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H^4-Solutions for the Olver–Benney equation

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Abstract

The Olver–Benney equation is a nonlinear fifth-order equation, which describes the interaction effects between short and long waves. In this paper, we prove the global existence of solutions of the Cauchy problem associated with this equation.

Keywords Existence · Olver–Benney equation · Cauchy problem

Mathematics Subject Classification 35G25 · 35K55

1 Introduction

In this study, we investigate the existence of solutions of the following Cauchy problem:

$$\begin{cases} \partial_t u + \alpha \partial_x u \partial_x^2 u + \beta u \partial_x^3 u + \gamma \partial_x^5 u = 0, & t > 0, \quad x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

with

$$\beta = -2\alpha, \quad \gamma \neq 0, \quad \text{or} \tag{1.2}$$

In memory of Professor Enrico Jannelli.

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$$\alpha = 2\beta, \quad \gamma \neq 0, \quad \text{or} \tag{1.3}$$

$$\beta = 2\alpha, \quad \gamma \neq 0. \tag{1.4}$$

On the initial datum, we assume

$$u_0(x) \in H^4(\mathbb{R}). \tag{1.5}$$

From a physical point of view, (1.1) was derived in the context of water waves by Olver [16, 17] (see also [10, 18]), using Hamiltonian perturbation theory, with further generalization given by Craig and Groves [8], while, under Assumption (1.3), (1.1) was derived by Benney [1] as a model to describe the interaction effects between short and long waves.

[17] shows that (1.1) is a particular case of the following equation:

$$\partial_t u + \kappa u \partial_x u + q u^2 \partial_x u + \tau \partial_x^3 u + \alpha \partial_x u \partial_x^2 u + \beta u \partial_x^3 u + \gamma \partial_x^5 u = 0. \tag{1.6}$$

If $q = \alpha = \beta = \gamma = 0$ in (1.6), (1.6) becomes the Korteweg–de Vries equation [11], whose the well-posedness is studied in [4].

Instead, if $\alpha = \beta = 0$, (1.6) becomes the Kawahara–Korteweg–de Vries type equation, which was derived by Kawahara [9] to describe small-amplitude gravity capillary waves on water of a finite depth when the Weber number is close to 1/3 (see [15]). In [2], the well-posedness of the Cauchy problem for the Kawahara–Korteweg–de Vries type equation is studied.

Moreover, assuming $\kappa = q = \gamma = 0$ and (1.2), (1.6) reads

$$\partial_t u + \tau \partial_x^3 u_\varepsilon + \alpha \partial_x u \partial_x^2 u - 2\alpha u \partial_x^3 u = 0. \tag{1.7}$$

It is a particular case of the Kudryashov–Sinelshchikov equation [5, 12], which describes pressure waves in liquids with gas bubbles taking into account heat transfer and viscosity. In [5], the existence of solutions of the Cauchy problem is proven.

From a mathematical point of view, under suitable assumptions on κ, q, τ, α, β, γ, the existence of the travelling waves solutions for (1.6) is proven in [14, 20], while a method to find exact solutions of (1.6) is given in [13]. Instead, in [19], the local well-posedness of the Cauchy problem of (1.1) is proven.

The main result of this paper is the following theorem.

Theorem 1.1 *Assume* (1.2), *or* (1.3), *or* (1.4) *and* (1.5). *There exists an unique solution u of* (1.1) *such that*

$$u \in L^\infty(0, T; H^4(\mathbb{R})), \quad T \geq 0. \tag{1.8}$$

We remind that [19] the local in time well-posedness is H^s, $s \geq 4$. Here, Theorem 1.1 gives the global well-posedness of the solution of the Cauchy problem of (1.1), under Assumptions (1.2), (1.3) and (1.4).

Since we are able to prove estimates only on the spatial derivatives of (1.1), the proof of Theorem 1.1 is based on the Aubin–Lions lemma (see [3, 6, 7, 21]), which requires only the H^{-1} boundedness of the time derivative.

One of the main point of our argument is the invariance of the energy space H^4 (see Lemma 2.4). The key point in that direction is the H^2 regularity. Assumptions (1.2), (1.3) and (1.4) are needed for that purpose. Indeed, assuming (1.2), (1.1) preserves (see Lemma 2.1):

$$t \to \int_{\mathbb{R}} (\partial_x^2 u)^2 \mathrm{d}x,$$

while assuming (1.3), the same equation preserves (see Lemma 3.1):

$$t \to \int_{\mathbb{R}} u^2 \mathrm{d}x,$$

and if (1.4) holds, it preserves (see Lemma 4.1):

$$t \to \int_{\mathbb{R}} (\partial_x u)^2 \mathrm{d}x.$$

So we can say that the assumptions on the constants are needed for the H^2 regularity of u and only indirectly for the H^4 one.

Observe again that, (1.5) is the same assumption to prove the well-posedness of the Cauchy problem for the Kawahara equation (see [2]). [2, Appendix A] shows that, for the Kawahara equation, it is possible also to assume on the initial datum also

$$u_0 \in H^5(\mathbb{R}). \tag{1.9}$$

and obtain the well-posedness of the classical solution (see [2, Theorem A.1]). It is not possible to show a similar result for (1.1), under Assumption (1.9), due to the term $\alpha\partial_x u \partial_x^2 u + \beta u \partial_x^3 u$.

The paper is organized as follows. In Sect. 2, we prove Theorem 1.1 under Assumption (1.2), while in Sects. 3 and 4, we prove Theorem 1.1, under Assumptions (1.3) and (1.4), respectively.

2 Proof of Theorem 1.1 under Assumption (1.2).

In section, we prove Theorem 1.1 under Assumption (1.2).

Our existence argument is based on passing to the limit in a vanishing viscosity approximation of (1.1).

Fix a small number $0 < \varepsilon < 1$ and let $u_\varepsilon = u_\varepsilon(t, x)$ be the unique classical solution of the following problem [2, 4, 19]:

$$\begin{cases} \partial_t u_\varepsilon + \alpha\partial_x u_\varepsilon \partial_x^2 u_\varepsilon + \beta u_\varepsilon \partial_x^3 u_\varepsilon + \gamma \partial_x^5 u_\varepsilon = \varepsilon \partial_x^6 u_\varepsilon, & t > 0, x \in \mathbb{R}, \\ u_\varepsilon(0, x) = u_{\varepsilon,0}(x), & x \in \mathbb{R}, \end{cases} \tag{2.1}$$

where $u_{\varepsilon,0}$ is a C^∞ approximation of u_0 such that

$$\|u_{\varepsilon,0}\|_{H^4(\mathbb{R})} \le \|u_0\|_{H^4(\mathbb{R})}. \tag{2.2}$$

Let us prove some a priori estimates on u_ε. We denote with C_0 the constants which depend only on the initial data, and with $C(T)$, the constants which depend also on T.

Lemma 2.1 *Assume* (1.2). *For each* $t \ge 0$,

$$\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\varepsilon\int_0^t \left\|\partial_x^5 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \le C_0. \tag{2.3}$$

Proof Multiplying (2.1) by $2\partial_x^4 u_\varepsilon$, an integration on $\mathbb{R}$ gives

$$\begin{aligned}\frac{d}{dt}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} =& 2\int_{\mathbb{R}} \partial_x^4 u_\varepsilon \partial_t u_\varepsilon dx\\ =& -2\alpha\int_{\mathbb{R}} \partial_x u_\varepsilon \partial_x^2 u_\varepsilon \partial_x^4 u_\varepsilon dx - 2\beta\int_{\mathbb{R}} u_\varepsilon \partial_x^3 u_\varepsilon \partial_x^4 u_\varepsilon dx\\ &- 2\gamma\int_{\mathbb{R}} \partial_x^4 u_\varepsilon \partial_x^5 u_\varepsilon dx + 2\varepsilon\int_{\mathbb{R}} \partial_x^4 u_\varepsilon \partial_x^6 u_\varepsilon dx\\ =& (2\alpha+\beta)\int_{\mathbb{R}} \partial_x u_\varepsilon (\partial_x^3 u_\varepsilon)^2 dx - 2\varepsilon\left\|\partial_x^5 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

Hence,

$$\frac{d}{dt}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\varepsilon\left\|\partial_x^5 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} = (2\alpha+\beta)\int_{\mathbb{R}} \partial_x u_\varepsilon (\partial_x^3 u_\varepsilon)^2 dx.$$

Integrating on $(0,t)$, thanks to (1.2) and (2.2), we have (2.3). □

Lemma 2.2 *Fix $T>0$ and assume* (1.2). *There exists a constant $C(T)>0$, independent on ε, such that*

$$\left\|u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\varepsilon e^{\frac{5t}{4}}\int_0^t e^{-\frac{5s}{4}}\left\|\partial_x^3 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \le C(T), \tag{2.4}$$

for every $0\le t\le T$. In particular, we have

$$\left\|u_\varepsilon\right\|_{L^\infty((0,T)\times\mathbb{R})}, \left\|\partial_x u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}, \left\|\partial_x u_\varepsilon\right\|_{L^\infty((0,T)\times\mathbb{R})} \le C(T). \tag{2.5}$$

The proof of the previous lemma is based on the regularity of the functions u_ε and the following result.

Lemma 2.3 *For each $t\ge 0$, we have that*

$$\int_{\mathbb{R}} (\partial_x u_\varepsilon)^2(\partial_x^2 u_\varepsilon)^2 dx \le 2\sqrt{\left\|u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}}\sqrt{\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^7_{L^2(\mathbb{R})}}. \tag{2.6}$$

Proof We begin by observing that, thanks to the regularity of u_ε and the Hölder inequality,

$$\begin{aligned}\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} =& \int_{\mathbb{R}} \partial_x u_\varepsilon \partial_x u_\varepsilon dx = -\int_{\mathbb{R}} u_\varepsilon \partial_x^2 u_\varepsilon dx\\ \le& \int_{\mathbb{R}} |u_\varepsilon||\partial_x u_\varepsilon| dx \le \left\|u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}.\end{aligned}$$

Consequently,

$$\left\|\partial_x u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})} \le \sqrt{\left\|u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}}\sqrt{\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}}. \tag{2.7}$$

Moreover, again by the regularity of u_ε and the Hölder inequality,

$$\begin{aligned}(\partial_x u_\varepsilon(t,x))^2 &= 2\int_{-\infty}^{x} \partial_x u_\varepsilon \partial_x^2 u_\varepsilon \mathrm{d}y \le 2\int_{\mathbb{R}} |\partial_x u_\varepsilon||\partial_x^2 u_\varepsilon| \mathrm{d}y\\ &\le 2\left\|\partial_x u_\varepsilon(t,\cdot)\right\|_{L^\infty(\mathbb{R})}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}.\end{aligned}$$

Hence,

$$\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \le 2\left\|\partial_x u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}. \tag{2.8}$$

It follows from (2.7) and (2.8) that

$$\begin{aligned}\int_{\mathbb{R}} (\partial_x u_\varepsilon)^2(\partial_x^2 u_\varepsilon)^2 \mathrm{d}x &\le \left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &\le 2\left\|\partial_x u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^3_{L^2(\mathbb{R})}\\ &\le 2\sqrt{\left\|u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}}\sqrt{\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^7_{L^2(\mathbb{R})}},\end{aligned}$$

which gives (2.6). □

Proof of Lemma 2.2 Let $0 \le t \le T$. Multiplying (2.1) by $2u$, an integration on $\mathbb{R}$ gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left\|u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} &= 2\int_{\mathbb{R}} u_\varepsilon \partial_t u_\varepsilon \mathrm{d}x\\ &= -2\alpha\int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon \partial_x^2 u_\varepsilon \mathrm{d}x - 2\beta\int_{\mathbb{R}} u_\varepsilon^2 \partial_x^3 u_\varepsilon \mathrm{d}x\\ &\quad - 2\gamma\int_{\mathbb{R}} u_\varepsilon \partial_x^5 u_\varepsilon \mathrm{d}x + 2\varepsilon\int_{\mathbb{R}} u_\varepsilon \partial_x^6 u_\varepsilon \mathrm{d}x\\ &= 2(\alpha-2\beta)\int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon \partial_x^2 u_\varepsilon \mathrm{d}x + 2\gamma\int_{\mathbb{R}} \partial_x u_\varepsilon \partial_x^4 u_\varepsilon \mathrm{d}x\\ &\quad - 2\varepsilon\int_{\mathbb{R}} \partial_x u_\varepsilon \partial_x^5 u_\varepsilon \mathrm{d}x\\ &= 2(\alpha-2\beta)\int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon \partial_x^2 u_\varepsilon \mathrm{d}x - 2\gamma\int_{\mathbb{R}} \partial_x^2 u_\varepsilon \partial_x^3 u_\varepsilon \mathrm{d}x\\ &\quad + 2\varepsilon\int_{\mathbb{R}} \partial_x^2 u_\varepsilon \partial_x^4 u_\varepsilon \mathrm{d}x\\ &= 2(\alpha-2\beta)\int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon \partial_x^2 u_\varepsilon \mathrm{d}x - 2\varepsilon\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+2\varepsilon\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}=2(\alpha-2\beta)\int_{\mathbb{R}}u_\varepsilon\partial_x u_\varepsilon\partial_x^2 u_\varepsilon\mathrm{d}x. \tag{2.9}$$

Due to (2.3), (2.6) and the Young inequality,

$$\begin{aligned}
2|\alpha-2\beta|\int_{\mathbb{R}}|u_\varepsilon||\partial_x u_\varepsilon\partial_x^2 u_\varepsilon|\mathrm{d}x=&2\int_{\mathbb{R}}|u_\varepsilon|\left|(\alpha-2\beta)\partial_x u_\varepsilon\partial_x^2 u_\varepsilon\right|\mathrm{d}x\\
\le&\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+(\alpha-2\beta)^2\int_{\mathbb{R}}(\partial_x u_\varepsilon)^2(\partial_x^2 u_\varepsilon)^2\mathrm{d}x\\
\le&\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+2(\alpha-2\beta)^2\sqrt{\|u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}}\sqrt{\|\partial_x^2 u_\varepsilon(t,\cdot)\|^7_{L^2(\mathbb{R})}}\\
\le&\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+C_0\sqrt{\|u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}}\\
\le&\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+\frac{1}{2}\|u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}+C_0\\
\le&\frac{5}{4}\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+C_0.
\end{aligned}$$

Consequently, by (2.9),

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+2\varepsilon\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\le\frac{5}{4}\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+C_0.$$

The Gronwall lemma and (2.2) give

$$\begin{aligned}
&\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+2\varepsilon e^{\frac{5t}{4}}\int_0^t e^{-\frac{5s}{4}}\left\|\partial_x^3 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\mathrm{d}s\\
&\qquad\le C_0e^{\frac{5t}{4}}+C_0e^{\frac{5t}{4}}\int_0^t e^{-\frac{5t}{4}}\mathrm{d}s\le C(T).
\end{aligned}$$

Therefore, (2.4) is proven.

Finally, we prove (2.5). Thanks to (2.4) and the Hölder inequality,

$$\begin{aligned}
u_\varepsilon^2(t,x)=&2\int_{-\infty}^x u_\varepsilon\partial_x u_\varepsilon\mathrm{d}y\le 2\int_{\mathbb{R}}|u_\varepsilon||\partial_x u_\varepsilon|\mathrm{d}x\\
\le&2\left\|u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}\left\|\partial_x u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}.
\end{aligned} \tag{2.10}$$

Hence, by (2.4),

$$\left\|u_\varepsilon(t,\cdot)\right\|^2_{L^\infty(\mathbb{R})}\le C(T)\left\|\partial_x u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}. \tag{2.11}$$

(2.5) follows from (2.3), (2.4), (2.7), (2.8) and (2.11). □

Lemma 2.4 *Fix $T>0$ and assume (1.2). There exists a constant $C(T)>0$, independent on ε, such that*

$$\left\|\partial_x^4 u_\varepsilon\right\|_{L^\infty(0,T;L^2(\mathbb{R}))} \leq C(T). \tag{2.12}$$

In particular, we have

$$\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{\mathbb{R}} + \varepsilon e^{C(T)t}\int_0^t e^{-C(T)s}\left\|\partial_x^7 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} \mathrm{d}s \leq C(T), \tag{2.13}$$

for every $0 \leq t \leq T$. *Moreover,*

$$\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}, \left\|\partial_x^2 u_\varepsilon\right\|_{L^\infty((0,T)\times\mathbb{R})}, \left\|\partial_x^3 u_\varepsilon\right\|_{L^\infty((0,T)\times\mathbb{R})} \leq C(T), \tag{2.14}$$

for every $0 \leq t \leq T$.

The proof of the previous lemma is based on the regularity of the functions u_ε and the following result.

Lemma 2.5 *For each* $t \geq 0$, *we have that*

$$\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})} \leq \sqrt{\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}}\sqrt{\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}}, \tag{2.15}$$

$$\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|_{L^\infty(\mathbb{R})} \leq \sqrt{2}\sqrt[4]{\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^3_{L^2(\mathbb{R})}}\sqrt[4]{\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}}, \tag{2.16}$$

$$\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|_{L^\infty(\mathbb{R})} \leq \sqrt{2}\sqrt[4]{\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}}\sqrt[4]{\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^3_{L^2(\mathbb{R})}}. \tag{2.17}$$

In particular, we have

$$\int_{\mathbb{R}} |\partial_x^3 u_\varepsilon|^3 \mathrm{d}x \leq \sqrt{2}\sqrt[4]{\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^5_{L^2(\mathbb{R})}}\sqrt[4]{\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^7_{L^2(\mathbb{R})}}. \tag{2.18}$$

Moreover, fixed $T > 0$, *there exists a constant* $C(T) > 0$, *independent on* ε, *such that*

$$\varepsilon\int_0^t \left\|\partial_x^4 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} \mathrm{d}s \leq C(T), \tag{2.19}$$

for every $0 \leq t \leq T$.

Proof Arguing as in [2, Lemma 2.5], we have (2.15), (2.16) and (2.17).

