Hamiltonian maps and normal forms for intense beams

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Abstract. The dynamics of a beam in a ring with a localized multipolar nonlinearity is described by a polynomial one turn map. The space charge forces act continuously along the ring, but their effect can be included by replacing the linear tune with the depressed tune which depends on the Courant Snyeder invariant. This approximation allows to use the normal forms to compute the nonlinear invariants, the nonlinear tune and the islands geometric parameters when a low order resonance is approached.

INTRODUCTION

The theory of normal forms for symplectic maps, introduced by Birkhoff in 1920[1], was first proposed by the Bologna group [2, 3, 4, 5] as the natural extension of Courant-Snyder theory to the nonlinear betatronic motion, and independently developed by E. Forest and others [6, 7, 8]. The resonant and quasi-resonant theory has been developed for dynamical applications [9, 10] and the asymptotic properties of the Birkhoff series has been analyzed to understand the nature of the underlying singularities [11, 12, 13] and to obtain Nekhoroshev-like stability estimates [14, 15], see [16] for an overall review. Numerical codes have been developed for the 2D [17] and 4D case [18] and libraries for the analysis of a nonlinear lattice [19] and the analysis of resonances[20]. These libraries include routines for polynomial algebra and Lie series manipulations, for which more more specific codes have been developed to this end, see [21] and [22] respectively.

Canonical perturbation methods were and still are currently used to deal with the polynomial nonlinearities (chromatic or extraction sextupoles, multipolar errors). The sextupoles or higher multipoles are usually treated in the thin lens approximation, which provides a symplectic map suitable for tracking. Canonical perturbation is based on a truncated Fourier expansion of the δ functions introduced by the thin lens approximation. The slow convergence and the low order truncation of the Fourier series render this theory quantitatively not accurate, especially in the description of resonance phenomena. The normal forms apply to the symplectic one turn map, which is evaluated by composing the polynomial maps associated to all the thin nonlinear lenses present in the ring. The normal form map, obtained by a polynomial transformation, is symplectic and is symmetric with respect to continuous (non resonant normal form) or discrete (resonant or quasi-resonant normal form) rotation group and can be written as the time one map of an interpolating Hamiltonian, which provides the invariants of the system[16].

The non-resonant normal forms allow to compute the nonlinear tune, which does not depend on the selected section of the ring, and the transformation, determining the nonlinear deformation of the beam envelope. This is the analogue of the Twiss parameters used to determine the beam profile in the linear case. When a resonance is approached, the change of topology in phase space, occurring with the appearance of a chain of islands in the 2D case, causes a divergence due to the resonant divisors. In this case the resonant normal forms must be used: the transformation and the interpolating Hamiltonian H are no longer affected by the resonant divisors, but an angular dependence is introduced in H. For a 2D map the pendulum like Hamiltonian H exhibits the chain of islands. For a 4D map this is still true for a single resonance, since a suitable combination of the actions is still invariant; for a double resonance H is the unique invariant and its level lines correspond to a 3D, rather than a 2D, manifold. At the lowest order the non-resonant normal forms allow to draw the resonance lines in the space actions (or any 2D manifold like $p_x = p_y = 0$) for any choice of the linear tunes (excluding the values corresponding to unstable resonances). In this case the tunes depend linearly on the actions. The method is applicable to the tunes of the next order normal forms, which have a quadratic dependence on the actions, or to the tunes and actions obtained numerically from the tracking data by

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Fourier analysis. The action-frequency map detects not only the presence but also the localization of a resonance, unlikely the standard frequency map analysis in the tunes space.

