ON THE EQUATION OF DEGREE 6.

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January 2003

Introduction.

If one wants to find the roots of an equation $x^m + a_1 x^{m-1} + \cdots = 0$ of degree m over $\mathbb{C}$ there are two immediate reductions. The first is to change variables and set $a_1 = 0$, the second is to reduce to the case of distinct roots or to a lower degree equation.

This leads to consider the roots as unramified covering of the open set $\mathcal{P}_m$ of $\mathbb{C}^{m-1}$ complement of the discriminant hypersurface.

At the end the problem is, to give a minimum number of monodromous functions of the coefficients (which necessarily are defined only in some open subsets of $\mathcal{P}_m$) with values the roots. It is a simple topological fact which we will recall presently, that one can always exhibit m such functions (covering the entire $\mathcal{P}_m$). It was known that, if $m = p^k$ is a prime power this is always the minimum possible ([Va]).

In general some weak lower bounds are known.

In this paper we will study the first non prime power case $m = 6$ and prove:

Theorem. *For an equation of degree 6 one can express the distinct roots through 5 functions of the coefficients (and 5 is the minimum).*

The methods in principle could be extended but at the moment we could overcome the computational difficulties only with the help of a computer program which seems too complex to perform in the next unknown case $m = 10$.

1 The Schwarz genus.

Let us formalize the discussion, $\mathcal{P}_{n+1}$ be the space of monic polynomials of degree $n+1$ over $\mathbb{C}$ with distinct roots, the complement of the *discriminant hypersurface* and let

$$\Delta := \{(z_1, z_2, \dots, z_{n+1}\} \in \mathbb{C}^{n+1} \mid z_i \neq z_j,\ \forall i \neq j\}, \quad \mathcal{P}_{n+1} = \mathbb{C}^{n+1} - \Delta/S_{n+1}$$

*Partially supported by M.U.R.S.T. 40%

(Δ is the big diagonal).

Problem. Compute the Schwarz genus $g(n+1) := g(\pi_{n+1})$ for the covering

$$\pi_{n+1} : \mathbb{C}^{n+1} - \Delta \to \mathcal{P}_{n+1} = \mathbb{C}^{n+1} - \Delta / S_{n+1}.$$

Definition 1.1 [Sc]. *For a Galois covering $\pi : A \to B$ the genus $g(\pi)$ is the minimum number for which one can cover the base with open sets U_i so that the covering, restricted to U_i is trivial.*

Remark The notion of genus is a generalization, to fibrations, of the Luisternik Schnirlemann category of a space.

This number is interpreted by Smale in the case of π_{n+1}, as a measure of topological complexity for any algorithm that should compute the roots of a polynomial (cf. [Sm]).

As already remarked we can normalize to the case in which the sum of the roots is 0, this is then the simplest example of *reflection arrangement*, and the question may be asked for all such arrangements. In [DS] it has been shown that, for all reflection arrangements except type $\mathcal{A}_n$ the genus of the associated covering is $n+1$ where n is the complex dimension of the arrangement and the cohomological dimension of the corresponding Artin braid group.

For type $\mathcal{A}_n$ the same statement is true when $n+1 = p^k$ is a prime power (cf. [Va]).

In this paper we show that the previous statement is not always true. In fact we will verify that, for the first case $n+1 = 6$ of non prime powers, the estimate given by the cohomological dimension is 6 instead $g(6) = 5$, (Theorem 7.4).

Let us recall briefly the main ideas of Schwarz for the computation of this genus, we will restrict our discussion to finite Galois coverings, although Schwarz treats a much more general case. The discussion that follows is extracted from ([Sc]) and it is included for convenience of the reader.

