Orbiting Frames and Satellite Attitudes in Relativistic Astrometry

Donato Bini†*, Maria T. Crosta+ and Fernando de Felice‡

† Istituto per le Applicazioni del Calcolo "M. Picone", C.N.R., I–00161 Roma, Italy

* ICRA International Center for Relativistic Astrophysics, University of Rome, I–00185 Rome, Italy

+ Osservatorio Astronomico di Torino, Pino Torinese, Torino, Italy

‡ Dipartimento di Fisica Università di Padova, I–35131 Padova, Italy

I.N.F.N. Istituto Nazionale di Fisica Nucleare, sezione di Padova, Italy

PACS numbers: 04.20.-q, 04.70.Bw, 04.90.+e

Abstract.

Space-born measuring devices require an accurate determination of the satellite rest frame.

This frame consists of a clock and a triad of orthonormal axes which provide a local Carthesian reference system. Aim of this paper is to find the mathematical representation of this triad in two cases which may correspond to actual satellite attitudes. First we construct a Fermi frame which can be operationally fixed by a set of three mutually orthogonal gyroscopes, then we find a frame which corresponds to the expected attitude of the satellite GAIA which was ESA approuved to fly not later than 2012. In the latter case we were able to find an analytical solution accurate to $(v/c)^3$. In order to exploit this solution in the treatment of GAIA's astrometrical observations, we illustrate all the steps needed to deduce the components of this triad vectors.

1. Introduction

New space technologies allow for astrometric accuracies of 1 microarcsecond (μas) in stellar positions. With such an accuracy we must model and interpret the observations of satellites like GAIA in a general relativistic context. Future astrometric catalogues will be based on new astrometric parameters derived from the solution of the observation equations which link GAIA observations to the astrometric unknowns. In order to write these equations, the satellite attitude needs to be given in the same general relativistic framework, namely one has to define a comoving frame which connects GAIA measurements to the satellite motion and attitude law. This frame will be termed *attitude frame*; it consists of a *spatial* triad of orthonormal axes which are operationally fixed according to the specific goals of the space mission. Indeed the mathematical description of an attitude frame is in general a non trivial task.

In this paper we shall first deduce a Fermi frame adapted to a satellite with arbitrary orbital motion (section 3). This type of frame consists of three mutually orthogonal unit space-like vectors which are Fermi-Walker transported [1] along the satellite trajectory and represents the closest approximation to a locally inertial frame. This solution was obtained exploiting a more general solution for a satelite rest-frame recently deduced in [2]. Then we construct the expected attitude frame of the satellite GAIA (section 4) deducing the mathematical expressions of its components in a way useable in any numerical astrometric model. The attitude frame, in fact, is essential to fix the boundary conditions for solving the ray tracing problem in relativistic astrometry, expressing them in terms of the satellite observations (see [2]); this will be shown in section 5. Finally in the conclusions we stress how essential is to link the satellite observations to the actual motion of the satellite with respect to a chosen general reference system.

In this paper Latin indeces run from 1 to 3, Greek indeces run from 0 to 3.

2. Mathematical preliminaries

The rest-frame of a satellite consists of a *clock* which measures the satellite proper-time and a triad of orthonormal axes. The latters are described by four-vectors which are referred to a coordinate system which in general is not connected to the satellite itself. The mathematical quantity which defines a rest-frame of a given observer (the satellite in our case) is a tetrad adapted to that observer, namely a set of four unitary mutually orthogonal four-vectors one of which is the time-like tangent to the observer's world-line; the parameter on this world-line is the observer's proper time. The remaining three space-like vectors of the tetrad are defined up to spatial rotations. There are infinitely many possible orientations of the spatial triad to be fixed in a satellite, therefore our task is to identify non ambiguously those which correspond to actual attitudes.

Since we have in mind applications to satellite missions, we fix the background geometry as that of the Solar System assuming that it is the only source of gravity; moreover we assume that it generates a weak gravitational field so we shall retain only terms of first order in the gravitational constant G and consider these terms only up to the order of $(v/c)^3$.

The space-time geometry is then given by the following line element

$$\mathrm{d}s^2 \equiv g_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta = (\eta_{\alpha\beta} + h_{\alpha\beta} + O(h^2))\mathrm{d}x^\alpha \mathrm{d}x^\beta \tag{2.1}$$

where $O(h^2)$ denotes non linear terms in h, the coordinates are $x^0 = t, x^1 = x, x^2 = y, x^3 = z$ the origin being fixed at the barycenter of the Solar System, $\eta_{\alpha\beta}$ is the Minkowski metric so that the metric components read:

$$g_{00} = -1 + \underset{2}{h}_{00} + O(4), \quad g_{0a} = \underset{3}{h}_{0a} + O(5), \quad g_{ab} = 1 + \underset{2}{h}_{00}\delta_{ab} + O(4). \tag{2.2}$$

Here $\underset{2}{h}_{00} = 2U$ where U is the gravitational potential generated by the sources of the Solar System and subscripts indicate the order of (v/c) (ex. $\underset{3}{h}_{0a} \sim O(3)$) and $O(n) = O[(v/c)^n]$. As in [2] we shall carry all the calculations without specifying the metric coefficients so to assure generality. Unless otherwise stated, we use units such that $c = 1 = G$.

Let us fix the satellite's trajectory in the above space-time geometry as the time-like, unitary four vector $\boldsymbol{u'}$ ($u'^\alpha u'_\alpha = -1$) given by:

$$\boldsymbol{u'} = T_s(\boldsymbol{\partial}_t + \beta_1\boldsymbol{\partial}_x + \beta_2\boldsymbol{\partial}_y + \beta_3\boldsymbol{\partial}_z), \tag{2.3}$$

where $\boldsymbol{\partial}_\alpha$'s are the coordinate basis vectors relative to the baricentric coordinate system, β_i are the *coordinate* components of the satellite three-velocity with respect to the baricenter of the Solar System recalling that we use here subscripts to refer to contravariant components not to confuse them with power indeces. Finally we define $T_s = 1 + (U + \frac{1}{2}\beta^2)$ and $\beta^2 = \beta_1^2 + \beta_2^2 + \beta_3^2$.

3. A Fermi frame

A Fermi frame adapted to a given observer can be obtained from *any* frame adapted to the same observer provided we know its Fermi coefficients. The latters, to be defined shortly, tell how much the tetrad spatial axes must rotate in order to be reduced to a Fermi frame. We shall apply this procedure to the tetrad adapted to (2.3) and deduced in [2]. In this case however the construction of a Fermi tetrad was unexpecteadly complicated; in fact, an algebric solution was possible only if we confined ourselves to terms of the order of $(v/c)^2$ and set $\beta_3 = 0$. In a more general case and to the order of $(v/c)^3$ the solution was so long and cumbersome to descourage any practical use if not just numerical. We shall therefore present a less accurate but analytically tractable solution as indicated.