Birkhoff theory is based on the Taylor series representation of the initial map, the normalized map and the coordinates transformation. The homologic equation of order n is obtained from the order n polynomial of the Taylor expansion of the conjugation equation. The perturbation parameter is the distance r from the origin in phase space and corresponds to an average of the linear oscillation amplitudes in the $x,\ p_x$ and $y,\ p_y$ phase planes: $r = (\varepsilon_x + \varepsilon_y)^{1/2}$, where $\varepsilon_x = 2j_x,\ \varepsilon_y = 2j_y$ are the emittances of the linear orbits and $j_x,\ j_y$ the corresponding actions. As a consequence, Birkhoff theory is accurate for a low emittance beam, in spite of its asymptotic character due to the presence of small divisors. No arbitrariness is involved, except for a group of gauge transformations, commuting with the symmetry group. Moreover it is sufficient to truncate the one turn map to the same order of the normal form to have correct results. This simple property allows to compute the normal forms even for rings with thousands of nonlinear elements. Indeed the one term map is obtained by repeatedly composing and truncating the contributions of all the nonlinear terms. Manageable analytic expressions are obtained (by hand or algebraic manipulators) up to fourth order; higher orders can be reached using the algorithmic approach, on which our codes are based[17, 18]. The limits, imposed by storage, are never reached for any practical application since the Birkhoff series are asymptotic (order 10 is typically the highest to be used, even though order 200 and 20 can be reached for 2D and 4D map respectively). The remainder for a truncation to order $N-1$ is given by $C_N\, r^N$ and analytical estimates, based on majorant series, show that C_N has a factorial growth. For any value of r the remainder diverges as $N \to \infty$ but an exponentially small value $\exp[-(r_*/r)^{\eta}]$ is reached at the optimal truncation order N_*. Observing that the error between the exact and the normal forms evolution after n iterates is just n times the remainder, an exponential stability estimate follows. The related scaling law was actually observed in the long term dynamic aperture simulations for the Henon map and the CERN Large Hadron Collider (LHC) one turn map [23]. The normal forms were used to develop the correction schemes for the multipolar errors of the preliminary version of LHC lattice [5]; by imposing that the lowest order tune shift vanishes, the sextupolar and octupolar corrector gradients are analytically determined. Information on the strength of multipoles can be extracted from the one turn map with an alternating current (a.c.) dipole or with measures at different sections of the ring [24, 25].

The quasi resonant normal forms provide some information on the short term dynamic aperture as well when the linear tune approaches an unstable resonance. The level curve of the interpolating Hamiltonian provides a good approximation to the triangular shaped dynamic aperture for integer resonances in the 4D map [26]. For the 1/3 resonance the hyperbolic normal forms were used to describe the slow extraction process [27]. A new application is the resonant extraction based on an accurate tailoring of the beam: the desired fraction of particles has to be captured into an island and transported up to the extraction region [28, 29]. The analytic dependence of the resonance shape and position in phase space from the lattice parameters allows a clear and accurate description of the extraction mechanism.

The Birkhoff normal forms theory is applicable to impulsive polynomial nonlinearities (thin multipoles) whereas canonical Hamiltonian theory is applicable to perturbations with a smooth periodic dependence on s and on the phase space coordinates. When space charge effects are present the nonlinear Coulomb forces act continuously on a test particle, out of the core for a Kapchinsky-Vladimirsky (KV) beam or anywhere for a Gaussian beam, and the use of canonical perturbation theory is appropriate if the charge density is not too high. When both space charge and magnetic multipoles are present, the use of normal forms faces a difficulty because the corresponding nonlinear forces require a different perturbation scheme. A possible solution consists in approximating the space charge force with a polynomial. The repeated composition and truncation of space charge kicks provides the transfer map between two thin lenses and finally a polynomial one turn map is obtained. This method works for smooth space charge forces but is applicable to a limited region around the beam core. A polynomial approximation to the space charge force accurate within the core cannot be extended beyond 1.5 times the core radius. As a consequence it is not suited to explore regions up to several times the core radius, where the most interesting effects such as the halo formation are present.

This limit on the applicability of normal forms theory is removed if we observe that the one turn map is just a a rotation of the depressed phase advance composed with the multipolar kick. We assume there is only one kick in the ring and we choose linearly normalized (Courant-Snyder) coordinates. In this case the phase advance depends on the orbit: since the space charge Hamiltonian is integrable in the 2D case, the tune is a function of the linear invariant $x^2 + p^2$. For a 4D map integrability is lost because the space charge potential couples the x, p_x and y, p_y phase planes, except for a KV beam within the core. This problem is overcome by replacing the exact tune with its first approximation, with respect to the depressed tune at the origin, provided by canonical perturbation theory. This recently proposed procedure[30] provides a symplectic one turn map whose accuracy decreases with the distance from the the origin; the largest relative error occurs at infinity where it is of order $(1-\nu_0/\nu)^2$. We propose to apply the Birkhoff procedure to this symplectic map in order to obtain its normal form representation. Unlikely the standard case, where the linearly frequency is fixed, now the normalizing transformations are locally defined. Indeed the normalizing transformation and the corresponding interpolating Hamiltonian depend on the space charge depressed phase advance which is a function of the linear invariants. The plan of the presentation is the following: in section 2 we show how to compute the one turn map, in section 3 we recall the relevant properties of the normal forms and in section 4 we apply the proposed method to a 2D map in a parameters range corresponding to the CERN PS experiment [31].