Let then $\pi : A \to B = A/W$ be a finite covering associated to the free action of a finite group W on a space A. To this covering and any positive integer k one can apply the *join construction*, set:

$$A_\pi^{k*} := A *_B *A_B * \cdots *_B A := \{\sum_{i=1}^{k} t_i a_i \,|\, 0 \le t_i, \sum_i t_i = 1,\ \pi(a_i) = \pi(a_j),\ \forall i, j\}$$

in particular A_π^{k*} is a subset of the $k-$fold join. We have a fibration $\pi_k : A_\pi^{k*} \to B$, with fibers the join F^{k*} of the fibers of π. Then we have (assume B paracompact):

Proposition 1.2. *The genus $g(\pi) \le k$ if and only if the fibration $\pi_k : A_\pi^{k*} \to B$ has a section.*

Proof. Let V_i be the open subset of A_π^{k*} where the i^{th} coordinate $t_i \neq 0$, in V_i we have a copy of A as the points where the i^{th} coordinate $t_i = 1$. By the standard properties of the join the space A is a deformation retract of V_i. If $s : B \to A_\pi^{k*}$ is a section of π_k define

$U_i := s^{-1}(V_i)$, the U_i cover B and by the previous remarks the fibration restricted to U_i has a section, since it is a Galois fibration it is then trivial.

Conversely if π restricted to U_i has a section s_i for some covering $U_1, \dots, U_k$ let f_i be a partition of unity relative to this covering, we have that the map $\sum_{i=1}^{k} f_i(b)s_i(b)$ is a well defined section of A_π^{k*}.

It is useful to think of the join construction as a base change, consider for a given k the join W^{*k} of the group W, $k-$times.

We act on the right with W on W^{*k} by $(\sum_i t_i w_i, w) \to \sum_i t_i w_i w$ and have

$$A_\pi^{k*} = W^{*k} \times_W A, \quad \rho_k : A_\pi^{k*} \to W^{*k}/W.$$

The fibration $W \to W^{*k} \to W^{*k}/W$ gives W^{*k}/W as the k^{th} step of Milnor's construction of a classifying space of W (cf. [Mi]). W^{*k}/W is itself a classifying space for W fibrations over CW-complexes of dimension $< k$ (if one wishes one can take W^{*k}/W as the $k-1$ skeleton of a classifying space).

It may be useful to think of the right action of W on W^{*k} as a left action by $w(\sum_i t_i w_i) := \sum_i t_i w_i w^{-1}$, then $W^{*k} \times_W A$ is the space of W orbits of the diagonal action.

Given a space X over which W acts on the left consider the quotient $(X \times A)/W$. $(X \times A)/W$ fibers over B, by $\pi_X(x, a) = \pi(a)$ and over X/W by $\rho(x, a) = [x]$, where $[x] := \overline{\pi}(x)$ denotes the orbit of x in X/W. We claim that:

Proposition 1.3. *i) There is a 1-1 correspondence between sections of $\pi_X : (X \times A)/W \to B$ and W equivariant maps $\phi : A \to X$.*

ii) If $s : B \to (X \times A)/W$ is a section of π_X and $\phi : A \to X$ the corresponding equivariant map we have a commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{\phi} & X \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\overline{\pi}} \\ B & \xrightarrow{\rho\circ s} & X/W \end{array}$$

and the covering $\pi : A \to B$ is the pullback, under $\rho \circ s$ of the covering $\overline{\pi} : X \to X/W$.

Proof. i) Given a pair $(x, a) \in X \times A$ let us denote by $[x, a]$ its class in the $W-$orbit space. Given an equivariant map $\phi : A \to X$ consider the map $\sigma : A \to X \times A \to (X \times A)/W$ given by $\sigma(a) = [\phi(a), a]$ we have that for $w \in W$, $\sigma(w(a)) = [\phi(wa), wa] = [w\phi(a), wa] = [\phi(a), a] = \sigma(a)$ therefore σ factors through $B = A/W$ to a section of π_X. Conversely given a section s and $a \in A$ we can choose for $s(\pi(a)) \in (X \times A)/W$ as representative in $X \times A$ a unique element $(\phi(a), a)$ and this defines the equivariant map.

ii) By definition $s(\pi(a)) = \sigma(a) = [\phi(a), a]$ which composed with ρ gives $[\phi(a)] = \overline{\pi}(\phi(a))$, or $\rho \circ s \circ \pi = \overline{\pi} \circ \phi$ as desired. □

In the case of $A_\pi^{k*} = (W^{*k} \times A)/W$ the previous Proposition gives

Corollary 1.4. *$\rho_k \circ s_k$ is a classifying map for the covering A.*

The homology $H_{k-1}(W^{*k}, \mathbb{Z})$ can be computed using the chain complex associated to the canonical cell decomposition of W^{*k} (given by the join construction), as the group of $k-1$-cycles. Notice that this chain complex is a free resolution of the trivial $W-$module $\mathbb{Z}$ up to degree $k-1$.

