43rd IEEE Conference on Decision and Control
December 14-17, 2004
Atlantis, Paradise Island, Bahamas

State Observation for systems with linear state dynamics and polynomial output

Domenico Di Martino Alfredo Germani Costanzo Manes Pasquale Palumbo

Abstract— **This paper investigates the problem of asymptotic state reconstruction for a class of continuous-time systems characterized by linear input-state dynamics and polynomial state-output function. It is shown that the dynamics of systems in this class can be embedded into the dynamics of systems of higher dimension, with time-varying linear state dynamics and linear state-output map. An asymptotic state observer for the original system is presented, whose design is based on the equations of the extended system. The observer gain is computed on-line by solving a Riccati differential equation. The interest in this observer is in its capability of state reconstruction also in cases in which the original system is not drift-observable (observable for zero input) nor uniformly observable (observable for any input).**

Index Terms— **nonlinear systems, state observers, Riccati equation, Kronecker algebra.**

I. INTRODUCTION

Linear time-invariant (LTI) descriptions of dynamical systems are widely used in control and identification theory [8], even though no real-life system can be considered exactly LTI. Nevertheless, LTI models are of enormous value in all of engineering fields, since LTI models may be good approximations of real systems and have proved to be very useful in control and/or identification algorithms design. One step toward reality is to consider dynamic systems described by linear input-state equations and nonlinear state-output functions. This paper deals with the state observation problem for such systems, in the particular case of output functions that are polynomials of the state vector. This class of systems is of particular interest in applications. Consider, as an example, an electrical network in which the powers at some terminals are measured. Such outputs are quadratic functions of the state of the network, typically made of currents and voltages. In electromechanical systems the torque is the product of magnetic fluxes and currents, and the counterelectromotive force is proportional to the product of the magnetic flux and rotor velocity. Note also that any smooth nonlinear output function can be approximated through polynomials.

This work is supported by *MIUR* (Italian Ministry for Education and Research), project 2003090328–003, by *CNR* (Italian National Research Council) and by *LNGS-INFN* (Italian National Institute of Nuclear Physics).

D. Di Martino is with Istituto Nazionale di Fisica Nucleare (LNGS-INFN), 67010 Assergi (AQ), Italy. E-mail: domdimar@lngs.infn.it

A. Germani and C. Manes are with the Department of Electrical Engineering, University of L'Aquila, Monteluco di Roio, 67040 L'Aquila, Italy. E-mail: germani@ing.univaq.it, manes@ing.univaq.it

P. Palumbo is with Istituto di Analisi dei Sistemi ed Informatica del CNR "A. Ruberti", Viale Manzoni 30, 00185 Roma, Italy. E-mail: palumbo@iasi.rm.cnr.it

The observability analysis of nonlinear systems can be made through the so called *drift-observability map* (see [4]): theoretical invertibility of such map guarantees the possibility of state reconstruction for systems with full relative degree or with zero input. However, drift-obsevability does not provide a complete observabilty analysis of a nonlinear system in the case of general relative degree, when the input is not identically zero. Differently from what happens for linear systems, it is well known that in nonlinear systems the forcing input plays an important role in the state observability. The observability for any input (*uniform observability* in [3]) is a strong property that in most applications is not satisfied.

The asymptotic state observer presented in this paper does not assume the drift-observability of the system, nor the uniform observability. This observer can be constructed in all cases in which the system, together with the applied input, satisfies an observability condition based on the observability Gramian of a suitably defined extended system.

The paper is organized as follows: section II introduces the class of systems considered and the concept of drift-obsevability. Also the formalism of Kronecker products and powers is introduced for the system description. In section III an extended state space and an extended system are defined. In section IV the observation algorithm is presented, whose construction is based on the extended system. Simulations results are reported in section V. Conclusions follow. Some results concerning the Kronecker algebra are reported in Appendix.