THE SINGLE MULTIPOLE MODEL

We consider a ring in the smooth focusing approximation with a radial charge distribution $Q\rho(r)$, where Q is the total charge per unit length and ρ is normalized according to

$$2\pi\int_0^\infty \rho(r)\,r\,dr=1 \tag{1}$$

Denoting by $Q\mathbf{E}(r)$ and $QV(r)$ the electric field and its potential, where $\mathbf{E}(r)=E(r)\,\mathbf{r}/r$ and $V(r)=-\int_0^r E(r')\,dr'$, from Gauss theorem we have

$$E(r)=\frac{2\mu(r)}{r} \qquad\qquad \mu(r)=2\pi\int_0^r \rho(r')\,r'\,dr' \tag{2}$$

Assuming that $\rho(0)>0$, we define the core radius and introduce an auxiliary function $g(r)$ according to

$$r_c^2=\frac{1}{\pi\,\rho(0)} \qquad\qquad \rho(r)=\rho(0)\,g\left(\frac{r^2}{r_c^2}\right) \tag{3}$$

where $g(0)=1$ and $\int_0^\infty g(t)dt=1$ from (1). The mean square radius is $\langle r^2\rangle=r_c^2\int_0^\infty t\,g(t)\,dt$. It is convenient to introduce two additional auxiliary functions

$$g_1(t)=\int_0^t g(u)\,du \qquad\qquad g_2(t)=\int_0^t \frac{g_1(u)}{u}\,du \tag{4}$$

such that $g_1(t)=t+O(t^2)$, $g_2(t)=t+O(t^2)$ for $t\to 0$ and $g_1(t)\simeq 1$, $g_2(t)\simeq \log t$ for $t\to\infty$. The potential becomes $V(r)=-g_2(r^2/r_c^2)$. The equations of the orbits in the smooth focusing approximation are

$$\frac{d^2x_i}{ds^2}=-\frac{\omega_0^2}{2}x_i-\frac{\partial V_m}{\partial x_i}\sum_l \delta(s-l)+\frac{\xi}{r}g_1\left(\frac{r^2}{r_c^2}\right)\frac{x_i}{r} \tag{5}$$

where $V_m(x,y)$ denotes the potential of the $2m$-pole and is a homogeneous polynomial of degree m. The linear force close to the origin is $-\omega^2 x_i$, where ω is the depressed phase advance defined by

$$\omega^2=\omega_0^2-\frac{\xi}{r_c^2} \tag{6}$$

As a consequence the Hamiltonian of the system reads

$$H=\frac{p_x^2+p_y^2}{2}+\omega^2\frac{r^2}{2}+\frac{\xi}{2}\left(\frac{r^2}{r_c^2}-g_2\left(\frac{r^2}{r_c^2}\right)\right)+V_m\sum_l \delta(s-l) \tag{7}$$

The sum of the focusing plus space charge potential behaves as $\frac{1}{2}\,\omega^2\,r^2$ for $r\to 0$ and $\frac{1}{2}\,\omega_0^2\,r^2$ up to a logarithmic correction as $r\to\infty$. After a change to the linearly normalized coordinates $x_i'=\omega^{1/2}x_i$ and $p_i'=\omega^{-1/2}p_i$ the Hamiltonian becomes

$$H=\omega\,\frac{p_x'^2+p_y'^2+x'^2+y'^2}{2}+\frac{\xi}{2}\left(\frac{r'^2}{r_c'^2}-g_2\left(\frac{r'^2}{r_c'^2}\right)\right) \tag{8}$$

