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Geometric Properties of the Pruning Front

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Monotonicity of the pruning front is proved for the Lozi map. A general expression for its Hausdorff dimension is also derived which takes into account multifractal fluctuations as well.

§ 1. Introduction

Generating partitions play a relevant role in the understanding of chaotic properties of strange attractors. In fact, by transforming trajectories in phase space into sequences of symbols, the problem of characterizing a chaotic dynamics is reduced to that of studying the grammatical rules of the associated language, or the statistical mechanics of a suitable spin system.[1] In the case of horseshoe-type 2-d maps, the generating partition is made of two elements, so that the transformation is particularly appealing as each trajectory is encoded as a doubly infinite sequence of bits (s_n). Starting from any time position, all the future and past bits can be read as the expansion of two real numbers (x_f, x_p) in the unit interval, so that all empty sets in the unit square $\{x_f, x_p | 0 \leq x_f, x_p \leq 1\}$ correspond to forbidden sequences. A more informative representation of trajectories is obtained by introducing the τ and δ sequences defined as

$$\tau_k = (\tau_{k-1} + s_k) \bmod 2\,,$$

$$\delta_k = (\delta_{k-1} + 1 - s_{1-k}) \bmod 2 \tag{1·1}$$

for $k > 1$, with $\tau_0 = 0$ and $\delta_0 = 1$. Again, such two sequences can be seen as the binary expansion of two numbers (τ, δ) confined to the unit interval. Analogously to the (x_f, x_p) square, holes in the (τ, δ) unit square correspond to forbidden sequences. However, at variance with the previous coding, now there is an ordering of trajectories which greatly simplifies the search of forbidden words. To be more precise, let us consider a two-dimensional Hénon-type map, i.e., a map which simulates the stretching and folding-over property that naturally appears in strongly chaotic systems like, for instance, the Duffing attractor. In particular, throughout the paper, we shall consider the Lozi map $T(x_n, y_n) = (x_{n+1}, y_{n+1})$,[2] where

$$y_{n+1} = 1 - a|y_n| + bx_n\,, \quad x_{n+1} = y_n\,, \tag{1·2}$$

and a, b are positive parameters. The map (1·2) has been shown to have strong statistical properties, being essentially equivalent to an Axiom-A system[3] and is perhaps the simplest known example which exhibits the aforementioned Hénon-type dynamics. We point out, however, that the difficulty of constructing a generating partition increases sharply when passing from the Lozi map (1·2), where it is simply obtained by dividing the plane according to the y-axis, to the actual Hénon map,[4] where homoclinic tangencies have to be determined.[5),6] Anyhow, once the partition has been constructed, the grammar of the map can be extracted from the *pruning front*:[7] To each intersection P of the unstable manifold with the line defining the partition (from now on such points will be referred to as 'tangencies', although for the Lozi map they are not such) are attributed two different vertical coordinates $\delta_+(P)$ and $\delta_-(P)=1-\delta_+(P)$ (since s_0 is undetermined), with the same horizontal coordinate $\tau(P)$. Then, the pruning front is obtained by cutting out the rectangles $\{\tau, \delta | \tau > \tau(P), \delta \in [\delta_-(P), \delta_+(P)]\}$ for all P. It has been conjectured in Ref. 7) that the union of these rectangles (i.e., the whole region S_0 to the right of the pruning front) gives the whole set of forbidden sequences. In other words, if S_i denotes the image of S_0 under the map T^i, no other empty regions besides $\bigcup_{i=-\infty}^{\infty} S_i$ can exist in the (τ, δ) plane. This conjecture is the natural extension of a rigorous result holding for unimodal one-dimensional maps (see, for instance, Ref. 8)), where the past plays no role and S_0 reduces to the vertical strip $\{\tau, \delta | \tau(C) < \tau < 1\}$ where $\tau(C)$ is obtained from the kneading sequence.

Notice that this picture can be correct only if the pruning front is monotonous in the half plane $\delta < 1/2$ (and so also for $\delta > 1/2$ by symmetry reasons). In the next section we show that this is true for the Lozi map. Monotonocity allows for a one-to-one correspondence between the points of the pruning front and their projection onto the τ-axis. The Hausdorff dimension $D(0)$ of this last set is estimated in § 3, by deriving an expression which relates $D(0)$ to the multifractal spectrum of Lyapunov exponents. Such a quantity, being essentially a measure of the roughness of the front, is related to the growth rate of the number of new forbidden words for increasing length.[9]

§ 2. Monotonicity of the pruning front

In this section we show that the pruning front of the Lozi map (1·2) is monotonous. Actually we will be able to say more, namely, that the ordering of the tangencies along the y-axis is preserved when passing to the space of symbols (τ and δ coordinates).