Finally, we prove (2.19). Fix $T > 0$. Thanks to the regularity of u_ε and the Hölder inequality,

$$\begin{aligned}\varepsilon\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} &= \varepsilon\int_{\mathbb{R}} \partial_x^4 u_\varepsilon \partial_x^4 u_\varepsilon \mathrm{d}x = -\varepsilon\int_{\mathbb{R}} \partial_x^3 u_\varepsilon \partial_x^5 u_\varepsilon \mathrm{d}x\\ &\le \varepsilon\int_{\mathbb{R}} |\partial_x^3 u_\varepsilon||\partial_x^4 u_\varepsilon|\mathrm{d}x \le \varepsilon\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}\left\|\partial_x^5 u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}.\end{aligned}$$

Consequently, by the Young inequality,

$$\varepsilon\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \le \frac{\varepsilon}{2}\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\varepsilon}{2}\left\|\partial_x^5 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

Integrating on $(0, t)$, by (2.3) and (2.4), we have that

$$\begin{aligned}\varepsilon\int_0^t \left\|\partial_x^4 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\mathrm{d}s &\le \frac{\varepsilon}{2}\int_0^t \left\|\partial_x^3 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\mathrm{d}s + \frac{\varepsilon}{2}\int_0^t \left\|\partial_x^5 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\mathrm{d}s\\ &\le \frac{\varepsilon}{2}e^{\frac{5t}{4}}\int_0^t e^{-\frac{5s}{4}}\left\|\partial_x^3 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\mathrm{d}s + C_0\\ &\le C(T) + C_0 \le C(T),\end{aligned}$$

that is (2.19). □

Proof of Lemma 2.4 Let $0 \le t \le T$. Consider five real constants A, B, C, D, E,, which will be specified later. Multiplying (2.1) by

$$2\partial_x^8 u_\varepsilon + A(\partial_x^3 u_\varepsilon)^2 + Bu_\varepsilon\partial_x^6 u_\varepsilon + C\partial_x^2 u_\varepsilon\partial_x^4 u_\varepsilon + D\partial_x u_\varepsilon\partial_x^5 u_\varepsilon + E\partial_x^2((\partial_x^2 u_\varepsilon)^2),$$

thanks to (1.2), we have

$$\begin{aligned}&\left(2\partial_x^8 u_\varepsilon + A(\partial_x^3 u_\varepsilon)^2 + Bu_\varepsilon\partial_x^6 u_\varepsilon + C\partial_x^2 u_\varepsilon\partial_x^4 u_\varepsilon\right)\partial_t u_\varepsilon\\ &\quad + \left(D\partial_x u_\varepsilon\partial_x^5 u_\varepsilon + E\partial_x^2((\partial_x^2 u_\varepsilon)^2)\right)\partial_t u_\varepsilon\\ &\quad + \alpha\left(2\partial_x^8 u_\varepsilon + A(\partial_x^3 u_\varepsilon)^2 + Bu_\varepsilon\partial_x^6 u_\varepsilon + C\partial_x^2 u_\varepsilon\partial_x^4 u_\varepsilon\right)\partial_x u_\varepsilon\partial_x^2 u_\varepsilon\\ &\quad + \alpha\left(D\partial_x u_\varepsilon\partial_x^5 u_\varepsilon + E\partial_x^2((\partial_x^2 u_\varepsilon)^2)\right)\partial_x u_\varepsilon\partial_x^2 u_\varepsilon\\ &\quad - 2\alpha\left(2\partial_x^8 u_\varepsilon + A(\partial_x^3 u_\varepsilon)^2 + Bu_\varepsilon\partial_x^6 u_\varepsilon + C\partial_x^2 u_\varepsilon\partial_x^4 u_\varepsilon\right)u_\varepsilon\partial_x^3 u_\varepsilon\\ &\quad - 2\alpha\left(D\partial_x u_\varepsilon\partial_x^5 u_\varepsilon + E\partial_x^2((\partial_x^2 u_\varepsilon)^2)\right)u_\varepsilon\partial_x^3 u_\varepsilon\\ &\quad + \gamma\left(2\partial_x^8 u_\varepsilon + A(\partial_x^3 u_\varepsilon)^2 + Bu_\varepsilon\partial_x^6 u_\varepsilon + C\partial_x^2 u_\varepsilon\partial_x^4 u_\varepsilon\right)\partial_x^5 u_\varepsilon\\ &\quad + \gamma\left(D\partial_x u_\varepsilon\partial_x^5 u_\varepsilon + E\partial_x^2((\partial_x^2 u_\varepsilon)^2)\right)\partial_x^5 u_\varepsilon\\ &= \varepsilon\left(2\partial_x^8 u_\varepsilon + A(\partial_x^3 u_\varepsilon)^2 + Bu_\varepsilon\partial_x^6 u_\varepsilon + C\partial_x^2 u_\varepsilon\partial_x^4 u_\varepsilon\right)\partial_x^6 u_\varepsilon\\ &\quad + \varepsilon\left(D\partial_x u_\varepsilon\partial_x^5 u_\varepsilon + E\partial_x^2((\partial_x^2 u_\varepsilon)^2)\right)\partial_x^6 u_\varepsilon.\end{aligned} \tag{2.20}$$

Observe that

$$
\begin{aligned}
&\int_{\mathbb{R}} \left(2\partial_x^8 u_\varepsilon + A(\partial_x^3 u_\varepsilon)^2 + B u_\varepsilon \partial_x^6 u_\varepsilon + C\partial_x^2 u_\varepsilon \partial_x^4 u_\varepsilon + D\partial_x u_\varepsilon \partial_x^5 u_\varepsilon\right)\partial_t u_\varepsilon \mathrm{d}x \\
&\quad = \frac{\mathrm{d}}{\mathrm{d}t}\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + A\int_{\mathbb{R}} (\partial_x^3 u_\varepsilon)^2 \partial_t u_\varepsilon \mathrm{d}x + B\int_{\mathbb{R}} u_\varepsilon \partial_x^6 u_\varepsilon \partial_t u_\varepsilon \mathrm{d}x \\
&\qquad + C\int_{\mathbb{R}} \partial_x^2 u_\varepsilon \partial_x^4 u_\varepsilon \partial_t u_\varepsilon \mathrm{d}x + D\int_{\mathbb{R}} \partial_x u_\varepsilon \partial_x^5 u_\varepsilon \partial_t u_\varepsilon \mathrm{d}x, \\
&\alpha\int_{\mathbb{R}} \left(2\partial_x^8 u_\varepsilon + A(\partial_x^3 u_\varepsilon)^2 + B u_\varepsilon \partial_x^6 u_\varepsilon + C\partial_x^2 u_\varepsilon \partial_x^4 u_\varepsilon + D\partial_x u_\varepsilon \partial_x^5 u_\varepsilon\right)\partial_x u_\varepsilon \partial_x^2 u_\varepsilon \mathrm{d}x \\
&\quad = -2\alpha\int_{\mathbb{R}} (\partial_x^2 u_\varepsilon)^2 \partial_x^7 u_\varepsilon \mathrm{d}x - 2\alpha\int_{\mathbb{R}} \partial_x u_\varepsilon \partial_x^3 u_\varepsilon \partial_x^7 u_\varepsilon \mathrm{d}x \\
&\qquad + A\alpha\int_{\mathbb{R}} \partial_x u_\varepsilon \partial_x^2 u_\varepsilon (\partial_x^3 u_\varepsilon)^2 \mathrm{d}x - B\alpha\int_{\mathbb{R}} (\partial_x u_\varepsilon)^2 \partial_x^2 u_\varepsilon \partial_x^5 u_\varepsilon \mathrm{d}x \\
&\qquad - B\alpha\int_{\mathbb{R}} u_\varepsilon (\partial_x^2 u_\varepsilon)^2 \partial_x^5 u_\varepsilon \mathrm{d}x - B\alpha\int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon \partial_x^3 u_\varepsilon \partial_x^5 u_\varepsilon \mathrm{d}x \\
&\qquad + (C-2D)\alpha\int_{\mathbb{R}} \partial_x u_\varepsilon (\partial_x^2 u_\varepsilon)^2 \partial_x^4 u_\varepsilon \mathrm{d}x - 2D\alpha\int_{\mathbb{R}} (\partial_x u_\varepsilon)^2 \partial_x^3 u_\varepsilon \partial_x^4 u_\varepsilon \mathrm{d}x \\
&\quad = 6\alpha\int_{\mathbb{R}} \partial_x^2 u_\varepsilon \partial_x^3 u_\varepsilon \partial_x^6 u_\varepsilon \mathrm{d}x + 2\alpha\int_{\mathbb{R}} \partial_x u_\varepsilon \partial_x^4 u_\varepsilon \partial_x^6 u_\varepsilon \mathrm{d}x \\
&\qquad + A\alpha\int_{\mathbb{R}} \partial_x u_\varepsilon \partial_x^2 u_\varepsilon (\partial_x^3 u_\varepsilon)^2 \mathrm{d}x + 3B\alpha\int_{\mathbb{R}} \partial_x u_\varepsilon (\partial_x^2 u_\varepsilon)^2 \partial_x^4 u_\varepsilon \mathrm{d}x \\
&\qquad + B\alpha\int_{\mathbb{R}} (\partial_x u_\varepsilon)^2 \partial_x^3 u_\varepsilon \partial_x^4 u_\varepsilon \mathrm{d}x + 2B\int_{\mathbb{R}} u_\varepsilon \partial_x^2 u_\varepsilon \partial_x^3 u_\varepsilon \partial_x^4 u_\varepsilon \mathrm{d}x \\
&\qquad + B\alpha\int_{\mathbb{R}} (\partial_x u_\varepsilon)^2 \partial_x^3 u_\varepsilon \partial_x^4 u_\varepsilon \mathrm{d}x + B\alpha\int_{\mathbb{R}} u_\varepsilon \partial_x^2 u_\varepsilon \partial_x^3 u_\varepsilon \partial_x^4 u_\varepsilon \mathrm{d}x \\
&\qquad + B\alpha\int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon (\partial_x^4 u_\varepsilon)^2 \mathrm{d}x + (C-2D)\alpha\int_{\mathbb{R}} \partial_x u_\varepsilon (\partial_x^2 u_\varepsilon)^2 \partial_x^4 u_\varepsilon \mathrm{d}x \\
&\qquad - 2D\alpha\int_{\mathbb{R}} (\partial_x u_\varepsilon)^2 \partial_x^3 u_\varepsilon \partial_x^4 u_\varepsilon \mathrm{d}x \\
&\quad = -6\alpha\int_{\mathbb{R}} (\partial_x^3 u_\varepsilon)^2 \partial_x^5 u_\varepsilon \mathrm{d}x - 8\alpha\int_{\mathbb{R}} \partial_x^2 u_\varepsilon \partial_x^3 u_\varepsilon \partial_x^5 u_\varepsilon \mathrm{d}x \\
&\qquad - 2\alpha\int_{\mathbb{R}} \partial_x u_\varepsilon (\partial_x^5 u_\varepsilon)^2 \mathrm{d}x + A\alpha\int_{\mathbb{R}} \partial_x u_\varepsilon \partial_x^2 u_\varepsilon (\partial_x^3 u_\varepsilon)^2 \mathrm{d}x \\
&\qquad + 3B\alpha\int_{\mathbb{R}} \partial_x u_\varepsilon (\partial_x^2 u_\varepsilon)^2 \partial_x^4 u_\varepsilon \mathrm{d}x + B\alpha\int_{\mathbb{R}} (\partial_x u_\varepsilon)^2 \partial_x^3 u_\varepsilon \partial_x^4 u_\varepsilon \mathrm{d}x \\
&\qquad + 2B\int_{\mathbb{R}} u_\varepsilon \partial_x^2 u_\varepsilon \partial_x^3 u_\varepsilon \partial_x^4 u_\varepsilon \mathrm{d}x + B\alpha\int_{\mathbb{R}} (\partial_x u_\varepsilon)^2 \partial_x^3 u_\varepsilon \partial_x^4 u_\varepsilon \mathrm{d}x \\
&\qquad + B\alpha\int_{\mathbb{R}} u_\varepsilon \partial_x^2 u_\varepsilon \partial_x^3 u_\varepsilon \partial_x^4 u_\varepsilon \mathrm{d}x + B\alpha\int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon (\partial_x^4 u_\varepsilon)^2 \mathrm{d}x \\
&\qquad + (C-2D)\alpha\int_{\mathbb{R}} \partial_x u_\varepsilon (\partial_x^2 u_\varepsilon)^2 \partial_x^4 u_\varepsilon \mathrm{d}x - 2D\alpha\int_{\mathbb{R}} (\partial_x u_\varepsilon)^2 \partial_x^3 u_\varepsilon \partial_x^4 u_\varepsilon \mathrm{d}x
\end{aligned} \tag{2.21}
$$

$$
\begin{aligned}
&= 10\alpha \int_{\mathbb{R}} \partial_x^3 u_\varepsilon (\partial_x^4 u_\varepsilon)^2 \mathrm{d}x - 2\alpha \int_{\mathbb{R}} \partial_x u_\varepsilon (\partial_x^5 u_\varepsilon)^2 \mathrm{d}x \\
&\quad + A\alpha \int_{\mathbb{R}} \partial_x u_\varepsilon \partial_x^2 u_\varepsilon (\partial_x^3 u_\varepsilon)^2 \mathrm{d}x + 3B\alpha \int_{\mathbb{R}} \partial_x u_\varepsilon (\partial_x^2 u_\varepsilon)^2 \partial_x^4 u_\varepsilon \mathrm{d}x \\
&\quad + 2B\alpha \int_{\mathbb{R}} (\partial_x u_\varepsilon)^2 \partial_x^3 u_\varepsilon \partial_x^4 u_\varepsilon \mathrm{d}x + 3B \int_{\mathbb{R}} u_\varepsilon \partial_x^2 u_\varepsilon \partial_x^3 u_\varepsilon \partial_x^4 u_\varepsilon \mathrm{d}x \\
&\quad + B\alpha \int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon (\partial_x^4 u_\varepsilon)^2 \mathrm{d}x + (C - 2D)\alpha \int_{\mathbb{R}} \partial_x u_\varepsilon (\partial_x^2 u_\varepsilon)^2 \partial_x^4 u_\varepsilon \mathrm{d}x \\
&\quad - 2D\alpha \int_{\mathbb{R}} (\partial_x u_\varepsilon)^2 \partial_x^3 u_\varepsilon \partial_x^4 u_\varepsilon \mathrm{d}x,
\end{aligned}
$$

$$
\begin{aligned}
&- 2\alpha \int_{\mathbb{R}} \left(2\partial_x^8 u_\varepsilon + A(\partial_x^3 u_\varepsilon)^2 + B u_\varepsilon \partial_x^6 u_\varepsilon + C \partial_x^2 u_\varepsilon \partial_x^4 u_\varepsilon + D \partial_x u_\varepsilon \partial_x^5 u_\varepsilon \right) u_\varepsilon \partial_x^3 u_\varepsilon \mathrm{d}x \\
&\quad = 4\alpha \int_{\mathbb{R}} \partial_x u_\varepsilon \partial_x^3 u_\varepsilon \partial_x^7 u_\varepsilon \mathrm{d}x + 4\alpha \int_{\mathbb{R}} u_\varepsilon \partial_x^4 u_\varepsilon \partial_x^7 u_\varepsilon \mathrm{d}x \\
&\quad - 2A\alpha \int_{\mathbb{R}} u_\varepsilon (\partial_x^3 u_\varepsilon)^3 \mathrm{d}x + 4B\alpha \int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon \partial_x^3 u_\varepsilon \partial_x^5 u_\varepsilon \mathrm{d}x \\
&\quad + 2B\alpha \int_{\mathbb{R}} u_\varepsilon^2 \partial_x^4 u_\varepsilon \partial_x^5 u_\varepsilon \mathrm{d}x - 2(C - D)\alpha \int_{\mathbb{R}} u_\varepsilon \partial_x^2 u_\varepsilon \partial_x^3 u_\varepsilon \partial_x^4 u_\varepsilon \mathrm{d}x \\
&\quad + 2D\alpha \int_{\mathbb{R}} (\partial_x u_\varepsilon)^2 \partial_x^3 u_\varepsilon \partial_x^4 u_\varepsilon \mathrm{d}x + 2D\alpha \int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon (\partial_x^4 u_\varepsilon)^2 \mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&= -4\alpha \int_{\mathbb{R}} \partial_x^2 u_\varepsilon \partial_x^3 u_\varepsilon \partial_x^6 u_\varepsilon \mathrm{d}x - 8\alpha \int_{\mathbb{R}} \partial_x u_\varepsilon \partial_x^4 u_\varepsilon \partial_x^6 u_\varepsilon \mathrm{d}x - 4\alpha \int_{\mathbb{R}} u_\varepsilon \partial_x^5 u_\varepsilon \partial_x^6 u_\varepsilon \mathrm{d}x \\
&\quad - 2A\alpha \int_{\mathbb{R}} u_\varepsilon (\partial_x^3 u_\varepsilon)^3 \mathrm{d}x - 4B\alpha \int_{\mathbb{R}} (\partial_x u_\varepsilon)^2 \partial_x^3 u_\varepsilon \partial_x^4 u_\varepsilon \mathrm{d}x \\
&\quad - 4B\alpha \int_{\mathbb{R}} u_\varepsilon \partial_x^2 u_\varepsilon \partial_x^3 u_\varepsilon \partial_x^4 u_\varepsilon \mathrm{d}x - 6B\alpha \int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon (\partial_x^4 u_\varepsilon)^2 \mathrm{d}x \\
&\quad - 2(C-D)\alpha \int_{\mathbb{R}} u_\varepsilon \partial_x^2 u_\varepsilon \partial_x^3 u_\varepsilon \partial_x^4 u_\varepsilon \mathrm{d}x + 2D\alpha \int_{\mathbb{R}} (\partial_x u_\varepsilon)^2 \partial_x^3 u_\varepsilon \partial_x^4 u_\varepsilon \mathrm{d}x \\
&\quad + 2D\alpha \int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon (\partial_x^4 u_\varepsilon)^2 \mathrm{d}x \\
&= 4\alpha \int_{\mathbb{R}} (\partial_x^3 u_\varepsilon)^2 \partial_x^5 u_\varepsilon \mathrm{d}x + 12\alpha \int_{\mathbb{R}} \partial_x^2 u_\varepsilon \partial_x^4 u_\varepsilon \partial_x^5 u_\varepsilon \mathrm{d}x + 10\alpha \int_{\mathbb{R}} \partial_x u_\varepsilon (\partial_x^5 u_\varepsilon)^2 dx \\
&\quad - 2A\alpha \int_{\mathbb{R}} u_\varepsilon (\partial_x^3 u_\varepsilon)^3 \mathrm{d}x - 4B\alpha \int_{\mathbb{R}} (\partial_x u_\varepsilon)^2 \partial_x^3 u_\varepsilon \partial_x^4 u_\varepsilon \mathrm{d}x \\
&\quad - 4B\alpha \int_{\mathbb{R}} u_\varepsilon \partial_x^2 u_\varepsilon \partial_x^3 u_\varepsilon \partial_x^4 u_\varepsilon \mathrm{d}x - 6B\alpha \int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon (\partial_x^4 u_\varepsilon)^2 \mathrm{d}x \\
&\quad - 2(C-D)\alpha \int_{\mathbb{R}} u_\varepsilon \partial_x^2 u_\varepsilon \partial_x^3 u_\varepsilon \partial_x^4 u_\varepsilon \mathrm{d}x + 2D\alpha \int_{\mathbb{R}} (\partial_x u_\varepsilon)^2 \partial_x^3 u_\varepsilon \partial_x^4 u_\varepsilon \mathrm{d}x \\
&\quad + 2D\alpha \int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon (\partial_x^4 u_\varepsilon)^2 \mathrm{d}x \\
&= -14\alpha \int_{\mathbb{R}} \partial_x^3 u_\varepsilon (\partial_x^4 u_\varepsilon)^2 \mathrm{d}x + 10\alpha \int_{\mathbb{R}} \partial_x u_\varepsilon (\partial_x^5 u_\varepsilon)^2 \mathrm{d}x \\
&\quad - 2A\alpha \int_{\mathbb{R}} u_\varepsilon (\partial_x^3 u_\varepsilon)^3 \mathrm{d}x - 4B\alpha \int_{\mathbb{R}} (\partial_x u_\varepsilon)^2 \partial_x^3 u_\varepsilon \partial_x^4 u_\varepsilon \mathrm{d}x \\
&\quad - 4B\alpha \int_{\mathbb{R}} u_\varepsilon \partial_x^2 u_\varepsilon \partial_x^3 u_\varepsilon \partial_x^4 u_\varepsilon \mathrm{d}x - 6B\alpha \int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon (\partial_x^4 u_\varepsilon)^2 \mathrm{d}x \\
&\quad - 2(C-D)\alpha \int_{\mathbb{R}} u_\varepsilon \partial_x^2 u_\varepsilon \partial_x^3 u_\varepsilon \partial_x^4 u_\varepsilon \mathrm{d}x + 2D\alpha \int_{\mathbb{R}} (\partial_x u_\varepsilon)^2 \partial_x^3 u_\varepsilon \partial_x^4 u_\varepsilon \mathrm{d}x \\
&\quad + 2D\alpha \int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon (\partial_x^4 u_\varepsilon)^2 \mathrm{d}x,
\end{aligned}
$$

$$
\begin{aligned}
&\gamma \int_{\mathbb{R}} \left(2\partial_x^8 u_\varepsilon + A(\partial_x^3 u_\varepsilon)^2 + B u_\varepsilon \partial_x^6 u_\varepsilon + C\partial_x^2 u_\varepsilon \partial_x^4 u_\varepsilon + D\partial_x u_\varepsilon \partial_x^5 u_\varepsilon\right) \partial_x^5 u_\varepsilon \mathrm{d}x \\
&\qquad = -\left(2A\gamma + \frac{C\gamma}{2}\right) \int_{\mathbb{R}} \partial_x^3 u_\varepsilon (\partial_x^4 u_\varepsilon)^2 \mathrm{d}x - \left(\frac{B\gamma}{2} + D\gamma\right) \int_{\mathbb{R}} \partial_x u_\varepsilon (\partial_x^5 u_\varepsilon)^2 \mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&\varepsilon \int_{\mathbb{R}} \left(2\partial_x^8 u_\varepsilon + A(\partial_x^3 u_\varepsilon)^2 + B u_\varepsilon \partial_x^6 u_\varepsilon + C\partial_x^2 u_\varepsilon \partial_x^4 u_\varepsilon + D\partial_x u_\varepsilon \partial_x^5 u_\varepsilon\right) \partial_x^6 u_\varepsilon \mathrm{d}x \\
&\qquad = -2\varepsilon \left\|\partial_x^7 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - 2A\varepsilon \int_{\mathbb{R}} \partial_x^3 u_\varepsilon \partial_x^4 u_\varepsilon \partial_x^5 u_\varepsilon \mathrm{d}x + B\varepsilon \int_{\mathbb{R}} u_\varepsilon (\partial_x^6 u_\varepsilon)^2 \mathrm{d}x \\
&\qquad\quad + C\varepsilon \int_{\mathbb{R}} \partial_x^2 u_\varepsilon \partial_x^4 u_\varepsilon \partial_x^6 u_\varepsilon \mathrm{d}x - \frac{D\varepsilon}{2} \int_{\mathbb{R}} \partial_x^2 u_\varepsilon (\partial_x^5 u_\varepsilon)^2 \mathrm{d}x.
\end{aligned}
$$