$$+\,\omega^{-m/2}V_m(x',y')\sum_l \delta(s-l)$$

Considering the motion in the invariant x,p_x plane, $2m$-pole potential reads $V_m(x)=-x^m\,k_{m-1}/m!$ and after the scaling $x'=\lambda X$, $p_x'=\lambda P_x$, $r_c'=\lambda R_c$, where the scaling factor is defined by $\lambda=[\omega\,(m-1)!/k_{m-1}]^{\frac{1}{m-2}}\,\omega^{\frac{1}{2}}$ and letting $\tilde{\xi}=\lambda^{-2}\xi$, the Hamiltonian $\tilde{H}=\lambda^{-2}H$ reads

$$\tilde{H}=\omega\frac{P_x^2+X^2}{2}+\frac{\tilde{\xi}}{2}\left(\frac{X^2}{R_c^2}-g_2\left(\frac{X^2}{R_c^2}\right)\right)-\frac{X^m}{m},\sum_l \delta(s-l) \tag{9}$$

Introducing the action-angle variables $X=(2J)^{1/2}\cos\theta$, $P_x=(-2J)^{1/2}\sin\theta$ the space charge Hamiltonian has an angular dependence, which disappears after averaging. As a consequence the space charge depressed phase advance reads

$$\omega_{\rm sc}=\omega+\frac{\omega_0^2-\omega^2}{2\omega}\left[1-\frac{R_c^2}{J}\frac{1}{2\pi}\int_0^{2\pi} g_1\left(\frac{2J\cos^2\theta}{R_c^2}\right)d\theta\right] \tag{10}$$

where we have used $\tilde{\xi}/R_c^2=\xi/{r_c'}^2=(\omega_0^2-\omega^2)/\omega$ and $\omega_{\rm sc}=\frac{\partial}{\partial J}\langle\tilde{H}(\Theta,J)\rangle$. We notice that $\omega_{\rm sc}=\omega+O(J)$ as $J\to 0$ and $\omega_{\rm sc}=\omega+(\omega_0^2-\omega^2)/(2\omega)+O(J^{-1}\log J)$ as $J\to\infty$.

We evaluate the integral for the KV distribution defined by $g(t)=\vartheta(1-t)$, where ϑ is the step function, and $g_1(t)=t\vartheta(1-t)+\vartheta(t-1)$. Denoting by $f(y)$ the expression between the square brackets in (10) where $y=2J/R_c^2$ we see that $f(y)=0$ if $y<1$ (within the core). If $y>1$ we define $\theta_0=\arccos y^{-1/2}$ and notice that

$$f(y)=1-\frac{1}{\pi y}\left[\int_{\theta_0}^{\pi-\theta_0} y\cos^2\theta\,d\theta+\int_{\pi+\theta_0}^{2\pi-\theta_0} y\cos^2\theta\,d\theta+\right.$$

$$+\int_{-\theta_0}^{\theta_0} d\theta + \int_{\pi-\theta_0}^{\pi+\theta_0} d\theta\Bigg] = \frac{2}{\pi y}\left[(y-2)\text{arcos}\,\frac{1}{\sqrt{y}} + \sqrt{y-1}\right] \tag{11}$$

The Gaussian distribution is defined by $g(t) = e^{-t}$ and $g_1(t) = 1 - e^{-t}$ and the function $f(y)$ is given by

$$f(y) = 1 - \frac{2}{y}\left[1 - e^{-y/2} I_0\left(\frac{y}{2}\right)\right] \tag{12}$$

The 4D case can be treated in a similar way. A numerical integration over the angles and a polynomial or cubic splines fitting of the action depending function f might be convenient in this case.

BIRKHOFF NORMAL FORMS

A map $U(\mathbf{x})$ of $\mathbb{R}^{2d}$ with a fixed point at the origin $U(\mathbf{0}) = \mathbf{0}$ is in normal form with respect to the group generated by a linear map L if it is invariant with respect to the group transformation: $U(L^n\mathbf{x}) = L^n\,U(\mathbf{x})$. A sufficient condition is that U commutes with the generator of the group

$$\Delta U \equiv UL - LU = 0 \tag{13}$$

A symplectic map U with a linear component L, in normal form with respect to the group generated by L, takes the form