This implies that, given any projective resolution (C_*, ∂) of the trivial $W-$module $\mathbb{Z}$, there are two projective W-modules P_1, P_2, such that:

$$H_{k-1}(W^{*k}, \mathbb{Z}) \oplus P_1 = \partial C_k \oplus P_2.$$

In particular $H^*(W, H_{k-1}(W^{*k})) = H^*(W, \partial C_k)$.

2 The Universal Obstruction.

Proposition 1.2 allows us to use obstruction theory to help to compute $g(\pi)$ when B is a finite CW-complex (cf. [St]). One uses the fact that $\pi_i(W^{*k}) = 0$, $\forall i < k-1$, to deduce that $g(\pi) \leq \dim B + 1$. Let $\dim B = n$. In order to decide whether $g(\pi) \leq n$ one has to compute the following obstruction class.

Consider $H^*(W, H_{n-1}(W^{*k})) = H^*(W, \partial C_n)$. Recall that this can be computed as the cohomology of the cochain complex $\mathrm{HOM}(C_*, \partial C_n)$.

In particular the map $\partial_n : C_n \to \partial C_n$ is clearly a cocycle $[\partial_n] \in H^n(W, H_{n-1}(W^{*k}))$ (it is immediate to remark that this class does not depend on the choice of the resolution (C_*, ∂)). We can then consider the local system $\mathcal{H}_{n-1}(W^{*n})$ on B, and we get the cocycle $c = \rho^*[\partial_n]$, where $\rho : B \to BW$ is a classifying map for our covering and a cohomology class $[c] \in H^n(B, \mathcal{H}_{n-1}(W^{*n}))$. One has (see [St] Theorem 34.2, Lemma 32.7):

Theorem 2.1. *∂_n and c are obstruction cocycles for the respective problems of extending a section to the $n-$dimensional skeleton.*

If B is $n-$dimensional then $g(\pi) \leq n$ if and only if $[c] = 0$.

When $B = B_G$ is the classifying space of a group G (as in our case in which $\mathcal{P}_{n+1}$ is the classifying space of the Braid group B_{n+1}), we have that the Galois covering is given by a homomorphism $\rho_* : G \to W$ (induced by the homotopy exact sequence of the fibration π) and one can think of the cohomology $H^n(B_G, \mathcal{H}_{n-1}(W^{*n})) = H^n(G, H_{n-1}(W^{*n}))$ as the cohomology of the group G with coefficients in the module $H_{n-1}(W^{*n})$ where the action is induced, via the homomorphism, by the W action.

If we now denote by (E_*, d) a projective resolution of the trivial $G-$module $\mathbb{Z}$, then the homomorphism ρ_* induces a map of chain complexes $\rho_* : E_*(n) \to C_*(n)$, which induces the map of complexes $\rho^* : \mathrm{HOM}_W(C_*, H_{n-1}(W^{*n})) \to \mathrm{HOM}_W(E_*, H_{n-1}(W^{*n}))$ and the map $\rho^* : H^*(W, M) \to H^*(G, M)$, in cohomology.

We may summarize the various ingredients introduced:

Theorem 2.2. *Let G and W be two groups, with W finite. Let $A \to B_G$ be a Galois covering with fiber W and denote by $\rho_* : G \to W$ the corresponding homomorphism. Assume* $\dim B_G \leq n$. *The following statements are equivalent:*

1) The genus of the covering $A \to B_G$ is less than or equal to n.

2) The fibration $W^{*k} \times_W A \to B$ has a section.

3) The classifying map of the covering A factors through the $n-1$ dimensional space W^{*n}/W.

4) For every W module M the map in cohomology $\rho^* : H^n(W, M) \to H^n(G, M)$ is 0.

5) The cohomology class of $\rho^*[\partial_n]$ is 0.

Proof. By the Proposition 1.2, 1) and 2) are equivalent. 2) implies 3) by Corollary 1.4.