To the order of $(v/c)^2$, the tetrad used in [2], simplifies to

$$\begin{aligned}
\boldsymbol{\lambda}_{\hat{0}} &= [1 + U + \frac{1}{2}\beta^2]\boldsymbol{\partial}_t + \beta[\cos\omega(t)\boldsymbol{\partial}_x + \sin\omega(t)\boldsymbol{\partial}_y] \\
\boldsymbol{\lambda}_{\hat{1}} &= (1 - U)[\sin\omega(t)\boldsymbol{\partial}_x - \cos\omega(t)\boldsymbol{\partial}_y] \\
\boldsymbol{\lambda}_{\hat{2}} &= \beta\boldsymbol{\partial}_t + \left[\frac{1}{2}\beta^2 - U + 1\right](\cos\omega(t)\boldsymbol{\partial}_x + \sin\omega(t)\boldsymbol{\partial}_y) \\
\boldsymbol{\lambda}_{\hat{3}} &= (1 - U)\boldsymbol{\partial}_z
\end{aligned} \tag{3.1}$$

where we have set $\beta_1 = \beta\cos\omega(t)$, $\beta_2 = \beta\sin\omega(t)$ and $\beta_3 = 0$, ω being the angular velocity of rotation of the frame under consideration. This tetrad is not Fermi transported because its Fermi coeficients are not all zero. The Fermi coefficients are defined as

$$C_{\hat{a}\hat{b}} = \lambda_{\hat{a}} \cdot \nabla_{\hat{0}}\lambda_{\hat{b}} \tag{3.2}$$

where the covariant derivative is meant with respect to the tetrad. In the case of (3.1) the only non zero coefficient is

$$C_{\hat{2}\hat{1}} = -\dot{\omega} \tag{3.3}$$

a dot meaning derivative with respect to coordinate time. Subtracting the Fermi rotation at each time, the triad $\{\boldsymbol{\lambda}_{\hat{a}}\}$ reduces to a Fermi triad:

$$\begin{aligned}
R_{\hat{1}} =& -\beta\sin\omega(t)\boldsymbol{\partial}_t - \frac{1}{4}\beta^2\sin 2\omega(t)\boldsymbol{\partial}_x \\
& - \left[\frac{\beta^2}{2}\sin^2\omega(t) - U + 1\right]\boldsymbol{\partial}_y \\
R_{\hat{2}} =& \beta\cos\omega(t)\boldsymbol{\partial}_t + \left[1 - U + \frac{\beta^2}{2}\cos^2\omega(t)\right]\boldsymbol{\partial}_x \\
& + \frac{1}{4}\sin 2\omega(t)\beta^2\boldsymbol{\partial}_y \\
R_{\hat{3}} =& (1 - U)\boldsymbol{\partial}_z.
\end{aligned} \tag{3.4}$$

It is easy to verify that the Fermi coefficients of (3.4) are all identically zero.

A Fermi triad is defined up to a constant rotation; this freedom corresponds to the arbitrariness in the operational setting of the Fermi frame by means of three mutually orthogonal gyroscopes. In most cases the motion of a satellite is complicated by spin and precession so, unless one is able to handle a stable system of three mutually orthogonal and freely rotating gyroscopes, a Fermi frame is difficult to set up. It is instead more convenient to define a frame which is fixed to the satellite and is constrained according to criteria of best efficiency for the mission goal.

4. GAIA's attitude frame

The astrometric satellite GAIA is expected to orbit the Earth-Sun System in the outer Lagrangian point L_2 following a trajectory modulated in the three spatial directions [3, 4]; all is referred to the Solar System baricentric reference system. Moreover the satellite rotates at a rate of 1 turn every 6 hours about an axis (its x axis) which forms a fixed angle $\alpha = 50°$ with the Sun direction; the spin axis then precesses about the Sun direction with a period of 70 days.

At the moment there are two main attempts to describe GAIA's attitude. The first is based on a rigorous formulation of the scanning law through the integration of two differential equations which express the condition for a uniformly revolving angle and an inertially constant scanning motion about the spin axis [5–7]. The main limitation of this approach is that the spin axis revolves around the direction to the Sun as seen from the point L2 and not from the actual position of the satellite. The second one [8] is an analytical solution that allows one to write a simple and compact code fully parametrized with the Sun aspect angle and the speed of the Sun centered cone. Starting from an ecliptic triad, this second approach uses Euler representation to obtain the spacecraft frame at the point L2. Both approaches need at first to fix the direction to the Sun as seen from within the satellite rest frame, (figure 1).

Aim of the following is to find GAIA's attitude frame keeping the approximation to the order of $(v/c)^3$ as in [2]. We first fix a coordinate system whose origin is located at the baricenter of the Solar System and the spatial axes are pointing to distance sources; the latters identify a *global* Carthesian-like spatial coordinate representation $(x,\, y,\, z)$. The world-line of the baricenter in the space-time(2.1) is given by the unit four-vector

$$u = (g_{tt})^{-1/2}\boldsymbol{\partial}_t \approx (1+U)\boldsymbol{\partial}_t \tag{4.1}$$

where t is a coordinate time. The observer $\boldsymbol{u}$ together with the spatial axes as specified is termed baricentric obsever and the parameter on its world-line is the baricentric proper-time. The reference frame so defined is termed Baricentric Celestial Reference System (BCRS, [9] [10]). Obviously, at each point in space-time there exists a baricentric observer u who carries a triad of spatial and mutually orthogonal unitary vectors which point to the same distant sources as for the BCRS. We shall term each of these frames a local BCRS.

As shown in [2], the spatial triad of a local BCRS at each space-time point is given to $(v/c)^3$ by the following vectors‡:

$$\begin{aligned}
\boldsymbol{\lambda}_{\hat{1}} &= h_{01}\boldsymbol{\partial}_t + (1-U)\boldsymbol{\partial}_x \\
\boldsymbol{\lambda}_{\hat{2}} &= h_{02}\boldsymbol{\partial}_t + (1-U)\boldsymbol{\partial}_y \\
\boldsymbol{\lambda}_{\hat{3}} &= h_{03}\boldsymbol{\partial}_t + (1-U)\boldsymbol{\partial}_z
\end{aligned} \tag{4.2}$$

We need to identify the spatial direction to the geometrical center of the Sun as seen from within the satellite. To this purpose we first identify this direction with respect

‡ Here we correct a sign misprint in [2].