II. SYSTEMS WITH LINEAR INPUT-STATE DYNAMICS AND POLYNOMIAL OUTPUT

The class of systems considered in this paper is described by the following state-space representation:

$$\begin{aligned} \dot{x}(t) &= f\big(x(t)\big) + Bu(t), \\ y(t) &= h\big(x(t)\big), \end{aligned} \qquad t \geq 0, \quad x(0) = x_0, \qquad (1)$$

where $x(t) \in I\!R^n$ is the system state, $y(t) \in I\!R^q$ is the measured output, $u(t) \in I\!R^p$ is a known input, $f(x) = Ax$, with $A \in I\!R^{n \times n}$, is the (linear) drift term of the system dynamics, $B \in I\!R^{n \times p}$ is the state-input link, and $h(x)$ is the state-output function, whose components are polynomials of the state.

In [4] it is shown that if the *drift-observability property* is satisfied (i.e. observability for $u(t) \equiv 0$), then the system turns out to be observable for a class of *suitably* bounded inputs (the bound depends on some Lipschitz constants of the system equations). If the observation relative degree of the system is *full*, then the drift-observability property allows asymptotic state reconstruction for *any* bounded input.

The observer described in [4] is based on the construction of a drift-observability map, a square function that, in the case of zero input, provides the vector output and a number of its derivatives as a function of the systems-state. In the simple case of scalar output the drift-obervability map provides the output and its derivatives up to the $n-1$ order, and is given by

$$\Phi(x) = \begin{pmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{n-1} h(x) \end{pmatrix}, \qquad (2)$$

where $L_f^k h(x)$ is the k-th repeated Lie derivative of the function $h(x)$ along the field $f(x)$. Global/local drift-observability coincides with the global/local invertibility of the drift-observability map. The observer in [4] can be implemented only if the Jacobian $d\Phi/dx$ is nonsingular and the convergence is ensured inside the region of invertibility of $\Phi(x)$.

It is important to stress that, differently from what happens to linear systems, the observability property for nonlinear systems may depend on the input applied: a *favorable input* may allow the state reconstruction for systems that are not drift-observable, while a *bad input* may forbid state reconstruction for systems that are drift-observable.

The approach described in this paper allows the construction of an observer for the class of systems of the type (1) that works well when the input is *favorable*.

Before discussing the observer construction, it is useful to describe how polynomial functions of vectors can be conveniently written using linear combinations of Kronecker powers. Recall that the Kronecker product of two matrices M and N of dimensions $r \times s$ and $p \times q$ respectively, is the $(r \cdot p) \times (s \cdot q)$ matrix

$$M \otimes N = \begin{bmatrix} m_{11}N & \dots & m_{1s}N \\ \vdots & \ddots & \vdots \\ m_{r1}N & \dots & m_{rs}N \end{bmatrix}, \qquad (3)$$

where the m_{ij} are the entries of M. The Kronecker power of a matrix M is recursively defined as

$$M^{[0]} = 1, \quad M^{[i]} = M \otimes M^{[i-1]}, \quad i \geq 1. \qquad (4)$$

Note that if $M \in I\!R^{a \times b}$, then $M^{[i]} \in I\!R^{a^i \times b^i}$. See the Appendix in [5] for a quick survey on the Kronecker algebra. The properties of the Kronecker product used throughout the paper are the following:

$$\begin{aligned} (A+B)\otimes(C+D) =& A\otimes C + A\otimes D + \\ & B\otimes C + B\otimes D \qquad (5) \\ A\otimes(B\otimes C) =& (A\otimes B)\otimes C \qquad (6) \\ (A\cdot C)\otimes(B\cdot D) =& (A\otimes B)\cdot(C\otimes D) \qquad (7) \end{aligned}$$

In particular, from property (7) it follows

$$(Ax)^{[i]} = A^{[i]}x^{[i]}, \qquad (8)$$

intensively used throughout the paper. See [9] for more properties.

A q components polynomial of degree not greater than m of a vector $x \in I\!R^n$ can be written as

$$\sum_{i=0}^{m} D_i v^i, \qquad (9)$$

where $D_i \in I\!R^{q \times n^i}$ are the coefficients of the polynomial.

Systems of the class (1) can be written in the following form:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= \sum_{i=1}^{m} C_i x^{[i]}(t), \end{aligned} \qquad t \geq 0, \quad x(0) = x_0. \qquad (10)$$

where $C_i \in I\!R^{q \times n^i}$, $i = 1, \dots, m$ are the coefficients of the output polynomial. Note that, without loss of generality, the constant term of the output polynomial is not considered.