First, let us introduce in the (x, y)-plane the ordering

$$(x, y) > (x', y') \quad \text{if and only if} \quad y > y' \tag{2·1}$$

and recall that in this plane the points of the pruning front correspond to the intersections of the unstable manifold with the line defining the generating partition (the y-axis for the Lozi map), i.e., those points that we have called "tangencies". We shall prove the

THEOREM

Let $P=(0, y_p)$ and $Q=(0, y_p)$ be any two intersections between the unstable *manifold of map* (1·2) *and the y-axis. Then* $P>Q$ *implies* $\tau(P)>\tau(Q)$ *and* $\delta(P)>\delta(Q)$.

Proof

To prove that the ordering is preserved under coding we simply need to show that

$$\tau_j(P)=1\,, \quad \tau_j(Q)=0\,, \tag{2·2}$$

where j is the first integer such that $\tau_j(P)\neq\tau_j(Q)$. A similar property shall then be proved for the δ-sequences.

As long as the symbol sequences arising from the iteration of the points P and Q are equal, the images of the initial segment PQ are still segments of the form $T^n(P)T^n(Q)$. In order to describe the evolution of these segments it is therefore convenient to introduce the linearization of (1·2),

$$v_{n+1}=(1-2s_{n+1})av_n+bu_n\,, \quad u_{n+1}=v_n\,, \tag{2·3}$$

where s_{n+1} is defined as

$$2s_{n+1}-1=\mathrm{sign}\,y_n\,. \tag{2·4}$$

By further introducing the slope

$$R_n\equiv\frac{v_n}{u_n}=\frac{v_n}{v_{n-1}}\,, \tag{2·5}$$

we obtain the recursion relation

$$R_{n+1}=(1-2s_{n+1})a+\frac{b}{R_n}\,. \tag{2·6}$$

Besides the information on the slope, we need to trace the evolution of the ordering along the segment. This can be done by introducing the variable θ_n which is equal to 0 whenever $T^n(P)>T^n(Q)$ and 1 otherwise. From the definition of R_n, it is immediately seen that θ_n satisfies the recursive equation

$$\theta_{n+1}=\left(\theta_n+\frac{1-\mathrm{sign}R_{n+1}}{2}\right)\mathrm{mod}\,2\,. \tag{2·7}$$

Accordingly, the study of the ordering problem is reduced to that of the sign of the slope R_n.

Equation (2·6) is invariant under the simultaneous exchange of the symbol s_n with its complement and of the sign of R. Therefore, we can limit the analysis of the evolution of the R-variable to its positive values only, and this can be done by taking the absolute value of the hyperbolae defined in (2·6), as shown in Fig. 1.

Lemma If $a\geq 2\sqrt{b}$, *then for all* $n\geq 1$,

$$|R_n|\in[C,D]\equiv\left[\frac{a-\sqrt{a^2-4b}}{2},\frac{3a-\sqrt{a^2-4b}}{2}\right]. \tag{2·8}$$

We point out that the hypotesis is always satisfied for those values of a and b for which a strange attractor exists.

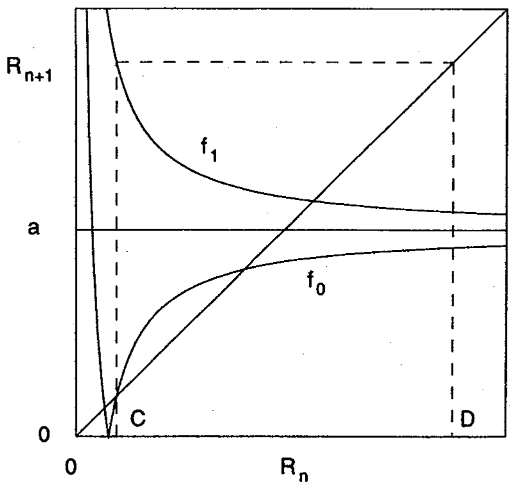

Fig. 1. Absolute values of the hyperbolae (2·6). Points C and D define an interval which is mapped into itself both by f_0 and f_1.