3) clearly implies 4) and also 4) clearly implies 5). Finally 5) implies 2) by Theorem 2.1. □

3 Braid and symmetric groups

We apply now the previous theory to $G = B_{n+1}$, $W = S_{n+1}$ and to the canonical quotient homomorphism $B_{n+1} \to S_{n+1}$, so that the fibration we get is, up to homotopy, the fibration $\pi_{n+1} : \mathbb{C}^{n+1} - \Delta \to \mathcal{P}_{n+1}$, considered in the introduction.

In this case $\mathcal{P}_{n+1}$ can be replaced by a finite CW-complex of dimension n and we will use the very explicit constructions given in ([Sa], [DS]). To say that $[c] = 0$ where $c = \rho^*[\partial_n]$ means that the cocycle ∂_n restricted to E_n is a coboundary, or in other words that there exists a map $g : E_{n-1} \to \partial C_n$ such that the diagram:

$$\begin{array}{ccc} E_n & \xrightarrow{\partial} & E_{n-1} \\ {\scriptstyle \rho_n}\downarrow & & \downarrow{\scriptstyle g} \\ C_n & \xrightarrow{\partial_n} & \partial C_n \end{array}$$

commutes.

All the maps are linear with respect to the group algebra S of the braid group, so it is convenient to reformulate this statement as follows. Let R be the integral group algebra of the symmetric group and define $D_* := E_* \otimes_S R$, then D_* is a subcomplex of C_* and denoting by i its inclusion the diagram becomes by abuse of notations:

$$\begin{array}{ccc} D_n & \xrightarrow{\partial} & D_{n-1} \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle g} \\ C_n & \xrightarrow{\partial_n} & \partial C_n \end{array}$$

where now all the modules are free R modules and the maps $R-$linear.

Recall from [DS] that $D_n = Re$, $D_{n-1} = \oplus_{k=1}^n Re_k$, $\partial e = \sum_k A_k e_k$ with the A_k's defined as follows. For each $1 \leq k \leq n$ set

$$\mathcal{A}_k := \{(i_1, i_2, \dots, i_{n+1}) \in S_{n+1} |\ i_1 < i_2 \cdots < i_{k-1} < i_k; \quad i_{k+1} < i_{k+2} \cdots < i_n < i_{n+1}\}$$

Then

$$A_k = \sum_{\sigma \in \mathcal{A}_k} \epsilon(\sigma)\sigma,$$

$\epsilon(\sigma)$ is the sign.

Therefore:

Lemma 3.1. *The map g exists if and only if there exist elements $x_k \in \partial C_n$ with $\sum_k A_k x_k = \sum_k A_k e_k$.*

Proof. A map $g : D_{n-1} \to \partial C_n$ is completely determined by the values $x_k = g(e_k)$ and, since D_n is generated by e the equality $g \circ \partial = a \circ i$ is equivalent to $g \circ \partial(e) = \partial_n \circ i(e)$ or $\sum_k A_k x_k = \sum_k A_k e_k$. □

We can further reformulate the last statement as follows.

By definition, given $x_k \in \partial C_n$ there exists $y_k \in C_n$ such that $x_k = \partial y_k$ so that the map g exists if and only if there exist elements $y_k \in C_n$ with $\sum_k A_k \partial y_k = \sum_k A_k e_k = \partial e$, or $\partial(e - \sum_k A_k y_k) = 0$. Again by exactness of C_* this means that there exists an element $b \in C_{n+1}$ with $\partial(b) = e - \sum_k A_k y_k$.

Finally this can be reinterpreted as follows. Let $I = \sum_k A_k R$ be the right ideal in R generated by the elements A_k. On the category of R modules define the functor

$$N \to T(N) := N/\sum_k A_k N = R/I \otimes_R N.$$

Take the complex $T(C_*) = R/I \otimes_R C_*$ whose homology is $H_*(S_{n+1}, R/I)$.