$$U = Le^{D_H} \qquad\qquad D_H F = [F, H] \tag{14}$$

where D_H is the Lie derivative, [ ] denotes the Poisson bracket and $e^{D_H} = \sum D_H^n/n!$ is the Lie series. The interpolating Hamiltonian H is symmetric $H(L\mathbf{x}) = H(\mathbf{x})$ and is invariant with respect to the map U namely $H(U(\mathbf{x})) = H(\mathbf{x})$. For simplicity we refer from now on to the two dimensional case ($d = 1$). The factorization of the linear map L is relevant when L generates a discrete group. Suppose $L = R(2\pi p/q)$ is the rotation of an angle commensurate with 2π for a 2D map. The symmetry group is discrete and if the linear part of the map is L we call *resonant* the normal form. If the linear part of the map is $R(2\pi p/q+\varepsilon)$ we may still choose $L = R(2\pi p/q)$ imposing a discrete symmetry. This *quasi-resonant* normal form is appropriate to investigate the limit $\varepsilon \to 0$ and the topology of resonances. The orbits of the Hamiltonian H have the pendulum topology. The factorization of the discrete rotation takes care of the jumps from one island to a next one. When $L = R(\omega)$ is a rotation of an angle incommensurate with 2π, the group is continuous, the orbits are circles and the normal form is *non-resonant.*

The linear map properties depend on its trace: if $|\mathrm{Tr}\,L| < 2$ the eigenvalues are complex conjugate with unit modulus $e^{\pm i\omega}$, the fixed point is *elliptic* and $L = W\,R(\omega)\,W^{-1}$, where W has the standard form in terms of Twiss parameters. It is convenient to use complex coordinates $z = x - ip$, $z^* = x + ip$. Since the transformation has determinant $2i$ the evolution equations read $\dot z = D_H\,z \equiv 2i\,[z, H]$ where the Poisson bracket is defined with respect to z, z^*. The linear map becomes $z' = e^{i\omega}\,z$. If $|\mathrm{Tr}\,L| > 2$ the eigenvalues are real $e^{\pm\alpha}$ and the fixed point is *hyperbolic*. Expanding $U(z, z^*)$ in a Taylor series, the normal form condition $\Delta U = 0$ defines its structure. In the non-resonant elliptic case the allowed monomials are $z^{n+1}z^{*n}$ and the map can be written as

$$U(z, z^*) = e^{i\omega}\,e^{D_H(zz^*)}\,z = e^{i\Omega(zz^*)}\,z \tag{15}$$

$$\Omega = \omega + 2H'(zz^*) = \omega + \sum \Omega_2 zz^* + \ldots + \Omega_{2n}\,(zz^*)^n + \ldots$$

In the resonant or quasi-resonant case where the linear phase advance is $\omega = 2\pi p/q + \varepsilon$ and $L = R(2\pi p/q)$, the normal form is

$$U(z, z^*) = e^{i\,2\pi\,p/q}\,e^{D_H}\,z$$

$$2H = h_0(zz^*) + \sum_{\ell\geq 1}\left[\,h_\ell(zz^*)\,z^{\ell q} + h_\ell^*(zz^*)\,z^{*\,\ell q}\,\right] \tag{16}$$

where $h_0 = \varepsilon zz^* + \frac{1}{2}\,\Omega_2\,(zz^*)^2 + \ldots$. The structure of the Hamiltonian H is more transparent in the action angle $z = \sqrt{2j}\,e^{i\theta}$ or polar coordinates $z = r\,e^{i\theta}$:

$$2H \equiv h = \sum_{\ell\geq 0} h_\ell(2j)\,\cos(\ell q\theta + \delta_\ell) = \sum_{n\geq 2} r^n\,C_n(\theta) \tag{17}$$

The level lines of H interpolate the orbits of the map U. Letting $H(\theta, j) = E$ be a level line diffeomorphic to a circle centered at the origin, the invariant action is defined by

$$J = \frac{1}{2\pi}\int_0^{2\pi} j(\theta, E)\,d\theta \tag{18}$$

where $j = 2r^2$ and r is a real positive root of the polynomial defined by the r.h.s. of (17). To compute the phase advance $\Omega(J)$ we notice that if H does not depend on θ then $J = j$ and $\Omega = \partial H/\partial J$. In the general case, letting $E = E(J)$ the inverse function of $J = J(E)$ defined by (18) we have

$$\Omega = \frac{\partial E}{\partial J} = \frac{1}{\dfrac{\partial J}{\partial E}} = \frac{2\pi}{\displaystyle\int_0^{2\pi} \frac{\partial j}{\partial E}(\theta, E)\,d\theta} =$$

$$= \frac{2\pi}{\displaystyle\int_0^{2\pi}\left(\frac{\partial H}{\partial j}(\theta, j(\theta, E))\right)^{-1} d\theta} \tag{19}$$

A level line usually consists of two curves diffeomorphic to circles centered at the origin corresponding to an internal and to an external orbit with respect to the chain of islands. A chain is easily detected since in some intervals of θ there are no real roots. Since the phase is locked, the corresponding value of the phase advance is $\Omega = 2\pi p/q$. Efficient codes implementing the algorithm (21) and computing the Hamiltonian H from U have been developed [18, 21].