Proof From Eq. (2·6) we have that if $R_n > C$, then

$$|R_{n+1}| = \begin{cases} f_0(R_n) = a + b/R_n\,, \\ f_1(R_n) = a - b/R_n\,. \end{cases} \tag{2·9}$$

Both are monotonous functions in the interval (2·8), so that we can limit ourselves to considering the evolution of the extrema. It is easily seen that $f_0(C) = D$ and $f_0(D) > C$, so that $f_0([C, D]) \subset [C, D]$. Moreover, since C is a fixed point of f_1 and $f_1(D) < D$, then also $f_1([C, D]) \subset [C, D]$. Thus, the two functions f_0 and f_1 map the interval $[C, D]$ into itself. As a consequence, since (2·8) is satisfied for $n=1$ (as $|R_1| = a$, and $C < a < D$), the same condition also holds for $n \geq 1$, and the lemma follows.

Note that from this lemma we have immediately that the ambiguous case of a horizontal segment does not occur. Now, taking again the sign of R_n into account, we see that the two hyperbolae obtained from (2·6) have strictly opposite sign inside both $[C, D]$ and its symmetric $[-D, -C]$. Therefore, from the above lemma we have that $s_n = 0$ and $s_n = 1$ imply $R_n > 0$ and $R_n < 0$ respectively, independently of the past R-values. Thus (2·7) becomes

$$\theta_{n+1} = (\theta_n + s_{n+1}) \bmod 2 \tag{2·10}$$

for all symbol sequences (and hence for all trajectories of the Lozi map). This is nothing but the definition of τ_{n+1}. If $j-1$ represents the largest integer such that $T^{j-1}(P)$ and $T^{j-1}(Q)$ are on the same side of the generating partition, then the segment $T^{j-1}(P)T^{j-1}(Q)$ crosses the x-axis. In principle, two distinct cases may occur: i) $\theta_{j-1} = 0$ and ii) $\theta_{j-1} = 1$. In the first case $T^{j-1}(P) > T^{j-1}(Q)$ and $\tau_j(P) = s_j(P) = 1$, whereas $\tau_j(Q) = s_j(Q) = 0$. In the second case $T^{j-1}(P) < T^{j-1}(Q)$ and $\tau_j(P) = 1 - s_j(P) = 1$, whereas $\tau_j(Q) = 1 - s_j(Q) = 0$. In both cases (2·2) is satisfied and the first part of the theorem is proved.

To prove that an analogous ordering holds for the δ-sequence, it is sufficient to recall that the inverse of map (1·2) reads as

$$\tilde{y}_{n-1} = 1 - a'|\tilde{y}_n| + b'\tilde{x}_n\,, \quad \tilde{x}_{n-1} = \tilde{y}_n\,, \tag{2·11}$$

where

$$\tilde{x}_n = -by_n\,, \quad \tilde{y}_n = -bx_n \tag{2·12}$$

and $a' = a/b$, $b' = 1/b$. Equation (2·11) is formally identical to Eq. (1·2) apart from the sign of time. Moreover, the preimage of the y-axis under map (1·2) is the x-axis, so that the dividing line defining the generating partition for map (2·12) is the $\tilde{y}$-axis.

This allows us to apply the same procedure as before. The assumption of the above lemma becomes $a'>2\sqrt{b'}$ which is fully equivalent to the previous inequality $a>2\sqrt{b}$, so that the result can be straightforwardly rephrased. Instead of θ_n we now define a variable η_n which satisfies the recursion

$$\eta_{n-1}=(\eta_n+\tilde{s}_{n-1})\bmod 2\,, \qquad (2\cdot13)$$

where

$$2\tilde{s}_{n-1}-1=\mathrm{sign}\ \tilde{y}_n\,. \qquad (2\cdot14)$$

From (2·4) and (2·12) we then find

$$\tilde{s}_n=1-s_{n+1} \qquad (2\cdot15)$$

so that

$$\eta_{n-1}=(\eta_n+1-s_n)\bmod 2\,. \qquad (2\cdot16)$$

This means that η_n coincides with δ_n, thus proving the second part of the theorem.

