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Algebraic entropy of endomorphisms of M-sets

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Abstract: The usual notion of algebraic entropy associates to every group (monoid) endomorphism a value estimating the chaos created by the self-map. In this paper, we study the extension of this notion to arbitrary sets endowed with monoid actions, providing properties and relating it with other entropy notions. In particular, we focus our attention on the relationship with the coarse entropy of bornologous self-maps of quasi-coarse spaces. While studying the connection, an extension of a classification result due to Protasov is provided.

Keywords: monoid actions, group actions, algebraic entropy, endomorphisms, coarse spaces, quasi-coarse spaces, coarse entropy

2020 MSC: 22F05, 20M75, 51F30, 54C70, 54E15, 20M99

1 Introduction

After Clausius' definition in thermodynamics in 1865, entropy in mathematics was firstly introduced by Shannon in information theory ([32]). Inspired by his definition, this notion was implemented in several other areas. In each of the latter settings, entropy associates to every self-map of a space a value in $\mathbb{R}_{\geq 0} \cup \{\infty\}$ estimating the chaos created by it. Let us mention, for example, Kolmogorov ([21]) and Sinai's ([33]) measure theoretic entropy in ergodic theory, Adler, Konheim and McAndrew's topological entropy ([1]), and, more recently, coarse entropy in coarse geometry ([37] and [14], where the authors of the latter were inspired by Bowen's definition of topological entropy in uniform spaces, [3]). We refer to [8] for a wide survey of many entropy notions, and we also cite [10] where the authors used normed semigroups to provide a unifying approach to study several entropy notions.

Let us now focus on the situation in group theory. The first idea of extending entropy to the realm of abelian groups is contained in the work of Adler, Konheim and MacAndrew ([1]), and this entropy was later studied by Weiss ([35]) in the class of torsion abelian groups. Let us cite also [11] for a more recent reference to Weiss' algebraic entropy, and [15] for the algebraic entropy of endomorphisms of locally finite groups. A different approach was provided by Peters in [23], and then it was generalised for all endomorphisms of abelian groups in [9]. The same definition can be readily extended to semigroup and monoid endomorphisms. Let us also address the interested reader again to the survey [8].

Algebraic entropy was later generalised to consider continuous endomorphisms of topological groups. Peters in [24] gave an extension of the algebraic entropy defined in [23] for topological automorphisms of locally compact abelian groups. Peter's definition was then further generalised by Virili ([34]) to all endomorphisms of locally compact abelian groups. Virili's notion can be found in [8] also for non-abelian groups. More recently, a growing interest was paid to entropies associated to actions of amenable semigroups and groups, instead of single morphisms. In the algebraic setting, Dikranjan, Fornasiero and Giordano Bruno studied the

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algebraic entropy of actions of amenable cancellative semigroups on abelian groups ([7]). The authors were inspired by the measure and topological counterparts (see [4] and [18])

In the sequel, as algebraic entropy of monoid endomorphisms we always refer to the definition contained in [10] and reported in Definition 4.1. In this paper we present a different way to generalise the algebraic entropy of a single monoid endomorphism. In order to do that, instead of using monoids, we consider sets endowed with monoid actions, which are called M-sets, for some monoid M. We then introduce in Definition 3.1 the entropy of their endomorphisms (Definition 2.3). In addition to its intrinsic interest and the connections to other known entropies that we shortly mention in the following paragraph, this generalisation is motivated since M-sets are very flexible objects, and so it has possible applications in different areas of mathematics where they appear.

This new notion of algebraic entropy extends the usual one in the following sense. Every monoid M can be canonically seen as an M-set, and an endomorphism of the corresponding M-set can be associated to every endomorphism of M. The algebraic entropy of the endomorphism of the M-set coincides with the algebraic entropy of the original endomorphism (Theorem 4.3). According to this observation, we extend several properties that hold for the usual algebraic entropy of group endomorphisms to this wider setting. Examples of these properties are the weak logarithmic law (Proposition 3.8), the invariance under conjugation (Corollary 3.11), the monotonicity for subspaces (Corollary 3.12) and quotients (Corollary 3.13), and the weak addition theorem (Theorem 3.18). We refer to [8] for their classical counterparts in the realm of group endomorphisms.