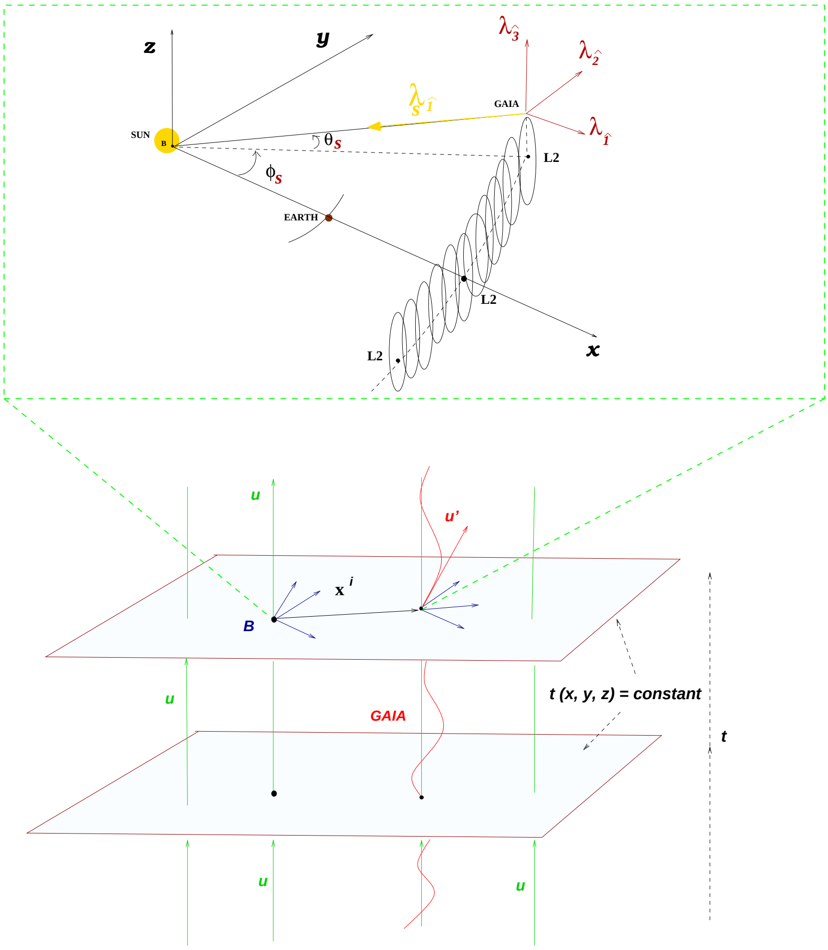

Figure 1. The spatial triad $\boldsymbol{\lambda}_{\hat{a}}$ is comoving with the local baricentric observer $\boldsymbol{u}$ (**B**= baricenter) defined at the GAIA center-of-mass at each point of its Lissajous orbit about L2. The $\underset{s}{\boldsymbol{\lambda}}_{\hat{1}}$ identifies the instantaneous Sun direction as seen by the local baricentric observer $\boldsymbol{u}$. This vector will be boosted to the satellite motion to obtain the Sun direction as seen from on-board of the satellite

to the local BCRS which is defined at each point of the satellite's trajectory, then we boost the corresponding triad to adapt it to the motion of the satellite.

Let $x_0(t)$, $y_0(t)$, $z_0(t)$ be the coordinates of the satellite's center of mass with respect to the baricenter of the Solar System and $x_\odot(t)$, $y_\odot(t)$, $z_\odot(t)$ those of the Sun at the same coordinate time t. Here the time dependence is assumed to be known. The relative spatial position of the Sun with respect to the satellite at the time t is then:

$$\begin{aligned} x'_\odot &= x_\odot - x_0 \\ y'_\odot &= y_\odot - y_0 \\ z'_\odot &= z_\odot - z_0 . \end{aligned} \tag{4.3}$$

We omit the time dependence to ease notation. With respect to a local BCRS, the

Sun direction is fixed rotating the triad (4.2) by an angle ϕ_s around $\boldsymbol{\lambda}_{\hat{3}}$ and then by an angle θ_s around the vector image of $\boldsymbol{\lambda}_{\hat{2}}$ under the above ϕ_s-rotation, where:

$$\phi_s = \tan^{-1}\frac{y'_\odot}{x'_\odot} \quad , \quad \theta_s = \tan^{-1}\frac{z'_\odot}{\sqrt{x'^2_\odot + y'^2_\odot}}. \tag{4.4}$$

Thus we have the new triad adapted to the observer $\boldsymbol{u}$:

$$\underset{s}{\boldsymbol{\lambda}}_{\hat{a}} = \mathcal{R}_2(\theta_s)\mathcal{R}_3(\phi_s)\boldsymbol{\lambda}_{\hat{a}} \tag{4.5}$$

where

$$\mathcal{R}_2(\theta_s) = \begin{pmatrix} \cos\theta_s & 0 & \sin\theta_s \\ 0 & 1 & 0 \\ -\sin\theta_s & 0 & \cos\theta_s \end{pmatrix} \tag{4.6}$$

and

$$\mathcal{R}_3(\phi_s) = \begin{pmatrix} \cos\phi_s & \sin\phi_s & 0 \\ -\sin\phi_s & \cos\phi_s & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.7}$$

It should be noted here that, since the Sun is an extended body, its geometrical center may be difficult to determine with great precision. The uncertainty in this measurement may affect the precision of fixing the angles ϕ_s and θ_s from on-board of the satellite. We assume here that space technology will cope with this problem satisfactorily.