III. THE EXTENDED SYSTEM

In this section it is shown that the state-output dynamics of the nonlinear stationary system (10) obeys linear time-varying equations in the state-space form if an extended state $X_m(t)$ is suitably defined as follows

$$X_m(t) = \begin{pmatrix} x(t) \\ x^{[2]}(t) \\ \vdots \\ x^{[m]}(t) \end{pmatrix} \in I\!R^{d(n,m)}, \qquad (11)$$

where $d(n,m) = \sum_{i=1}^{m} n^i$. With this definition the output $y(t)$ can be written as a linear function of the extended state $X_m(t)$

$$\begin{aligned} y(t) &= \mathcal{C}X_m(t), \\ \text{where } \mathcal{C} &= \begin{bmatrix} C_1 & C_2 & \cdots & C_m \end{bmatrix} \end{aligned} \qquad (12)$$

It is interesting to show that the dynamics of the extended state obeys the equation described by the following lemma:

Lemma III.1. *The dynamics of the extended state* $X_m(t)$ *defined by* (11) *is given by:*

$$\dot{X}_m(t) = \mathcal{A}\big(u(t)\big)X_m(t) + \mathcal{B}u(t), \qquad (13)$$

with matrix $\mathcal{A}(u)$ *is defined as*

$$\begin{bmatrix} \mathcal{A}_{1,1} & O & \cdots & 0 & 0 \\ \mathcal{A}_{2,1}(u) & \mathcal{A}_{2,2} & \cdots & 0 & 0 \\ O & \mathcal{A}_{3,2}(u) & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{A}_{m,m-1}(u) & \mathcal{A}_{m,m} \end{bmatrix}, \qquad (14)$$

where matrices $\mathcal{A}_{i,i},\ i = 1,\dots,m$ and $\mathcal{A}_{i,i-1}(u),\ 2 \leq i \leq m$ are recursively defined by

$$\begin{aligned} \mathcal{A}_{i,i} &= A \otimes I_{n^{i-1}} + I_n \otimes \mathcal{A}_{i-1,i-1}, \\ \mathcal{A}_{1,1} &= A, \\ \mathcal{A}_{i,i-1}(u) &= (Bu) \otimes I_{n^{i-1}} + I_n \otimes \mathcal{A}_{i-1,i-2}(u), \\ \mathcal{A}_{2,1}(u) &= Bu, \end{aligned} \tag{15}$$

I_{n^k} is the identity matrix of dimension n^k and

$$\mathcal{B} = \begin{bmatrix} B \\ O \\ \vdots \\ O \end{bmatrix}. \tag{16}$$

Proof: The state dynamics (13) is equivalent to the following equations

$$\begin{aligned} \dot{x}(t) &= \mathcal{A}_{1,1}x(t) + Bu(t), \\ \frac{d}{dt}x^{[i]}(t) &= \mathcal{A}_{i,i}x^{[i]}(t) + \mathcal{A}_{i,i-1}\big(u(t)\big)x^{[i-1]}(t), \\ i &= 2,\dots,m. \end{aligned} \tag{17}$$

The first of (17) is readily proved by observing that, by definition, $\mathcal{A}_{1,1} = A$. The second equation, for $i = 2$, is proved with the following passages, by using the Kronecker product properties:

$$\begin{aligned} \frac{d}{dt}x^{[2]}(t) &= x \otimes \dot{x} + \dot{x} \otimes x \\ &= x \otimes (Ax + Bu) + (Ax + Bu) \otimes x \\ &= (A \otimes I_n + I_n \otimes A)\, x^{[2]} \\ &\quad + \big((Bu) \otimes I_n + I_n \otimes (Bu)\big)\, x \end{aligned} \tag{18}$$

By definitions (15)

$$\begin{aligned} \mathcal{A}_{2,2} &= A \otimes I_n + I_n \otimes A \\ \mathcal{A}_{2,1}(u) &= (Bu) \otimes I_n + I_n \otimes (Bu) \end{aligned} \tag{19}$$

so that the second of (17) is proved for $i = 2$.

