x))^2 &= 2 \int_{-\infty}^x \partial_x u \partial_x^2 u dy \le 2 \int_{\mathbb{R}} |\partial_x u||\partial_x^2 u| dx \\
&\le 2 \|\partial_x u(t,\cdot)\|_{L^2(\mathbb{R})} \left\| \partial_x^2 u(t,\cdot) \right\|_{L^2(\mathbb{R})} \le C.
\end{aligned}
$$

Hence,

$$
\|\partial_x u\|^2_{L^\infty((0,T)\times\mathbb{R})} \le C,
$$

which gives (2.46). □

Lemma 2.6 *Assume that one within (1.3), (1.4), (1.5), (1.6) holds. There exist a constant $C > 0$, such that*

$$
\int_0^t \|\partial_t u(s,\cdot)\|^2_{L^2(\mathbb{R})} ds \le C, \tag{2.48}
$$

for every $0 \le t \le T$.

Proof Let $0 \le t \le T$. Multiplying (1.1) by $2\partial_t u$, an integration on $\mathbb{R}$ gives

$$\begin{aligned}2\left\|\partial_t u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} =& -2\kappa\int_{\mathbb{R}} u\partial_x u\partial_t u dx - 2\nu\int_{\mathbb{R}} \partial_x^2 u\partial_t u dx - 2\delta\int_{\mathbb{R}} \partial_x^3 u\partial_t u dx\\ &- 2\beta^2\int_{\mathbb{R}} \partial_x^4 u\partial_t u dx - 2\gamma\int_{\mathbb{R}} (\partial_x u)^2\partial_t u dx - 2\alpha\int_{\mathbb{R}} u\partial_x^2 u\partial_t u dx.\end{aligned}$$

Therefore,

$$\begin{aligned}2\left\|\partial_t u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} =& -2\kappa\int_{\mathbb{R}} u\partial_x u\partial_t u dx - 2\nu\int_{\mathbb{R}} \partial_x^2 u\partial_t u dx - 2\delta\int_{\mathbb{R}} \partial_x^3 u\partial_t u dx\\ &- 2\beta^2\int_{\mathbb{R}} \partial_x^4 u\partial_t u dx - 2\gamma\int_{\mathbb{R}} (\partial_x u)^2\partial_t u dx - 2\alpha\int_{\mathbb{R}} u\partial_x^2 u\partial_t u dx.\end{aligned} \tag{2.49}$$

Due to the (2.2), (2.3), (2.45) and the Young inequality,

$$\begin{aligned}
2|\kappa|\int_{\mathbb{R}} |u||\partial_x u||\partial_t u| dx \le& 2|\kappa|\left\|u\right\|_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}} |\partial_x u||\partial_t u|dx\\
\le& 2C\int_{\mathbb{R}} |\partial_x u||\partial_t u|dx \le 2\int_{\mathbb{R}}\left|\frac{C\partial_x u}{\sqrt{D_1}}\right|\left|\sqrt{D_1}\partial_t u\right|dx\\
\le& \frac{C}{D_1}\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + D_1\left\|\partial_t u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
\le& \frac{C}{D_1} + D_1\left\|\partial_t u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
2|\nu|\int_{\mathbb{R}} |\partial_x^2 u||\partial_t u| dx =& 2\int_{\mathbb{R}}\left|\frac{\nu\partial_x^2 u}{\sqrt{D_1}}\right|\left|\sqrt{D_1}\partial_t u\right|dx\\
\le& \frac{\nu^2}{D_1}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + D_1\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
\le& \frac{C}{D_1} + D_1\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
2|\delta|\int_{\mathbb{R}} |\partial_x^3 u||\partial_t u| dx =& 2\int_{\mathbb{R}}\left|\frac{\delta\partial_x^3 u}{\sqrt{D_1}}\right|\left|\sqrt{D_1}\partial_t u\right|dx\\
\le& \frac{\delta^2}{D_1}\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + D_1\left\|\partial_t u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
2\beta^2\int_{\mathbb{R}} |\partial_x^4 u||\partial_t u| dx =& 2\int_{\mathbb{R}}\left|\frac{\beta^2\partial_x^4 u}{\sqrt{D_1}}\right|\left|\sqrt{D_1}\partial_t u\right|dx\\
\le& \frac{\beta^4}{D_1}\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + D_1\left\|\partial_t u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
2|\gamma|\int_{\mathbb{R}} (\partial_x u)^2|\partial_t u| dx =& 2\int_{\mathbb{R}}\left|\frac{\gamma(\partial_x u)^2}{\sqrt{D_1}}\right|\left|\sqrt{D_1}\partial_t u\right|dx\\
\le& \frac{\gamma^2}{D_1}\left\|\partial_x u(t,\cdot)\right\|^4_{L^4(\mathbb{R})} + D_1\left\|\partial_t u(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
2|\alpha|\int_{\mathbb{R}} |u||\partial_x^2 u||\partial_t u| dx \le& 2|\alpha|\left\|u\right\|_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}} |\partial_x^2 u||\partial_t u|dx\\
\le& 2C\int_{\mathbb{R}} |\partial_x^2 u||\partial_t u|dx \le 2\int_{\mathbb{R}}\left|\frac{C\partial_x^2 u}{\sqrt{D_1}}\right|\left|\sqrt{D_1}\partial_t u\right|dx
\end{aligned}$$