From (4.5), (4.6) and (4.7), the explicit expressions of the coordinate components of the vectors of the new triad are given by:

$$\begin{aligned} \underset{s}{\boldsymbol{\lambda}}_{\hat{1}} = {} & [\cos\theta_s(\cos\phi_s h_{01} + \sin\phi_s h_{02}) + \sin\theta_s h_{03}]\boldsymbol{\partial}_t \\ & + \cos\phi_s\cos\theta_s(1-U)\boldsymbol{\partial}_x \\ & + \sin\phi_s\cos\theta_s(1-U)\boldsymbol{\partial}_y \\ & + \sin\theta_s(1-U)\boldsymbol{\partial}_z \end{aligned} \tag{4.8}$$

$$\begin{aligned} \underset{s}{\boldsymbol{\lambda}}_{\hat{2}} = {} & -(\sin\phi_s h_{01} + \cos\phi_s h_{02})\boldsymbol{\partial}_t \\ & - \sin\phi_s(1-U)\boldsymbol{\partial}_x \\ & - \cos\phi_s(1-U)\boldsymbol{\partial}_y \end{aligned} \tag{4.9}$$

$$\begin{aligned} \underset{s}{\boldsymbol{\lambda}}_{\hat{3}} = {} & -[\sin\theta_s(\cos\phi_s h_{01} + \sin\phi_s h_{02}) - \cos\theta_s h_{03}]\boldsymbol{\partial}_t \\ & - \cos\theta_s\sin\theta_s(1-U)\boldsymbol{\partial}_x \\ & - \sin\phi_s\sin\theta_s(1-U)\boldsymbol{\partial}_y \\ & + \cos\theta_s(1-U)\boldsymbol{\partial}_z \end{aligned} \tag{4.10}$$

It is easy to verify that the set $\{\boldsymbol{u}, \underset{s}{\boldsymbol{\lambda}}_{\hat{a}}\}$ forms an orthonormal tetrad; moreover it is equally straightforward to see that:

$$\cos\theta_s\, \underset{s}{\boldsymbol{\lambda}}_{\hat{2}} = \frac{d}{d\phi_s}\underset{s}{\boldsymbol{\lambda}}_{\hat{1}} \quad , \quad \underset{s}{\boldsymbol{\lambda}}_{\hat{3}} = \frac{d}{d\theta_s}\underset{s}{\boldsymbol{\lambda}}_{\hat{1}}. \tag{4.11}$$

All quantities in (4.8) to (4.10) are defined at the position (x_0, y_0, z_0) of the satellite at time t.

Let us recall that our aim here is to identify a tetrad frame which is adapted to the satellite and whose spatial triad mirrors its attitude. Recalling that the satellite four velocity is given by $\boldsymbol{u'}$ as in (2.3), we boost the vectors of the triad $\{\underset{s}{\boldsymbol{\lambda}}_{\hat{a}}\}$ along the satellite relative motion to obtain the following boosted triad (see [11] and fig.2):

$$\underset{bs}{\lambda}{}^{\alpha}_{\hat{a}} = P(u')^{\alpha}{}_{\sigma}\left[\underset{s}{\lambda}{}^{\sigma}_{\hat{a}} - \frac{\gamma}{\gamma+1}\nu^{\sigma}\left(\nu^{\rho}\underset{s}{\lambda}_{\rho\hat{a}}\right)\right]_{\hat{a}=1,2,3} \tag{4.12}$$

where:

$$P(u')^{\alpha}{}_{\sigma} = \delta^{\alpha}_{\sigma} + u'^{\alpha}u'_{\sigma} \tag{4.13}$$

is the operator which projects to the rest-frame of $\boldsymbol{u'}$, ν^{α} is the relative spatial four-velocity of $\boldsymbol{u'}$ with respect to the local BCRS observer $\boldsymbol{u}$ and it is defined as

$$\nu^{\alpha} = \frac{1}{\gamma}(u'^{\alpha} - \gamma u^{\alpha}) \tag{4.14}$$

and $\gamma = -u'^{\alpha}u_{\alpha}$ is the relative Lorentz factor. The vector $\underset{bs}{\boldsymbol{\lambda}}_{\hat{1}}$ identifies the direction to the Sun as seen from within the satellite. The other vectors of the boosted triad are related to $\underset{bs}{\boldsymbol{\lambda}}_{\hat{1}}$ by the simple relations:

$$\cos\theta_s\,\underset{bs}{\boldsymbol{\lambda}}_{\hat{2}} = \frac{d}{d\phi_s}\underset{bs}{\boldsymbol{\lambda}}_{\hat{1}} \quad , \quad \underset{bs}{\boldsymbol{\lambda}}_{\hat{3}} = \frac{d}{d\theta_s}\underset{bs}{\boldsymbol{\lambda}}_{\hat{1}}. \tag{4.15}$$

The tetrad $\{\underset{bs}{\boldsymbol{\lambda}}_{\hat{0}} \equiv \boldsymbol{u'}, \underset{bs}{\boldsymbol{\lambda}}_{\hat{a}}\}$ will be referred to as the *Sun-locked frame*. The relation between the components ν^{α} of the spatial four-velocity $\boldsymbol{\nu}$ and the components β_i appearing in (2.3) is easily established from (2.3) itself and (4.14) and read:

$$\nu^{\alpha} = \frac{1}{\gamma}\left[T_s\left(\beta_i\delta^{i\alpha} + \delta^{0\alpha}\right) - u^{\alpha}\gamma\right]. \tag{4.16}$$

Due to their complexity, the explicit expressions of the components of the vectors $\underset{bs}{\boldsymbol{\lambda}}_{\hat{a}}$ are given in appendix A.

To deduce GAIA's attitude frame, which we remember is our main goal, we have to make the following final steps.

i) Rotate the Sun-locked triad by an angle $\omega_p t$ about the vector $\underset{bs}{\boldsymbol{\lambda}}_{\hat{1}}$ which constantly points to the Sun; ω_p is the angular velocity of precession.

ii) Rotate the resulting triad by a fixed angle α about the image of the vector $\underset{bs}{\boldsymbol{\lambda}}_{\hat{2}}$ under rotation i).

iii) Rotate the triad obtained after step ii) by an angle $\omega_r t$ about the image of the vector $\underset{bs}{\boldsymbol{\lambda}}_{\hat{1}}$ under the previous two rotations; ω_r is now the angular velocity of the satellite spin.

The triad resulting from these three steps will be the satellite *attitude triad*; this is given by:

$$\boldsymbol{E}_{\hat{a}} = \mathcal{R}_1(\omega_r t)\mathcal{R}_2(\alpha)\mathcal{R}_1(\omega_p t)\underset{bs}{\boldsymbol{\lambda}}_{\hat{a}} \qquad \hat{a} = 1,\, 2,\, 3 \tag{4.17}$$

where:

$$\mathcal{R}_1(\omega_p t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\omega_p t & \sin\omega_p t \\ 0 & -\sin\omega_p t & \cos\omega_p t \end{pmatrix} \tag{4.18}$$

$$\mathcal{R}_2(\alpha) = \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix} \tag{4.19}$$

$$\mathcal{R}_1(\omega_r t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\omega_r t & \sin\omega_r t \\ 0 & -\sin\omega_r t & \cos\omega_r t \end{pmatrix} \tag{4.20}$$

The explicit expressions of the triad vectors $\boldsymbol{E}_{\hat{a}}$ are shown in Appendix B. It is clear that most of the terms entering the attitude triad components are of the order of $(v/c)^3$ however, despite their cumbersome expressions, we expect that this attitude triad can be exploited without difficulties in a numerical code since all quantities entering the $\boldsymbol{E}_{\hat{a}}$'s are well defined.