Moreover, since

$$\partial_x((\partial_x^2 u_\varepsilon)^2) = 2\partial_x^2 u_\varepsilon \partial_x^3 u_\varepsilon, \quad \partial_x^2((\partial_x^2 u_\varepsilon)^2) = 2(\partial_x^3 u_\varepsilon)^2 + 2\partial_x^2 u_\varepsilon \partial_x^4 u_\varepsilon,$$

we have that

$$\begin{aligned}
E\alpha \int_{\mathbb{R}} \partial_x^2((\partial_x^2 u_\varepsilon)^2)\partial_x u_\varepsilon \partial_x^2 u_\varepsilon \mathrm{d}x =& 2E\alpha \int_{\mathbb{R}} \partial_x u_\varepsilon \partial_x^2 u_\varepsilon (\partial_x^3 u_\varepsilon)^2 \mathrm{d}x \\
&+ 2E\alpha \int_{\mathbb{R}} \partial_x u_\varepsilon (\partial_x^2 u_\varepsilon)^2 \partial_x^4 u_\varepsilon \mathrm{d}x \\
=& - 2E\alpha \int_{\mathbb{R}} \partial_x u_\varepsilon \partial_x^2 u_\varepsilon (\partial_x^3 u_\varepsilon)^2 \mathrm{d}x, \\
-2E\alpha \int_{\mathbb{R}} \partial_x^2((\partial_x^2 u_\varepsilon)^2) u_\varepsilon \partial_x^3 u_\varepsilon \mathrm{d}x =& - 4E\alpha \int_{\mathbb{R}} u_\varepsilon (\partial_x^3 u_\varepsilon)^3 \mathrm{d}x - 4E\alpha \int_{\mathbb{R}} u_\varepsilon \partial_x^2 u_\varepsilon \partial_x^3 u_\varepsilon \partial_x^4 u_\varepsilon \mathrm{d}x \\
=& - 4E\alpha \int_{\mathbb{R}} u_\varepsilon (\partial_x^3 u_\varepsilon)^3 \mathrm{d}x + 2E\alpha \int_{\mathbb{R}} \partial_x u_\varepsilon \partial_x^2 u_\varepsilon (\partial_x^3 u_\varepsilon)^2 \mathrm{d}x \\
&+ 2E\alpha \int_{\mathbb{R}} u_\varepsilon (\partial_x^3 u_\varepsilon)^3 \mathrm{d}x + 2E\alpha \int_{\mathbb{R}} \partial_x u_\varepsilon \partial_x^2 u_\varepsilon (\partial_x^3 u_\varepsilon)^2 \mathrm{d}x, \\
E\gamma \int_{\mathbb{R}} \partial_x^2((\partial_x^2 u_\varepsilon)^2)\partial_x^5 u_\varepsilon \mathrm{d}x =& 2E\gamma \int_{\mathbb{R}} (\partial_x^3 u_\varepsilon)^2 \partial_x^5 u_\varepsilon \mathrm{d}x + 2E\gamma \int_{\mathbb{R}} \partial_x^2 \partial_x^4 u_\varepsilon \partial_x^5 u_\varepsilon \mathrm{d}x \\
=& - 5E\gamma \int_{\mathbb{R}} \partial_x^3 u_\varepsilon (\partial_x^4 u_\varepsilon)^2 \mathrm{d}x, \\
E\varepsilon \int_{\mathbb{R}} \partial_x^2((\partial_x^2 u_\varepsilon)^2)\partial_x^6 u_\varepsilon \mathrm{d}x =& 2E\varepsilon \int_{\mathbb{R}} (\partial_x^3 u_\varepsilon)^2 \partial_x^6 u_\varepsilon \mathrm{d}x + 2E\varepsilon \int_{\mathbb{R}} \partial_x^2 u_\varepsilon \partial_x^4 u_\varepsilon \partial_x^6 u_\varepsilon \mathrm{d}x \\
=& - 6E\varepsilon \int_{\mathbb{R}} \partial_x^3 u_\varepsilon \partial_x^4 u_\varepsilon \partial_x^5 u_\varepsilon \mathrm{d}x - 2E\varepsilon \int_{\mathbb{R}} \partial_x^2 u_\varepsilon (\partial_x^5 u_\varepsilon)^2 \mathrm{d}x.
\end{aligned} \tag{2.22}$$

It follows from (2.21), (2.22) and an integration of (2.20) that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + A\int_{\mathbb{R}} (\partial_x^3 u_\varepsilon)^2 \partial_t u_\varepsilon \mathrm{d}x + B\int_{\mathbb{R}} u_\varepsilon \partial_x^6 u_\varepsilon \partial_t u_\varepsilon \mathrm{d}x \\
&\quad + C\int_{\mathbb{R}} \partial_x^2 u_\varepsilon \partial_x^4 u_\varepsilon \partial_t u_\varepsilon \mathrm{d}x + D\int_{\mathbb{R}} \partial_x u_\varepsilon \partial_x^5 u_\varepsilon \partial_t u_\varepsilon \mathrm{d}x \\
&\quad + E\int_{\mathbb{R}} \partial_x^2((\partial_x^2 u_\varepsilon)^2)\partial_t u_\varepsilon \mathrm{d}x + 2\varepsilon\left\|\partial_x^6 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&= \left(4\alpha + 2A\gamma + \frac{C\gamma}{2} + 5E\gamma\right)\int_{\mathbb{R}} \partial_x^3 u_\varepsilon (\partial_x^4 u_\varepsilon)^2 \mathrm{d}x \\
&\quad + \left(\frac{B\gamma}{2} + D\gamma - 8\alpha\right)\int_{\mathbb{R}} \partial_x u_\varepsilon (\partial_x^5 u_\varepsilon)^2 \mathrm{d}x \\
&\quad - A\alpha\int_{\mathbb{R}} \partial_x u_\varepsilon \partial_x^2 u_\varepsilon (\partial_x^3 u_\varepsilon)^2 \mathrm{d}x - (3B + C - 2D)\alpha\int_{\mathbb{R}} \partial_x u_\varepsilon (\partial_x^2 u_\varepsilon)^2 \partial_x^4 u_\varepsilon \mathrm{d}x \\
&\quad + (2B - 2D)\alpha\int_{\mathbb{R}} (\partial_x u_\varepsilon)^2 \partial_x^3 u_\varepsilon \partial_x^4 u_\varepsilon \mathrm{d}x - 3B\alpha\int_{\mathbb{R}} u_\varepsilon \partial_x^2 u_\varepsilon \partial_x^3 u_\varepsilon \partial_x^4 u_\varepsilon \mathrm{d}x \\
&\quad + (5B - 2D)\alpha\int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon (\partial_x^4 u_\varepsilon)^2 \mathrm{d}x + (2A - 2E)\alpha\int_{\mathbb{R}} u_\varepsilon (\partial_x^3 u_\varepsilon)^3 \mathrm{d}x \\
&\quad + (4B + 2C - 2D)\alpha\int_{\mathbb{R}} u_\varepsilon \partial_x^2 u_\varepsilon \partial_x^3 u_\varepsilon \partial_x^4 u_\varepsilon \mathrm{d}x \\
&\quad - (2A + 6E)\varepsilon\int_{\mathbb{R}} \partial_x^3 u_\varepsilon \partial_x^4 u_\varepsilon \partial_x^5 u_\varepsilon \mathrm{d}x + C\varepsilon\int_{\mathbb{R}} \partial_x^2 u_\varepsilon \partial_x^4 u_\varepsilon \partial_x^5 u_\varepsilon \mathrm{d}x \\
&\quad - \left(\frac{D}{2} + 2E\right)\varepsilon\int_{\mathbb{R}} \partial_x^2 u_\varepsilon (\partial_x^5 u_\varepsilon)^2 \mathrm{d}x + B\varepsilon\int_{\mathbb{R}} u_\varepsilon (\partial_x^6 u_\varepsilon)^2 \mathrm{d}x.
\end{aligned} \tag{2.23}$$

Observe that

$$\begin{aligned}
B\int_{\mathbb{R}} u_\varepsilon\partial_x^6u_\varepsilon\partial_t u_\varepsilon dx =& -B\int_{\mathbb{R}}\partial_x u_\varepsilon\partial_x^5u_\varepsilon\partial_t u_\varepsilon dx - B\int_{\mathbb{R}} u_\varepsilon\partial_x^5u_\varepsilon\partial_t\partial_x u_\varepsilon dx\\
=& -B\int_{\mathbb{R}}\partial_x u_\varepsilon\partial_x^5u_\varepsilon\partial_t u_\varepsilon dx + B\int_{\mathbb{R}}\partial_x u_\varepsilon\partial_x^4u_\varepsilon\partial_t\partial_x u_\varepsilon dx\\
&+ B\int_{\mathbb{R}} u_\varepsilon\partial_x^4u_\varepsilon\partial_t\partial_x^2 u_\varepsilon dx\\
=& -B\int_{\mathbb{R}}\partial_x u_\varepsilon\partial_x^5u_\varepsilon\partial_t u_\varepsilon dx + B\int_{\mathbb{R}}\partial_x u_\varepsilon\partial_x^4u_\varepsilon\partial_t\partial_x u_\varepsilon dx\\
&- B\int_{\mathbb{R}}\partial_x u_\varepsilon\partial_x^3u_\varepsilon\partial_t\partial_x^2 u_\varepsilon dx - \frac{B}{2}\int_{\mathbb{R}} u_\varepsilon\partial_t((\partial_x^3u_\varepsilon)^2)dx\\
=& -2B\int_{\mathbb{R}}\partial_x u_\varepsilon\partial_x^5u_\varepsilon dx - B\int_{\mathbb{R}}\partial_x^2 u_\varepsilon\partial_x^4u_\varepsilon\partial_t u_\varepsilon dx\\
&+ B\int_{\mathbb{R}}\partial_x^2 u_\varepsilon\partial_x^3u_\varepsilon\partial_t\partial_x u_\varepsilon + B\int_{\mathbb{R}}\partial_x u_\varepsilon\partial_x^4u_\varepsilon\partial_t\partial_x u_\varepsilon dx\\
&- \frac{B}{2}\int_{\mathbb{R}} u_\varepsilon\partial_t((\partial_x^3u_\varepsilon)^2)dx\\
=& -3B\int_{\mathbb{R}}\partial_x u_\varepsilon\partial_x^5u_\varepsilon dx - 3B\int_{\mathbb{R}}\partial_x^2 u_\varepsilon\partial_x^4u_\varepsilon\partial_t u_\varepsilon dx\\
&- B\int_{\mathbb{R}}(\partial_x^3u_\varepsilon)^2\partial_t u_\varepsilon dx - \frac{B}{2}\int_{\mathbb{R}} u_\varepsilon\partial_t((\partial_x^3u_\varepsilon)^2)dx.
\end{aligned}$$

Consequently, by (2.23),

$$\begin{aligned}
&\frac{d}{dt}\left\|\partial_x^4u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + (A-B)\int_{\mathbb{R}}(\partial_x^3u_\varepsilon)^2\partial_t u_\varepsilon dx - \frac{B}{2}\int_{\mathbb{R}} u_\varepsilon\partial_t((\partial_x^3u_\varepsilon)^2)dx\\
&\quad+(C-3B)\int_{\mathbb{R}}\partial_x^2u_\varepsilon\partial_x^4u_\varepsilon\partial_t u_\varepsilon dx + (D-3B)\int_{\mathbb{R}}\partial_x u_\varepsilon\partial_x^5u_\varepsilon\partial_t u_\varepsilon dx\\
&\quad+E\int_{\mathbb{R}}\partial_x^2((\partial_x^2u_\varepsilon)^2)\partial_t u_\varepsilon dx + 2\varepsilon\left\|\partial_x^7u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&=\left(4\alpha+2A\gamma+\frac{C\gamma}{2}+5E\gamma\right)\int_{\mathbb{R}}\partial_x^3u_\varepsilon(\partial_x^4u_\varepsilon)^2dx\\
&\quad+\left(\frac{B\gamma}{2}+D\gamma-8\alpha\right)\int_{\mathbb{R}}\partial_x u_\varepsilon(\partial_x^5u_\varepsilon)^2dx\\
&\quad-A\alpha\int_{\mathbb{R}}\partial_x u_\varepsilon\partial_x^2u_\varepsilon(\partial_x^3u_\varepsilon)^2dx-(3B+C-2D)\alpha\int_{\mathbb{R}}\partial_x u_\varepsilon(\partial_x^2u_\varepsilon)^2\partial_x^4u_\varepsilon dx\\
&\quad+(2B-2D)\alpha\int_{\mathbb{R}}(\partial_x u_\varepsilon)^2\partial_x^3u_\varepsilon\partial_x^4u_\varepsilon dx\\
&\quad+(B+2C-2D)\alpha\int_{\mathbb{R}} u_\varepsilon\partial_x^2u_\varepsilon\partial_x^3u_\varepsilon\partial_x^4u_\varepsilon dx\\
&\quad+(5B-2D)\alpha\int_{\mathbb{R}} u_\varepsilon\partial_x u_\varepsilon(\partial_x^4u_\varepsilon)^2dx\\
&\quad+(2A-2E)\alpha\int_{\mathbb{R}} u_\varepsilon(\partial_x^3u_\varepsilon)^3dx-(2A+6E)\varepsilon\int_{\mathbb{R}}\partial_x^3u_\varepsilon\partial_x^4u_\varepsilon\partial_x^5u_\varepsilon dx\\
&\quad+C\varepsilon\int_{\mathbb{R}}\partial_x^2u_\varepsilon\partial_x^4u_\varepsilon\partial_x^5u_\varepsilon dx-\left(\frac{D}{2}+2E\right)\varepsilon\int_{\mathbb{R}}\partial_x^2u_\varepsilon(\partial_x^5u_\varepsilon)^2dx.\\
&\quad+B\varepsilon\int_{\mathbb{R}} u_\varepsilon(\partial_x^6u_\varepsilon)^2dx.
\end{aligned} \tag{2.24}$$

Observe that

$$
\begin{aligned}
(C-3B)\int_{\mathbb{R}}\partial_x^2 u_\varepsilon \partial_x^4 u_\varepsilon \partial_t u_\varepsilon \mathrm{d}x =& -(C-3B)\int_{\mathbb{R}}(\partial_x^3 u_\varepsilon)^2 \partial_t u_\varepsilon \mathrm{d}x\\
&-(C-3B)\int_{\mathbb{R}}\partial_x^2 u_\varepsilon \partial_x^3 u_\varepsilon \partial_t \partial_x u_\varepsilon \mathrm{d}x\\
=& -(C-3B)\int_{\mathbb{R}}(\partial_x^3 u_\varepsilon)^2 \partial_t u_\varepsilon \mathrm{d}x\\
&-\frac{C-3B}{2}\int_{\mathbb{R}}\partial_x((\partial_x^2 u_\varepsilon)^2)\partial_t \partial_x u_\varepsilon \mathrm{d}x\\
=& -(C-3B)\int_{\mathbb{R}}(\partial_x^3 u_\varepsilon)^2 \partial_t u_\varepsilon \mathrm{d}x\\
&+\frac{C-3B}{2}\int_{\mathbb{R}}\partial_x^2((\partial_x^2 u_\varepsilon)^2)\partial_t \partial_x u_\varepsilon \mathrm{d}x.
\end{aligned}
$$

Therefore, by (2.24),

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + (A+2B-C)\int_{\mathbb{R}}(\partial_x^3 u_\varepsilon)^2\partial_t u_\varepsilon \mathrm{d}x - \frac{B}{2}\int_{\mathbb{R}} u_\varepsilon \partial_t((\partial_x^3 u_\varepsilon)^2)\mathrm{d}x\\
&\quad + (D-3B)\int_{\mathbb{R}}\partial_x u_\varepsilon \partial_x^5 u_\varepsilon \partial_t u_\varepsilon \mathrm{d}x + \frac{2E+C-3B}{2}\int_{\mathbb{R}}\partial_x^2((\partial_x^2 u_\varepsilon)^2)\partial_t u_\varepsilon \mathrm{d}x\\
&\quad + 2\varepsilon\left\|\partial_x^7 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&= \left(4\alpha + 2A\gamma + \frac{C\gamma}{2} + 5E\gamma\right)\int_{\mathbb{R}}\partial_x^3 u_\varepsilon(\partial_x^4 u_\varepsilon)^2\mathrm{d}x\\
&\quad + \left(\frac{B\gamma}{2} + D\gamma - 8\alpha\right)\int_{\mathbb{R}}\partial_x u_\varepsilon(\partial_x^5 u_\varepsilon)^2\mathrm{d}x\\
&\quad - A\alpha\int_{\mathbb{R}}\partial_x u_\varepsilon \partial_x^2 u_\varepsilon(\partial_x^3 u_\varepsilon)^2\mathrm{d}x - (3B+C-2D)\alpha\int_{\mathbb{R}}\partial_x u_\varepsilon(\partial_x^2 u_\varepsilon)^2\partial_x^4 u_\varepsilon \mathrm{d}x\\
&\quad + (2B-2D)\alpha\int_{\mathbb{R}}(\partial_x u_\varepsilon)^2\partial_x^3 u_\varepsilon \partial_x^4 u_\varepsilon \mathrm{d}x\\
&\quad + (B+2C-2D)\alpha\int_{\mathbb{R}} u_\varepsilon \partial_x^2 u_\varepsilon \partial_x^3 u_\varepsilon \partial_x^4 u_\varepsilon \mathrm{d}x\\
&\quad + (5B-2D)\alpha\int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon(\partial_x^4 u_\varepsilon)^2\mathrm{d}x + (2A-2E)\alpha\int_{\mathbb{R}} u_\varepsilon(\partial_x^3 u_\varepsilon)^3\mathrm{d}x\\
&\quad - (2A+6E)\varepsilon\int_{\mathbb{R}}\partial_x^3 u_\varepsilon \partial_x^4 u_\varepsilon \partial_x^5 u_\varepsilon \mathrm{d}x + C\varepsilon\int_{\mathbb{R}}\partial_x^2 u_\varepsilon \partial_x^4 u_\varepsilon \partial_x^5 u_\varepsilon \mathrm{d}x\\
&\quad - \left(\frac{D}{2} + 2E\right)\varepsilon\int_{\mathbb{R}}\partial_x^2 u_\varepsilon(\partial_x^5 u_\varepsilon)^2\mathrm{d}x + B\varepsilon\int_{\mathbb{R}} u_\varepsilon(\partial_x^6 u_\varepsilon)^2\mathrm{d}x.
\end{aligned}
\tag{2.25}
$$