Now, proceed by induction: assume that (17) is true for a given $i \geq 2$ and prove that it is also true for $i+1$. Indeed

$$\begin{aligned} \frac{d}{dt}x^{[i+1]}(t) &= x \otimes \frac{d}{dt}x^{[i]} + \dot{x} \otimes x^{[i]} \\ &= x \otimes \big(\mathcal{A}_{i,i}x^{[i]} + \mathcal{A}_{i,i-1}(u)x^{[i-1]}\big) + (Ax + Bu) \otimes x^{[i]} \\ &= \big(A \otimes I_{n^i} + I_n \otimes \mathcal{A}_{i,i}\big)\, x^{[i+1]} + \\ &\quad \big((Bu) \otimes I_{n^i} + I_n \otimes \mathcal{A}_{i,i-1}(u)\big)x^{[i]}. \end{aligned} \tag{20}$$

From definitions (15) it follows

$$\begin{aligned} \mathcal{A}_{i+1,i+1} &= A \otimes I_{n^i} + I_n \otimes \mathcal{A}_{i,i} \\ \mathcal{A}_{i+1,i}(u) &= (Bu) \otimes I_{n^i} + I_n \otimes \mathcal{A}_{i,i-1}(u) \end{aligned} \tag{21}$$

so that

$$\frac{d}{dt}x^{[i+1]}(t) = \mathcal{A}_{i+1,i+1}x^{[i+1]}(t) + \mathcal{A}_{i+1,i}\big(u(t)\big)x^{[i]}(t), \tag{22}$$

and the induction is proved. ∎

Remark III.2. Note that the extended state matrix (14) is time-varying, due to its dependence on the known input $u(t)$.

Since redundant terms are present in the Kronecker powers, redundant state components are present in the extended state vector X_m, so that the extended state space results to be output-indistinguishable. Such redundancy can be eliminated by considering suitably defined reduction matrices. First of all note that $x^{[i]}$, the i-th Kronecker power of $x \in \mathbb{R}^n$, has n^i components, but only $\binom{n+i-1}{i}$ are distinct terms (the number of ways to choose i elements from a set of n, with repetitions allowed). Defining the following functions of pairs of integers

$$d(n,m) = n\frac{1-n^m}{1-n} = \sum_{i=1}^{m} n^i, \tag{23}$$

$$c(n,m) = \binom{n+m}{m} - 1 = \sum_{i=1}^{m} \binom{n+i-1}{i}, \tag{24}$$

it is easy to see that the vector X_m has $d(n,m)$ components, but only $c(n,m)$ are distinct (obviously $c(n,m) < d(n,m)$).

A block-diagonal reduction matrix $\overline{T}_{n,m} \in \mathbb{R}^{c(n,m)\times d(n,m)}$ can be suitably defined, as described in detail in [6], for the selection of a nonredundant subvector $\overline{X}_m \in \mathbb{R}^{c(n,m)}$ from $X_m \in \mathbb{R}^{d(n,m)}$. A block-diagonal matrix $T_{n,m} \in \mathbb{R}^{d(n,m)\times c(n,m)}$ allows to reconstruct the redundant vector X_m from $\overline{X}_m$. In formulas

$$\overline{X}_m(k) = \overline{T}_{n,m}X_m, \quad X_m = T_{n,m}\overline{X}_m(k). \tag{25}$$

Using Lemma III.1 and the reduction matrices (25), system (10) can be embedded in the following system of larger dimension

$$\begin{aligned} \dot{\mathcal{X}}(t) &= \bar{\mathcal{A}}(u)\big(u(t)\big)\mathcal{X}(t) + \overline{\mathcal{B}}u(t) \\ y(t) &= \bar{\mathcal{C}}\mathcal{X}(t), \end{aligned} \tag{26}$$

where $\mathcal{X}(t) \in \mathbb{R}^{c(n,m)}$ and

$$\begin{aligned} \bar{\mathcal{A}}(u) &= T_{n,m}\mathcal{A}(u)\overline{T}_{n,m}, \qquad \overline{\mathcal{B}} = T_{n,m}\mathcal{B}, \\ \bar{\mathcal{C}} &= \mathcal{C}\overline{T}_{n,m}. \end{aligned} \tag{27}$$