5. Observations in GAIA's attitude frame

The mathematical characterization of GAIA's attitude triad is essential to solve the boundary value problem in the process of reconstructing the light trajectory (see [12–17] and references therein) which connects the satellite to the emitting star. Although this problem has been illustrated in [2], we shall briefly recall some of the considerations made in that paper.

A light ray is described by a geodesic whose tangent vector field $\boldsymbol{k}$ satisfies the light like condition $k_\alpha k^\alpha = 0$ and the geodetic equation:

$$k^\beta \nabla_\beta k^\alpha = 0 \tag{5.1}$$

where ∇_β is the covariant derivative with respect to the coordinate x^β in the metric (2.1). At each point of its trajectory the light signal strikes the local BCRS $\boldsymbol{u}$; in this frame the light signal would be seen propagating along a spatial direction (the local line of sight) given by a vector $\boldsymbol{\ell}$ defined as:

$$\ell^\rho = P(u)^\rho{}_\sigma k^\sigma \tag{5.2}$$

where $P(u)^\rho{}_\sigma = \delta^\rho_\sigma + u^\rho u_\sigma$ is the operator which projects into the rest space of $\boldsymbol{u}$. The space-like vector $\boldsymbol{\ell}$ is not unitary hence we can always normalize it to $\bar{\boldsymbol{\ell}} = -\boldsymbol{\ell}/(u^\alpha k_\alpha)$, such that $\bar{\ell}^\alpha \bar{\ell}_\alpha = 1$. This operation corresponds to parametrize the light curve with the proper-time σ of the observer $\boldsymbol{u}$ which it crosses at each of its points or equivalently to fix equal to 1 the photon energy as measured by $\boldsymbol{u}$, namely $\mathcal{E}(u) = -k^\alpha u_\alpha = 1$. Since the vector field $\bar{\boldsymbol{\ell}}$ is everywhere orthogonal to $\boldsymbol{u}$, namely $\ell_\alpha u^\alpha = 0$, then it satisfies the conditions:

$$\bar{\ell}_0 = 0, \qquad \bar{\ell}^0 = \underset{3}{h}{}_{0i}\bar{\ell}^i + O(5). \tag{5.3}$$

From (5.1), (5.2) and (5.3), the differential equation of light propagation which is basic to the problem of determining the astrometric parameters of a star has a general form (see [13]):

$$\frac{d\bar{\ell}^\alpha}{d\sigma} = \mathcal{F}^\alpha(\partial_\beta h(x,\, y,\, z,\, t), \bar{\ell}^i(\sigma(x)) \tag{5.4}$$

where $\mathcal{F}^\alpha$ are real, non singular, smooth functions of their arguments.

A general solution of (5.4) is

$$\bar{\ell}^i(\sigma) = f^i(\sigma, \bar{\ell}^k_0) \tag{5.5}$$

where $\bar{\ell}^k_0$ are the components of the vector $\bar{\boldsymbol{\ell}}$ at the observation and represent the baundary values which are necessary to integrate (5.4). These boundary conditions can only be expressed in terms of the satellite observations. In case of GAIA, the

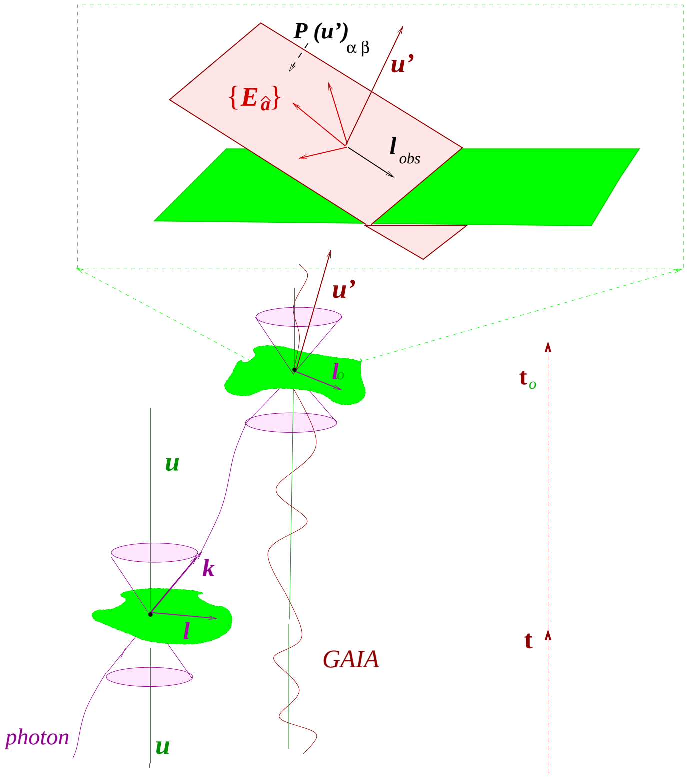

Figure 2. The rest frame of GAIA is Lorentz boosted with respect to the rest frame of the local baricentric observer $\boldsymbol{u}$. The GAIA's attitude triad $\boldsymbol{E}_{\hat{a}}$ is defined in the staellite's rest frame.

latters are the angles that the incoming light ray forms with the axes of the attitude triad and defined as:

$$\cos\psi_{(E_{\hat{a}},\ell)} \equiv \mathbf{e}_{\hat{a}} = \frac{P(u')_{\alpha\beta}k^{\alpha}E^{\beta}_{\hat{a}}}{(P(u')_{\alpha\beta}k^{\alpha}k^{\beta})^{1/2}} \tag{5.6}$$

where no sum is meant over $\hat{a}$; $P(u')_{\alpha\beta}$, is the operator which projects into the satellite's rest-frame defined as:

$$P(u')_{\alpha\beta} = g_{\alpha\beta} + u'_{\alpha}u'_{\beta}. \tag{5.7}$$

Recalling (5.2), we easily see that all quantities contained in (5.6) are known except ℓ^i_0 which obviously are the unknowns boundary conditions, as stated. In this case an analytical solution of (5.6) in terms of ℓ^i_0 and up tp $(v/c)^3$ is too long to be written explicitely. The purpose of this work, however, is to provide all the *ingredients* needed to implement a numerical code which will routinely solve the boundary values problem allowing integration of (5.4) and then the whole relativisitic astrometric model [13] to become fully operative.