We establish a strong connection also with another notion of entropy, the coarse entropy introduced in [37] in the realm of coarse geometry. This branch of mathematics, also known as large-scale geometry, studies the large-scale, global, properties of spaces, ignoring their topological ones. It was at first developed for metric spaces (see [17] and [22] for applications to geometric group theory and geometric topology, respectively) and then Roe introduced coarse spaces to deal with non-metrisable spaces ([31]). Other structures that are equivalent to coarse spaces are balleans ([28]), asymptotic proximities ([26]), and large-scale structures ([13]). More recently, this approach was generalised to non-symmetric spaces with the introduction in [36] of quasi-coarse spaces (Definition 4.6). The morphisms between those spaces are called bornologous maps. Every monoid action on a set induces a quasi-coarse space, which is a coarse space if the monoid is a group ([27]). In [27] and [25] several classes of such coarse spaces are characterised. Moreover, every endomorphism of an M-set induces a bornologous self-map of the corresponding quasi-coarse space, and we show that, provided that this morphism is surjective, its coarse entropy coincides with the algebraic entropy of the original endomorphism (Theorem 4.12). We also prove that, for a large class of coarse spaces, every injective bornologous self-map can be induced by an endomorphism of G-sets, for some subgroup G of permutations (Theorem 4.16), generalising a result due to Protasov ([27, Theorem 1]).

This paper is organised as follows. In Section 2 we recall some basic definitions in the realm of M-sets, such as monoid and group actions, orbits and endomorphisms of M-sets, and prove results concerning these objects. Moreover, we define the category **FlowMon-Set** of M-sets endowed with endomorphisms. Then in Section 3 we define the algebraic entropy, present some examples of easy computations and discuss consequences of the results proved in §2.1. More standard properties of this entropy (e.g., weak logarithmic law, invariance under conjugation, monotocity for subspaces and quotients) are collected in §3.1, while in §3.2 we provide the weak addition theorem and the coproduct formula. Section 4 is devoted to the comparison of the algebraic entropy with other entropy notions. In particular, in §4.1 we focus on the relationship with the algebraic entropy of monoid endomorphisms, while in §4.2 with the coarse entropy. More precisely, in the latter subsection, we provide all the necessary background in coarse geometry, show when the coarse entropy and the algebraic entropy coincide, and generalise Protasov's result.

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2 Endomorphisms of M-sets

The theory of M-sets and their homomorphisms that we use in this paper is self-contained. For the interested reader, we refer to [5], [6] and [20].

Let M be a monoid and X be a set. In the sequel, we denote by e_M the neutral element of M. If there is no risk of ambiguity, we write e. A *right action of M on X* is a map $\alpha\colon M \to X^X$ satisfying the following properties:
- $\alpha(e) = id_X$;
- $\alpha(b) \circ \alpha(a) = \alpha(ab)$, for every $a, b \in M$.

Since in the sequel we always refer to right actions, we call them actions. If the action involved is clear, we usually simplify the notation by writing xa instead of $\alpha(a)(x)$, for every $x \in X$ and $a \in M$. It is easy to check that, if $a \in M$ is *invertible* (i.e., there exists $a^{-1} \in M$ such that $aa^{-1} = a^{-1}a = e$) then $\alpha(a)$ is a bijection. In particular, if M is a group, then it acts via bijections, which means that $\alpha(M) \subseteq S_X$, where S_X denotes the group of permutations of X.