Observe that

$$\begin{aligned}\frac{2E+C-3B}{2}\int_{\mathbb{R}}\partial_x^2((\partial_x^2u_\varepsilon)^2)\partial_tu_\varepsilon\mathrm{d}x=&-\frac{2E+C-3B}{2}\int_{\mathbb{R}}\partial_x((\partial_x^2u_\varepsilon)^2)\partial_t\partial_xu_\varepsilon\mathrm{d}x\\=&\frac{2E+C-3B}{2}\int_{\mathbb{R}}(\partial_x^2u_\varepsilon)^2\partial_t\partial_x^2u_\varepsilon\mathrm{d}x\\=&\frac{2E+C-3B}{6}\frac{d}{dt}\int_{\mathbb{R}}(\partial_x^2u_\varepsilon)^3\mathrm{d}x.\end{aligned}$$

Consequently, by (2.25),

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left\|\partial_x^4u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+(A+2B-C)\int_{\mathbb{R}}(\partial_x^3u_\varepsilon)^2\partial_tu_\varepsilon\mathrm{d}x-\frac{B}{2}\int_{\mathbb{R}}u_\varepsilon\partial_t((\partial_x^3u_\varepsilon)^2)\mathrm{d}x\\&\quad+(D-3B)\int_{\mathbb{R}}\partial_xu_\varepsilon\partial_x^5u_\varepsilon\partial_tu_\varepsilon\mathrm{d}x+\frac{2E+C-3B}{6}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}(\partial_x^2u_\varepsilon)^3\mathrm{d}x\\&\quad+2\varepsilon\left\|\partial_x^7u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\&=\left(4\alpha+2A\gamma+\frac{C\gamma}{2}+5E\gamma\right)\int_{\mathbb{R}}\partial_x^3u_\varepsilon(\partial_x^4u_\varepsilon)^2\mathrm{d}x\\&\quad+\left(\frac{B\gamma}{2}+D\gamma-8\alpha\right)\int_{\mathbb{R}}\partial_xu_\varepsilon(\partial_x^5u_\varepsilon)^2\mathrm{d}x\\&\quad-A\alpha\int_{\mathbb{R}}\partial_xu_\varepsilon\partial_x^2u_\varepsilon(\partial_x^3u_\varepsilon)^2\mathrm{d}x-(3B+C-2D)\alpha\int_{\mathbb{R}}\partial_xu_\varepsilon(\partial_x^2u_\varepsilon)^2\partial_x^4u_\varepsilon\mathrm{d}x\\&\quad+(2B-2D)\alpha\int_{\mathbb{R}}(\partial_xu_\varepsilon)^2\partial_x^3u_\varepsilon\partial_x^4u_\varepsilon\mathrm{d}x\\&\quad+(B+2C-2D)\alpha\int_{\mathbb{R}}u_\varepsilon\partial_x^2u_\varepsilon\partial_x^3u_\varepsilon\partial_x^4u_\varepsilon\mathrm{d}x\\&\quad+(5B-2D)\alpha\int_{\mathbb{R}}u_\varepsilon\partial_xu_\varepsilon(\partial_x^4u_\varepsilon)^2\mathrm{d}x+(2A-2E)\alpha\int_{\mathbb{R}}u_\varepsilon(\partial_x^3u_\varepsilon)^3\mathrm{d}x\\&\quad-(2A+6E)\varepsilon\int_{\mathbb{R}}\partial_x^3u_\varepsilon\partial_x^4u_\varepsilon\partial_x^5u_\varepsilon\mathrm{d}x+C\varepsilon\int_{\mathbb{R}}\partial_x^2u_\varepsilon\partial_x^4u_\varepsilon\partial_x^5u_\varepsilon\mathrm{d}x\\&\quad-\left(\frac{D}{2}+2E\right)\varepsilon\int_{\mathbb{R}}\partial_x^2u_\varepsilon(\partial_x^5u_\varepsilon)^2\mathrm{d}x+B\varepsilon\int_{\mathbb{R}}u_\varepsilon(\partial_x^6u_\varepsilon)^2\mathrm{d}x.\end{aligned}\tag{2.26}$$

We search A, B, C, D, E such that

$$\begin{aligned}&A+2B-C=-\frac{B}{2},\quad D-3B=0,\quad E=-\frac{3C}{10}\\&4\alpha+2A\gamma+\frac{C\gamma}{2}+3E\gamma=0,\quad \frac{B\gamma}{2}+D\gamma-8\alpha=0,\end{aligned}$$

that is

$$\begin{aligned}&2A+5B-2C=0,\quad D=3B,\quad E=-\frac{3C}{10},\\&4A\gamma+C\gamma+10E\gamma=-8\alpha,\quad B\gamma+2D\gamma=16\alpha.\end{aligned}\tag{2.27}$$

Since

$$(A, B, C, D, E) = \left(\frac{12\alpha}{7\gamma}, \frac{16\alpha}{7\gamma}, \frac{52\alpha}{7\gamma}, \frac{48\alpha}{7\gamma}, -\frac{78\alpha}{35\gamma}\right),$$

is the unique solution of (2.27), it follows from (2.26) that

$$\begin{aligned}
&\frac{d}{dt}\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - \frac{8\alpha}{7\gamma}\int_{\mathbb{R}}(\partial_x^3 u_\varepsilon)^2\partial_t u_\varepsilon dx - \frac{8\alpha}{7\gamma}\int_{\mathbb{R}} u_\varepsilon\partial_t((\partial_x^3 u_\varepsilon)^2)dx + \ell_1\frac{d}{dt}\int_{\mathbb{R}}(\partial_x^2 u_\varepsilon)^3 dx\\
&\quad + 2\varepsilon\left\|\partial_x^7 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&= \frac{12\alpha^2}{7\gamma}\int_{\mathbb{R}}\partial_x u_\varepsilon\partial_x^2 u_\varepsilon(\partial_x^3 u_\varepsilon)^2 dx + \frac{4\alpha^2}{7\gamma}\int_{\mathbb{R}}\partial_x u_\varepsilon(\partial_x^2 u_\varepsilon)^2\partial_x^4 u_\varepsilon dx\\
&\quad - \frac{64\alpha^2}{7\gamma}\int_{\mathbb{R}}(\partial_x u_\varepsilon)^2\partial_x^3 u_\varepsilon\partial_x^4 u_\varepsilon dx - \frac{120\alpha^2}{7\gamma}\int_{\mathbb{R}} u_\varepsilon\partial_x^2 u_\varepsilon\partial_x^3 u_\varepsilon\partial_x^4 u_\varepsilon dx\\
&\quad - \frac{16\alpha^2}{7\gamma}\int_{\mathbb{R}} u_\varepsilon\partial_x u_\varepsilon(\partial_x^4 u_\varepsilon)^2 dx + \ell_2\int_{\mathbb{R}} u_\varepsilon(\partial_x^3 u_\varepsilon)^3 dx\\
&\quad + \ell_3\varepsilon\int_{\mathbb{R}}\partial_x^3 u_\varepsilon\partial_x^4 u_\varepsilon\partial_x^5 u_\varepsilon dx - \frac{20\alpha\varepsilon}{7\gamma}\int_{\mathbb{R}}\partial_x^2 u_\varepsilon\partial_x^4 u_\varepsilon\partial_x^5 u_\varepsilon dx\\
&\quad + \ell_4\int_{\mathbb{R}}\partial_x^2 u_\varepsilon(\partial_x^5 u_\varepsilon)^2 dx + \frac{16\alpha\varepsilon}{7\gamma}\int_{\mathbb{R}} u_\varepsilon(\partial_x^6 u_\varepsilon)^2 dx,
\end{aligned}$$

where

$$\ell_1 := \frac{2E + C - 3B}{6}, \quad \ell_2 := 2A - 2E, \quad \ell_3 := 2A + 6E, \quad \ell_4 := \frac{D}{2} + 2E.$$

Consequently, we get

$$\begin{aligned}
&\frac{dG(t)}{dt} + 2\varepsilon\left\|\partial_x^7 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&= \frac{12\alpha^2}{7\gamma}\int_{\mathbb{R}}\partial_x u_\varepsilon\partial_x^2 u_\varepsilon(\partial_x^3 u_\varepsilon)^2 dx + \frac{4\alpha^2}{7\gamma}\int_{\mathbb{R}}\partial_x u_\varepsilon(\partial_x^2 u_\varepsilon)^2\partial_x^4 u_\varepsilon dx\\
&\quad - \frac{64\alpha^2}{7\gamma}\int_{\mathbb{R}}(\partial_x u_\varepsilon)^2\partial_x^3 u_\varepsilon\partial_x^4 u_\varepsilon dx - \frac{120\alpha^2}{7\gamma}\int_{\mathbb{R}} u_\varepsilon\partial_x^2 u_\varepsilon\partial_x^3 u_\varepsilon\partial_x^4 u_\varepsilon dx\\
&\quad - \frac{16\alpha^2}{7\gamma}\int_{\mathbb{R}} u_\varepsilon\partial_x u_\varepsilon(\partial_x^4 u_\varepsilon)^2 dx + \ell_2\int_{\mathbb{R}} u_\varepsilon(\partial_x^3 u_\varepsilon)^3 dx\\
&\quad + \ell_3\varepsilon\int_{\mathbb{R}}\partial_x^3 u_\varepsilon\partial_x^4 u_\varepsilon\partial_x^5 u_\varepsilon dx - \frac{20\alpha\varepsilon}{7\gamma}\int_{\mathbb{R}}\partial_x^2 u_\varepsilon\partial_x^4 u_\varepsilon\partial_x^5 u_\varepsilon dx\\
&\quad + \ell_4\int_{\mathbb{R}}\partial_x^2 u_\varepsilon(\partial_x^5 u_\varepsilon)^2 dx + \frac{16\alpha\varepsilon}{7\gamma}\int_{\mathbb{R}} u_\varepsilon(\partial_x^6 u_\varepsilon)^2 dx,
\end{aligned} \tag{2.28}$$

where

$$G(t) := \left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - \frac{8\alpha}{7\gamma}\int_{\mathbb{R}} u_\varepsilon(\partial_x^3 u_\varepsilon)^2 dx + \ell_1\int_{\mathbb{R}}(\partial_x^2 u_\varepsilon)^3 dx. \tag{2.29}$$

Due to (2.3), (2.5), (2.15), (2.16), (2.17), (2.18), the Hölder inequality and the Young inequality,

$$
\begin{aligned}
&\left|\frac{12\alpha^2}{7\gamma}\right| \int_{\mathbb{R}} |\partial_x u_\varepsilon||\partial_x^2 u_\varepsilon|(\partial_x^3 u_\varepsilon)^2 \mathrm{d}x \\
&\quad \le \left|\frac{12\alpha^2}{7\gamma}\right| \left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} |\partial_x u_\varepsilon||\partial_x^2 u_\varepsilon| \mathrm{d}x \\
&\quad \le \left|\frac{12\alpha^2}{7\gamma}\right| \left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^\infty(\mathbb{R})} \|\partial_x u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})} \left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})} \\
&\quad \le C(T)\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^\infty(\mathbb{R})} \le C(T)\sqrt{\|\partial_x^2 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}}\sqrt{\|\partial_x^4 u_\varepsilon(t,\cdot)\|^3_{L^2(\mathbb{R})}} \\
&\quad \le C(T)\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^{\frac{1}{2}} \\
&\quad \le C(T)\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + C(T)\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})} \\
&\quad \le C(T)\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + C(T), \\
&\left|\frac{4\alpha^2}{7\gamma}\right| \int_{\mathbb{R}} |\partial_x u_\varepsilon(\partial_x^2 u_\varepsilon)^2||\partial_x^4 u_\varepsilon| \mathrm{d}x \\
&\quad \le \frac{16\alpha^4}{49\gamma^2}\int_{\mathbb{R}} (\partial_x u_\varepsilon)^2(\partial_x^2 u_\varepsilon)^4 \mathrm{d}x + \left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&\quad \le \frac{16\alpha^4}{49\gamma^2}\|\partial_x u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^\infty(\mathbb{R})}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&\quad \le C(T)\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^\infty(\mathbb{R})} + \left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&\quad \le C(T)\sqrt{\|\partial_x^2 u_\varepsilon(t,\cdot)\|^3_{L^2(\mathbb{R})}}\sqrt{\|\partial_x^4 u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}} + \left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&\quad \le C(T)\sqrt{\|\partial_x^4 u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}} + \left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&\quad \le C(T)\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})} + \left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + C(T) \\
&\quad \le C(T)\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + C(T) \\
&\quad \le C(T)\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + C(T), \\
&\left|\frac{64\alpha^2}{7\gamma}\right| \int_{\mathbb{R}} (\partial_x u_\varepsilon)^2|\partial_x^3 u_\varepsilon||\partial_x^4 u_\varepsilon| \mathrm{d}x \\
&\quad \le \left|\frac{64\alpha^2}{7\gamma}\right| \|\partial_x u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}} |\partial_x^3 u_\varepsilon||\partial_x^4 u_\varepsilon| \mathrm{d}x \\
&\quad \le C(T)\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})} \\
&\quad \le C(T)\sqrt{\|\partial_x^2 u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}}\sqrt{\|\partial_x^4 u_\varepsilon(t,\cdot)\|^3_{L^2(\mathbb{R})}} \\
&\quad \le C(T)\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}\sqrt{\|\partial_x^4 u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}} \\
&\quad \le C(T)\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + C(T)\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})} \\
&\quad \le C(T)\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + C(T),
\end{aligned}
$$

$$
\begin{aligned}
&\left|\frac{120\alpha^2}{7\gamma}\right|\int_{\mathbb{R}}|u_\varepsilon\partial_x^2u_\varepsilon\partial_x^3u_\varepsilon||\partial_x^4u_\varepsilon|\mathrm{d}x\\
&\quad\le\frac{60\alpha^4}{49\gamma^2}\int_{\mathbb{R}}u_\varepsilon^2(\partial_x^3u_\varepsilon)^2(\partial_x^2u_\varepsilon)^2\mathrm{d}x+\left\|\partial_x^4u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad\le\frac{60\alpha^4}{49\gamma^2}\|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^3u_\varepsilon(t,\cdot)\right\|^2_{L^\infty(\mathbb{R})}\left\|\partial_x^2u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\left\|\partial_x^4u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad\le C(T)\left\|\partial_x^3u_\varepsilon(t,\cdot)\right\|^2_{L^\infty(\mathbb{R})}+\left\|\partial_x^4u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad\le C(T)\sqrt{\|\partial_x^2u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}}\sqrt{\|\partial_x^4u_\varepsilon(t,\cdot)\|^3_{L^2(\mathbb{R})}}+\left\|\partial_x^4u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad\le C(T)\left\|\partial_x^4u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}\sqrt{\|\partial_x^4u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}}+\left\|\partial_x^4u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad\le C(T)\left\|\partial_x^4u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+C(T)\left\|\partial_x^4u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}+\left\|\partial_x^4u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad\le C(T)\left\|\partial_x^4u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+C(T),\\
&\left|\frac{16\alpha^2}{7\gamma}\right|\int_{\mathbb{R}}|u_\varepsilon\partial_xu_\varepsilon|(\partial_x^4u_\varepsilon)^2\mathrm{d}x\\
&\quad\le\left|\frac{16\alpha^2}{7\gamma}\right|\|u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})}\|\partial_xu_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^4u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad\le C(T)\left\|\partial_x^4u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
&|\ell_2|\int_{\mathbb{R}}|u_\varepsilon||\partial_x^3u_\varepsilon|^3\mathrm{d}x\\
&\quad\le|\ell_2|\|u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}}|\partial_x^3u_\varepsilon|^3\mathrm{d}x\\
&\quad\le C(T)\sqrt[4]{\|\partial_x^2u_\varepsilon(t,\cdot)\|^5_{L^2(\mathbb{R})}}\sqrt[4]{\|\partial_x^4u_\varepsilon(t,\cdot)\|^7_{L^2(\mathbb{R})}}\\
&\quad\le C(T)\left\|\partial_x^4u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}\sqrt[4]{\|\partial_x^4u_\varepsilon(t,\cdot)\|^3_{L^2(\mathbb{R})}}\\
&\quad\le C(T)\left\|\partial_x^4u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+C(T)\left\|\partial_x^4u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}\sqrt{\|\partial_x^4u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}}\\
&\quad\le C(T)\left\|\partial_x^4u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+C(T)\left\|\partial_x^4u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}\\
&\quad\le C(T)\left\|\partial_x^4u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+C(T),\\
&|\ell_3|\varepsilon\int_{\mathbb{R}}|\partial_x^3u_\varepsilon||\partial_x^4u_\varepsilon||\partial_x^5u_\varepsilon|\mathrm{d}x\\
&\quad=2\int_{\mathbb{R}}\left|\frac{\ell_3\partial_x^4u_\varepsilon}{2}\right|\left|\partial_x^3u_\varepsilon\partial_x^5u_\varepsilon\right|\mathrm{d}x\\
&\quad\le\frac{\ell_3^2\varepsilon}{2}\left\|\partial_x^4u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\varepsilon\int_{\mathbb{R}}(\partial_x^3u_\varepsilon)^2(\partial_x^5u_\varepsilon)^2\mathrm{d}x\\
&\quad\le\frac{\ell_3^2\varepsilon}{4}\left\|\partial_x^4u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\varepsilon\left\|\partial_x^3u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^5u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})},
\end{aligned}
$$

$$
\begin{aligned}
&\left|\frac{20\alpha\varepsilon}{7\gamma}\right|\int_{\mathbb{R}}|\partial_x^2 u_\varepsilon||\partial_x^4 u_\varepsilon|\partial_x^5 u_\varepsilon|\mathrm{d}x\\
&\quad\le \frac{100\alpha^2\varepsilon}{49\gamma^2}\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\varepsilon\int_{\mathbb{R}}(\partial_x^2 u_\varepsilon)^2(\partial_x^5 u_\varepsilon)^2\mathrm{d}x\\
&\quad\le \frac{100\alpha^2\varepsilon}{49\gamma^2}\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\varepsilon\left\|\partial_x^2 u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^5 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
&|\ell_4|\varepsilon\int_{\mathbb{R}}\partial_x^2 u_\varepsilon(\partial_x^5 u_\varepsilon)^2\mathrm{d}x\\
&\quad=2\varepsilon\int_{\mathbb{R}}\left|\frac{\ell_4\partial_x^2 u_\varepsilon\partial_x^5 u_\varepsilon}{2}\right|\left|\partial_x^5 u_\varepsilon\right|\mathrm{d}x\\
&\quad\le \frac{\ell_4^2\varepsilon}{4}\left\|\partial_x^5 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\varepsilon\int_{\mathbb{R}}(\partial_x^2 u_\varepsilon)^2(\partial_x^5 u_\varepsilon)^2\mathrm{d}x\\
&\quad\le \frac{\ell_4^2\varepsilon}{4}\left\|\partial_x^5 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\varepsilon\left\|\partial_x^2 u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^5 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
&\left|\frac{16\alpha\varepsilon}{7\gamma}\right|\int_{\mathbb{R}}|u_\varepsilon|(\partial_x^6 u_\varepsilon)^2\mathrm{d}x\\
&\quad\le\left|\frac{16\alpha\varepsilon}{7\gamma}\right|\|u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^6 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad\le C(T)\varepsilon\left\|\partial_x^6 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

