A NEW FRACTAL VISCOELASTIC ELEMENT: PROMISE AND APPLICATIONS TO MAXWELL-RHEOLOGICAL MODEL

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This paper proposes a fractal viscoelastic element via He's fractal derivative, its properties are analyzed in details by the two-scale transform for the first time. The element is used to establish a fractal Maxwell-rheological model(FMRM), which unifies the fractal creep equation and relaxation equation, and includes the classic elastic model and the classical Maxwell-rheological model as two special cases. This paper sheds a bright light on viscoelasticity, and the model can find wide applications in rock mechanics, plastic mechanics, and non-continuum mechanics.

Keywords: *Fractal viscoelastic element; Two-scale method; He's fractal derivative; fractal Maxwell-rheological model*

1. Introduction

The rheological property plays an important role in rock, and both the long-term stability and durability of rock machines are closely related to the rock's rheological property. For example, the surrounding rock mass is stable at the beginning of tunnel formation, however, as time goes on, the deformation of rock mass develops continuously, and after some time, the tunnel may lose stability or collapse suddenly, and the surrounding rock has the obvious characteristics of slow deformation with the increase of time. With people's attention to the long-term safety of geotechnical engineering, more and more attention has been paid to the rheological study of geotechnical engineering, however, the focus was put mainly on what kind of constitutive equations was suitable for the relationship between stress, strain and time of rock materials[1-3].

As we all know that the stress-strain relationship of an ideal elastic element satisfies Hooke's law is (see Fig. 1)

$$\sigma(\tau) = E\varepsilon(\tau), \tag{1.1}$$

where $\sigma(\tau)$, $\varepsilon(\tau)$ and E are the stress, strain and modulus of elasticity of ideal elastic element respectively.

As shown in Fig 2, the ideal viscous element satisfies Newton's law gives

$$\sigma(\tau) = \eta \frac{d\varepsilon(\tau)}{d\tau}, \tag{1.2}$$

where η is the viscosity coefficient.

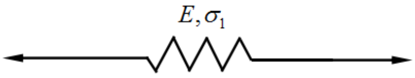

Fig 1. The model of elastic element

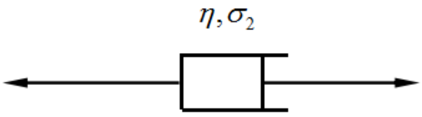

Fig 2. The model of viscous element

As a powerful mathematical analysis tool, the fractal derivative has been widely used in the description of various complex phenomena[4-22]. Now we use the He's fractal derivative to correct the Eqs.(1.1) and (1.2). The He's fractal derivative is defined as follows[22-25]:

$$\frac{Df}{Dt^{\alpha}}(t_0) = \Gamma(1+\alpha) \lim_{\substack{t-t_0 \to \Delta t \\ \Delta t \neq 0}} \frac{f - f_0}{(t-t_0)^{\alpha}}, \tag{1.3}$$

where α is the fractal dimension. The fractal derivative is a powerful tool to establishment of complex models in fractal space or discontinuous media. The geometric physical interpretation of fractal derivatives and the process of establishing mathematical models are described in Ref[22].

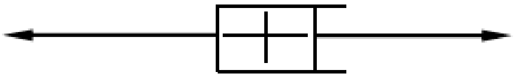

Fig 3. The model of viscoelastic element using He's fractal derivative

By comparing Eq.(1.1) and Eq.(1.2), we propose a common expression, which reads:

$$\sigma(\tau) = \Im \frac{d^{\zeta}\varepsilon(\tau)}{d\tau^{\zeta}}, \qquad 0 \le \zeta \le 1 \tag{1.4}$$

The above equation can be used to describe the relationship between force and strain of the viscoelastic body(see fig.3), where $\Im$ is the viscoelasticity coefficient. For example, the Eq.(1.3) is used to describe the elastic element when $\zeta = 0$, and the viscous element when $\zeta = 1$. For $0 < \zeta < 1$, it can be used to describe the viscoelastic element. Now we plan to use the two-scale transform method[23-25] to analyze the creep properties in details.

2. The two-scale transform method

The two-scale method[23-25], as a new transformation method, is an extension of the He's fractional complex transformation[21]. The two-scale transform can be used to convert the fractal calculus into the traditional partner and successfully applied to solve many fractal problems.