An action α of a monoid M on a set X is said to be:
- *free* if, for every $a, b \in M$, $a = b$ provided that there exists $x \in X$ such that $\alpha(a)(x) = \alpha(b)(x)$;
- *weakly free* if, for every $a, b \in M$, $\alpha(a) = \alpha(b)$ provided that there exists $x \in X$ such that $\alpha(a)(x) = \alpha(b)(x)$.

Of course, an action is weakly free if it is free.

Let M be a monoid. A *right M-set*, briefly, for the purpose of this paper, an *M-set*, is a set X endowed with an action of M on it, and we write $X \curvearrowleft M$.

Let M be a monoid and X be an M-set. The *orbit of a point* $x \in X$ is the subset $xM = \{xa \in X \mid a \in M\}$ of X. A subset Y of X is called a *ceiling of X* if $YM := \bigcup\{yM \mid y \in Y\} = X$. Moreover, an element $x \in X$ is a *top element* if $\{x\}$ is a ceiling.

Remark 2.1. Let G be a group and X be a G-set. Then the notion of orbit coincides with the usual one (see, for example, [19]). Thus, the family of orbits $\{xG \mid x \in X\}$ creates a partition of X. A ceiling of X has to contain at least one point for each orbit. Moreover, the following properties are trivially equivalent:
- X consists of just one orbit (i.e., the action is *transitive*);
- every point of X is a top element.

Example 2.2. Let M be a monoid. Then the *right regular action ρ of M on itself* is defined as follows: for every $a \in M$, $\rho(a)$ is the *right shift s_a^ρ by a*, i.e., $\rho(a) = s_a^\rho\colon M \to M$, where $s_a^\rho(b) = ba$. Moreover, the following properties are equivalent:
(a) M is *left-cancellative* (i.e., for every $a, b, c \in M$, if $ab = ac$ then $b = c$);
(b) ρ is free;
(c) ρ is weakly free.

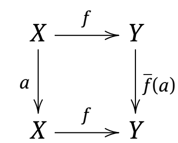
The implications (a)→(b)→(c) are trivial. Suppose that ρ is weakly free and $a, b, c \in M$ satisfying $ab = ac$. Since ρ is weakly free, for every $x \in M$, $\rho(b)(x) = \rho(c)(x)$, and thus, $b = \rho(b)(e) = \rho(c)(e) = c$.

Hence, in particular, for a group G, the right regular action is free.

Note that the right regular action always has a top element: the neutral element $e \in M$. However, this top element may not be unique. In fact, every invertible element of M is a top element, and so, if M is a group, every element is a top element.

If M and N are two monoids, a map $f\colon M \to N$ is a *homomorphism* if $f(e_M) = e_N$ and, for every $a, b \in M$, $f(ab) = f(a)f(b)$.

Definition 2.3. Let M and N be two monoids, and X and Y be an M-set and an N-set, respectively. A *homomorphism* from the M-set X to the N-set Y is a pair $(f, \overline{f})$ consisting of a map $f\colon X \to Y$ and a homomorphism of monoids $\overline{f}\colon M \to N$ such that, for every $a \in M$, the diagram

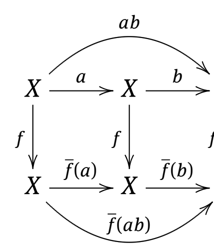


commutes. More explicitly, we require that, for every $x \in X$ and $a \in M$, $f(xa) = f(x)\overline{f}(a)$. Moreover, we say that $(f, \overline{f})$ is

- an *endomorphism* if $X \curvearrowleft M = Y \curvearrowleft N$;
- an *isomorphism* if both f and $\overline{f}$ are bijections;
- an *automorphism* if it is both an endomorphism and an isomorphism.

Note that, if M and N are two monoids and $(f, \overline{f})$ is an isomorphism from the M-set X to the N-set Y, then $(f^{-1}, \overline{f}^{-1})$ is an isomorphism from Y to X.