THE VARIABLE FREQUENCY MAP

The model described in section 2 is a ring with space charge and a thin $2m$-pole and its one turn map reads

$$\begin{pmatrix} x' \\ p'_x \end{pmatrix} = R\left(\omega_{\,\mathrm{sc}}\left(\frac{x^2+p_x^2}{2}\right)\right)\begin{pmatrix} x \\ p_x + x^{m-1} \end{pmatrix} \tag{23}$$

The main difference with respect to the standard case previously considered is that the unperturbed frequency varies. However chosen any point on one orbit of the unperturbed map, its frequency is defined and the normal form transformation can be carried out. Since $\omega_{\,\mathrm{sc}}$ varies between ω and ω_0, if no low order resonance appears in that interval we can use the non-resonant normal form. If there is a resonance $2\pi p/q$ with $q \le 6$ then the corresponding quasi-resonant normal form should be used. The presence of two low order resonances is excluded by the rather small depression of the phase advance necessary to insure a good accuracy of the perturbative approximation to $\omega_{\,\mathrm{sc}}$. The analytical formula remain unchanged, except for the functional dependence of $\omega_{\,\mathrm{sc}}$ on $x^2+p_x^2$. In the algorithmic approach, the normal form transformation needs to be recomputed for every orbit. We quote the analytical result at the lowest order of normal forms $N = m+1$ for m even and $N = m$ for m odd. If the multipole is even $m = 2(n+1)$, letting $J = \frac{1}{2}(x^2+p_x^2)$ we have

$$\Phi = I + [\Phi]_{2n+1} + \dots \qquad \Omega = \omega_{\,\mathrm{sc}}\,(J) - J^n \frac{1}{2^{n+1}}\binom{2n+1}{n} \tag{24}$$

If the multipole is odd $m = 2n+1$

$$\Phi = I + [\Phi]_{2n} + [\Phi]_{4n-1} + \dots \qquad \Omega = \omega_{\,\mathrm{sc}}\,(J) + J^{2n-1}\,c_{2n-1}$$

$$c_{2n-1} = \frac{n}{2^{2n+1}}\sum_{k=0}^{2n-1}\binom{2n-1}{k}\binom{2n}{k}\left(\cot\left(n-k-\frac{1}{2}\right)\omega_{\,\mathrm{sc}} - \right.$$
$$\left. -\cot\left(n-k+\frac{1}{2}\right)\omega_{\,\mathrm{sc}}\right) \tag{25}$$

The coefficient c_{2n-1} is not defined when $\omega_{\,\mathrm{sc}}\,/(2\pi) = 0, \frac{1}{3}, \dots, \frac{1}{2n+1}$. As a specific example we consider a map whose bare tune $\nu_0 = \omega_0/(2\pi)$ varies on the range $[6.25, 6.3]$, the tune depression $\Delta\nu = \nu_0 - \nu$ being 0.05 with a normalized core radius $R_c = 0.1$. These values correspond to the dynamic aperture experiment performed at PS.

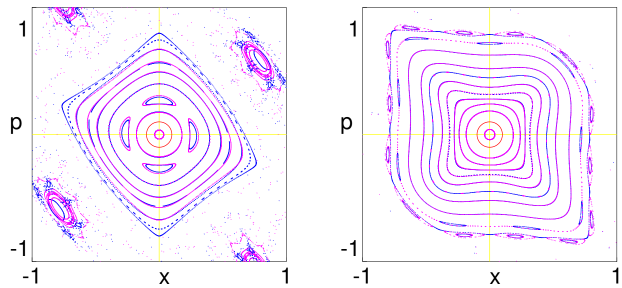

Figure 1 Phase plot for a bare tune ν_0=6.3 (left), and a tune shift $\Delta\nu$=0.2 due to space charge for a KV beam of normalized radius R_c=0.1 (left) in presence of a thin octupole m=4. Same plot for ν_0=6.26, $\Delta\nu$=0.05 (right). The blue dots correspond to the map (23), the purple dots to the exact space charge map (2×10^3 space charge kicks per turn). In the printed version the difference in the grey tones cannot be appreciated