6. Summary and Conclusions

We have found the mathematical reprentation of a satellite attitude frame in two different cases which may correspond to possible satellite sets-up; a Fermi frame and the axpected GAIA's attitude.

We show that the formal characterization of a Fermi frame is analytically very difficult and indeed we were able to find a solution only to the order of $(v/c)^2$ in the post-Newtonian approximation of the background metric. This seems to indicate that use of a Fermi frame as attitude frame for a satellite may meet with dificulties at high orders of accuracy.

A better result we obtaind with constrained (not Fermi) frames. The Relativistic Astrometric Model developed in [13] and ready to be tested to the order of $(v/c)^3$, requires that GAIA's attitude is well defined in terms of a spatial triad of orthonormal vectors adapted to the satellite's composite motion. This triad, together with an on-board clock which measures the satellite's proper-time, forms the *attitude frame* of the satellite. The coordinate components of this frame are relative to a *global* Baricentric Celestial Reference System (BCRS) which is identified by three spatial axes centered at the baricenter of the Solar System and pointing to distant cosmic sources chosen so to assure that the system is kinematically non rotating. The coordinate axes then define a Carthesian like coordinate system $(x,\ y,\ z)$ and we assume that an everywhere space-like hypersurface exists with equation $t(x, y, z)$ =contant. The function t is chosen as a coordinate time§ hence, togheter with the set $(x,\ y,\ z)$, it provides a coordinate representation of the space-time. This coordinate system is assumed to fix the space-time metric form (2.1). At any space-time point there exists an observer which is at rest with respect to the BCRS since its world-line is parallel to the local coordinate time axis. The tangent to the observer's whorld-line is a unitary vector labelled as $\boldsymbol{u}$ and given by (4.1). Assume first that this observer is located at the origin of the BCRS, then this system can be locally identified by a spatial triad of unitary and orthonormal vectors given by (4.2) up to $(v/c)^3$. They are orthogonal to $\boldsymbol{u}$ and point to the corresponding coordinate directions. In this case the proper time of $\boldsymbol{u}$ is the baricentric proper-time. As said, such an observer can be defined at each space-time point and again we can adapt to this observer a local triad of space-like vectors which point to the local coordinate directions. This frame will be termed local BCRS; evidently the local BCRS proper-time varies as a function of the position as can be deduced from (4.1).

The physical observations made within the satellite can only be referred to its attitude frame, namely that adapted to the satellite composite motion. Thus, in order to exploit the observations as boundary data essential to the solution of the ray tracing problem in the astrometric model, one must be able to relate attitude frame quantities to local BCRS components. This must be done consistently with the requirements of general relativity; in this paper we show how to do this providing the mathematical reprentation of the attitude frame of GAIA in a form ready for applications.

§ In [13] we illustrate how one can make the choice of the coordinate time not arbitrary.

Appendix A

We give here explicitly the boosted triad $\underset{bs}{\boldsymbol{\lambda}}{}_{\hat{a}}$, specifying the coordinate components of each vector. Using the notation

$$
\begin{aligned}
\underset{bs}{\boldsymbol{\lambda}}{}_{\hat{1}} &= \underset{bs}{\lambda}{}^{t}_{\hat{1}}\boldsymbol{\partial}_t + \underset{bs}{\lambda}{}^{x}_{\hat{1}}\boldsymbol{\partial}_x + \underset{bs}{\lambda}{}^{y}_{\hat{1}}\boldsymbol{\partial}_y + \underset{bs}{\lambda}{}^{z}_{\hat{1}}\boldsymbol{\partial}_z \\
\underset{bs}{\boldsymbol{\lambda}}{}_{\hat{2}} &= \underset{bs}{\lambda}{}^{t}_{\hat{2}}\boldsymbol{\partial}_t + \underset{bs}{\lambda}{}^{x}_{\hat{2}}\boldsymbol{\partial}_x + \underset{bs}{\lambda}{}^{y}_{\hat{2}}\boldsymbol{\partial}_y + \underset{bs}{\lambda}{}^{z}_{\hat{2}}\boldsymbol{\partial}_z \\
\underset{bs}{\boldsymbol{\lambda}}{}_{\hat{3}} &= \underset{bs}{\lambda}{}^{t}_{\hat{3}}\boldsymbol{\partial}_t + \underset{bs}{\lambda}{}^{x}_{\hat{3}}\boldsymbol{\partial}_x + \underset{bs}{\lambda}{}^{y}_{\hat{3}}\boldsymbol{\partial}_y + \underset{bs}{\lambda}{}^{z}_{\hat{3}}\boldsymbol{\partial}_z
\end{aligned}
\tag{6.1}
$$

we have

$$
\begin{aligned}
\underset{bs}{\lambda}{}^{t}_{\hat{1}} &= \{3(\cos\theta_s\cos\phi_s\beta_1 + \cos\theta_s\sin\phi_s\beta_2 + \sin\theta_s\beta_3)U \\
&\quad + \frac{1}{2}\beta^2(\cos\theta_s\cos\phi_s\beta_1 + \cos\theta_s\sin\phi_s\beta_2 + \sin\theta_s\beta_3) \\
&\quad + (\cos\theta_s\cos\phi_s h_{01} + \cos\theta_s\sin\phi_s h_{02} + \sin\theta_s h_{03}) \\
&\quad + [\cos\theta_s\cos\phi_s\beta_1 + \cos\theta_s\sin\phi_s\beta_2 + \sin\theta_s\beta_3]\} \\
\underset{bs}{\lambda}{}^{x}_{\hat{1}} &= \left\{\cos\theta_s\cos\phi_s + \frac{1}{2}\beta_1(\cos\theta_s\cos\phi_s\beta_1 + \cos\theta_s\sin\phi_s\beta_2 + \sin\theta_s\beta_3) - U\cos\theta_s\cos\phi_s\right\} \\
\underset{bs}{\lambda}{}^{y}_{\hat{1}} &= \left\{\cos\theta_s\cos\phi_s + \frac{1}{2}\beta_2(\cos\theta_s\cos\phi_s\beta_1 + \cos\theta_s\sin\phi_s\beta_2 + \sin\theta_s\beta_3) - U\cos\theta_s\sin\phi_s\right\} \\
\underset{bs}{\lambda}{}^{z}_{\hat{1}} &= \left\{\sin\theta_s + \frac{1}{2}\beta_3(\cos\theta_s\cos\phi_s\beta_1 + \cos\theta_s\sin\phi_s\beta_2 + \sin\theta_s\beta_3) - U\sin\theta_s\right\}
\end{aligned}
$$