It follows from (2.28) that

$$
\begin{aligned}
&\frac{\mathrm{d}G(t)}{\mathrm{d}t}+2\varepsilon\left\|\partial_x^7 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\qquad\le C(T)\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+C(T)+C_0\varepsilon\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}
\end{aligned}
\tag{2.30}
$$

$$
\begin{aligned}
&+\varepsilon\left\|\partial_x^3 u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^5 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{\ell_4^2\varepsilon}{4}\left\|\partial_x^5 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&+2\varepsilon\left\|\partial_x^2 u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^5 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+C(T)\varepsilon\left\|\partial_x^6 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
\tag{2.31}
$$

Thanks to the Young inequality,

$$
\begin{aligned}
C(T)\varepsilon\left\|\partial_x^6 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}&=C(T)\varepsilon\int_{\mathbb{R}}\partial_x^6 u_\varepsilon\partial_x^6 u_\varepsilon\mathrm{d}x=-C(T)\varepsilon\int_{\mathbb{R}}\partial_x^5 u_\varepsilon\partial_x^7 u_\varepsilon\mathrm{d}x\\
&\le C(T)\varepsilon\int_{\mathbb{R}}|\partial_x^5 u_\varepsilon||\partial_x^7 u_\varepsilon\mathrm{d}x\\
&\le C(T)\varepsilon\left\|\partial_x^5 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\varepsilon\left\|\partial_x^7 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

Consequently, by (2.30),

$$
\begin{aligned}
&\frac{\mathrm{d}G(t)}{\mathrm{d}t}+\varepsilon\left\|\partial_x^7u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad\le C(T)\left\|\partial_x^4u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+C(T)+C_0\varepsilon\left\|\partial_x^4u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\qquad+\varepsilon\left\|\partial_x^3u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^5u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+C(T)\varepsilon\left\|\partial_x^5u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\qquad+2\varepsilon\left\|\partial_x^2u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^5u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
\tag{2.32}
$$

By (2.29), we have that

$$
\begin{aligned}
C(T)\left\|\partial_x^4u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}&=C(T)G(t)+\frac{8C(T)\alpha}{7\gamma}\int_{\mathbb{R}}u_\varepsilon(\partial_x^3u_\varepsilon)^2\mathrm{d}x\\
&\quad+C(T)\ell_1\int_{\mathbb{R}}(\partial_x^2u_\varepsilon)^3\mathrm{d}x.
\end{aligned}
\tag{2.33}
$$

Moreover, by (2.3), (2.5), (2.15) and the Young inequality,

$$
\begin{aligned}
&\left|\frac{8C(T)\alpha}{7\gamma}\right|\int_{\mathbb{R}}|u_\varepsilon|(\partial_x^3u_\varepsilon)^2\mathrm{d}x\\
&\quad\le\left|\frac{8C(T)\alpha}{7\gamma}\right|\left\|u_\varepsilon\right\|_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^3u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad\le C(T)\left\|\partial_x^2u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}\left\|\partial_x^4u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}\\
&\quad\le C(T)\left\|\partial_x^4u_\varepsilon\right\|_{L^\infty(0,T;L^2(\mathbb{R}))},\\
&\left|C(T)\ell_1\right|\int_{\mathbb{R}}|\partial_x^2u_\varepsilon|^3\mathrm{d}x\\
&\quad\le\left|C(T)\ell_1\right|\left\|\partial_x^2u_\varepsilon(t,\cdot)\right\|_{L^\infty(\mathbb{R})}\left\|\partial_x^2u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad\le C(T)\left\|\partial_x^2u_\varepsilon(t,\cdot)\right\|_{L^\infty(\mathbb{R})}\\
&\quad\le C(T)+C(T)\left\|\partial_x^2u_\varepsilon(t,\cdot)\right\|^2_{L^\infty(\mathbb{R})}\\
&\quad\le C(T)+C(T)\left\|\partial_x^2u_\varepsilon(t,\cdot)\right\|^4_{L^\infty(\mathbb{R})}\\
&\quad\le C(T)+C(T)\left\|\partial_x^2u_\varepsilon(t,\cdot)\right\|^3_{L^2(\mathbb{R})}\left\|\partial_x^4u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}\\
&\quad\le C(T)+C(T)\left\|\partial_x^4u_\varepsilon\right\|_{L^\infty(0,T;L^2(\mathbb{R}))}.
\end{aligned}
\tag{2.34}
$$

Therefore, by (2.32), (2.33) and (2.34),

$$
\begin{aligned}
&\frac{\mathrm{d}G(t)}{\mathrm{d}t}+\varepsilon\left\|\partial_x^7u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad\le C(T)G(t)+C(T)\left\|\partial_x^4u_\varepsilon\right\|_{L^\infty(0,T;L^2(\mathbb{R}))}+C_0\varepsilon\left\|\partial_x^4u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+C(T)\\
&\qquad+C(T)\left(1+\left\|\partial_x^3u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}+\left\|\partial_x^2u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\right)\varepsilon\left\|\partial_x^5u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

The Gronwall lemma, (2.2), (2.3), (2.19) and (2.29) give

$$
\begin{aligned}
&\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - \frac{8\alpha}{7\gamma}\int_{\mathbb{R}} u_\varepsilon(\partial_x^3 u_\varepsilon)^2 \mathrm{d}x + \ell_1 \int_{\mathbb{R}} (\partial_x^2 u_\varepsilon)^3 \mathrm{d}x \\
&\qquad + \varepsilon e^{C(T)t}\int_0^t e^{-C(T)s}\left\|\partial_x^7 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\mathrm{d}s \\
&\quad \le C_0 e^{C(T)t} + C_0 e^{C(T)t}\varepsilon \int_0^t e^{-C_0 s}\left\|\partial_x^4 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\mathrm{d}s \\
&\qquad + C(T)\left\|\partial_x^4 u_\varepsilon\right\|_{L^\infty(0,T;L^2(\mathbb{R}))} e^{C(T)t}\int_0^t e^{-C(T)s} ds + C(T)e^{C(T)t}\int_0^t e^{-C(T)s}\mathrm{d}s \\
&\qquad + C(T)\left(1 + \left\|\partial_x^3 u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\right) e^{C(T)t}\varepsilon\int_0^t e^{-C(T)s}\left\|\partial_x^5 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\mathrm{d}s \\
&\qquad + \left\|\partial_x^2 u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})} e^{C(T)t}\varepsilon\int_0^t e^{-C(T)s}\left\|\partial_x^5 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\mathrm{d}s \\
&\quad \le C(T)\left(1 + \left\|\partial_x^4 u_\varepsilon\right\|_{L^\infty(0,T;L^2(\mathbb{R}))}\right) + C(T)\varepsilon\int_0^t \left\|\partial_x^4 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\mathrm{d}s \\
&\qquad + C(T)\left(1 + \left\|\partial_x^3 u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})} + \left\|\partial_x^2 u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\right)\varepsilon\int_0^t \left\|\partial_x^5 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\mathrm{d}s \\
&\quad \le C(T)\left(1 + \left\|\partial_x^3 u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})} + \left\|\partial_x^2 u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})} + \left\|\partial_x^4 u_\varepsilon\right\|_{L^\infty(0,T;L^2(\mathbb{R}))}\right).
\end{aligned}
$$

Consequently, by (2.34),

$$
\begin{aligned}
&\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \varepsilon e^{C(T)t}\int_0^t e^{-C(T)s}\left\|\partial_x^7 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\mathrm{d}s \\
&\qquad \le C(T)\left(1 + \left\|\partial_x^3 u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})} + \left\|\partial_x^2 u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})} \right.\\
&\qquad\quad \left. + \left\|\partial_x^4 u_\varepsilon\right\|_{L^\infty(0,T;L^2(\mathbb{R}))}\right) \\
&\qquad\quad + \frac{8\alpha}{7\gamma}\int_{\mathbb{R}} u_\varepsilon(\partial_x^3 u_\varepsilon)^2 \mathrm{d}x + \ell_1 \int_{\mathbb{R}} (\partial_x^2 u_\varepsilon)^3 \mathrm{d}x \\
&\qquad \le C(T)\left(1 + \left\|\partial_x^3 u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})} + \left\|\partial_x^2 u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})} \right.\\
&\qquad\quad \left. + \left\|\partial_x^4 u_\varepsilon\right\|_{L^\infty(0,T;L^2(\mathbb{R}))}\right).
\end{aligned}
\tag{2.35}
$$

Observe that, by (2.3), (2.16) and the Young inequality,

$$
\begin{aligned}
\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^\infty(\mathbb{R})} &\leq 2\sqrt{\|\partial_x^2 u_\varepsilon(t,\cdot)\|^3_{L^2(\mathbb{R})}}\sqrt{\|\partial_x^4 u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}}\\
&\leq C_0\sqrt{\|\partial_x^4 u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}}\\
&\leq C_0 + C_0\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}\\
&\leq C_0\left(1+\left\|\partial_x^4 u_\varepsilon\right\|_{L^\infty(0,T;L^2(\mathbb{R}))}\right),\\
\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^\infty(\mathbb{R})} &\leq 2\sqrt{\|\partial_x^2 u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}}\sqrt{\|\partial_x^4 u_\varepsilon(t,\cdot)\|^3_{L^2(\mathbb{R})}}\\
&\leq C(T)\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}\sqrt{\|\partial_x^4 u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}}\\
&\leq \frac{C(T)}{D_6}\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + C(T)D_6\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\leq \frac{C(T)}{D_6}\left\|\partial_x^4 u_\varepsilon\right\|^2_{L^\infty(0,T;L^2(\mathbb{R}))} + C(T)D_6\left\|\partial_x^4 u_\varepsilon\right\|_{L^\infty(0,T;L^2(\mathbb{R}))}.
\end{aligned}
$$

where D_6 is a positive constant, which will be specified later. Therefore,

$$
\begin{aligned}
\left\|\partial_x^2 u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})} &\leq C_0\left(1+\left\|\partial_x^4 u_\varepsilon\right\|_{L^\infty(0,T;L^2(\mathbb{R}))}\right),\\
\left\|\partial_x^3 u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})} &\leq \frac{C(T)}{D_6}\left\|\partial_x^4 u_\varepsilon\right\|^2_{L^\infty(0,T;L^2(\mathbb{R}))} + C(T)D_6\left\|\partial_x^4 u_\varepsilon\right\|_{L^\infty(0,T;L^2(\mathbb{R}))}.
\end{aligned}
$$

Consequently, by (2.35),

$$
\begin{aligned}
&\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \varepsilon e^{C(T)t}\int_0^t e^{-C(T)s}\left\|\partial_x^7 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\\
&\quad\leq C(T)\left(1+\frac{C(T)}{D_6}\left\|\partial_x^4 u_\varepsilon\right\|^2_{L^\infty(0,T;L^2(\mathbb{R}))} + (1+D_6)\left\|\partial_x^4 u_\varepsilon\right\|_{L^\infty(0,T;L^2(\mathbb{R}))}\right).
\end{aligned}
\tag{2.36}
$$

Hence, by (2.36), we have

$$
\left(1-\frac{C(T)}{D_6}\right)\left\|\partial_x^4 u_\varepsilon\right\|^2_{L^\infty(0,T;L^2(\mathbb{R}))} - C(T)(1+D_6)\left\|\partial_x^4 u_\varepsilon\right\|_{L^\infty(0,T;L^2(\mathbb{R}))} - C(T) \leq 0.
$$

Choosing

$$
D_6 = \frac{1}{2C(T)},
\tag{2.37}
$$

we have that

$$
\frac{1}{2}\left\|\partial_x^4 u_\varepsilon\right\|^2_{L^\infty(0,T;L^2(\mathbb{R}))} - C(T)\left\|\partial_x^4 u_\varepsilon\right\|_{L^\infty(0,T;L^2(\mathbb{R}))} - C(T) \leq 0,
$$

which give (2.12).

Finally, (2.13) follows from (2.12), (2.36) and (2.37), while (2.3), (2.13), (2.15), (2.16) and (2.17) give (2.14). □

Now, we prove the following lemma.

Lemma 2.6 *Fix* $T > 0$. *Then*,

$$\{u_\varepsilon\}_{\varepsilon>0} \text{is compact in } L^2_{loc}((0,\infty)\times\mathbb{R}). \tag{2.38}$$

Consequently, there exists a subsequence $\{u_{\varepsilon_k}\}_{k\in\mathbb{N}}$ of $\{u_\varepsilon\}_{\varepsilon>0}$ and $u \in L^2_{loc}((0,\infty)\times\mathbb{R})$ *such that, for each compact subset* K of $(0,\infty)\times\mathbb{R})$,

$$u_{\varepsilon_k} \to u \text{ in } L^2(K) \text{ and a.e.} \tag{2.39}$$

Moreover, u *is a solution of* (1.1) *satisfying* (1.8).

Proof To prove (2.38), we rely on the Aubin–Lions lemma (see [3, 6, 7, 21]). We recall that

$$H^1_{loc}(\mathbb{R}) \hookrightarrow\hookrightarrow L^2_{loc}(\mathbb{R}) \hookrightarrow H^{-1}_{loc}(\mathbb{R}),$$

where the first inclusion is compact and the second one is continuous. Owing to the Aubin–Lions lemma [21], to prove (2.38), it suffices to show that

$$\{u_\varepsilon\}_{\varepsilon>0} \text{ is uniformly bounded } \text{ in } L^2(0,T;H^1_{loc}(\mathbb{R})), \tag{2.40}$$

$$\{\partial_t u_\varepsilon\}_{\varepsilon>0} \text{ is uniformly bounded in } L^2(0,T;H^{-1}_{loc}(\mathbb{R})). \tag{2.41}$$

We prove (2.40). Thanks to Lemmas 2.1, 2.4 and 2.4,

$$\begin{aligned}\|u_\varepsilon(t,\cdot)\|^2_{H^4(\mathbb{R})} = \|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ + \left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \le C(T).\end{aligned}$$

Therefore,

$$\{u_\varepsilon\}_{\varepsilon>0} \text{ is uniformly bounded in } L^\infty(0,T;H^4(\mathbb{R})),$$

which gives (2.40).

We prove (2.41). We begin by observing that

$$\beta u_\varepsilon \partial_x^3 u_\varepsilon = \beta\partial_x(u_\varepsilon\partial_x^2 u_\varepsilon) - \frac{\beta}{2}\partial_x((\partial_x u_\varepsilon)^2).$$

Therefore, by (2.1),

$$\partial_t u_\varepsilon = \partial_x\left(\frac{\beta-\alpha}{2}(\partial_x u_\varepsilon)^2 - \beta u_\varepsilon\partial_x^2 u_\varepsilon - \gamma\partial_x^4 u_\varepsilon + \varepsilon\partial_x^5 u_\varepsilon\right). \tag{2.42}$$

We have that

$$\frac{(\beta-\alpha)^2}{4}\|\partial_x u_\varepsilon\|^4_{L^4((0,T)\times\mathbb{R})} \le C(T). \tag{2.43}$$

Thanks to Lemma 2.5,

$$\frac{(\beta-\alpha)^2}{4}\int_0^T\int_{\mathbb{R}}(\partial_x u_\varepsilon)^4 \mathrm{d}t\mathrm{d}x \le \frac{(\beta-\alpha)^2}{4}\|\partial_x u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\int_0^T\int_{\mathbb{R}}(\partial_x u_\varepsilon)^2\mathrm{d}t\mathrm{d}x$$
$$\le C(T)\int_0^T\int_{\mathbb{R}}(\partial_x u_\varepsilon)^2\mathrm{d}t\mathrm{d}x \le C(T).$$

We claim that

$$\beta^2\left\|u_\varepsilon\partial_x^2 u_\varepsilon\right\|^2_{L^2((0,T)\times\mathbb{R})} \le C(T). \tag{2.44}$$

Thanks to (2.3) and (2.5),

$$\beta^2\int_0^T\int_{\mathbb{R}}u_\varepsilon^2(\partial_x^2 u_\varepsilon)^2\mathrm{d}t\mathrm{d}x \le \beta^2\|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\int_0^T\int_{\mathbb{R}}(\partial_x^2 u_\varepsilon)^2\mathrm{d}t\mathrm{d}x$$
$$\le C(T)\int_0^T\int_{\mathbb{R}}(\partial_x^2 u_\varepsilon)^2\mathrm{d}t\mathrm{d}x \le C(T).$$

Moreover, since $0<\varepsilon<1$, by Lemmas 2.3 and 2.4,

$$\gamma^2\left\|\partial_x^4 u_\varepsilon\right\|^2_{L^2((0,T)\times\mathbb{R})},\ \varepsilon^2\left\|\partial_x^5 u_\varepsilon\right\|^2_{L^2((0,T)\times\mathbb{R})} \le C(T). \tag{2.45}$$

Therefore, by (2.43), (2.44) and (2.45),

$$\left\{\frac{\beta-\alpha}{2}(\partial_x u_\varepsilon)^2 - \beta u_\varepsilon\partial_x^2 u_\varepsilon - \gamma\partial_x^4 u_\varepsilon + \varepsilon\partial_x^5 u_\varepsilon\right\}_{\varepsilon>0}$$
$$\text{is bounded in } L^2((0,T)\times\mathbb{R}).$$

Thanks to the Aubin–Lions lemma, (2.38) and (2.39) hold.

Consequently, u is solution of (1.1) and (1.8) holds. □

Proof of Theorem 1.1 Lemma (2.6) says that there exists a solution u of (1.1) such that (1.8) holds. Since $H^2(\mathbb{R})\subset H^4(\mathbb{R})$, thanks to [19, Theorem 2.2], u is unique. □

3 Proof of Theorem 1.1 under Assumption (1.3).

In section, we prove Theorem 1.1 under Assumption (1.3).

We consider approximation (2.1) of (1.1), where $u_{\varepsilon,0}$ is a C^∞ approximation of u_0 such that (2.2) holds.

Let us prove some a priori estimates on u_ε.

Lemma 3.1 *Assume* (1.3). *For each* $t\ge 0$,

$$\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\varepsilon\int_0^t\left\|\partial_x^3 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\mathrm{d}s \le C_0. \tag{3.1}$$

Proof Multiplying (2.1) by $2u_\varepsilon$, an integration on $\mathbb{R}$ give

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} =& 2\int_{\mathbb{R}} u_\varepsilon \partial_t u_\varepsilon \mathrm{d}x\\
=& -2\alpha\int_{\mathbb{R}} u_\varepsilon\partial_x u_\varepsilon \partial_x^2 u_\varepsilon \mathrm{d}x - 2\beta\int_{\mathbb{R}} u_\varepsilon^2 \partial_x^3 u_\varepsilon \mathrm{d}x\\
&-2\gamma\int_{\mathbb{R}} u_\varepsilon \partial_x^5 u_\varepsilon \mathrm{d}x + 2\varepsilon\int_{\mathbb{R}} u_\varepsilon\partial_x^6 u_\varepsilon \mathrm{d}x\\
=& -2(\alpha-2\beta)\int_{\mathbb{R}} u_\varepsilon\partial_x u_\varepsilon \partial_x^2 u_\varepsilon \mathrm{d}x\\
&+2\gamma\int_{\mathbb{R}} \partial_x u_\varepsilon \partial_x^4 u_\varepsilon \mathrm{d}x - 2\varepsilon\int_{\mathbb{R}} \partial_x u_\varepsilon\partial_x^5 u_\varepsilon \mathrm{d}x\\
=& -2(\alpha-2\beta)\int_{\mathbb{R}} u_\varepsilon\partial_x u_\varepsilon \partial_x^2 u_\varepsilon \mathrm{d}x\\
&-2\gamma\int_{\mathbb{R}} \partial_x^2 u_\varepsilon \partial_x^3 u_\varepsilon \mathrm{d}x + 2\varepsilon\int_{\mathbb{R}} \partial_x^2 u_\varepsilon\partial_x^4 u_\varepsilon \mathrm{d}x\\
=& -2(\alpha-2\beta)\int_{\mathbb{R}} u_\varepsilon\partial_x u_\varepsilon \partial_x^2 u_\varepsilon \mathrm{d}x - 2\varepsilon\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

Hence,

$$
\frac{\mathrm{d}}{\mathrm{d}t}\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\varepsilon\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2(\alpha-2\beta)\int_{\mathbb{R}} u_\varepsilon\partial_x u_\varepsilon\partial_x^2 u_\varepsilon \mathrm{d}x = 0. \tag{3.2}
$$

Integrating on $(0, t)$, by (1.2) and (2.2), we have (2.4). ◻

Lemma 3.2 *Assume* (1.3) *and fix* $T > 0$. *There exists a constant* $C(T) > 0$, *independent on* ε, *such that*

$$
\varepsilon\int_0^t \left\|\partial_x^2 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\mathrm{d}s \le C(T), \tag{3.3}
$$

for every $0 \le t \le T$. *In particular, we have that*

$$
\varepsilon\int_0^t \|\partial_x u_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})}\mathrm{d}s \le C(T), \tag{3.4}
$$

for every $0 \le t \le T$.