In that experiment $\nu_0 = 6.26$, the dynamics aperture is 6σ for a Gaussian beam when the resonance is crossed and the tune depression is comparable with the values we choose. For these values the agreement of the orbit of the map (23) with the exact one is such that small discrepancies are visible only in the chaotic region and can hardly be appreciated on the plot of the tune along any line $p = Cx$.

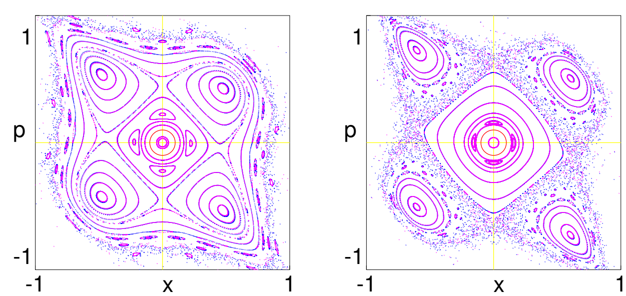

Figure 2 The same as Figure 1 for bare tune ν_0=6.27 (left), ν_0=6.28 (right) and $\Delta\nu$=0.05, R_c=0.1, m=4

For a larger tune depression such as $\Delta\nu = 0.2$ the discrepancy is visible far away from core, see figure 1 left, due to the error in the approximation of the tune $\nu_{\,\mathrm{sc}}$. In figure 1 and 2 we show the phase portraits when the bare tune varies from 6.26 to 6.28, with the birth of the four islands and their detachment from the region of stable orbits. The lowest order (($N = 3$)) approximation $\Omega = \omega_{\rm sc} + (x^2 + p^2)\Omega_2$ and the resonant normal form with respect to the resonance $q = 4$ at order $N = 8$ are compared with the exact result from tracking in figures 3 and 4.

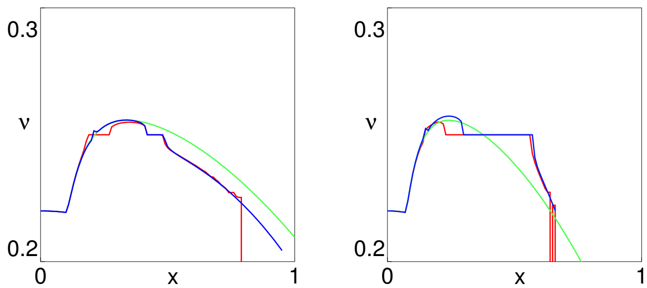

Figure 3 Left: comparison of the exact tune obtained from tracking for initial points along the x axis (red curve) with the resonant perturbation theory at order N=8 (blue line) and the lowest order perturbation theory N=3 (light green line) In the printed version the first curve appears dark grey, the second black the last one light grey. The map parameters are ν_0=6.27, ν=6.22, R_c=0.1, m=4 Right: the same for initial points on the line p=x.

It may be noticed that the non-resonant normal form at order $N > 3$ exhibits some wild oscillations due to the small divisors associate to the resonance $q = 4$. The resonant normal form is stable and no significant differences are observed from order 4 to order 8, see figure 4. The cases considered here are quite challenging because the tune is non monotonic and a low order resonance is present. The conclusion is that the third order non-resonant approximation, which has a very simple expression, is reasonably accurate before the chain of four islands (indeed there are two chains) is met.