$$\leq \frac{C}{D_1} \left\| \partial_x^2 u(t,\cdot) \right\|^2_{L^2(\mathbb{R})} + D_1 \left\| \partial_t u(t,\cdot) \right\|^2_{L^2(\mathbb{R})}$$
$$\leq \frac{C}{D_1} + D_1 \left\| \partial_t u(t,\cdot) \right\|^2_{L^2(\mathbb{R})},$$

where D_1 is a positive constant, which will specified later. Consequently, by (2.49),

$$\begin{aligned} 2\left(1-3D_1\right) \left\| \partial_t u(t,\cdot) \right\|^2_{L^2(\mathbb{R})} \leq & \frac{C}{D_1} + \frac{\delta^2}{D_1} \left\| \partial_x^3 u(t,\cdot) \right\|^2_{L^2(\mathbb{R})} \\ & + \frac{\beta^4}{D_1} \left\| \partial_x^4 u(t,\cdot) \right\|^2_{L^2(\mathbb{R})} + \frac{\gamma^2}{D_1} \left\| \partial_x u(t,\cdot) \right\|^4_{L^4(\mathbb{R})}. \end{aligned}$$

Taking $D_1 = \frac{1}{4}$, we have that

$$\begin{aligned} \frac{1}{2} \left\| \partial_t u(t,\cdot) \right\|^2_{L^2(\mathbb{R})} \leq & C + 4\delta^2 \left\| \partial_x^3 u(t,\cdot) \right\|^2_{L^2(\mathbb{R})} \\ & + 4\beta^4 \left\| \partial_x^4 u(t,\cdot) \right\|^2_{L^2(\mathbb{R})} + 4\gamma^2 \left\| \partial_x u(t,\cdot) \right\|^4_{L^4(\mathbb{R})}. \end{aligned}$$

Integrating on $(0, t)$, by (2.5), (2.6) and (2.45), we get

$$\begin{aligned} \frac{1}{2} \int_0^t \left\| \partial_t u(s,\cdot) \right\|^2_{L^2(\mathbb{R})} ds \leq & Ct + 4\delta^2 \int_0^t \left\| \partial_x^3 u(s,\cdot) \right\|^2_{L^2(\mathbb{R})} ds \\ & + 4\beta^4 \int_0^t \left\| \partial_x^4 u(s,\cdot) \right\|^2_{L^2(\mathbb{R})} ds + 4\gamma^2 \int_0^t \left\| \partial_x u(s,\cdot) \right\|^4_{L^4(\mathbb{R})} ds \\ \leq & C, \end{aligned}$$

which gives (2.48). □

3 Proof of Theorem 1.1

This section devoted to the proof of Theorem 1.1.

Proof of Theorem 1.1 Thanks to Lemmas 2.1, or 2.3, or 2.4, 2.5, 2.6 and the Cauchy-Kovalevskaya Theorem [63], we have that u is solution of (1.1) and (1.10) holds.