In view of a generalization of the above proof to other cases, let us comment about the relevant facts which guarantee the monotonicity of the pruning front. First of all, the piecewise linearity of the map is not a necessary condition. It just simplifies some technical point. More in general, let us express by Γ the dividing line which defines the generating partition. Roughly speaking, suppose that any forward iterate of each short piece of Γ, not crossing Γ, has everywhere a negative slope when it lies to the right of Γ and positive in the opposite case, then (2·2) holds. Unfortunately, this cannot be used to extend the proof to the Hénon map because in this case it is possible to find pieces of the unstable manifold which are hook-shaped but nevertheless characterized by a unique symbol sequence. On the other hand, this is not sufficient to prove the contrary, i.e., that the pruning front is not monotonous. Indeed, condition (2·2) simply requires that the extrema of the j-th iterate of a segment PQ have to lie in different elements of the partition. Therefore, also on the basis of numerical evidence,[5] it seems very plausible that the pruning front is monotonous for the Hénon map as well. Be as it may, to obtain a rigorous proof one should be first able to make a precise statement on the position of the homoclinic tangencies, which does not seem an easy task at all.

§ 3. The fractal dimension of the pruning front

In this section we derive an expression for the fractal dimension of the pruning front, including multifractal corrections as well. The formula improves the result obtained in 5), which is first rederived to introduce the notations and clarify the problem. The reasoning applies to a general strange attractor.

Let us start by covering the (2-d) phase space with the cylinders S_i, defined as the set of points displaying the same sequence of n symbols $s_{-n+1}\cdots s_{-1}s_0$ in the past. Such regions are, essentially, narrow strips, characterized by a width ε of the order of

$$\varepsilon\simeq\exp(\lambda_{-}n)\,, \qquad (3\cdot1)$$

where λ_- is the negative Lyapunov exponent computed in the past. The upper extremum of each region S_i contains either primary homoclinic tangencies, or forward iterates of them. Tangencies belonging to the same cylinder S_i are at most ε apart. Therefore, they exhibit the same symbol sequence in the future for k iterates, where k is implicitly given by

$$\varepsilon \simeq \exp(-\lambda_+ k), \tag{3·2}$$

where λ_+ is the positive Lyapunov exponent computed in the future. From Eqs. (3·1) and (3·2), we find the relation between the numbers of common bits in the past and in the future,

$$n = -\frac{\lambda_+}{\lambda_-} k. \tag{3·3}$$

If we now look at the representation in the symbol plane, the fractal dimension of the projection of the pruning front on the τ-axis can be formally estimated by means of a box-counting procedure. A resolution $\delta = 2^{-k}$ allows us to discriminate among tangencies lying in different cylinders S_i. Accordingly, the number of boxes $N(\delta)$ needed to cover such a projection is of the order of the number of sets S_i, that is,

$$\delta^D = N(\delta) \simeq \exp(hn), \tag{3·4}$$

where h is the topological entropy. Thus, from Eqs. (3·3) and (3·4) we obtain

$$D = -\frac{h}{\log 2} \frac{\lambda_+}{\lambda_-} \tag{3·5}$$

which is the expression obtained in Ref. 5). However, in general, the Lyapunov exponents depend on the initial condition, so that not all the strips characterized by the same n do exhibit the same thickness, or the same k. In order to include such fluctuations, it is necessary to introduce the number $N(n, \lambda_-, \lambda_+)$ of cylinders characterized by the same Lyapunov exponents, with the convention that the positive and negative exponents are evaluated in the future and in the past, respectively. This number can be formally written as the global number of cylinders, times the probability to find any such pair of values,

$$N(n, \lambda_-, \lambda_+) = e^{hn} Q(n, \lambda_-, \lambda_+). \tag{3·6}$$

Moreover, we assume that, in the limit $n \to \infty$ (and thus $k \to \infty$), the above probability factorizes as follows:

$$Q(n, \lambda_-, \lambda_+) = P(n, \lambda_-) P(n, \lambda_+). \tag{3·7}$$

It is worth mentioning that the probability $Q(n, \lambda_-, \lambda_+)$ does not refer to a specific point (around which two well-defined and perfectly correlated λ_- and λ_+ values are detected) but globally to the ensemble of all points. In this sense, it is a perfectly plausible statistical assumption, justified by the presence of a chaotic dynamics (one has indeed to remember that Eq. (3·7) is conjectured to hold in the limit $n \to \infty$). Finally, we assume that the "homoclinic tangencies" are typical points of the attractor, so that the probability distribution of the Lyapunov exponents coincides with the standard distribution on the attractor. This assumption should be reasonably true for generic parameter values, although it possibly fails in special cases, when the tangencies are eventually mapped onto periodic orbits. As the covering defined at the beginning of this section is made according to the Lebsegue measure, the probability to find a negative exponent for maps with constant Jacobian J, is given by[1]