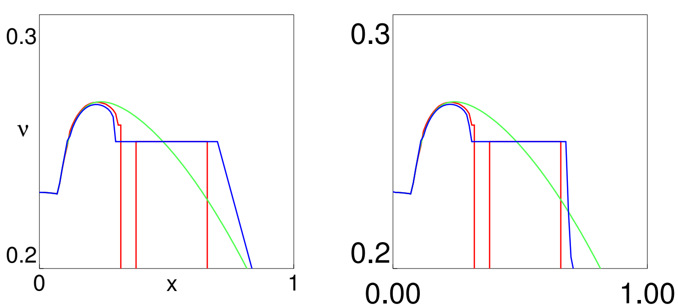

Figure 4 Left: the same as figure 3 for resonant normal form of order N=4 and map parameters ν_0=6.28, ν=6.23, R_c=0.1, m=4 Right: the tune for the same map and resonant normal form of order N=8

The $q = 4$ resonance can be dealt with the lowest order $N = 4$ resonant normal form for which a readable analytic expression can still be written. The case $\nu_0 = 6.3$, $\nu = 6.1$ where the tune shift is four times bigger is simpler because the islands do not show up for a sextupole $m = 2$ or a decapole or are far beyond the dynamic aperture for the octupole. In this case the non-resonant normal forms are free of divisors in the region within the dynamic aperture and are quite accurate. In Figure 5 we compare the tracking *exact* result for the tune with the lowest order $N = 3$ for the sextupole, and $N = 5$ for the decapole with the normal form at order $N = 8$. A similar result holds for the octupole.

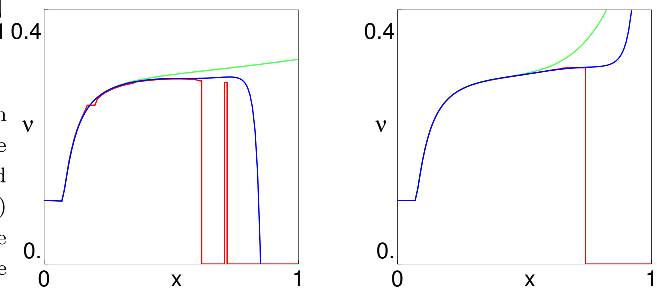

Figure 5 Left: comparison of the the exact tune, lowest order N=3 and N=8 non-resonant normal form for a sextupolar map m=3 with higher tune shift ν_0=6.28, ν=6.23, R_c=0.1. Right: comparison of the lowest order $N = 5$ and $N = 8$ non-resonant normal form for a decapolar map m=5 with the same parameters and ν_0=6.28, ν=6.23, R_c=0.1

EXTENSIONS AND CONCLUSIONS

If there are several thin multipoles the normal forms approach can be extended as follows. Suppose we have two sections of normalized length s_1, $s_2 = 1 - s_1$. Letting K_1 and K_2 be the multipolar kicks and supposing $\omega_{\rm sc}$ is the space charge depressed phase advance on the ring, the map can be written as

$$M = \Big(R(s_1\,\omega_{\rm sc}) \circ K_1 \Big) \circ \Big(R(s_2\,\omega_{\rm sc}) \circ K_2 \Big)$$

$$= \Big(\Phi_1 \circ R(\Omega_1) \circ \Phi_1^{-1} \Big) \circ \Big(\Phi_2 \circ R(\Omega_2) \circ \Phi_2^{-1} \Big) =$$

$$= \Phi_2 \left[\Phi_2^{-1} \circ \Phi_1 \circ R(\Omega_1) \circ \Phi_1^{-1} \circ \Phi_2 \circ R(\Omega_2) \right] \circ \Phi_2^{-1} =$$

$$= \Phi_2 \circ [R(\Omega_1 + \Omega_2) \circ K_{12}] \circ \Phi_2^{-1} = \Phi_2 \circ \Phi_{12} \circ R(\Omega_{12}) \circ \Phi_{12}^{-1} \circ \Phi_2^{-1}$$

where we have used repeatedly the normal form transformation (in the non-resonant case for simplicity) and in the step before the last one we have simply expressed the mapping factorizing the integrable part. This procedure can be iterated to any number of thin lenses. To summarize, we propose to use the normal form on a symplectic map which describes accurately the space charge effects combined with a multipolar kick. Even though normal forms are locally defined, the result is as accurate as in the standard case. The limits are imposed by its asymptotic character due to the small divisors. The extension to the 4D case is straightforward and the proposed method might be used to draw the resonance lines in action space or in any 2D section like the xy plane for any chosen working point.

Acknowledgments I would like to tank L. Rossi for valuable advice on the numerical procedures The present work was supported by the Italian M.I.U.R. project COFIN-03 *Order and Chaos in Nonlinear Extended Systems*

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