Proof Let $0 \le t \le T$. We begin by observing that, thanks to the Hölder inequality,

$$
\begin{aligned}
\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} &= \int_{\mathbb{R}} \partial_x u_\varepsilon \partial_x u_\varepsilon \mathrm{d}x = -\int_{\mathbb{R}} u_\varepsilon \partial_x^2 u_\varepsilon \mathrm{d}x\\
&\le \int_{\mathbb{R}} |u_\varepsilon||\partial_x^2 u_\varepsilon|\mathrm{d}x \le \|u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}.
\end{aligned} \tag{3.5}
$$

Again by the Hölder inequality,

$$\begin{aligned}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} &= \int_{\mathbb{R}} \partial_x^2 u_\varepsilon \partial_x^2 u_\varepsilon \mathrm{d}x = -\int_{\mathbb{R}} \partial_x u_\varepsilon \partial_x^3 u_\varepsilon \mathrm{d}x \\ &\leq \int_{\mathbb{R}} |\partial_x u_\varepsilon||\partial_x^3 u_\varepsilon| \mathrm{d}x \leq \|\partial_x u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})} \left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}.\end{aligned} \tag{3.6}$$

Therefore, by (3.5) and (3.6),

$$\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \leq \sqrt{\|u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}} \sqrt{\|\partial_x^2 u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}} \left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}. \tag{3.7}$$

Due to (2.4) and the Young inequality,

$$\begin{aligned}&\sqrt{\|u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}} \sqrt{\|\partial_x^2 u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}} \left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})} \\ &\leq \frac{1}{2}\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} \left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\ &\leq \frac{1}{8}\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{1}{2}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\ &\leq C_0 + \frac{1}{2}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

Consequently, by (3.7),

$$\frac{1}{2}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \leq C_0 + \left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

Since $0 < \varepsilon < 1$, we have that

$$\frac{\varepsilon}{2}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \leq \varepsilon C_0 + \varepsilon\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \leq C_0 + \varepsilon\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

If follows from an integration on $(0, t)$ and (2.4) that

$$\frac{\varepsilon}{2}\int_0^t \left\|\partial_x^2 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} \mathrm{d}s \leq C_0 t + \varepsilon \int_0^t \left\|\partial_x^3 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} \mathrm{d}s \leq C(T),$$

which gives (3.3).

Finally, we prove (3.4). By (2.4), (3.5) and the Young inequality,

$$\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} \leq \frac{1}{2}\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{1}{2}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \leq C_0 + \frac{1}{2}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

Since $0 < \varepsilon < 1$, we have that

$$\varepsilon \left\| \partial_x u_\varepsilon(t,\cdot) \right\|^2_{L^2(\mathbb{R})} \leq \varepsilon C_0 + \frac{\varepsilon}{2} \left\| \partial_x^2 u_\varepsilon(t,\cdot) \right\|^2_{L^2(\mathbb{R})} \leq C_0 + \frac{\varepsilon}{2} \left\| \partial_x^2 u_\varepsilon(t,\cdot) \right\|^2_{L^2(\mathbb{R})}.$$

Integrating on $(0, t)$, by (3.3), we get

$$\varepsilon \int_0^t \left\| \partial_x u_\varepsilon(t,\cdot) \right\|^2_{L^2(\mathbb{R})} \leq C_0 t + \frac{\varepsilon}{2} \int_0^t \left\| \partial_x^2 u_\varepsilon(s,\cdot) \right\|^2_{L^2(\mathbb{R})} \mathrm{d}s \leq C(T),$$

which gives (3.4). □

Lemma 3.3 *Assume* (1.2) *and fix* $T > 0$. *There exists a constant* $C(T) > 0$, *independent on* ε, *such that*

$$\left\| \partial_x^2 u_\varepsilon \right\|_{L^\infty(0,T;L^2(\mathbb{R}))} \leq C(T). \tag{3.8}$$

In particular, we have

$$\left\| \partial_x^2 u_\varepsilon(t,\cdot) \right\|^2_{\mathbb{R}} + \varepsilon \int_0^t \left\| \partial_x^5 u_\varepsilon(s,\cdot) \right\|^2_{L^2(\mathbb{R})} \mathrm{d}s \leq C(T), \tag{3.9}$$

for every $0 \leq t \leq T$. *Moreover,*

$$\left\| \partial_x u_\varepsilon(t,\cdot) \right\|_{L^2(\mathbb{R})}, \left\| u_\varepsilon \right\|_{L^\infty((0,T)\times\mathbb{R})}, \left\| \partial_x u_\varepsilon \right\|_{L^\infty((0,T)\times\mathbb{R})} \leq C(T), \tag{3.10}$$

for every $0 \leq t \leq T$.

Proof Let $0 \leq t \leq T$. Consider two real constants F, G, which will be specified later. Multiplying (2.1) by

$$2\partial_x^4 u_\varepsilon + F(\partial_x u_\varepsilon)^2 + G u_\varepsilon \partial_x^2 u_\varepsilon,$$

we have that

$$\begin{aligned}
&\left(2\partial_x^4 u_\varepsilon + F(\partial_x u_\varepsilon)^2 + G u_\varepsilon \partial_x^2 u_\varepsilon\right)\partial_t u_\varepsilon \\
&\quad + \alpha\left(2\partial_x^4 u_\varepsilon + F(\partial_x u_\varepsilon)^2 + G u_\varepsilon \partial_x^2 u_\varepsilon\right)\partial_x u_\varepsilon \partial_x^2 u_\varepsilon \\
&\quad + \beta\left(2\partial_x^4 u_\varepsilon + F(\partial_x u_\varepsilon)^2 + G u_\varepsilon \partial_x^2 u_\varepsilon\right) u_\varepsilon \partial_x^3 u_\varepsilon \\
&\quad + \gamma\left(2\partial_x^4 u_\varepsilon + F(\partial_x u_\varepsilon)^2 + G u_\varepsilon \partial_x^2 u_\varepsilon\right)\partial_x^5 u_\varepsilon \\
&= \varepsilon\left(2\partial_x^4 u_\varepsilon + F(\partial_x u_\varepsilon)^2 + G u_\varepsilon \partial_x^2 u_\varepsilon\right)\partial_x^6 u_\varepsilon.
\end{aligned} \tag{3.11}$$

Observe that

$$
\begin{aligned}
&\int_{\mathbb{R}} \left(2\partial_x^4 u_\varepsilon + F(\partial_x u_\varepsilon)^2 + G u_\varepsilon \partial_x^2 u_\varepsilon\right)\partial_t u_\varepsilon \mathrm{d}x\\
&\qquad= \frac{\mathrm{d}}{\mathrm{d}t}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + F\int_{\mathbb{R}}(\partial_x u_\varepsilon)^2\partial_t u_\varepsilon \mathrm{d}x + G\int_{\mathbb{R}} u_\varepsilon \partial_x^2 u_\varepsilon \partial_t u_\varepsilon \mathrm{d}x,\\
&\alpha\int_{\mathbb{R}} \left(2\partial_x^4 u_\varepsilon + F(\partial_x u_\varepsilon)^2 + G u_\varepsilon \partial_x^2 u_\varepsilon\right)\partial_x u_\varepsilon \partial_x^2 u_\varepsilon \mathrm{d}x\\
&\qquad= -2\alpha\int_{\mathbb{R}} \partial_x u_\varepsilon (\partial_x^3 u_\varepsilon)^2 dx + \alpha G\int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon (\partial_x^2 u_\varepsilon)^2 \mathrm{d}x,\\
&\beta\int_{\mathbb{R}} \left(2\partial_x^4 u_\varepsilon + F(\partial_x u_\varepsilon)^2 + G u_\varepsilon \partial_x^2 u_\varepsilon\right) u_\varepsilon \partial_x^3 u_\varepsilon \mathrm{d}x\\
&\qquad= -\beta\int_{\mathbb{R}} \partial_x u_\varepsilon (\partial_x^3 u_\varepsilon)^2 dx - 2\beta F\int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon (\partial_x^2 u_\varepsilon)^2 \mathrm{d}x\\
&\qquad\quad - \beta G\int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon (\partial_x^2 u_\varepsilon)^2 \mathrm{d}x,\\
&\gamma\int_{\mathbb{R}} \left(2\partial_x^4 u_\varepsilon + F(\partial_x u_\varepsilon)^2 + G u_\varepsilon \partial_x^2 u_\varepsilon\right)\partial_x^5 u_\varepsilon \mathrm{d}x\\
&\qquad= -\gamma(2F+G)\int_{\mathbb{R}} \partial_x u_\varepsilon \partial_x^2 u_\varepsilon \partial_x^4 u_\varepsilon \mathrm{d}x - \gamma G\int_{\mathbb{R}} u_\varepsilon \partial_x^3 u_\varepsilon \partial_x^4 u_\varepsilon \mathrm{d}x\\
&\qquad= \gamma\left(F + \frac{3G}{2}\right)\int_{\mathbb{R}} \partial_x u_\varepsilon (\partial_x^3 u_\varepsilon)^2 \mathrm{d}x\\
&\qquad= \gamma\left(F + \frac{3G}{2}\right)\int_{\mathbb{R}} \partial_x u_\varepsilon (\partial_x^3 u_\varepsilon)^2 \mathrm{d}x,\\
&\varepsilon\int_{\mathbb{R}} \left(2\partial_x^4 u_\varepsilon + F(\partial_x u_\varepsilon)^2 + G u_\varepsilon \partial_x^2 u_\varepsilon\right)\partial_x^6 u_\varepsilon \mathrm{d}x\\
&\qquad= -2\varepsilon\left\|\partial_x^5 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - \varepsilon(2F+G)\int_{\mathbb{R}} \partial_x u_\varepsilon \partial_x^2 u_\varepsilon \partial_x^5 u_\varepsilon \mathrm{d}x\\
&\qquad\quad - G\varepsilon\int_{\mathbb{R}} u_\varepsilon \partial_x^3 u_\varepsilon \partial_x^5 u_\varepsilon \mathrm{d}x.
\end{aligned}
\tag{3.12}
$$

It follows from (1.3), (3.12) and an integration on $\mathbb{R}$ of (3.11) that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + F\int_{\mathbb{R}}(\partial_x u_\varepsilon)^2\partial_t u_\varepsilon \mathrm{d}x + G\int_{\mathbb{R}} u_\varepsilon \partial_x^2 u_\varepsilon \partial_t u_\varepsilon \mathrm{d}x\\
&\qquad + 2\varepsilon\left\|\partial_x^5 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad= \left(5\beta - F\gamma - \frac{3\gamma G}{2}\right)\int_{\mathbb{R}} \partial_x u_\varepsilon (\partial_x^3 u_\varepsilon)^2 \mathrm{d}x\\
&\qquad + (2F-G)\beta\int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon (\partial_x^2 u_\varepsilon)^2 \mathrm{d}x\\
&\qquad - \varepsilon(2F+G)\int_{\mathbb{R}} \partial_x u_\varepsilon \partial_x^2 u_\varepsilon \partial_x^5 u_\varepsilon \mathrm{d}x - G\varepsilon\int_{\mathbb{R}} u_\varepsilon \partial_x^3 u_\varepsilon \partial_x^5 u_\varepsilon \mathrm{d}x.
\end{aligned}
\tag{3.13}
$$

Observe that

$$
\begin{aligned}
&F\int_{\mathbb{R}}(\partial_x u_\varepsilon)^2\partial_t u_\varepsilon \mathrm{d}x + G\int_{\mathbb{R}} u_\varepsilon\partial_x^2 u_\varepsilon\partial_t u_\varepsilon \mathrm{d}x\\
&\qquad = (F-G)\int_{\mathbb{R}}(\partial_x u_\varepsilon)^2\partial_t u_\varepsilon \mathrm{d}x - \frac{G}{2}\int_{\mathbb{R}} u_\varepsilon\partial_t((\partial_x u_\varepsilon)^2)\mathrm{d}x.
\end{aligned}
\tag{3.14}
$$

Consequently, by (3.13),

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + (F-G)\int_{\mathbb{R}}(\partial_x u_\varepsilon)^2\partial_t u_\varepsilon \mathrm{d}x - \frac{G}{2}\int_{\mathbb{R}} u_\varepsilon\partial_t((\partial_x u_\varepsilon)^2)\mathrm{d}x\\
&\quad + 2\varepsilon\left\|\partial_x^5 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&= \left(5\beta - F\gamma - \frac{3\gamma G}{2}\right)\int_{\mathbb{R}}\partial_x u_\varepsilon(\partial_x^3 u_\varepsilon)^2\mathrm{d}x\\
&\quad + (2F-G)\beta\int_{\mathbb{R}} u_\varepsilon\partial_x u_\varepsilon(\partial_x^2 u_\varepsilon)^2\mathrm{d}x\\
&\quad - \varepsilon(2F+G)\int_{\mathbb{R}}\partial_x u_\varepsilon\partial_x^2 u_\varepsilon\partial_x^5 u_\varepsilon \mathrm{d}x - G\varepsilon\int_{\mathbb{R}} u_\varepsilon\partial_x^3 u_\varepsilon\partial_x^5 u_\varepsilon \mathrm{d}x.
\end{aligned}
\tag{3.15}
$$

We search F, G such that

$$
F-G = -\frac{G}{2},\quad 5\beta - F\gamma - \frac{3\gamma G}{2} = 0,\quad 2F-G=0,
$$

that is

$$
F = \frac{G}{2},\quad 5\beta - 2\gamma G = 0. \tag{3.16}
$$

Since

$$
(F,G) = \left(\frac{5\beta}{4\gamma}, \frac{5\beta}{2\gamma}\right).
$$

is the unique solution of (3.16), it follows from (3.15) that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - \frac{5\beta}{4\gamma}\int_{\mathbb{R}}(\partial_x u_\varepsilon)^2\partial_t u_\varepsilon \mathrm{d}x\\
&\qquad - \frac{5\beta}{4\gamma}\int_{\mathbb{R}} u_\varepsilon\partial_t((\partial_x u_\varepsilon)^2)\mathrm{d}x + 2\varepsilon\left\|\partial_x^5 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad = -\frac{5\beta\varepsilon}{\gamma}\int_{\mathbb{R}}\partial_x u_\varepsilon\partial_x^2 u_\varepsilon\partial_x^5 u_\varepsilon \mathrm{d}x - \frac{5\beta\varepsilon}{2\gamma}\int_{\mathbb{R}} u_\varepsilon\partial_x^3 u_\varepsilon\partial_x^5 u_\varepsilon \mathrm{d}x,
\end{aligned}
$$

that is

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - \frac{5\beta}{4\gamma}\int_{\mathbb{R}} u_\varepsilon(\partial_x u_\varepsilon)^2\mathrm{d}x\right) + 2\varepsilon\left\|\partial_x^5 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\qquad = -\frac{5\beta\varepsilon}{\gamma}\int_{\mathbb{R}}\partial_x u_\varepsilon\partial_x^2 u_\varepsilon\partial_x^5 u_\varepsilon \mathrm{d}x - \frac{5\beta\varepsilon}{2\gamma}\int_{\mathbb{R}} u_\varepsilon\partial_x^3 u_\varepsilon\partial_x^5 u_\varepsilon \mathrm{d}x.
\end{aligned}
\tag{3.17}
$$

Due to the Young inequality,

$$
\begin{aligned}
&\left|\frac{5\beta\varepsilon}{\gamma}\right|\int_{\mathbb{R}}|\partial_x u_\varepsilon\partial_x^2 u_\varepsilon||\partial_x^5 u_\varepsilon|\mathrm{d}x\\
&\quad\le\frac{25\beta^2\varepsilon}{2\gamma^2}\int_{\mathbb{R}}(\partial_x u_\varepsilon)^2(\partial_x^2 u_\varepsilon)^2 dx+\frac{\varepsilon}{2}\left\|\partial_x^5 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad\le\frac{25\beta^2\varepsilon}{2\gamma^2}\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^\infty(\mathbb{R})}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{\varepsilon}{2}\left\|\partial_x^5 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad\le\frac{25\beta^2\varepsilon}{2\gamma^2}\left\|\partial_x u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{\varepsilon}{2}\left\|\partial_x^5 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
&\left|\frac{5\beta\varepsilon}{2\gamma}\right|\int_{\mathbb{R}}|u_\varepsilon\partial_x^3 u_\varepsilon||\partial_x^5 u_\varepsilon|\mathrm{d}x\\
&\quad\le\frac{25\beta^2\varepsilon}{8\gamma^2}\int_{\mathbb{R}}u_\varepsilon^2(\partial_x^3 u_\varepsilon)^2\mathrm{d}x+\frac{\varepsilon}{2}\left\|\partial_x^5 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad\le\frac{25\beta^2\varepsilon}{8\gamma^2}\left\|u_\varepsilon(t,\cdot)\right\|^2_{L^\infty(\mathbb{R})}\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{\varepsilon}{2}\left\|\partial_x^5 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad\le\frac{25\beta^2\varepsilon}{8\gamma^2}\left\|u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\frac{\varepsilon}{2}\left\|\partial_x^5 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

It follows from (3.17) that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}-\frac{5\beta}{4\gamma}\int_{\mathbb{R}}u_\varepsilon(\partial_x u_\varepsilon)^2\mathrm{d}x\right)\\
&\qquad+\varepsilon\left\|\partial_x^5 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad\le\frac{25\beta^2\varepsilon}{2\gamma^2}\left\|\partial_x u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\qquad+\frac{25\beta^2\varepsilon}{8\gamma^2}\left\|u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
\tag{3.18}
$$

Integration on $(0, t)$, by (2.2), (3.1) and (3.3), we have that

$$
\begin{aligned}
&\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}-\frac{5\beta}{4\gamma}\int_{\mathbb{R}}u_\varepsilon(\partial_x u_\varepsilon)^2\mathrm{d}x+\varepsilon\int_0^t\left\|\partial_x^5 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\mathrm{d}s\\
&\quad\le C_0+\frac{25\beta^2\varepsilon}{2\gamma^2}\left\|\partial_x u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\int_0^t\left\|\partial_x^2 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\mathrm{d}s\\
&\qquad+\frac{25\beta^2\varepsilon}{8\gamma^2}\left\|u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\int_0^t\left\|\partial_x^3 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\mathrm{d}s\\
&\quad\le C_0\left(1+\left\|\partial_x u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}+\left\|u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\right).
\end{aligned}
\tag{3.19}
$$