$$P(n, \lambda_-) \simeq \exp[(h - f(\tilde{\lambda}_+))n], \tag{3·8}$$

where

$$\tilde{\lambda}_+ = \log|J| - \lambda_- . \tag{3·9}$$

On the other hand, the probability to find a positive exponent λ_+ is

$$P(n, \lambda_+) \simeq \exp[(h - f(\lambda_+))k(n)], \tag{3·10}$$

where $k(n)$ is determined using Eq. (3·3). From Eqs. (3·6)~(3·8) and (3·10) we then find

$$N(n, \lambda_-, \lambda_+) = e^{hn} e^{(h - f(\tilde{\lambda}_+))n} e^{(h - f(\lambda_+))k} . \tag{3·11}$$

In order to estimate the fractal dimension of the pruning front, it is more convenient to express the number of cylinders in terms of k:

$$N(k, \lambda_-, \lambda_+) = \exp\left[-h + f(\lambda_+) - \frac{\lambda_+}{\lambda_-} f(\tilde{\lambda}_+) \right] k , \tag{3·12}$$

so that the total number of boxes needed to cover the pruning front is obtained by integrating (3·12) over all λ-values. Finally, the scaling behaviour of the number of boxes is given by the maximum value of the exponent in (3·12). By equating to zero the derivatives of the exponent in (3·12) with respect to λ_+ and λ_-, we find that the solutions $\lambda_+{}^*$, $\lambda_-{}^*$ ($\tilde{\lambda}_+{}^* \equiv \log J - \lambda_-{}^*$) are given by

$$\begin{aligned} &f'(\tilde{\lambda}_+{}^*)\lambda_-{}^* + f(\tilde{\lambda}_+{}^*) = 0 , \\ &f'(\lambda_+{}^*) + f'(\tilde{\lambda}_+{}^*) = 0 . \end{aligned} \tag{3·13}$$

The fractal dimension is then obtained by dividing the corresponding value of the exponent in (3·12) by log2, as for Eq. (3·5),

$$D(0) = \frac{1}{\log 2} \left(f(\lambda_+{}^*) - h - \frac{\lambda_+{}^* f(\tilde{\lambda}_+{}^*)}{\lambda_-{}^*} \right). \tag{3·14}$$

This formula can be written in a more compact form by introducing the Legendre transform $L(q)$ (i.e., the generalized Lyapunov exponents[10]) of $f(\lambda_+)$,

$$(q-1)L(q) = q\lambda_+ - f(\lambda_+) , \quad q = f'(\lambda_+) , \tag{3·15}$$

so that

$$D(0) = \frac{(1-q^*)L(q^*) - h}{\log 2} , \tag{3·16}$$

where $q^*=f'(\lambda_+^*)$.

In order to better discuss the role of multifractal fluctuations and to find a relation with the simple equation (3·5), we derive an explicit expression for $D(0)$, by approximating the $f(\lambda)$-curve with the parabola

$$f(\lambda)=h-\alpha(\lambda-\lambda_+^0)^2\,, \tag{3·17}$$

where λ_+^0 is such that $f(\lambda_+^0)=h$. The substitution of Eq. (3·17) in the first of Eq. (3·13) yields a second degree equation for $\tilde{\lambda}_+^*$ and the first order approximation in $1/\alpha$ (taken as a smallness parameter) gives

$$\tilde{\lambda}_+^*=\lambda_+'+\frac{h}{2\alpha\lambda_-'}\,, \tag{3·18}$$

where $\lambda_-'=\log|J|-\lambda_+'$. A further substitution of (3·18) and the subsequent use of (3·13) and (3·14) finally leads to

$$D(0)=-\frac{1}{\log 2}\frac{h\lambda_+'}{\lambda_-'}\left[1+\frac{h}{4\alpha\lambda_-'}(1-\lambda_+'\lambda_-')\right]. \tag{3·19}$$

We recognize that the leading term has exactly the structure obtained in Eq. (3·5), so that now we can give a more precise interpretation to the positive and negative exponents in that equation, which was derived in the absence of fluctuations. We see that the best approximation of the analytic result by means of the simple formula (3·5) is obtained by interpreting the Lyapunov exponents as the most probable ones, when the partition implicitly used to cover the attractor is constructed according to the Lebsegue measure.