Consider the following fractal equation:

$$\frac{D\phi}{D\tau^{\zeta}} + F(\phi) = 0, \tag{2.1}$$

For using the two-scale transform method[23-25], we let

$$T = \tau^{\zeta}. \tag{2.2}$$

By Substituting Eq.(2.2) into Eq.(2.1), the Eq.(2.1) is converted into the following form:

$$\frac{D\varphi}{DT} + F(\varphi) = 0. \tag{2.3}$$

So, the fractal equation is successfully converted into an integral order differential equation, which can be solved by many classical methods, such as the homotopy perturbation method[26-28], variational iteration method[29,30] and so on[31].

3. The analysis of the viscoelastic body

In order to study the creep properties, Eq.(1.3) can be rewritten as the following form by letting $\sigma(\tau) = \sigma_0$.

$$\sigma_0 = \Im \frac{d\varepsilon(\tau)}{d\tau^{\zeta}}, \qquad 0 \le \zeta \le 1 \tag{3.1}$$

Taking the two-scale transform as:

$$T = \tau^{\zeta} \tag{3.2}$$

Applying the two-scale transform to Eq.(3.1), yields:

$$\sigma_0 = \Im \frac{d\varepsilon(T)}{dT}, \tag{3.3}$$

The solution of the above is given as:

$$\varepsilon(T) = \frac{\sigma_0}{\Im} T + C, \tag{3.4}$$

where C is a constant. Thus we get the solution of $\varepsilon(\tau)$ with the help of Eq.(3.2), which reads:

$$\varepsilon(\tau) = \frac{\sigma_0}{\Im} \tau^{\zeta} + C, \tag{3.5}$$

Let $\sigma_0 = 1$, $\Im = 1$ and $C = 0$, we plot the curves of $\varepsilon(\tau)$ with different orders ζ in fig. 3. Obviously, for ζ =0 and 1, the $\varepsilon(\tau)$ represents the creep properties of the elastic element and the viscous element respectively. The larger the value of ζ is, the closer it is to the characteristics of elastic element, correspondingly, the smaller the value is, the closer it is to the characteristics of the viscous element. In other words, the fractional order ζ represents whether the element is mainly elastic or viscous.

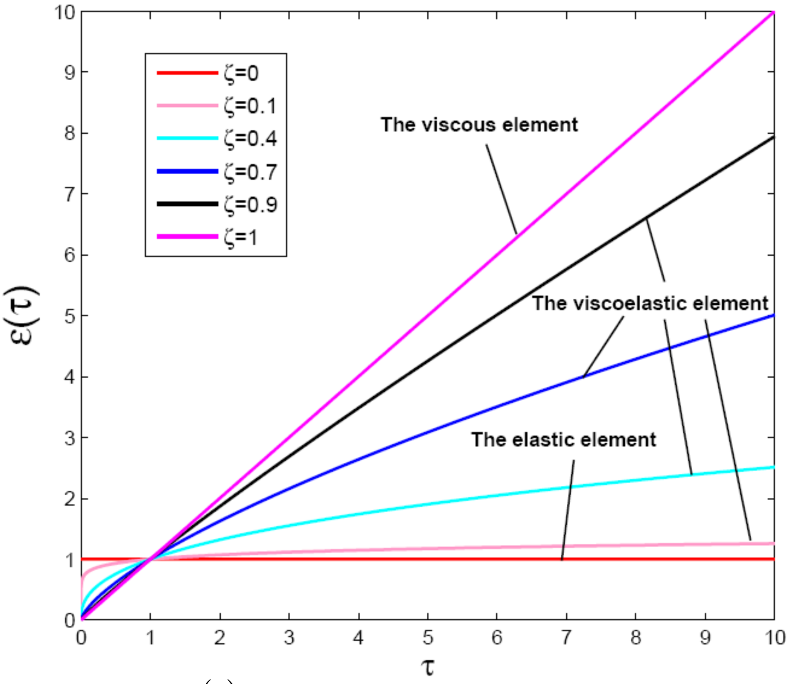

Fig 3. The curves of $\varepsilon(\tau)$ with different orders ζ =0, 0.1, 0.4, 0.7, 0.9 and 1.