The embedding of the original system (10) into the extended system (26) should be intended as follows: if the initial state value of the extended state of (26) is set to

$$\mathcal{X}(0) = T_{n,m}\begin{pmatrix} x(0) \\ x^{[2]}(0) \\ \vdots \\ x^{[m]}(0) \end{pmatrix} = T_{n,m}X_m(0), \tag{28}$$

then the outputs of the two systems is the same for any input, and the state $x(t)$ of the original system (10) is recovered by selecting the first n components of the extended state $\mathcal{X}(t)$:

$$\begin{aligned} x(t) &= \Sigma\mathcal{X}(t), \\ \text{where } \Sigma &= \big[\, I_n \ 0_{n\times(n^2+\cdots n^m)} \,\big]. \end{aligned} \tag{29}$$

IV. THE ASYMPTOTIC STATE OBSERVER

In this section an asymptotic observer for nonlinear systems of the type (10) is presented and a sufficient condition for the asymtotic convergence to zero of the state observation error is given. The observer is constructed on the basis of the extended system (26), that has the simpler structure of a time-varying linear system.

Theorem IV.1. *Consider the system* (10) *and the extended system* (26)*. Assume that the pair* $(\bar{\mathcal{A}}(u), \bar{\mathcal{C}})$ *and the input function* $u(t)$ *are such that there exist positive scalars* α, β, δ*, with* $\alpha < \beta$*, such that for all* $t \geq 0$

$$\alpha I_{c(n,m)} \leq \int_t^{t+\delta} e^{\bar{\mathcal{A}}_u^T(\tau)} \bar{\mathcal{C}}^T \bar{\mathcal{C}} e^{\bar{\mathcal{A}}_u(\tau)} d\tau \leq \beta I_{c(n,m)}, \quad (30)$$

where $\bar{\mathcal{A}}_u(t)$ *denotes the matrix* $\bar{\mathcal{A}}(u(\tau))$.

Then, for any $\widehat{\mathcal{X}}(0)$*, the system*

$$\dot{\widehat{\mathcal{X}}}(t) = \bar{\mathcal{A}}_u(t)\widehat{\mathcal{X}}(t) + \overline{\mathcal{B}}u(t) + P(t)\bar{\mathcal{C}}^T \left(y(t) - \hat{y}(t)\right), \quad (31)$$

$$\dot{P}(t) = \left(\bar{\mathcal{A}}_u(t) - P(t)\bar{\mathcal{C}}^T\bar{\mathcal{C}}\right)P(t) + P(t)\left(\bar{\mathcal{A}}_u(t) - P(t)\bar{\mathcal{C}}^T\bar{\mathcal{C}}\right)^T + Q(t), \quad (32)$$

$$\hat{x}(t) = \Sigma\widehat{\mathcal{X}}(t) \quad (33)$$

with $Q(t)$ *and* $P(0)$ *symmetric positive definite,* $Q(t) \geq q_m I_{c(n,m)}$ *for some positive* q_m*, is an asymptotic observer for the system* (10)*, i.e.*

$$\lim_{t\to\infty} \|x(t) - \hat{x}(t)\| = 0. \quad (34)$$

Proof: It is sufficient to show that equations (31)–(33) are an asymptotic observer for the extended system (26), i.e.

$$\lim_{t\to\infty} \|\mathcal{X}(t) - \widehat{\mathcal{X}}(t)\| = 0. \quad (35)$$

First, note that the assumption (30) coincides with the *uniform observability* of a linear time-varying system, and implies that $P(t)$ admits upper and lower bounds (see [1], [2]), i.e. there exist positive scalars p_m and p_M such that

$$p_m I_{c(n,m)} \leq P(t) \leq p_M I_{c(n,m)}. \quad (36)$$

Let $\varepsilon(t) = \mathcal{X}(t) - \widehat{\mathcal{X}}(t)$ be the observation error of the extended system. Subtracting equation (26) from (31), the error dynamics is obtained

$$\dot{\varepsilon}(t) = \left(\bar{\mathcal{A}}_u(t) - K(t)\bar{\mathcal{C}}\right)\varepsilon(t), \qquad t \geq 0. \quad (37)$$