We prove (1.11). Let u_1 and u_2 be two solutions of (1.1), which verify (1.10), that is

$$\begin{cases} \partial_t u_1 + \frac{\kappa}{2} \partial_x u_1^2 + \nu \partial_x^2 u_1 + \delta \partial_x^3 u_1 & \\ \qquad + \beta^2 \partial_x^4 u_2 + \gamma (\partial_x u_1)^2 + \alpha u_1 \partial_x^2 u_1, & t > 0,\ x \in \mathbb{R}, \\ u_1(0,x) = u_{1,0}(x), & x \in \mathbb{R}, \end{cases}$$
$$\begin{cases} \partial_t u_2 + \frac{\kappa}{2} \partial_x u_2^2 + \nu \partial_x^2 u_2 + \delta \partial_x^3 u_2 & \\ \qquad + \beta^2 \partial_x^4 u_2 + \gamma (\partial_x u_2)^2 + \alpha u_2 \partial_x^2 u_2, & t > 0,\ x \in \mathbb{R}, \\ u_2(0,x) = u_{2,0}(x), & x \in \mathbb{R}. \end{cases}$$

Then, the function

$$\omega = u_1 - u_2 \tag{3.1}$$

is the solution of the following Cauchy problem:

$$\begin{cases} \partial_t\omega+\dfrac{\kappa}{2}\partial_x\left(u_1^2-u_2^2\right)+\nu\partial_x^2\omega+\delta\partial_x^3\omega+\beta^2\partial_x^4\omega \\ \qquad +\gamma\left[(\partial_x u_1)^2-(\partial_x u_2)^2\right]+\alpha\left(u_1\partial_x^2u_1-u_2\partial_x^2u_2\right)=0, & t>0,\ x\in\mathbb{R}, \\ \omega(0,x)=u_{1,0}(x)-u_{2,0}(x), & x\in\mathbb{R}. \end{cases} \tag{3.2}$$

Observe the, thanks to (3.1),

$$u_1\partial_x^2u_1-u_2\partial_x^2u_2=u_1\partial_x^2u_1-u_2\partial_x^2u_1+u_2\partial_x^2u_1-u_2\partial_x^2u_2=\omega\partial_x^2u_1+u_2\partial_x^2\omega.$$

Therefore, (3.2) reads

$$\begin{aligned} &\partial_t\omega+\frac{\kappa}{2}\partial_x\left(u_1^2-u_2^2\right)+\nu\partial_x^2\omega+\delta\partial_x^3\omega+\beta^2\partial_x^4\omega \\ &\quad+\gamma\left[(\partial_x u_1)^2-(\partial_x u_2)^2\right]+\alpha\omega\partial_x^2u_1+\alpha u_2\partial_x^2\omega=0. \end{aligned} \tag{3.3}$$

Since, thanks to (3.1),

$$\begin{aligned} 2\int_{\mathbb{R}}\omega\partial_t\omega dx=&\frac{d}{dt}\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}, \\ \kappa\int_{\mathbb{R}}\partial_x\left(u_1^2-u_2^2\right)\omega=&\kappa\int_{\mathbb{R}}(u_1+u_2)\omega\partial_x\omega dx, \\ 2\nu\int_{\mathbb{R}}\omega\partial_x^2\omega=&-2\nu\|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}, \\ 2\delta\int_{\mathbb{R}}\omega\partial_x^3\omega dx=&-2\delta\int_{\mathbb{R}}\partial_x\omega\partial_x^2\omega dx=0, \\ 2\beta^2\int_{\mathbb{R}}\omega\partial_x^4\omega=&-2\beta\int_{\mathbb{R}}\partial_x\omega\partial_x^3\omega dx=2\beta^2\left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}, \\ 2\gamma\int_{\mathbb{R}}\left[(\partial_x u_1)^2-(\partial_x u_2)^2\right]\omega dx=&2\gamma\int_{\mathbb{R}}(\partial_x u_1+\partial_x u_2)\,\omega\partial_x\omega dx, \end{aligned}$$

multiplying (3.2) by 2ω, an integration on $\mathbb{R}$ gives

$$\begin{aligned} &\frac{d}{dt}\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}+2\beta^2\left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\ &\quad=-\kappa\int_{\mathbb{R}}(u_1+u_2)\omega\partial_x\omega dx+2\nu\|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} \\ &\qquad-2\gamma\int_{\mathbb{R}}(\partial_x u_1+\partial_x u_2)\,\omega\partial_x\omega dx+2\alpha\int_{\mathbb{R}}\omega^2\partial_x^2u_1 dx \\ &\qquad+2\alpha\int_{\mathbb{R}}u_2\omega\partial_x^2\omega dx. \end{aligned} \tag{3.4}$$

Observe that, since $u_1,\ u_2\in H^2(\mathbb{R})$, for every $0\le t\le T$, we have that

$$\begin{aligned} &\|u_1\|_{L^\infty((0,T)\times\mathbb{R})},\ \|\partial_x u_1\|_{L^\infty((0,T)\times\mathbb{R})}\le C, \\ &\|u_2\|_{L^\infty((0,T)\times\mathbb{R})},\ \|\partial_x u_2\|_{L^\infty((0,T)\times\mathbb{R})}\le C, \\ &\left\|\partial_x^2u_1(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\le C. \end{aligned} \tag{3.5}$$