By construction, the element $\overline{e} \in T(D_n) \subset T(C_n)$, class of e, is a cycle and so it gives a homology class $[\overline{e}] \in H_n(T(D_*))$. Denoting by j the inclusion of $T(D_n)$ into $T(C_n)$, take $j[\overline{e}] \in H_n(S_{n+1}, R/I)$. We can thus interpret the previous discussion as:

Theorem 3.2. *$\rho^*[\partial_n]$ is a coboundary if and only if $j[\overline{e}] = 0$.*

Notice now that the exact sequence $0 \to D_* \to C_* \to C_*/D_* \to 0$ of complexes of free R modules induces an exact sequence $0 \to T(D_*) \to T(C_*) \to T(C_*/D_*) \to 0$. From the long exact sequence of homology $H_{n+1}(T(C_*/D_*)) \xrightarrow{\partial} H_n(T(D_*)) \xrightarrow{j} H_n(T(C_*)) \dots$. Therefore $j[\overline{e}] = 0$ if and only if $[\overline{e}] = \partial(u),\ u \in H_{n+1}(T(C_*/D_*))$.

Since $D_{n+1} = 0$ we have that $H_n(T(D_*))$ are the cycles of the map $T(D_n) \xrightarrow{d} T(D_{n-1})$ and most of the computational difficulty lies in computing $H_{n+1}(T(C_*/D_*))$. In the next sections we will start to describe in more detail the module R/I and make a few simplifications which will allow us at the end to reduce the computational complexity, in case $n = 5$ to a manageable size.

4 The module R/I.

In order to continue we must deduce some properties of R/I and to describe algorithmically the complex $T(C_*/D_*)$.

Theorem 4.1. *We have a direct sum decomposition, as S_n modules, $R = \mathbb{Z}[S_{n+1}] = I \oplus \mathbb{Z}[S_n]$.*

Proof. First of all let us make the following remarks. By our description of the complex D_*, we can identify $H_n(D_*) = \{r \in R | rA_k = 0,\ \forall k\}$. Also from its very definition, the complex D_* computes the homology of the total space $A := \mathbb{C}^{n+1} - \Delta$, of our fibration and we get [O-S] that $H_n(D_*)$ is a free $\mathbb{Z}$ module of rank n!.

Let us now use the (integral) trace form $t(a) := \frac{1}{n!}Tr(a_L)$ associated to the regular representation, $a_L : \mathbb{Z}[S_{n+1}] \to \mathbb{Z}[S_{n+1}],\ a_L(u) := au$. This is non degenerate over $\mathbb{Z}$.

Under this form given $r \in R$:

$$r \in H_n(D_*), \iff 0 = t(xrA_k) = t(rA_kx)\ \forall k,$$

thus $H_n(D_*)$ is the orthogonal, under the form $t(ab)$, to the right ideal $I = \sum_{k=1}^{n} A_k\mathbb{Z}[S_{n+1}]$.

Since $\mathrm{rk}H_n(D_*) = n!$, $\mathrm{rk}R/I = n!$. Therefore in order to prove our claim, it suffices to show that $\mathbb{Z}[S_{n+1}] = I + \mathbb{Z}[S_n]$. Given a permutation $\sigma \in S_{n+1}$, consider $k := n+1-\sigma(n+1)$. If $k = 0$, we have nothing to show since $\sigma \in S_n$. Otherwise we proceed by induction on k.

Given a permutation $\tau = (i_1, i_2, \ldots, i_{n+1}) \in \mathcal{A}_k$ we clearly have that $i_{n+1} \geq n + 1 - k$. Furthermore, if $i_{n+1} = n + 1 - k$, we must necessarily have that

$$\tau = (n + 1 - (k - 1), \ldots, n + 1, 1, 2, 3, \ldots, n - k, n + 1 - k).$$

Thus, we can write $\tau = (-1)^{k(n+1-k)}A_k + b$, with b a linear combination of permutations γ for which $n+1-\gamma(n+1) < k$. Our inductive hypothesis then implies that $\tau \in I + \mathbb{Z}[S_n]$.

Take now an arbitrary permutation σ with $\sigma(n + 1) = n + 1 - k$. We can write $\sigma = \tau\gamma,\ \gamma \in S_n$. Since $I + \mathbb{Z}[S_n]$, is clearly stable under right multiplication by elements in S_n, everything follows. □

Remark We have that, R/I as a $\mathbb{Z}[S_n]$-module is free of rank 1.