Consider the following function of the observation error

$$V(\varepsilon, t) = \varepsilon^T P^{-1}(t)\varepsilon, \quad (38)$$

positive definite for all $t \geq 0$ because (36) implies

$$\frac{1}{p_M} I_{c(n,m)} \leq P^{-1}(t) \leq \frac{1}{p_m} I_{c(n,m)}. \quad (39)$$

Let $v(t) = V(\varepsilon(t), t)$. The time derivative of $v(t)$ along the error trajectories is

$$\begin{aligned}\dot{v}(t) &= \dot{\varepsilon}^T(t)P^{-1}(t)\varepsilon(t) + \varepsilon^T(t)\dot{P}^{-1}(t)\varepsilon(t) + \varepsilon^T(t)P^{-1}(t)\dot{\varepsilon}(t)\\ &= \varepsilon^T(t)\Big[\left(\bar{\mathcal{A}}_u(t) - P(t)\mathcal{C}^T\mathcal{C}\right)^T P^{-1}(t) + \dot{P}^{-1}(t) + P^{-1}(t)\left(\bar{\mathcal{A}}_u(t) - P(t)\mathcal{C}^T\mathcal{C}\right)\Big]\varepsilon(t). \quad (40)\end{aligned}$$

Recalling that $\dot{P}^{-1}(t) = -P^{-1}(t)\dot{P}(t)P^{-1}(t)$, it follows

$$\dot{v}(t) = \varepsilon^T(t)P^{-1}(t)\Big[P(t)\left(\bar{\mathcal{A}}_u(t) - P(t)\mathcal{C}^T\mathcal{C}\right)^T - \dot{P}(t) + \left(\bar{\mathcal{A}}_u(t) - P(t)\mathcal{C}^T\mathcal{C}\right)P(t)\Big].P^{-1}(t)\varepsilon(t). \quad (41)$$

From this, recalling (32),

$$\dot{v}(t) = -\varepsilon^T(t)P^{-1}(t)Q(t)P^{-1}(t)\varepsilon(t). \quad (42)$$

Since, by (39), it is

$$\dot{v}(t) \leq -\frac{1}{p_M^2} q_m \|\varepsilon(t)\|^2, \quad (43)$$

$$v(t) \geq \frac{1}{p_M}\|\varepsilon(t)\|^2. \quad (44)$$

It follows

$$\dot{v}(t) \leq -\frac{q_m}{p_M} v(t), \quad (45)$$

from which

$$v(t) \leq e^{-\frac{q_m}{p_M}t} v(0) = \frac{1}{p_m} e^{-\frac{q_m}{p_M}t}\|\varepsilon(0)\|^2, \quad (46)$$

and finally

$$\|\varepsilon(t)\|^2 \leq \frac{p_M}{p_m} e^{-\frac{q_m}{p_M}t}\|\varepsilon(0)\|^2. \quad (47)$$

This proves the convergence (34), and therefore (35). ■

Remark IV.2. Note that it may be difficult to test condition (30) before the construction of the observer. Moreover, in the cases in which the input applied to the system is measured, the condition (30) can only be checked *on-line*. In practice, the observer (31)–(33) can be applied without a preliminary check of the condition (30). A positive definite initial value of $P(t)$ for $t = 0$ ensures that for small t the matrix $P(t)$ remains nonsingular and bounded. However, it is convenient to monitor the minimum and maximum eigenvalues of $P(t)$ during its evolution. The divergence of $\lambda_{\text{Max}}(P(t))$ or the approach to zero of $\lambda_{\min}(P(t))$ are caused by a loss of observability of the extended system (26), due to a *bad input*. In these cases it may be necessary to reset $P(t)$ to some well conditioned P_0, waiting for the input to become *favorable* again.