4. An Application to fractal Maxwell rheological model (FMRM)

The fractal Maxwell-rheological model is plotted in fig.4, we have the following relation according to the series theory:

$$\sigma(\tau) = \sigma_1(\tau) = \sigma_2(\tau), \tag{4.1}$$

and

$$\varepsilon(\tau) = \varepsilon_1(\tau) + \varepsilon_2(\tau), \tag{4.2}$$

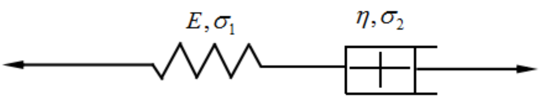

Fig 4. The fractal Maxwell-rheological model

Taking ζ -order differentiation of the above formula, we get:

$$\varepsilon^{(\zeta)}(\tau) = \varepsilon_1^{(\zeta)}(\tau) + \varepsilon_2^{(\zeta)}(\tau), \tag{4.3}$$

For the elastic element, there is

$$\sigma_1(\tau) = E\varepsilon_1(\tau), \tag{4.4}$$

And for the viscoelastic element, we have

$$\sigma_2(\tau)=\eta\varepsilon_2^{(\zeta)}(\tau), \tag{4.5}$$

Eqs.(4.1-4.4) may now be combined to produce the constitutive equation of the fractal Maxwell rheological model as

$$\varepsilon^{(\zeta)}(\tau)=\frac{1}{E}\sigma^{(\zeta)}(\tau)+\frac{1}{\eta}\sigma(\tau), \tag{4.6}$$

Recalling the two-scale transform

$$T=\tau^{\zeta} \tag{4.7}$$

We replace τ with T for Eq.(4.6), converting the FMRM constitutive equation into the classical partner as:

$$\frac{d\varepsilon(T)}{dT}=\frac{1}{E}\frac{d\sigma(T)}{dT}+\frac{1}{\eta}\sigma(T), \tag{4.8}$$

We obtain the creep equation under constant load of $\sigma(T)=\sigma_0$ as:

$$\varepsilon(T)=\frac{\sigma_0}{\eta}T+\frac{\sigma_0}{E}, \tag{4.9}$$

Correspondingly, the creep equation of the fractal Maxwell rheological model (**FMRM)** is given as

$$\varepsilon(\tau)=\frac{\sigma_0}{\eta}\tau^{\zeta}+\frac{\sigma_0}{E}, \tag{4.10}$$

The creep curves of the FMRM is plotted in Fig.5 for different ζ by using $\sigma_0=1$, $\eta=1$ and $E=1$. The curve indicates the creep curve of the classical Maxwell rheological model(CMRM) when $\zeta=1$, which is because the viscoelastic element is a pure viscous element at $\zeta=1$. In addition, when $\zeta=0$, the curved edge represents the elastic element, which is precisely because the element is a pure elastic element for $\zeta=0$, and the FMRM is equivalent to two pure elastic elements in series. In the other cases for $0<\zeta<1$, the curve is between pure elastic element and classical Maxwell rheological model(CMRM). The larger the value of ζ, the closer it is to CMRM, the smaller it is, the closer it is to pure elastic element, which is related to the characteristics of the viscoelastic element.

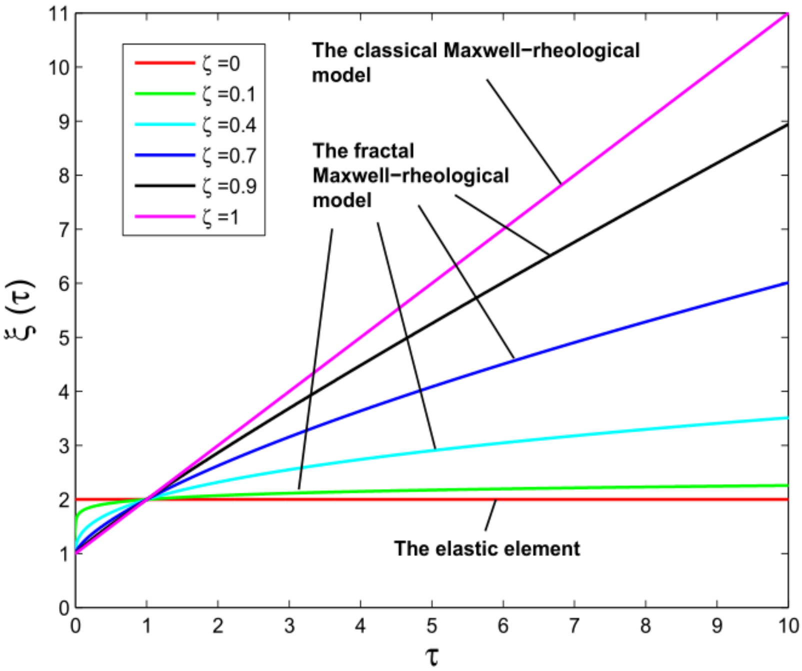

Fig. 5 The creep curves of the FMRM with different ζ.