Thus $H_i(S_n, R/I) = 0,\ \forall i > 0$. Since $n + 1 = [S_{n+1} : S_n]$, multiplication by $n + 1$ on $H_i(S_{n+1}, R/I)$ can be written as the composition of the restriction and corestriction

$$n + 1 : H_i(S_{n+1}, R/I) \xrightarrow{res} H_i(S_n, R/I) \xrightarrow{cores} H_i(S_{n+1}, R/I),$$

we deduce that $(n + 1)H_i(S_{n+1}, R/I) = 0, \forall i > 0$ (cf. [B], Prop. 9.5).

The previous discussion gives an explicit algorithm to perform the projection:

$$\mathbb{Z}[S_{n+1}] = \mathbb{Z}[S_n] \oplus I \xrightarrow{\pi} \mathbb{Z}[S_n]. \tag{Pr}$$

With this explicit map the computation of the complex $R/I \otimes_R C_*$ can be developed.

5 The complex.

In [DS] one describes a free resolution C_* of $\mathbb{Z}$ by R modules. Denote by $\mathcal{N}$ the set of n nodes of A_n which we identify with the set of simple transpositions $(h, h+1)$ and we order linearly. Let $\mathcal{B}_k$ denote the set of *flags* of k subsets of $\mathcal{N}$ with a total of k elements, i.e. sequences $b := \Gamma_1 \supseteq \Gamma_2 \supseteq \cdots \supseteq \Gamma_k \supseteq \mathcal{N}$, such that $\sum |\Gamma_i| = k$.

Then C_k is a direct sum $\oplus_{b\in\mathcal{B}_k}\mathbb{Z}[S_{n+1}]e_b$ and the differential ∂e_b is explicitly described as follows. Let S_{Γ_i} the parabolic subgroup generated by Γ_i and $S_{\Gamma_i}^{\Gamma_i\setminus\{\tau\}}$ the coset representatives of shortest length of $\Gamma_i \setminus \{\tau\}$ in Γ_i. Define a $\mathbb{Z}[S_{n+1}]$-linear differential $\partial_k : C_k \to C_{k-1}$ by

$$\partial_k\ e_b = \sum_{\substack{1\le i\le k\\ |\Gamma_i|>|\Gamma_{i+1}|}} \sum_{\tau\in\Gamma_i} \sum_{\substack{\beta\in S_{\Gamma_i}^{\Gamma_i\setminus\{\tau\}}\\ \beta^{-1}\Gamma_{i+1}\beta\ \subset\ \Gamma_i\setminus\{\tau\}}} (-1)^{\alpha(\Gamma,i,\tau,\beta)}\ \beta e_{b'}$$

where $b' = (\Gamma_1, \ldots, \Gamma_{i-1}, \Gamma_i \setminus \{\tau\}, \beta^{-1}\Gamma_{i+1}\beta, \ldots, \beta^{-1}\Gamma_k\beta)$ and

$$\alpha(\Gamma, i, \tau, \beta) = i\ell(\beta) \ + \ \sum_{j=1}^{i-1} |\Gamma_j| \ + \ \mu(\Gamma_i, \tau) \ + \ \sum_{j=i+1}^{d} \sigma(\beta, \Gamma_j).$$

Here ℓ is the standard length function, $\sigma(\beta, \Gamma_j)$ is the number of inversions in the map $\Gamma_j \to \beta^{-1}\Gamma_j\beta \subset \Gamma_i \setminus \tau$ and $\mu(\Gamma_i, \tau)$ is the number of reflections in Γ_i which are less than or equal to τ in the linear ordering.