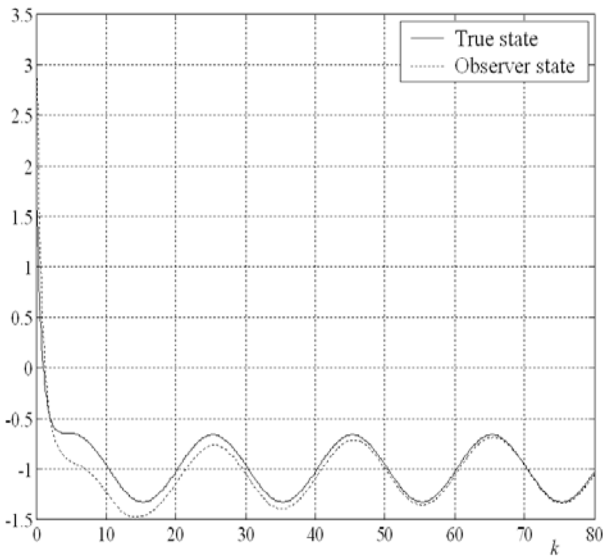

Fig. 1. **True and estimated state: the first component.**

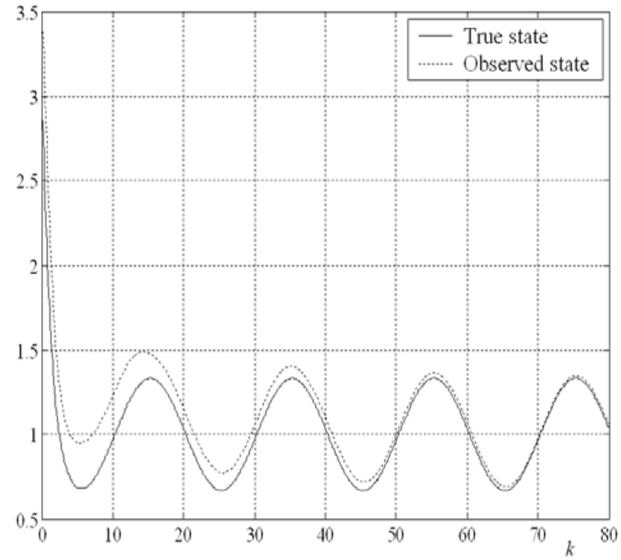

Fig. 2. **True and estimated state: the second component.**

Remark IV.3. Equation (47) provides a suggestion for the choice of matrix $Q(t)$ in equation (32): a faster convergence of the observer can be obtained by increasing q_m, the lower bound of $Q(t)$.

V. SIMULATION RESULTS

Simulations results are here reported in order to show the effectiveness of the proposed observer. Consider the following linear system with cubic output w.r.t. the state:

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x \in I\!R^3 \tag{48}$$

$$y(t) = \sum_{i=1}^{3} C_i x^{[i]}(t), \quad y \in I\!R^2. \tag{49}$$

The values chosen for the simulation are:

$$\begin{aligned} A &= \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & 1 \\ -1 & 1 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \\ C_1 &= \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \end{bmatrix} \\ C_2 &= \begin{bmatrix} 1 & 0 & 1 & 2 & -1 & 0 & 1 & -2 & 1 \\ 0 & 1 & 1 & 3 & 0 & 1 & 4 & 0 & -2 \end{bmatrix} \\ C_3 &= C_1 \otimes C_2. \end{aligned} \tag{50}$$

The linear approximation of this system around the origin (the pair (A, C_1)) is not observable, not even detectable: A has an eigenvalue in 0 whose eigenvector u_0 lies in the nullspace of C_1:

$$u_0 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} : \quad \begin{bmatrix} A \\ C_1 \end{bmatrix} u_0 = 0 \tag{51}$$