Thanks to (3.1), [2, Lemma 2.5] and the Young inequality,

$$\begin{aligned}
\|u_\varepsilon(t,\cdot)\|^2_{L^\infty(\mathbb{R})} &\le \sqrt{\|u_\varepsilon(t,\cdot)\|^3_{L^2(\mathbb{R})}}\sqrt{\|\partial_x^2 u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}}\\
&\le C_0\sqrt{\|\partial_x^2 u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}}\\
&\le C_0 + C_0\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}\\
&\le C_0 + C_0\left\|\partial_x^2 u_\varepsilon\right\|_{L^\infty(0,T;L^2(\mathbb{R}))},\\
\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^\infty(\mathbb{R})} &\le \sqrt{\|u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}}\sqrt{\|\partial_x^2 u_\varepsilon(t,\cdot)\|^3_{L^2(\mathbb{R})}},\\
&\le C_0\sqrt{\|\partial_x^2 u_\varepsilon(t,\cdot)\|^3_{L^2(\mathbb{R})}}\\
&= C_0\frac{\|\partial_x^2 u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}}{\sqrt{D_1}}\sqrt{D_1\|\partial_x^2 u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}}\\
&\le \frac{C_0}{D_1}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + D_1C_0\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\le \frac{C_0}{D_1}\left\|\partial_x^2 u_\varepsilon\right\|^2_{L^\infty(0,T;L^2(\mathbb{R}))} + D_1C_0\left\|\partial_x^2 u_\varepsilon\right\|_{L^\infty(0,T;L^2(\mathbb{R}))},
\end{aligned}$$

where D_1 is a positive constant, which will be specified later. Consequently,

$$\begin{aligned}
\|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})} &\le C_0\left(1 + \left\|\partial_x^2 u_\varepsilon\right\|_{L^\infty(0,T;L^2(\mathbb{R}))}\right),\\
\|\partial_x u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})} &\le \frac{C_0}{D_1}\left\|\partial_x^2 u_\varepsilon\right\|^2_{L^\infty(0,T;L^2(\mathbb{R}))} + D_1C_0\left\|\partial_x^2 u_\varepsilon\right\|_{L^\infty(0,T;L^2(\mathbb{R}))}.
\end{aligned}$$

Thus, by (3.19),

$$\begin{aligned}
&\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - \frac{5\beta}{4\gamma}\int_{\mathbb{R}} u_\varepsilon(\partial_x u_\varepsilon)^2 dx + \varepsilon\int_0^t \left\|\partial_x^5 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds\\
&\quad\le C_0\left(1 + \frac{1}{D_1}\left\|\partial_x^2 u_\varepsilon\right\|^2_{L^\infty(0,T;L^2(\mathbb{R}))} + \left(1 + D_1\right)\left\|\partial_x^2 u_\varepsilon\right\|_{L^\infty(0,T;L^2(\mathbb{R}))}\right).
\end{aligned} \tag{3.20}$$

Thanks to (3.1), (3.5), [4, Lemma 2.6] and the Young inequality,

$$
\begin{aligned}
&\left|\frac{5\beta}{4\gamma}\right|\int_{\mathbb{R}}|u_\varepsilon|(\partial_x u_\varepsilon)^2 dx\\
&\quad\le \frac{25\beta^2}{32\gamma^2}\int_{\mathbb{R}}u_\varepsilon^2(\partial_x u_\varepsilon)^2 dx+\frac{25\beta^2}{32\gamma^2}\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\quad\le \frac{25\beta^2}{16\gamma^2}\sqrt{\|u_\varepsilon(t,\cdot)\|^5_{L^2(\mathbb{R})}}\sqrt{\|\partial_x^2 u_\varepsilon(t,\cdot)\|^3_{L^2(\mathbb{R})}}\\
&\qquad+\frac{25\beta^2}{32\gamma^2}\|u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}\\
&\quad\le C_0\sqrt{\|\partial_x^2 u_\varepsilon(t,\cdot)\|^3_{L^2(\mathbb{R})}}+C_0\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}\\
&\quad\le C_0\frac{\|\partial_x^2 u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}}{\sqrt{D_1}}\sqrt{D_1\|\partial_x^2 u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}}+C_0\left\|\partial_x^2 u_\varepsilon\right\|_{L^\infty(0,T;L^2(\mathbb{R}))}\\
&\quad\le \frac{C_0}{D_1}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+D_1C_0\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|_{L^2(\mathbb{R})}+C_0\left\|\partial_x^2 u_\varepsilon\right\|_{L^\infty(0,T;L^2(\mathbb{R}))}\\
&\quad\le \frac{C_0}{D_1}\left\|\partial_x^2 u_\varepsilon\right\|^2_{L^\infty(0,T;L^2(\mathbb{R}))}+C_0\big(1+D_1\big)\left\|\partial_x^2 u_\varepsilon\right\|_{L^\infty(0,T;L^2(\mathbb{R}))}.
\end{aligned}
\tag{3.21}
$$

Consequently, by (3.20) and (3.21),

$$
\begin{aligned}
&\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+\varepsilon\int_0^t\left\|\partial_x^5 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\\
&\quad\le C_0\left(1+\frac{1}{D_1}\left\|\partial_x^2 u_\varepsilon\right\|^2_{L^\infty(0,T;L^2(\mathbb{R}))}+\big(1+D_1\big)\left\|\partial_x^2 u_\varepsilon\right\|_{L^\infty(0,T;L^2(\mathbb{R}))}\right)\\
&\qquad+\frac{5\beta}{4\gamma}\int_{\mathbb{R}}u_\varepsilon(\partial_x u_\varepsilon)^2 dx\\
&\quad\le C_0\left(1+\frac{1}{D_1}\left\|\partial_x^2 u_\varepsilon\right\|^2_{L^\infty(0,T;L^2(\mathbb{R}))}+\big(1+D_1\big)\left\|\partial_x^2 u_\varepsilon\right\|_{L^\infty(0,T;L^2(\mathbb{R}))}\right)\\
&\qquad+\left|\frac{5\beta}{4\gamma}\right|\int_{\mathbb{R}}|u_\varepsilon|(\partial_x u_\varepsilon)^2 dx\\
&\quad\le C_0\left(1+\frac{1}{D_1}\left\|\partial_x^2 u_\varepsilon\right\|^2_{L^\infty(0,T;L^2(\mathbb{R}))}+\big(1+D_1\big)\left\|\partial_x^2 u_\varepsilon\right\|_{L^\infty(0,T;L^2(\mathbb{R}))}\right).
\end{aligned}
\tag{3.22}
$$

It follows from (3.22) that

$$
\left(1-\frac{C_0}{D_1}\right)\left\|\partial_x^2 u_\varepsilon\right\|^2_{L^\infty(0,T;L^2(\mathbb{R}))}-C_0\big(1+D_1\big)\left\|\partial_x^2 u_\varepsilon\right\|_{L^\infty(0,T;L^2(\mathbb{R}))}-C_0\le 0.
$$

Choosing

$$
D_1=2C_0, \tag{3.23}
$$

we obtain that

$$
\frac{1}{2}\left\|\partial_x^2 u_\varepsilon\right\|^2_{L^\infty(0,T;L^2(\mathbb{R}))}-C_0\big(1+D_1\big)\left\|\partial_x^2 u_\varepsilon\right\|_{L^\infty(0,T;L^2(\mathbb{R}))}-C_0\le 0,
$$

which gives (3.8).

Finally, thanks to (3.8), (3.22) and (3.23), we have (3.9), while (3.1), (3.9) and [2, Lemma 2.3] give (3.10). □

Observe that, arguing as in Lemma 2.4, we have (2.13) and (2.14). Therefore, arguing as in Sect. 2, we have Theorem 1.1.

4 Proof of Theorem 1.1 under Assumption (1.4).

In section, we prove Theorem 1.1 under Assumption (1.4).

We consider approximation (2.1) of (1.1), where $u_{\varepsilon,0}$ is a C^∞ approximation of u_0 such that (2.2) holds.

Let us prove some a priori estimates on u_ε.

Lemma 4.1 *Assume* (1.4). *For each* $t \geq 0$,

$$\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\varepsilon \int_0^t \left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \leq C_0. \tag{4.1}$$

Proof Multiplying (2.1) by $-2\partial_x^2 u_\varepsilon$, an integration on $\mathbb{R}$ gives

$$\begin{aligned}
\frac{d}{dt}\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} =& -2\int_{\mathbb{R}} \partial_t u_\varepsilon \partial_x^2 u_\varepsilon dx\\
=& \, 2\alpha \int_{\mathbb{R}} \partial_x u_\varepsilon (\partial_x^2 u_\varepsilon)^2 dx + 2\beta \int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon \partial_x^2 u_\varepsilon dx\\
&+ 2\gamma \int_{\mathbb{R}} \partial_x^2 u_\varepsilon \partial_x^5 u_\varepsilon dx - 2\varepsilon \int_{\mathbb{R}} \partial_x^2 u_\varepsilon \partial_x^6 u_\varepsilon dx\\
=& \, (2\alpha - \beta) \int_{\mathbb{R}} \partial_x u_\varepsilon (\partial_x^2 u_\varepsilon)^2 dx - 2\gamma \int_{\mathbb{R}} \partial_x^3 u_\varepsilon \partial_x^4 u_\varepsilon dx\\
&+ 2\varepsilon \int_{\mathbb{R}} \partial_x^3 u_\varepsilon \partial_x^5 u_\varepsilon dx\\
=& \, (2\alpha - \beta) \int_{\mathbb{R}} \partial_x u_\varepsilon (\partial_x^2 u_\varepsilon)^2 dx - 2\varepsilon \left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}$$

Therefore, we have that

$$\frac{d}{dt}\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\varepsilon \left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} = (2\alpha - \beta) \int_{\mathbb{R}} \partial_x u_\varepsilon (\partial_x^2 u_\varepsilon)^2 dx.$$

Integrating on $(0, t)$, thanks to (1.4) and (2.2), we get (4.1). □

Lemma 4.2 *Assume* (1.4) *and fix* $T > 0$. *There exists a constant* $C(T) > 0$, *independent on* ε, *such that*

$$\varepsilon \int_0^t \left\|\partial_x^2 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \leq C(T), \tag{4.2}$$

for every $0 \leq t \leq T$. *In particular, we have that*

$$\varepsilon \int_0^t \left\| \partial_x^3 u_\varepsilon(s,\cdot) \right\|^2_{L^2(\mathbb{R})} \mathrm{d}s \le C(T), \tag{4.3}$$

for every $0 \le t \le T$.

Proof Let $0 \le t \le T$. We begin by observing that, thanks to the Hölder inequality,

$$\begin{aligned} \left\| \partial_x^2 u_\varepsilon(t,\cdot) \right\|^2_{L^2(\mathbb{R})} &= \int_{\mathbb{R}} \partial_x^2 u_\varepsilon \partial_x^2 u_\varepsilon \mathrm{d}x = - \int_{\mathbb{R}} \partial_x u_\varepsilon \partial_x^3 u_\varepsilon \mathrm{d}x \\ &\le \int_{\mathbb{R}} |\partial_x u_\varepsilon||\partial_x^3 u_\varepsilon| \mathrm{d}x \le \left\| \partial_x u_\varepsilon(t,\cdot) \right\|_{L^2(\mathbb{R})} \left\| \partial_x^3 u_\varepsilon(t,\cdot) \right\|^2_{L^2(\mathbb{R})}. \end{aligned}$$

Hence, by (4.1),

$$\varepsilon \left\| \partial_x^2 u_\varepsilon(t,\cdot) \right\|^2_{L^2(\mathbb{R})} \le \varepsilon \left\| \partial_x u_\varepsilon(t,\cdot) \right\|_{L^2(\mathbb{R})} \left\| \partial_x^3 u_\varepsilon(t,\cdot) \right\|_{L^2(\mathbb{R})} \le C_0 \varepsilon \left\| \partial_x^3 u_\varepsilon(t,\cdot) \right\|_{L^2(\mathbb{R})}. \tag{4.4}$$

Moreover, by the Hölder inequality and the Young one,

$$\begin{aligned} \left\| \partial_x^3 u_\varepsilon(t,\cdot) \right\|^2_{L^2(\mathbb{R})} &= \int_{\mathbb{R}} \partial_x^3 u_\varepsilon \partial_x^3 u_\varepsilon \mathrm{d}x = - \int_{\mathbb{R}} \partial_x^2 u_\varepsilon \partial_x^4 u_\varepsilon \mathrm{d}x \\ &\le \int_{\mathbb{R}} |\partial_x^2 u_\varepsilon||\partial_x^4 u_\varepsilon| \mathrm{d}x \le \left\| \partial_x^2 u_\varepsilon(t,\cdot) \right\|_{L^2(\mathbb{R})} \left\| \partial_x^4 u_\varepsilon(t,\cdot) \right\|^2_{L^2(\mathbb{R})} \\ &= \frac{\left\| \partial_x^2 u_\varepsilon(t,\cdot) \right\|_{L^2(\mathbb{R})}}{\sqrt{D_2}} \sqrt{D_2} \left\| \partial_x^4 u_\varepsilon(t,\cdot) \right\|^2_{L^2(\mathbb{R})} \\ &\le \frac{\left\| \partial_x^2 u_\varepsilon(t,\cdot) \right\|^2_{L^2(\mathbb{R})}}{2D_2} + \frac{D_2}{2} \left\| \partial_x^4 u_\varepsilon(t,\cdot) \right\|^2_{L^2(\mathbb{R})}, \end{aligned} \tag{4.5}$$

where D_2 is positive constant, which will be specified later. Since $0 < \varepsilon < 1$, it follows from (4.4), (4.5) and the Young inequality,

$$\begin{aligned} \varepsilon \left\| \partial_x^2 u_\varepsilon(t,\cdot) \right\|^2_{L^2(\mathbb{R})} &\le C_0 \varepsilon + C_0 \varepsilon \left\| \partial_x^3 u_\varepsilon(t,\cdot) \right\|^2_{L^2(\mathbb{R})} \\ &\le C_0 + \frac{C_0 \varepsilon \left\| \partial_x^2 u_\varepsilon(t,\cdot) \right\|^2_{L^2(\mathbb{R})}}{2D_2} + \frac{C_0 D_2 \varepsilon}{2} \left\| \partial_x^4 u_\varepsilon(t,\cdot) \right\|^2_{L^2(\mathbb{R})}. \end{aligned}$$

Therefore,

$$\left(1 - \frac{C_0}{2D_2}\right) \varepsilon \left\| \partial_x^2 u_\varepsilon(t,\cdot) \right\|^2_{L^2(\mathbb{R})} \le C_0 + \frac{C_0 D_2 \varepsilon}{2} \left\| \partial_x^4 u_\varepsilon(t,\cdot) \right\|^2_{L^2(\mathbb{R})}.$$

Choosing

$$D_2 = C_0, \tag{4.6}$$

we have that

$$\frac{\varepsilon}{2} \left\| \partial_x^2 u_\varepsilon(t,\cdot) \right\|^2_{L^2(\mathbb{R})} \le C_0 + C_0 \varepsilon \left\| \partial_x^4 u_\varepsilon(t,\cdot) \right\|^2_{L^2(\mathbb{R})}.$$

It follows from an integration on $(0, t)$ and (4.1) that

$$\frac{\varepsilon}{2}\int_0^t \left\|\partial_x^2 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} \mathrm{d}s \leq C_0 t + C_0\varepsilon \int_0^t \left\|\partial_x^4 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} \mathrm{d}s$$
$$\leq C(T) + C_0 \leq C(T),$$

which gives (4.2).

Finally, we prove (4.3). Thanks to (4.5) and (4.6),

$$\varepsilon\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \leq C_0\varepsilon\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + C_0\varepsilon\left\|\partial_x^4 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

Therefore, (4.3) follows from (4.1), (4.2) and an integration on $(0, t)$. □

Lemma 4.3 *Assume* (1.4) *and fix* $T > 0$. *There exists a constant* $C(T) > 0$, *independent on* ε, *such that*

$$\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\varepsilon\int_0^t \left\|\partial_x^3 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} \mathrm{d}s \leq C(T)\left(1 + \sqrt{\|\partial_x^2 u_\varepsilon\|_{L^\infty(0,T;L^2(\mathbb{R}))}}\right), \quad (4.7)$$

for every $0 \leq t \leq T$. *In particular, we have that*

$$\|u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})} \leq C(T)\sqrt{\left(1 + \sqrt{\|\partial_x^2 u_\varepsilon\|_{L^\infty(0,T;L^2(\mathbb{R}))}}\right)}, \quad (4.8)$$

$$\|u_\varepsilon\|_{L^\infty(0,T;L^4(\mathbb{R}))} \leq C(T)\sqrt[4]{\left(1 + \|\partial_x^2 u_\varepsilon\|_{L^\infty(0,T;L^2(\mathbb{R}))}\right)}. \quad (4.9)$$

Proof Let $0 \leq t \leq T$. We begin by observing that

$$2\int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon \partial_x^2 u_\varepsilon \mathrm{d}x = -\int_{\mathbb{R}} (\partial_x u_\varepsilon)^3 \mathrm{d}x. \quad (4.10)$$

Consequently, by (1.4) and (3.2), we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\varepsilon\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} = -\alpha\int_{\mathbb{R}} (\partial_x u_\varepsilon)^3 \mathrm{d}x. \quad (4.11)$$

Due to (2.8) and (4.1),

$$|\alpha|\int_{\mathbb{R}} |\partial_x u_\varepsilon|^3 \mathrm{d}x \leq |\alpha|\|\partial_x u_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})}\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}$$
$$\leq |\alpha|\sqrt{2}\sqrt{\|\partial_x u_\varepsilon(t,\cdot)\|^3_{L^2(\mathbb{R})}}\sqrt{\|\partial_x^2 u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}}$$
$$\leq C_0\sqrt{\|\partial_x^2 u_\varepsilon\|_{L^\infty(0,T;L^2(\mathbb{R}))}}.$$

Consequently, by (4.11),

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\varepsilon\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \leq C_0\sqrt{\|\partial_x^2 u_\varepsilon\|_{L^\infty(0,T;L^2(\mathbb{R}))}}.$$

It follows from (2.2) and an integration on $\mathbb{R}$ that

$$\begin{aligned}\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} &+ 2\varepsilon\int_0^t \left\|\partial_x^3 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds\\ &\le C_0 + C_0\sqrt{\|\partial_x^2 u_\varepsilon\|_{L^\infty(0,T;L^2(\mathbb{R}))}}\,t\\ &\le C(T)\left(1+\sqrt{\|\partial_x^2 u_\varepsilon\|_{L^\infty(0,T;L^2(\mathbb{R}))}}\right),\end{aligned}$$

which gives (4.7).

We prove (4.8). Thanks to (2.10) and (4.1),

$$\|u_\varepsilon(t,\cdot)\|^2_{L^\infty(\mathbb{R})} \le C(T)\left(1+\sqrt{\|\partial_x^2 u_\varepsilon\|_{L^\infty(0,T;L^2(\mathbb{R}))}}\right),$$

which gives (4.8).