Using the decomposition and projection (Pr) we identify $R/I \otimes_R C_k$ with $\oplus_{b\in\mathcal{B}_k}\mathbb{Z}[S_n]e_b$. As for the boundary, $\mathbb{Z}[S_{n+1}] = \mathbb{Z}[S_n] \oplus I \xrightarrow{\pi} \mathbb{Z}[S_n]$ induces termwise a map

$$\oplus_{b\in\mathcal{B}_k}\mathbb{Z}[S_{n+1}]e_b = \oplus_{b\in\mathcal{B}_k}(\mathbb{Z}[S_n] \oplus I)e_b \xrightarrow{\pi} \oplus_{b\in\mathcal{B}_k}\mathbb{Z}[S_n]e_b.$$

So the boundary in $R/I \otimes_R C_*$ is the composition:

$$\text{(Co)} \qquad \Delta_k : \oplus_{b\in\mathcal{B}_k}\mathbb{Z}[S_n]e_b \xrightarrow{\partial} \oplus_{c\in\mathcal{B}_{k-1}}(\mathbb{Z}[S_n] \oplus I)e_c \xrightarrow{\pi} \oplus_{c\in\mathcal{B}_{k-1}}\mathbb{Z}[S_n]e_c.$$

Further simplification comes from the following remark.

Proposition 5.1. *Let $\mathcal{B}_k^h \subset \mathcal{B}_k$ denote the set of flags of subsets of $\{1, \ldots, \hat{h}, \ldots, n\}$ with cardinality k. Then the two subcomplexes $C_*(1)$, $C_*(n)$ of $R/I \otimes_R C_*$ which are generated respectively by*

$$\oplus_{b\in\mathcal{B}_k^1}\ R/I\ e_b, \quad \oplus_{b\in\mathcal{B}_k^n}\ R/I\ e_b$$

are acyclic.

The same statement is true in degree ≥ 2 *for the subcomplex given by their sum.*

Proof. From formulas of [DS] the subcomplex $C_*(n)$ which is generated by

$$\oplus_{b\in\mathcal{B}_k^n}\ \mathbb{Z}[S_n]e_b$$

equals the acyclic complex C_*^{n-1} of S_n. Clearly the theory is symmetric, so the same is true for the other subcomplex.

Now we see that the intersection $C_*(1)\cap C_*(n)$ is a subcomplex of free $\mathbb{Z}[S_n]$-modules with bases corresponding to flags with no nodes $1,\ n$. Again from formulas in [DS] it is clear that this subcomplex splits (in degree $>\ 0$) as a sum of n copies of the acyclic complex C_*^{n-2} of the group S_{n-1}. Therefore by looking at the Mayer-Vietoris sequence one concludes. □

So we reduce to make computations on the quotient

$$\tilde{C}_* :=\ R/I\otimes_R\ C_*/(C_*(1)+C_*(n))$$

which is a complex of free $\mathbb{Z}[S_n]-$modules with bases e_b, where the flag b contains both 1 and n. Let $\tilde{\partial}$ be the boundary induced by $\overline{\partial}$.

Theorem 3.2 translates here to

Theorem 5.2. *Let* $b_0\in\mathcal{B}_n$ *be the flag*

$$b_0 := (\{1,\dots,n\}\supset\emptyset\supset\dots).$$

The class $[\partial_n]$ *is* 0 *iff* e_{b_0} *belongs to the image of* $\tilde{\partial}_{n+1}$.

6 Computations.

Recall from Section 3 that the vanishing of the obstruction class $[\partial_n]$ is equivalent to that of a certain homology class $j(\overline{e})\ \in\ H_n(S_{n+1},R/I)$.

We want to compute $H_i(S_6,R/I)$ and show that $H_5(S_6,R/I)=0$.

As above, identify R/I with $\mathbb{Z}[S_n]$ as $\mathbb{Z}[S_n]$-modules.

The projection $\pi:\ \mathbb{Z}[S_{n+1}]\ \to\ \mathbb{Z}[S_n]$ is given by a matrix

$$P\in\mathcal{M}(n!\times(n+1)!;\ \mathbb{Z})$$

which, can be determined following the algorithm described in the proof of Theorem 4.1.