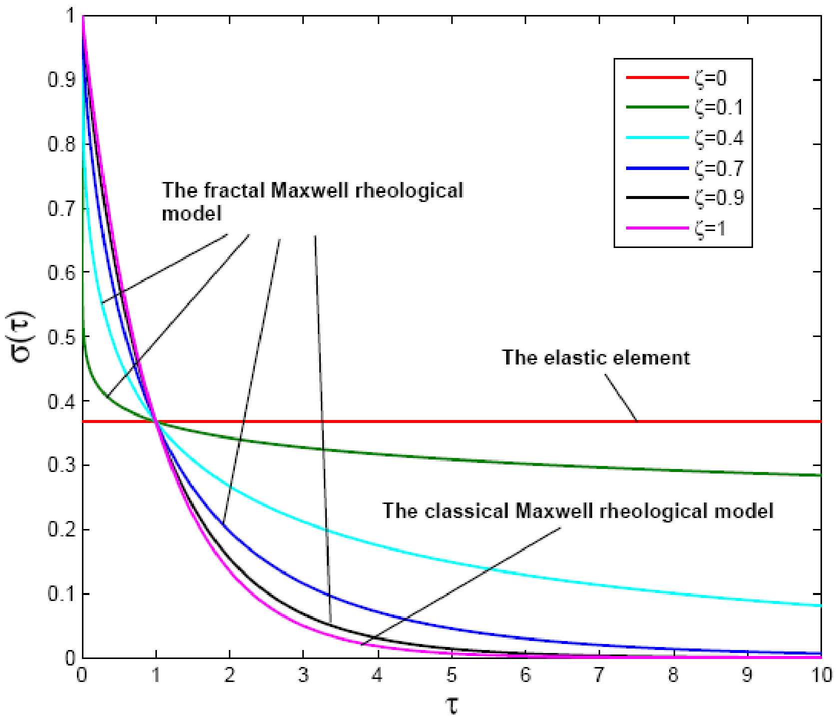

Fig. 6. The relaxation curves of the FMRM with different ζ .

Recalling Eq.(4.8) and letting $\varepsilon(T)=$constant, we get the relaxation equation:

$$\frac{1}{E}\frac{d\sigma(T)}{dT}+\frac{1}{\eta}\sigma(T)=0, \tag{4.11}$$

The application of the initial condition $\sigma=\sigma_0$ yields

$$\sigma(T)=\sigma_0 e^{-\frac{E}{\eta}T}, \tag{4.12}$$

By replacing T with τ^{ζ}, there is

$$\sigma(\tau)=\sigma_0 e^{-\frac{E}{\eta}\tau^{\zeta}} \tag{4.13}$$

We draw the relaxation curves of the FMRM as shown in the Fig.6. Obviously, the viscoelastic body of the FMRM becomes to the a pure elastic element when $\zeta=0$, which leads to the elastic element properties in Fig.6(red line). As for $\zeta=1$, the viscoelastic body changes into the viscosity element, so the FMRM becomes the CMRM. For $0<\zeta<1$, we can come to a similar conclusion by recalling the creep properties. Generally speaking, when the strain $\varepsilon(T)$is a constant, the stress decreases with the increase of time for $0<\zeta\le 1$. By carefully analyzing different curves, we find that the larger the fractional order ζ is, the faster the curve decays.

5. Conclusions

In this paper, for the first time ever, the fractal viscoelastic element is proposed by using He's fractal derivative, and analyzed by applying the two-scale transform method in details. Then we use the fractal viscoelastic element to model the FMRM, and study the creep characteristic and relaxation characteristic with different orders ζ . As expected for ζ =0, the FMRM is equivalent to two pure elastic elements in series, and when ζ =1, the FMRM becomes the CMRM. The obtained results in this paper are expected to open some new perspectives towards the characterization of the fractal rheological model. This paper sheds a bright light on viscoelasticity, and the model given in this paper can find wide applications in rock mechanics and plastic mechanics.

Acknowledgment: This work is supported by Program of Henan Polytechnic University (No.B2018-40).

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Received: March 1, 2020
Revised: June 12, 2020
Accepted: June 12, 2020