Remark 2.4. Let $(f, \overline{f})$ be an endomorphism of the M-set X, where M is a monoid. Since $\overline{f}$ is a homomorphism, for every $a, b \in M$, the subdiagrams in the following diagram commute:



Moreover, with a routine argument we can show that, for every $x \in X$, $n \in \mathbb{N}$, and $a \in M$, $f^n(xa) = f^n(x)\overline{f}^n(a)$.

Example 2.5. (a) Let $X \curvearrowleft \{e\}$ be the action of the trivial group on a set X. Then, for every self-map $f\colon X \to X$, the trivial endomorphism $\overline{f}\colon \{e\} \to \{e\}$ shows that $(f, \overline{f})$ is an endomorphism of $X \curvearrowleft \{e\}$.

(b) Let $f\colon M \to N$ be a homomorphism of monoids. In Example 2.2 we described the construction of the right regular action. Let us show how f can induce a homomorphism from $M \curvearrowleft M$ to $N \curvearrowleft N$ in a canonical way. It is enough to consider the pair (f, f) and the desired properties are fulfilled because f is a homomorphism.

(c) Let M be a (left-)cancellative commutative monoid, $G(M)$ be the abelian group generated by M ([5]), and $\imath\colon M \to G(M)$ the inclusion homomorphism. Then the right regular action of M on itself induces an action of M on $G(M)$ as follows: for every $a \in M$, $a\colon g \mapsto g + \imath(a)$. The action on $G(M)$ is free. Moreover, for every endomorphism $f\colon G(M) \to G(M)$ such that $f(\imath(M)) \subseteq \imath(M)$ (i.e., $\imath(M)$ is *f-invariant*), there exists an endomorphism $\overline{f}\colon M \to M$ such that $(f, \overline{f})$ is an endomorphism of $G(M) \curvearrowleft M$. In fact, define, for every $a \in M$, $\overline{f}(a) = \imath^{-1}(f(\imath(a))) \in \imath(M)$, which satisfies, for every $g \in G(M)$,

$$f(ga) = f(g + \imath(a)) = f(g) + f(\imath(a)) = f(g) + \imath(\overline{f}(a)) = f(g)\overline{f}(a).$$

Remark 2.6. Let M and N be two monoids, X be an M-set, Y be an N-set, and $(f, \overline{f})$ be a homomorphism from X to Y. Denote by α_M and α_N the actions on X and Y, respectively. Let Z be a ceiling of X. Then, for every $x \in X$, there exists $z \in Z$ and $a \in M$ such that $x = za$, and so $f(x) = f(za) = f(z)\overline{f}(a)$. Thus $f|_Z$ and $\overline{f}$ uniquely determine the map f.

Conversely, given a map $h\colon Z \to Y$ and a homomorphism $\overline{g}\colon M \to N$, the existence of a map $g\colon X \to Y$ such that $g|_Z = h$ and $(g, \overline{g})$ is a homomorphism from X to Y is not granted in general. In fact, consider the group $X = Z = Y = \mathbb{Z}_2 = \{0, 1\}$, endowed with its right regular action, and the homomorphisms $h = id_X$ and $\overline{g}(G) = \{0\}$. Then the only possible extension g of h is h itself, but $0 = h(0) = h(1 + 1)$, while $h(1) + \overline{g}(1) = 1$.

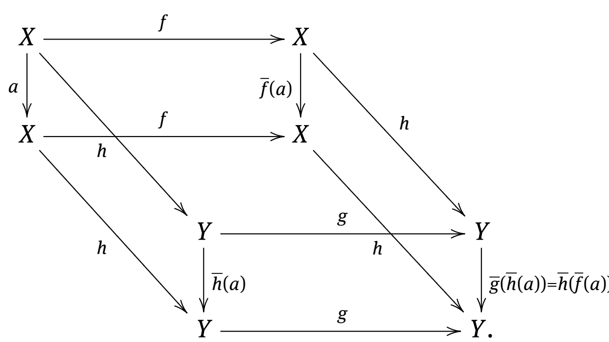
Let us now suppose that for every pair of distinct points of Z, their orbits are disjoint and the homomorphism $\overline{g}$ satisfies the following property: for every $a, b \in M$, $\alpha_N(\overline{g}(a)) = \alpha_N(\overline{g}(b))$ provided that there exists