More precisely, let us give an ordering $\sigma_1,\ \sigma_2,\dots$ to the $n+1!$ permutations so that the last $n!$ of them fix $n+1$. Then each element in $\mathbb{Z}[S_{n+1}]$ correspond to an integral

vector with $n+1!$ entries. Construct the $(n+1)! \times (n \cdot (n+1)!)$ integral matrix A whose columns are the $A_k \sigma_j,\ k = 1, \dots, n,\ j = 1, \dots, (n+1)!$. Decompose A as

$$A \ = \ \begin{bmatrix} B \\ C \end{bmatrix}$$

where B is of order $(n \cdot n!) \times (n \cdot (n+1)!)$ while C is of order $(n!) \times (n \cdot (n+1)!)$. Then Theorem 4.1 is equivalent to

Theorem 6.1. *B is right invertible over $\mathbb{Z}$, i.e. there is an integral matrix H with $BH = I_{n\,n!}$.*

There exists a unique projection $\pi : \ \mathbb{Z}[S_{n+1}] \ \to \mathbb{Z}[S_n]$ given by the matrix

$$P \ = \ [\, -CH \quad I_{n!}\,]$$

where we indicate by $I_{n!}$ the identity matrix of order $n!$, with:

$$[\, -CH \quad I_{n!}\,] \begin{bmatrix} B \\ C \end{bmatrix} = 0$$

Computationally, one can find CH by (integral) Gauss reduction of A^T.

We calculate (once for all) the matrix P. We also have algorithms which compute the boundary ∂e_b and, plugging into the formula (Co) we compute the integral matrix Δ_k.

7 The case $n = 5$.

We now consider the case $n = 5$ and compute $g(6) = g(\pi_6)$.

We know by the general dimension argument that $g(6) \leq 6$, on the other hand the covering π_6 contains a subcovering homeomorphic to π_5 (by considering the polynomials with one given fixed root), therefore from the results of Vassiliev $g(6) \geq 5$, In order to determine whether $g(6) = 6$ or $g(6) = 5$ we have to compute the obstruction class.

Lemma 7.1. *For $n = 5$ the ranks of the free $\mathbb{Z}[S_6]$-modules C_*, resp. the free $\mathbb{Z}[S_5]$-modules $\tilde{C}_*$, in dimensions $i : 0, 1, 2, 3, 4, 5, 6$ are:*

$$rk(C_i) = 1,\ 5,\ 15,\ 35,\ 70, 126, 210,$$

$$rk(\tilde{C}_i) = 0,\ 0,\ 1,\ 5,\ 15,\ 35,\ 70.$$

Proof. We just enumerate all flags with given cardinality, and those which contain both 1 and 5. □

Now what we do is the following:

1) By using the general algorithm which produces the boundaries in C_*, with coefficients in $\mathbb{Z}[S_6]$, we compute the $(35 \times 70)-$submatrix (with values in $\mathbb{Z}[S_6]$) of the boundary ∂_6 which corresponds to those flags containing both 1 and 5.

2) We multiply each column (on the left) by $\sigma \in S_5$ and transform that 35-vector over $\mathbb{Z}[S_6]$ into a $(35 \times 120)-$vector over $\mathbb{Z}$ by using the projection matrix P.

So we obtain a $(35 \times 120 = 4200) \times (70 \times 120 = 8400)-$matrix which represents the boundary $\tilde{\partial}_6$ as $\mathbb{Z}-$modules.

3) We find the (Smith) normal form of such a matrix.

By general results ([B]) all homology groups are finite in positive dimension, in our case by the Remark of section 4, the number 6 kills homology. From the normal form of $\tilde{\partial}_6$ one soon has $H_5(\tilde{C}_*)$ which equals $H_5(S_6, R/I)$ by proposition 5.1. By repeating the previous steps in lower dimension we find:

Theorem 7.2. *The homology $H_i(S_6;\ R/I)$ is for $i = 0, \ldots, 5$*

$$H_i(S_6;\ R/I): \quad \mathbb{Z}/3\mathbb{Z},\ 0,\ 0,\ 0,\ \mathbb{Z}/3\mathbb{Z},\ 0.$$

Proof. The cases $H_k(S_6;\ R/I)$ with $k \leq 2$ are computed directly. For the remaining ones we make computations using $\tilde{C}_*$ as said above, the computations have been made with a program written using the package AXIOM. □

Corollary 7.3. *For $n = 5$ the induced map ρ^* in cohomology vanishes.*

We have thus the main Theorem:

Theorem 7.4. *The genus $g(6) = 5$ so 5 holomorphic functions suffice to compute the roots of a polynomial of degree 6.*

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