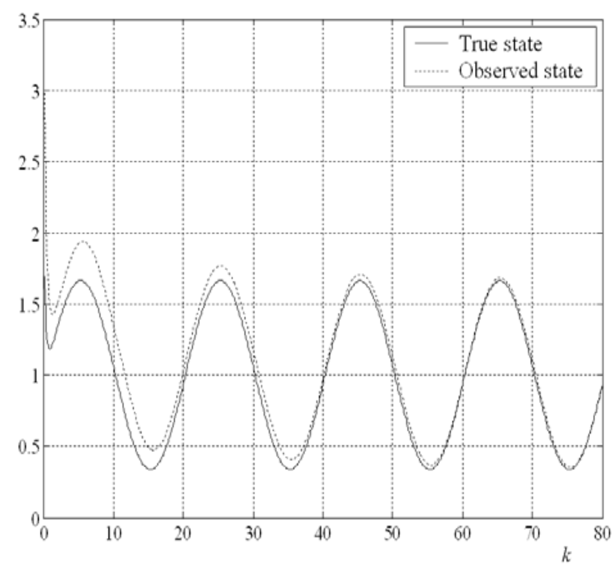

Fig. 3. **True and estimated state: the third component.**

This implies that in this example the drift observability map $\Phi(x)$, computed as defined in (2), loses rank at the origin, and therefore the considered system is not drift-observable in a neighborhood of the origin. Nevertheless, the presence of an input can allow the state reconstruction.

In the simulations performed the matrix $Q(t)$ and the initial value for the matrix $P(t)$ in the Riccati equation (32) have been chosen of the type

$$P(0) = \alpha I_{c(n,m)}, \qquad Q(t) = \beta I_{c(n,m)} \tag{52}$$

(remember that in our example $m = n = 3$, so that $c(3,3) = 29$)). The numerical simulations have shown that the convergence speed can be improved by increasing the value of the parameter β, thus confirming what is claimed in Remark (IV.3) (note that in our example $\beta = q_m$).

The simulations reported in this section are performed with a sinusoidal input

$$u(t) = \sin(\frac{2\pi}{T}t), \qquad T = 20. \tag{53}$$

Figs. 1-3 show the true and the observed state components.

VI. CONCLUSIONS

The problem of the state observation for the class of systems with linear input-state dynamics and polynomial state-output function is investigated in this paper, and an asymptotic observer is presented. It is shown how the original system can be embedded into an extended system, whose state is made of the original state and of some of its Kronecker powers. Next, an observation algorithm is presented, whose structure is derived from the extended system. The observer gain is time varying and is obtained as the solution of a differential Riccati equation. An interesting property of the proposed observer is that it can be implemented and works well also in cases in which the system is not drift-observable nor uniformly-observable, provided that the input applied to the system is *favorable* in a sense that is formalized in theorem IV.1.

The observer behavior has been numerically tested on some examples and has always given good results. The simulations here reported refer to a system whose linear approximation around the origin is not observable, not even detectable.

References

[1] R.S. Bucy, “Global theory of the Riccati equation,” *J. Comput. Syst. Sci.,* Vol. 1, pp. 349–361, 1967.

[2] A.H. Jazwinski, *Stochastic Processes and Filtering Theory.* New York: Academic, 1970.

[3] J.P. Gauthier, and G. Bornard, “Observability for any $u(t)$ of a class of nonlinear systems,” *IEEE Trans. A. C.*, Vol. 26, no. 4, pp. 922–926, 1981.

[4] M. Dalla Mora, A. Germani, C. Manes, “Design of State Observers from a Drift-Observability Property,” *IEEE Trans. A. C.*, Vol. 45, No. 6, pp. 1536–1540, August 2000.

[5] F. Carravetta, A. Germani, M. Raimondi, “Polynomial filtering of discrete-time stochastic linear systems with multiplicative state noise,” *IEEE Trans. A. C.*, Vol. 42, NO. 8, pp. 1106–1126, 1997.

[6] F. Carravetta, A. Germani, M. Raimondi, “Polynomial filtering for linear discrete-time non-Gaussian systems,” *SIAM J. Contr. Optim.*, Vol. 34, No. 5, pp. 1666–1690, 1996.

[7] C. Bruni, A. Gandolfi, A. Germani, “Observability of linear-in-the-state systems: a functional approach,” *IEEE Trans. A. C.*, Vol. AC–25, No. 3, pp. 566–567, June 1980.

[8] L. Ljung, “Linear time-invariant models of non-linear time-varying Systems,” *European Journal of Control,* Vol. 7, issue 2-3, pp. 203–219, 2001.

[9] A. Graham, *Kronecker Products and Matrix Calculus with Applications,* Ellis Horwood ltd. Publishers, Chichester, 1981.