Consequently, thanks to (3.5), we obtain that

$$|u_1+u_2|\le C,\qquad |\partial_x u_1+\partial_x u_2|\le C. \tag{3.6}$$

Due to (3.5), (3.6) and the Young inequality,

$$\begin{aligned}
|\kappa|\int_{\mathbb{R}}|u_1+u_2|\,|\omega||\partial_x\omega|dx &\le C(T)\int_{\mathbb{R}}|\omega|\partial_x\omega|dx\\
&\le C\left\|\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+C\left\|\partial_x\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
2|\gamma|\int_{\mathbb{R}}|\partial_x u_1+\partial_x u_2||\omega||\partial_x\omega|dx &\le C\int_{\mathbb{R}}|\omega|\partial_x\omega|dx\\
&\le C\left\|\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+C\left\|\partial_x\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
2|\alpha|\int_{\mathbb{R}}\omega^2|\partial_x^2u_1|dx &\le \left\|\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\alpha^2\int_{\mathbb{R}}\omega^2(\partial_x^2u_1)^2dx\\
&\le \left\|\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\alpha^2\left\|\omega(t,\cdot)\right\|^2_{L^\infty(\mathbb{R})}\left\|\partial_x^2u_1(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\le \left\|\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+C\left\|\omega(t,\cdot)\right\|^2_{L^\infty(\mathbb{R})},\\
2|\alpha|\int_{\mathbb{R}}|u_2||\omega||\partial_x^2\omega|dx &\le 2|\alpha|\left\|u_2\right\|_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}}|\omega||\partial_x^2\omega|dx\\
&\le 2C\int_{\mathbb{R}}|\omega||\partial_x^2\omega|dx=2\int_{\mathbb{R}}\left|\frac{C(T)\omega}{\beta}\right|\left|\beta\partial_x^2\omega\right|dx\\
&\le C\left\|\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\beta^2\left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}$$

It follows from (3.4) that

$$\begin{aligned}
&\frac{d}{dt}\left\|\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+2\beta^2\left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad\le C\left\|\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+C\left\|\partial_x\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+C\left\|\omega(t,\cdot)\right\|^2_{L^\infty(\mathbb{R})}.
\end{aligned}\tag{3.7}$$

Observe that, by the Hölder inequality,

$$\omega^2(t,x)=2\int_{-\infty}^{x}\omega\partial_x\omega dy\le 2\int_{\mathbb{R}}|\omega||\partial_x\omega|dx\le 2\left\|\omega(t,\cdot)\right\|_{L^2(\mathbb{R})}\left\|\partial_x\omega(t,\cdot)\right\|_{L^2(\mathbb{R})}.$$

Hence,

$$\left\|\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\le 2\left\|\omega(t,\cdot)\right\|_{L^2(\mathbb{R})}\left\|\partial_x\omega(t,\cdot)\right\|_{L^2(\mathbb{R})}.\tag{3.8}$$

Due to the Young inequality,

$$\left\|\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\le\left\|\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\left\|\partial_x\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

Consequently, by (3.7),

$$\begin{aligned}
&\frac{d}{dt}\left\|\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\beta^2\left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\qquad\le C\left\|\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+C\left\|\partial_x\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}\tag{3.9}$$

Observe that

$$C\left\|\partial_x\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}=C\int_{\mathbb{R}}\partial_x\omega\partial_x\omega dx=-C\int_{\mathbb{R}}\omega\partial_x^2\omega dx$$

Therefore, by the Young inequality,

$$C\left\|\partial_x\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\le 2\int_{\mathbb{R}}\left|\frac{C\omega}{2\beta}\right|\left|\beta\partial_x^2\omega\right|dx$$

$$\le C \left\|\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\beta^2}{2}\left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

It follows from (3.9) that

$$\frac{d}{dt}\left\|\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\beta^2}{2}\left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \le C\left\|\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

The Gronwall Lemma and (3.2) gives

$$\left\|\omega(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\beta^2 e^{Ct}}{2}\int_0^t e^{-Cs}\left\|\partial_x^2\omega(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \le e^{Ct}\left\|\omega_0\right\|^2_{L^2(\mathbb{R})}. \tag{3.10}$$

(1.10) follows from (3.1) and (3.10). □

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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