$$
\begin{aligned}
\underset{bs}{\lambda}{}^{t}_{\hat{2}} &= \{(-\sin\phi_s\beta_1 + \cos\phi_s\beta_2) \\
&\quad + \frac{1}{2}\beta^2(-\sin\phi_s\beta_1 + \cos\phi_s\beta_2) \\
&\quad + (-\sin\phi_s h_{01} + \cos\phi_s h_{02}) + 3U(-\sin\phi_s\beta_1 + \cos\phi_s\beta_2)\} \\
\underset{bs}{\lambda}{}^{x}_{\hat{2}} &= \left\{-\sin\phi_s + \frac{1}{2}\beta_1(-\sin\phi_s\underline{1} + \cos\phi_s\beta_2) + U\sin\phi_s\right\} \\
\underset{bs}{\lambda}{}^{y}_{\hat{2}} &= \left\{\cos\phi_s + \frac{1}{2}\beta_2(-\sin\phi_s\beta_1 + \cos\phi_s\beta_2) - U\cos\phi_s\right\} \\
\underset{bs}{\lambda}{}^{z}_{\hat{2}} &= \frac{1}{2}\beta_3(-\sin\phi_s\beta_1 + \cos\phi_s\beta_2)
\end{aligned}
$$

$$
\begin{aligned}
\underset{bs}{\lambda}{}^{t}_{\hat{3}} &= \{(-\cos\phi_s\sin\theta_s\beta_1 - \sin\phi_s\sin\theta_s\beta_2 + \cos\theta_s\beta_3) \\
&\quad + 3U(\cos\phi_s\sin\theta_s\beta_1 - \sin\phi_s\sin\theta_s\beta_2 + \cos\theta_s\beta_3) \\
&\quad + (-\cos\phi_s\sin\theta_s h_{01} - \sin\phi_s\sin\theta_s h_{02} + \cos\theta_s h_{03}) \\
&\quad + \frac{1}{2}\beta^2(-\cos\phi_s\sin\theta_s\beta_1 - \sin\phi_s\sin\theta_s\beta_2 + \cos\theta_s\beta_3)\} \\
\underset{bs}{\lambda}{}^{x}_{\hat{3}} &= \left\{-\cos\phi_s\sin\theta_s + \frac{1}{2}\beta_1(-\cos\phi_s\sin\theta_s\beta_1 - \sin\phi_s\sin\theta_s\beta_2 + \cos\theta_s\beta_3) + U\cos\phi_s\sin\theta_s\right\} \\
\underset{bs}{\lambda}{}^{y}_{\hat{3}} &= \left\{-\sin\phi_s\sin\theta_s + \frac{1}{2}\beta_2(-\cos\phi_s\sin\theta_s\beta_1 - \sin\phi_s\sin\theta_s\beta_2 + \cos\theta_s\beta_3) + U\sin\phi_s\sin\theta_s\right\}
\end{aligned}
$$

$$\underset{bs}{\lambda}{}^{z}_{\hat{3}} = \left\{ \cos\theta_s + \frac{1}{2}\beta_3(-\cos\phi_s\sin\theta_s\beta_1 - \sin\phi_s\sin\theta_s\beta_2 + \cos\theta_s\beta_3) - U\cos\theta_s \right\} \tag{6.2}$$

Appendix B

Here we present the GAIA's attitude triad $\boldsymbol{E}_{\hat{a}}$ given by (4.17), with coordinate components expressed in termes of those of the boosted triad $\underset{bs}{\boldsymbol{\lambda}}{}_{\hat{a}}$:

$$\begin{aligned}
E^t_{\hat{1}} &= \cos\alpha \underset{bs}{\lambda}{}^{t}_{\hat{1}} - \sin\alpha\sin(\omega_p t)\underset{bs}{\lambda}{}^{t}_{\hat{2}} + \sin\alpha\cos(\omega_p t)\underset{bs}{\lambda}{}^{t}_{\hat{3}}\\
E^x_{\hat{1}} &= \cos\alpha \underset{bs}{\lambda}{}^{x}_{\hat{1}} - \sin\alpha\sin(\omega_p t)\underset{bs}{\lambda}{}^{x}_{\hat{2}} + \sin\alpha\cos(\omega_p t)\underset{bs}{\lambda}{}^{x}_{\hat{3}}\\
E^y_{\hat{1}} &= \cos\alpha \underset{bs}{\lambda}{}^{y}_{\hat{1}} - \sin\alpha\sin(\omega_p t)\underset{bs}{\lambda}{}^{y}_{\hat{2}} + \sin\alpha\cos(\omega_p t)\underset{bs}{\lambda}{}^{y}_{\hat{3}}\\
E^z_{\hat{1}} &= \cos\alpha \underset{bs}{\lambda}{}^{z}_{\hat{1}} - \sin\alpha\sin(\omega_p t)\underset{bs}{\lambda}{}^{z}_{\hat{2}} + \sin\alpha\cos(\omega_p t)\underset{bs}{\lambda}{}^{z}_{\hat{3}}\\[1ex]
E^t_{\hat{2}} &= -\sin\alpha\sin(\omega_r t)\underset{bs}{\lambda}{}^{t}_{\hat{1}} + (\cos(\omega_r t)\cos(\omega_p t) - \sin(\omega_r t)\sin(\omega_p t)\cos\alpha)\underset{bs}{\lambda}{}^{t}_{\hat{2}}\\
&\quad + (\cos(\omega_r t)\sin(\omega_p t) + \sin(\omega_r t)\cos(\omega_p t)\cos\alpha)\underset{bs}{\lambda}{}^{t}_{\hat{3}}\\
E^x_{\hat{2}} &= -\sin\alpha\sin(\omega_r t)\underset{bs}{\lambda}{}^{x}_{\hat{1}} + (\cos(\omega_r t)\cos(\omega_p t) - \sin(\omega_r t)\sin(\omega_p t)\cos\alpha)\underset{bs}{\lambda}{}^{x}_{\hat{2}}\\
&\quad + (\cos(\omega_r t)\sin(\omega_p t) + \sin(\omega_r t)\cos(\omega_p t)\cos\alpha)\underset{bs}{\lambda}{}^{x}_{\hat{3}}\\
E^y_{\hat{2}} &= -\sin\alpha\sin(\omega_r t)\underset{bs}{\lambda}{}^{y}_{\hat{1}} + (\cos(\omega_r t)\cos(\omega_p t) - \sin(\omega_r t)\sin(\omega_p t)\cos\alpha)\underset{bs}{\lambda}{}^{y}_{\hat{2}}\\
&\quad + (\cos(\omega_r t)\sin(\omega_p t) + \sin(\omega_r t)\cos(\omega_p t)\cos\alpha)\underset{bs}{\lambda}{}^{y}_{\hat{3}}\\
E^z_{\hat{2}} &= -\sin\alpha\sin(\omega_r t)\underset{bs}{\lambda}{}^{z}_{\hat{1}} + (\cos(\omega_r t)\cos(\omega_p t) - \sin(\omega_r t)\sin(\omega_p t)\cos\alpha)\underset{bs}{\lambda}{}^{z}_{\hat{2}}\\
&\quad + (\cos(\omega_r t)\sin(\omega_p t) + \sin(\omega_r t)\cos(\omega_p t)\cos\alpha)\underset{bs}{\lambda}{}^{z}_{\hat{3}}\\[1ex]
E^t_{\hat{3}} &= -\sin\alpha\cos(\omega_r t)\underset{bs}{\lambda}{}^{t}_{\hat{1}} - (\sin(\omega_r t)\cos(\omega_p t) + \cos(\omega_r t)\sin(\omega_p t)\cos\alpha)\underset{bs}{\lambda}{}^{t}_{\hat{2}}\\
&\quad + (-\sin(\omega_r t)\sin(\omega_p t) + \cos(\omega_r t)\cos(\omega_p t)\cos\alpha)\underset{bs}{\lambda}{}^{t}_{\hat{3}}\\
E^x_{\hat{3}} &= -\sin\alpha\cos(\omega_r t)\underset{bs}{\lambda}{}^{x}_{\hat{1}} - (\sin(\omega_r t)\cos(\omega_p t) + \cos(\omega_r t)\sin(\omega_p t)\cos\alpha)\underset{bs}{\lambda}{}^{x}_{\hat{2}}\\
&\quad + (-\sin(\omega_r t)\sin(\omega_p t) + \cos(\omega_r t)\cos(\omega_p t)\cos\alpha)\underset{bs}{\lambda}{}^{x}_{\hat{3}}\\
E^y_{\hat{3}} &= -\sin\alpha\cos(\omega_r t)\underset{bs}{\lambda}{}^{y}_{\hat{1}} - (\sin(\omega_r t)\cos(\omega_p t) + \cos(\omega_r t)\sin(\omega_p t)\cos\alpha)\underset{bs}{\lambda}{}^{y}_{\hat{2}}\\
&\quad + (-\sin(\omega_r t)\sin(\omega_p t) + \cos(\omega_r t)\cos(\omega_p t)\cos\alpha)\underset{bs}{\lambda}{}^{y}_{\hat{3}}\\
E^z_{\hat{3}} &= -\sin\alpha\cos(\omega_r t)\underset{bs}{\lambda}{}^{z}_{\hat{1}} - (\sin(\omega_r t)\cos(\omega_p t) + \cos(\omega_r t)\sin(\omega_p t)\cos\alpha)\underset{bs}{\lambda}{}^{z}_{\hat{2}}\\
&\quad + (-\sin(\omega_r t)\sin(\omega_p t) + \cos(\omega_r t)\cos(\omega_p t)\cos\alpha)\underset{bs}{\lambda}{}^{z}_{\hat{3}}
\end{aligned} \tag{6.3}$$

Acknowledgments

This work has been partially supported by the Italian Space Agency (ASI) under contracts ASI I/R/117/01, and by the Italian Ministry for Research (MIUR) through the COFIN 2001 program. We wish to thank Mario Lattanzi for his helpful comments.

References

[1] de Felice F and Clark C J S 1990 *Relativity on Curved Manifolds* Cambridge University Press
[2] Bini D and de Felice F 2003 *Class. Quantum Grav.* **20** 2251
[3] de Boer K S Gilmore G Hoeg E Lattanzi M G Lindegren L Luri X Mignard F and de Zeeuw P T 2000 *Composition, Formation, and Evolution of the Galaxy. Concept and Technology Study Report* **ESA-SCI (2000)4**
[4] Mignard F 2002 *Considerations on the orbit of GAIA for simulations* **GAIA-FM-011**
[5] Lindegren L 1998 *The scanning law for GAIA* **SAG-LL-014**
[6] Lindegren L 2000 *Attiude parameterization for GAIA* **SAG-LL-30**
[7] Lindegren L 2001 *Calculating the GAIA nominal scanning law* **SAG-LL-35**
[8] Mignard F 2001 *A pratical scanning law for the GAIA simulations* **GAIA-FM-010**
[9] IAU: 2000 *Definition of Barycentric Celestial Reference System and Geocentric Celestial Reference System.* IAU Resolution B1.3 adopted at the 24th general Assembly, Manchester, August 2000
[10] Soffel M Klioner S A Petit G Wolf P Kopeikin S M Bretagnon P Brumberg V A Capitaine N Damour T Fukushima T Guinot B Huong T Lindegreen L Ma C Nordtwedt K Ries J Seidelmann P K Vokrouhlicky D Will C and Xu Ch 2003 *The IAU 2000 resolutions for astrometry, celestial mechanics and metrology in the relativistic framework: explanatory supplement* astro-ph/0303376
[11] Jantzen R T Carini P and Bini D 1992 *Annals of Physics* **215** 1
[12] de Felice F Crosta M T Vecchiato A Bucciarelli B and Lattanzi M 2003 *A General Relativistic Model of Light Propagation in the Gravitational Field of the Solar System: the Static Case* (submitted)
[13] de Felice F Crosta M T Vecchiato A Bucciarelli B and Lattanzi M 2003 *General Relativistic Satellite Astrometry. A many-body model for micro-arcsecond astrometry* (in preparation)
[14] Kopeikin S M and Mashhoon B 2002 *Phys. Rev.* **D 65** 064025
[15] Kopeikin S M 2002 Proceedings of the XXIII Spanish Relativity Meeting, held 6-9 September, 2000 in Valladolid, Spain. Edited by J. F. Pascual-Snchez, L. Flora, A. San Miguel, and F. Vicente. Singapore: World Scientific, p.79
[16] Kopeikin S Gwinn C 2000 Proceedings of IAU Colloquium 180 Edited by Kenneth J Johnston, Dennis D McCarthy, Brian J Luzum and George H Kaplan p. 303
[17] Klioner S 2003 *Astron. J.* **125** 1580