Finally, we prove (4.9). Thanks to (4.7) and (4.8),

$$\begin{aligned}\|u_\varepsilon(t,\cdot)\|^4_{L^4(\mathbb{R})} = \int_{\mathbb{R}} u_\varepsilon^4 dx &\le \|u_\varepsilon(t,\cdot)\|^2_{L^\infty(\mathbb{R})}\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\ &\le C(T)\left(1+\left\|\partial_x^2 u_\varepsilon\right\|_{L^\infty(0,T;L^2(\mathbb{R}))}\right).\end{aligned}$$

Consequently, we have that

$$\|u_\varepsilon\|^4_{L^\infty(0,T;L^4(\mathbb{R}))} \le C(T)\left(1+\left\|\partial_x^2 u_\varepsilon\right\|_{L^\infty(0,T;L^2(\mathbb{R}))}\right),$$

which gives (4.9). □

Lemma 4.4 *Assume* (1.4) *and fix* $T>0$. *There exists a constant* $C(T)>0$, *independent on* ε, *such that* (3.8) *holds. In particular, we have*

$$\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\varepsilon}{2}\int_0^t \left\|\partial_x^5 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds \le C(T), \tag{4.12}$$

for every $0\le t\le T$. *Moreover*,

$$\|u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})},\ \|u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})},\ \|\partial_x u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})} \le C(T). \tag{4.13}$$

Proof Let $0\le t\le T$. Consider three real constants H, I, L, which will be specified later. Multiplying (2.1) by

$$2\partial_x^4 u_\varepsilon + H(\partial_x u_\varepsilon)^2 + I u_\varepsilon\partial_x^2 u_\varepsilon + L u_\varepsilon^3,$$

we have that

$$\begin{aligned}&\left(2\partial_x^4 u_\varepsilon + H(\partial_x u_\varepsilon)^2 + I u_\varepsilon\partial_x^2 u_\varepsilon + L u_\varepsilon^3\right)\partial_t u_\varepsilon\\ &\quad+\alpha\left(2\partial_x^4 u_\varepsilon + H(\partial_x u_\varepsilon)^2 + I u_\varepsilon\partial_x^2 u_\varepsilon + L u_\varepsilon^3\right)\partial_x u_\varepsilon\partial_x^2 u_\varepsilon\\ &\quad+\beta\left(2\partial_x^4 u_\varepsilon + H(\partial_x u_\varepsilon)^2 + I u_\varepsilon\partial_x^2 u_\varepsilon + L u_\varepsilon^3\right)u_\varepsilon\partial_x^3 u_\varepsilon\\ &\quad+\gamma\left(2\partial_x^4 u_\varepsilon + H(\partial_x u_\varepsilon)^2 + I u_\varepsilon\partial_x^2 u_\varepsilon + L u_\varepsilon^3\right)\partial_x^5 u_\varepsilon\\ &=\varepsilon\left(2\partial_x^4 u_\varepsilon + H(\partial_x u_\varepsilon)^2 + I u_\varepsilon\partial_x^2 u_\varepsilon + L u_\varepsilon^3\right)\partial_x^6 u_\varepsilon.\end{aligned} \tag{4.14}$$

Observe that, thanks to (1.4),

$$
\begin{aligned}
&L\int_{\mathbb{R}} u_\varepsilon^3 \partial_t u_\varepsilon dx = \frac{L}{4}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} u_\varepsilon^4 \mathrm{d}x,\\
&\alpha L\int_{\mathbb{R}} u_\varepsilon^3\partial_x u_\varepsilon \partial_x^2 u_\varepsilon \mathrm{d}x + \beta L\int_{\mathbb{R}} u_\varepsilon^4\partial_x^3 u_\varepsilon \mathrm{d}x\\
&\quad = -\frac{3\alpha L}{2}\int_{\mathbb{R}} u_\varepsilon^2(\partial_x u_\varepsilon)^3\mathrm{d}x - 4\beta L\int_{\mathbb{R}} u_\varepsilon^3\partial_x u_\varepsilon\partial_x^2 u_\varepsilon \mathrm{d}x\\
&\quad = -\frac{3\alpha L}{2}\int_{\mathbb{R}} u_\varepsilon^2(\partial_x u_\varepsilon)^3\mathrm{d}x + 6\beta L\int_{\mathbb{R}} u_\varepsilon^2(\partial_x u_\varepsilon)^3\mathrm{d}x\\
&\quad = \frac{21\alpha L}{2}\int_{\mathbb{R}} u_\varepsilon^2(\partial_x u_\varepsilon)^3\mathrm{d}x,\\
&L\gamma\int_{\mathbb{R}} u_\varepsilon^3\partial_x^5 u_\varepsilon \mathrm{d}x = -3L\gamma\int_{\mathbb{R}} u_\varepsilon^2\partial_x u_\varepsilon\partial_x^4 u_\varepsilon \mathrm{d}x\\
&\quad = 6L\gamma\int_{\mathbb{R}} u_\varepsilon(\partial_x u_\varepsilon)^2\partial_x^3 u_\varepsilon \mathrm{d}x + 3L\gamma\int_{\mathbb{R}} u_\varepsilon^2\partial_x^2 u_\varepsilon\partial_x^3 u_\varepsilon \mathrm{d}x\\
&\quad = -15L\gamma\int_{\mathbb{R}} u_\varepsilon\partial_x u_\varepsilon(\partial_x^2 u_\varepsilon)^2\mathrm{d}x,\\
&L\varepsilon\int_{\mathbb{R}} u_\varepsilon^3\partial_x^6 u_\varepsilon \mathrm{d}x = -3L\varepsilon\int_{\mathbb{R}} u_\varepsilon^2\partial_x u_\varepsilon\partial_x^5 u_\varepsilon \mathrm{d}x.
\end{aligned}
\tag{4.15}
$$

Consequently, by (1.4), (3.12) with H, I instead of F, G, (3.14) with H, I instead of F, G, (4.15) and an integration on $\mathbb{R}$ of (4.14), we have that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{L}{4}\int_{\mathbb{R}} u_\varepsilon^4\mathrm{d}x\right) + (H-I)\int_{\mathbb{R}}(\partial_x u_\varepsilon)^2\partial_t u_\varepsilon \mathrm{d}x\\
&\quad -\frac{I}{2}\int_{\mathbb{R}} u_\varepsilon\partial_t((\partial_x u_\varepsilon)^2)\mathrm{d}x + 2\varepsilon\left\|\partial_x^5 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&= \left(4\alpha - \gamma H - \frac{3I\gamma}{2}\right)\int_{\mathbb{R}}\partial_x u_\varepsilon(\partial_x^3 u_\varepsilon)^2\mathrm{d}x\\
&\quad + (\alpha I + 4\alpha H + 15\gamma L)\int_{\mathbb{R}} u_\varepsilon\partial_x u_\varepsilon(\partial_x^2 u_\varepsilon)^2\mathrm{d}x\\
&\quad - \frac{21\alpha L}{2}\int_{\mathbb{R}} u_\varepsilon^2(\partial_x u_\varepsilon)^3\mathrm{d}x - \varepsilon(2H+I)\int_{\mathbb{R}}\partial_x u_\varepsilon\partial_x^2 u_\varepsilon\partial_x^5 u_\varepsilon \mathrm{d}x\\
&\quad - I\varepsilon\int_{\mathbb{R}} u_\varepsilon\partial_x^3 u_\varepsilon\partial_x^5 u_\varepsilon \mathrm{d}x - 3L\varepsilon\int_{\mathbb{R}} u_\varepsilon^2\partial_x u_\varepsilon\partial_x^5 u_\varepsilon \mathrm{d}x.
\end{aligned}
\tag{4.16}
$$

We search H, I, L such that

$$
H - I = -\frac{I}{2}, \quad 4\alpha - \gamma H - \frac{3I\gamma}{2} = 0, \quad \alpha I + 4\alpha H + 15\gamma L = 0,
$$

that is

$$
2H - 2I = -I, \quad 8\alpha - 2\gamma H - 3I\gamma = 0, \quad \alpha I + 4\alpha H + 15\gamma L = 0. \tag{4.17}
$$

Since

$$
(H, I, L) = \left(\frac{\alpha}{\gamma}, \frac{2\alpha}{\gamma}, -\frac{2\alpha^2}{5\gamma^2}\right)
$$

is the unique solution of (4.17), it follows from (4.16) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - \frac{\alpha^2}{10\gamma^2}\int_{\mathbb{R}} u_\varepsilon^4 \mathrm{d}x - \frac{\alpha}{\gamma}\int_{\mathbb{R}} u_\varepsilon(\partial_x u_\varepsilon)^2 \mathrm{d}x\right) + \frac{\varepsilon}{2}\left\|\partial_x^5 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}$$
$$= \frac{21\alpha^3}{5\gamma^2}\int_{\mathbb{R}} u_\varepsilon^2(\partial_x u_\varepsilon)^3 \mathrm{d}x - \frac{4\alpha\varepsilon}{\gamma}\int_{\mathbb{R}} \partial_x u_\varepsilon \partial_x^2 u_\varepsilon \partial_x^5 u_\varepsilon \mathrm{d}x$$
$$- \frac{2\alpha\varepsilon}{\gamma}\int_{\mathbb{R}} u_\varepsilon \partial_x^3 u_\varepsilon \partial_x^5 u_\varepsilon \mathrm{d}x + \frac{6\alpha^2\varepsilon}{5\gamma^2}\int_{\mathbb{R}} u_\varepsilon^2 \partial_x u_\varepsilon \partial_x^5 u_\varepsilon \mathrm{d}x. \tag{4.18}$$

Since $0 < \varepsilon < 1$, due to (2.8), (4.1), (4.9) and the Young inequality,

$$\begin{aligned}
&\left|\frac{21\alpha^3}{5\gamma^2}\right|\int_{\mathbb{R}} u_\varepsilon^2|\partial_x u_\varepsilon|^3 \mathrm{d}x\\
&\quad= \left|\frac{21\alpha^3}{5\gamma^2}\right|\int_{\mathbb{R}} |\sqrt{D_3}u_\varepsilon^2|\left|\frac{(\partial_x u_\varepsilon)^3}{\sqrt{D_3}}\right|\mathrm{d}x\\
&\quad\le \left|\frac{21\alpha^3 D_3}{10\gamma^2}\right|\|u_\varepsilon(t,\cdot)\|^4_{L^4(\mathbb{R})} + \left|\frac{21\alpha^3}{10\gamma^2 D_3}\right|\int_{\mathbb{R}}(\partial_x u_\varepsilon)^6 \mathrm{d}x\\
&\quad\le \left|\frac{21\alpha^3 D_3}{10\gamma^2}\right|\|u_\varepsilon\|^4_{L^\infty(0,T;L^4(\mathbb{R}))} + \left|\frac{21\alpha^3}{10\gamma^2 D_3}\right|\|\partial_x u_\varepsilon(t,\cdot)\|^4_{L^\infty((0,T)\times\mathbb{R})}\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\quad\le C(T)D_3\left(1+\left\|\partial_x^2 u_\varepsilon\right\|_{L^\infty(0,T;L^2(\mathbb{R}))}\right) + \frac{C_0}{D_3}\|\partial_x u_\varepsilon\|^4_{L^\infty((0,T)\times\mathbb{R})}\\
&\quad\le C(T)D_3\left(1+\left\|\partial_x^2 u_\varepsilon\right\|_{L^\infty(0,T;L^2(\mathbb{R}))}\right) + \frac{C_0}{D_3}\left\|\partial_x^2 u_\varepsilon\right\|^2_{L^\infty(0,T;L^2(\mathbb{R}))},\\
&\left|\frac{4\alpha\varepsilon}{\gamma}\right|\int_{\mathbb{R}} |\partial_x u_\varepsilon \partial_x^2 u_\varepsilon||\partial_x^5 u_\varepsilon| \mathrm{d}x\\
&\quad\le \frac{8\alpha^2\varepsilon}{\gamma^2}\int_{\mathbb{R}}(\partial_x u_\varepsilon)^2(\partial_x^2 u_\varepsilon)^2 \mathrm{d}x + \frac{\varepsilon}{2}\left\|\partial_x^5 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad\le \frac{8\alpha^2\varepsilon}{\gamma^2}\|\partial_x u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\varepsilon}{2}\left\|\partial_x^5 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
&\left|\frac{2\alpha\varepsilon}{\gamma}\right|\int_{\mathbb{R}} |u_\varepsilon \partial_x^3 u_\varepsilon||\partial_x^5 u_\varepsilon| \mathrm{d}x\\
&\quad\le \frac{4\alpha^2\varepsilon}{\gamma^2}\int_{\mathbb{R}} u_\varepsilon^2(\partial_x^3 u_\varepsilon)^2 \mathrm{d}x + \frac{\varepsilon}{2}\left\|\partial_x^5 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad\le \frac{4\alpha^2\varepsilon}{\gamma^2}\|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\varepsilon}{2}\left\|\partial_x^5 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
&\left|\frac{6\alpha^2\varepsilon}{5\gamma^2}\right|\int_{\mathbb{R}} u_\varepsilon^2 \partial_x u_\varepsilon \partial_x^5 u_\varepsilon \mathrm{d}x\\
&\quad\le \frac{18\alpha^4\varepsilon}{25\gamma^4}\int_{\mathbb{R}} u_\varepsilon^4(\partial_x u_\varepsilon)^2 \mathrm{d}x + \frac{\varepsilon}{2}\left\|\partial_x^5 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad\le \frac{18\alpha^4\varepsilon}{25\gamma^4}\|u_\varepsilon\|^4_{L^\infty((0,T)\times\mathbb{R})}\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{\varepsilon}{2}\left\|\partial_x^5 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad\le C_0\|u_\varepsilon\|^4_{L^\infty((0,T)\times\mathbb{R})} + \frac{\varepsilon}{2}\left\|\partial_x^5 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad\le C(T)\left(1+\left\|\partial_x^2 u_\varepsilon\right\|_{L^\infty(0,T;L^2(\mathbb{R}))}\right) + \frac{\varepsilon}{2}\left\|\partial_x^5 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})},
\end{aligned}$$

where D_3 is a positive constant, which will be specified later. Therefore, by (4.18),

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - \frac{\alpha^2}{10\gamma^2}\int_{\mathbb{R}} u_\varepsilon^4 dx - \frac{\alpha}{\gamma}\int_{\mathbb{R}} u_\varepsilon(\partial_x u_\varepsilon)^2 \mathrm{d}x\right)\\
&\qquad + \frac{\varepsilon}{2}\left\|\partial_x^5 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad \le C(T)\left(1 + D_3 + D_3\left\|\partial_x^2 u_\varepsilon\right\|_{L^\infty(0,T;L^2(\mathbb{R}))} + \frac{1}{D_3}\left\|\partial_x^2 u_\varepsilon\right\|^2_{L^\infty(0,T;L^2(\mathbb{R}))}\right)\\
&\qquad + \frac{8\alpha^2\varepsilon}{\gamma^2}\left\|\partial_x u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\qquad + \frac{4\alpha^2\varepsilon}{\gamma^2}\left\|u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\qquad + C(T)\left(1 + \left\|\partial_x^2 u_\varepsilon\right\|_{L^\infty(0,T;L^2(\mathbb{R}))}\right).
\end{aligned}
$$

It follows from (2.2), (4.2), (4.3) and an integration on $(0,t)$ that

$$
\begin{aligned}
&\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - \frac{\alpha^2}{10\gamma^2}\int_{\mathbb{R}} u_\varepsilon^4 \mathrm{d}x\\
&\qquad - \frac{\alpha}{\gamma}\int_{\mathbb{R}} u_\varepsilon(\partial_x u_\varepsilon)^2 \mathrm{d}x + \frac{\varepsilon}{2}\left\|\partial_x^5 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad \le C_0 + C(T)\left(1 + D_3 + D_3\left\|\partial_x^2 u_\varepsilon\right\|_{L^\infty(0,T;L^2(\mathbb{R}))} + \frac{1}{D_3}\left\|\partial_x^2 u_\varepsilon\right\|^2_{L^\infty(0,T;L^2(\mathbb{R}))}\right)\\
&\qquad + \frac{8\alpha^2\varepsilon}{\gamma^2}\left\|\partial_x u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\int_0^t\left\|\partial_x^2 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\mathrm{d}s\\
&\qquad + \frac{4\alpha^2\varepsilon}{\gamma^2}\left\|u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\int_0^t\left\|\partial_x^3 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}\mathrm{d}s\\
&\qquad + C(T)\left(1 + \left\|\partial_x^2 u_\varepsilon\right\|_{L^\infty(0,T;L^2(\mathbb{R}))}\right)t\\
&\quad \le C(T)\left(1 + D_3 + D_3\left\|\partial_x^2 u_\varepsilon\right\|_{L^\infty(0,T;L^2(\mathbb{R}))} + \frac{1}{D_3}\left\|\partial_x^2 u_\varepsilon\right\|^2_{L^\infty(0,T;L^2(\mathbb{R}))}\right)\\
&\qquad + C(T)\left(1 + \left\|\partial_x u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})} + \left\|u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})} + \left\|\partial_x^2 u_\varepsilon\right\|_{L^\infty(0,T;L^2(\mathbb{R}))}\right).
\end{aligned}
\tag{4.19}
$$

Due to (4.1), (4.8) and the Young inequality,

$$
\begin{aligned}
&\left|\frac{\alpha}{\gamma}\right|\int_{\mathbb{R}}|u_\varepsilon(\partial_x u_\varepsilon)^2|\mathrm{d}x\\
&\qquad \le \frac{\alpha^2}{2\gamma^2}\int_{\mathbb{R}} u_\varepsilon^2(\partial_x u_\varepsilon)^2\mathrm{d}x + \frac{1}{2}\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\qquad \le \frac{\alpha^2}{2\gamma^2}\left\|u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + C_0\\
&\qquad \le C_0\left\|u_\varepsilon\right\|^2_{L^\infty((0,T)\times\mathbb{R})} + C_0\\
&\qquad \le C_0\left\|u_\varepsilon\right\|^4_{L^\infty((0,T)\times\mathbb{R})} + C_0\\
&\qquad \le C(T)\left(1 + \left\|\partial_x^2 u_\varepsilon\right\|_{L^\infty(0,T;L^2(\mathbb{R}))}\right).
\end{aligned}
$$

Moreover, by (2.8) and (4.1),

$$\|\partial_x u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})} \le C_0 \left\|\partial_x^2 u_\varepsilon\right\|_{L^\infty(0,T;L^2(\mathbb{R}))}.$$

Consequently, by (4.9) and (4.19),

$$\begin{aligned}
&\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + \frac{\varepsilon}{2}\left\|\partial_x^5 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad \le C(T)\left(1 + D_3 + D_3\left\|\partial_x^2 u_\varepsilon\right\|_{L^\infty(0,T;L^2(\mathbb{R}))} + \frac{1}{D_3}\left\|\partial_x^2 u_\varepsilon\right\|^2_{L^\infty(0,T;L^2(\mathbb{R}))}\right)\\
&\qquad + C(T)\left(1 + \left\|\partial_x^2 u_\varepsilon\right\|_{L^\infty(0,T;L^2(\mathbb{R}))}\right) + \frac{\alpha^2}{10\gamma^2}\|u_\varepsilon(t,\cdot)\|^4_{L^4(\mathbb{R})}\\
&\qquad + \left|\frac{\alpha}{\gamma}\right|\int_{\mathbb{R}} |u_\varepsilon(\partial_x u_\varepsilon)^2| \mathrm{d}x\\
&\quad \le C(T)\left(1 + D_3 + D_3\left\|\partial_x^2 u_\varepsilon\right\|_{L^\infty(0,T;L^2(\mathbb{R}))} + \frac{1}{D_3}\left\|\partial_x^2 u_\varepsilon\right\|^2_{L^\infty(0,T;L^2(\mathbb{R}))}\right)\\
&\qquad + C(T)\left(1 + \left\|\partial_x^2 u_\varepsilon\right\|_{L^\infty(0,T;L^2(\mathbb{R}))}\right).
\end{aligned} \tag{4.20}$$

Therefore, by (4.20), we obtain that

$$\begin{aligned}
\left(1 - \frac{C(T)}{D_3}\right)\left\|\partial_x^2 u_\varepsilon\right\|^2_{L^\infty(0,T;L^2(\mathbb{R}))} &- C(T)\big(1 + D_3\big)\left\|\partial_x^2 u_\varepsilon\right\|_{L^\infty(0,T;L^2(\mathbb{R}))}\\
&- C(T)\big(1 + D_3\big) \le 0.
\end{aligned}$$

Choosing

$$D_3 = 2C(T), \tag{4.21}$$

we have that

$$\frac{1}{2}\left\|\partial_x^2 u_\varepsilon\right\|^2_{L^\infty(0,T;L^2(\mathbb{R}))} - C(T)\left\|\partial_x^2 u_\varepsilon\right\|_{L^\infty(0,T;L^2(\mathbb{R}))} - C(T) \le 0,$$

which give (3.8).

Finally, (4.12) follows from (3.8), (4.20) and (4.21), while (2.8), (3.8), (4.1), (4.7), (4.8) and (4.12) give (4.13). □

Observe that, arguing as in Lemma 2.4, we have (2.13) and (2.14). Therefore, arguing as in Sect. 2, we have Theorem 1.1.

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