Finally, we have tested numerically Eq. (3·16) for the Lozi map. The homoclinic tangencies have been computed by constructing the unstable manifold of the positive fixed point and determining successive intersections with the y-axis. By further iterating such points, we have determined their τ-coordinates. The most efficient way of computing the Hausdorff dimension with a few thousand of points available is by means of the nearest-neighbour algorithm. By calling $\Delta_j(n)$ the distance of the j-th point from its n-th neighbour, it is well known that[11]

$$S(n,\gamma)=\sum_j \Delta_j(n)^\gamma \simeq n^{(1-q)}\,, \tag{3·20}$$

where $\gamma\equiv(1-q)D(q)$ and $D(q)$ are the generalized Renyi dimensions. As we are interested in $q=0$, from Eq. (3·20) it turns out that $D(0)$ corresponds to the γ-value such that the scaling behaviour of $S(n,\gamma)$ is linear with n. We report in Fig. 2 the results of simulations per-

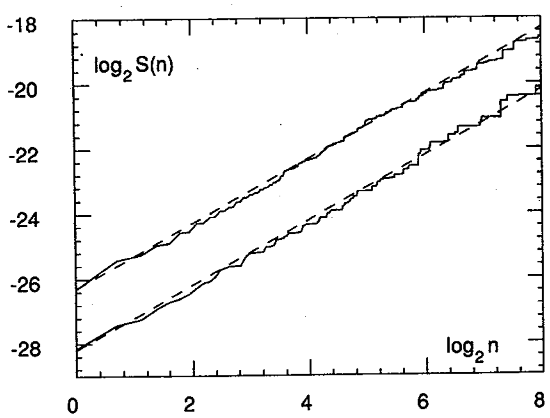

Fig. 2. $\log_2 S(n)$ (see Eq. (3·20)) versus the logarithm of the number of neighbours, for the pruning front of the Lozi map. Upper and lower curves refer to $a=1.7$, $b=0.6$, resp. $a=1.6$, $b=0.3$. The dashed straight lines with slope 1 are drawn for comparison.

formed for i) $a=1.6$, $b=0.3$ and ii) $a=1.7$, $b=0.6$. Lower and upper curves refer to cases i) and ii), respectively. They have been estimated for $\gamma=0.18$ resp. $\gamma=0.36$. The slope is in both cases very close to one, indicating that γ coincides with the Hausdorff dimension. Such results have to be compared with the "theoretical" estimates, obtained through Eq. (3·16), $D(0)=0.186\pm.004$ and $D(0)=0.38\pm.02$. The spectrum $f(\lambda)$ of the Lozi map has been obtained by using the cycle-expansion[12] and going up to orbits of period 14 in i) and period 18 in ii). Notice that in this last case the convergence is still rather poor even for such a long period. In both cases, the agreement between direct and indirect estimations of the fractal dimensions is rather good, thus implicitly confirming the crucial assumptions made in deriving Eq. (3·16) on the factorization of probabilities and on the "generic nature" of the points of the pruning front.

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References

1) V. M. Alekseev and M. V. Jakobson, Phys. Rep. **75** (1981), 287.
2) R. Lozi, J. Phys. Colloque **39** (1978), C5.
3) P. Collet and Y. Levy, Commun. Math. Phys. **93** (1984), 461.
4) M. Hénon, Commun. Math. Phys. **50** (1976), 69.
5) P. Grassberger and H. Kantz, Phys. Lett. **113A** (1985), 235.
6) G. D'Alessandro, P. Grassberger, S. Isola and A. Politi, J. of Phys. **A 23** (1990), 5285.
7) P. Cvitanović, G. Gunaratne and I. Procaccia, Phys. Rev **A38** (1988), 1503.
8) S. Isola and A. Politi, J. Stat. Phys. **61** (1990), 259.
9) G. D'Alessandro and A. Politi, Phys. Rev. Lett. **64** (1990), 1609.
10) H. Fujisaka, Prog. Theor. Phys. **70** (1983), 1264.
11) P. Grassberger, R. Badii and A. Politi, J. Stat. Phys. **51** (1988), 135.
12) P. Cvitanović, Phys. Rev. Lett. **61** (1988), 2729.