$x \in X$ such that $\alpha_M(a)(x) = \alpha_M(b)(x)$. Then the desired extension g of h can be defined. In fact, for every $x \in X$, there exists an element $a_x \in M$ and a unique $z_x \in Z$ such that $x = z_x a_x$, and thus we can define $g(x) = h(z_x)\overline{g}(a_x)$. Note that, for every other $a'_x \in M$ such that $x = z_x a'_x$, $\alpha_N(\overline{g}(a_x)) = \alpha_N(\overline{g}(a'_x))$ thanks to the required property on $\overline{g}$.

Let us now introduce the category **FlowMon-Set** of flows of sets endowed with monoid actions. Its objects are quintuplets $(M, X, \alpha, f, \overline{f})$, where M is a monoid, X is an M-set, α is the action of M on X, and $(f, \overline{f})$ is an endomorphism of X. Given two objects $(M, X, \alpha, f, \overline{f})$ and $(N, Y, \beta, g, \overline{g})$ of **FlowMon-Set**, a morphism between them is a homomorphism $(h, \overline{h})$ from $X \curvearrowleft M$ to $Y \curvearrowleft N$ such that the following two squares commute:

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ Y & \xrightarrow{g} & Y, \end{array} \quad \text{and} \quad \begin{array}{ccc} M & \xrightarrow{\overline{f}} & M \\ {\scriptstyle \overline{h}}\downarrow & & \downarrow{\scriptstyle \overline{h}} \\ N & \xrightarrow{\overline{g}} & N. \end{array} \tag{1}$$

Since $(f, \overline{f})$ and $(g, \overline{g})$ are endomorphisms of X and Y, respectively, (1) implies that, for every $a \in M$, all subdiagrams in the following diagram commute:



In the previous notation, the pair $(h, \overline{h})$ is an isomorphism of **FlowMon-Set** if $(h, \overline{h})$ is an isomorphism from $X \curvearrowleft M$ to $Y \curvearrowleft N$, i.e., both h and $\overline{h}$ are bijective.

If there is no risk of ambiguity, in the sequel we denote the objects of **FlowMon-Set** as quadruplets $(M, X, f, \overline{f})$, not explicitly mentioning the action of M on X.

2.1 Relationship between the components of a homomorphism

In this subsection we study the relationship between the two maps f and $\overline{f}$ composing a homomorphism $(f, \overline{f})$ between an M-set and an N-set.

Proposition 2.7. *Let M and N be two monoids, X be an M-set, and Y be an N-set. Let $(f_1, \overline{f})$ and $(f_2, \overline{f})$ be two homomorphisms from X to Y.*
(a) If $x \in X$ satisfies $f_1(x) = f_2(x)$, then $f_1|_{xM} = f_2|_{xM}$.
(b) Let $Z \subseteq X$. Then the following properties are equivalent:
(b_1) Z is a ceiling of X;
(b_2) $f_1|_Z = f_2|_Z$ if and only if $f_1 = f_2$.

Proof. For every $a \in M$, $f_1(xa) = f_1(x)\overline{f}(a) = f_2(x)\overline{f}(a) = f_2(xa)$. Thus item (a) and the implication (b_1)→(b_2) follow.

Let now Z be a subset of X. Define $Y = ZM$. Let i_1 and i_2 be the canonical injections of X in the disjoint union $X \sqcup X$, and $\sim$ be the equivalence relation on $X \sqcup X$ defined as follows: for every $x, y \in X$, $i_1(x) \sim i_2(y)$ if and only if $x = y \in Y$. Let $X \sqcup_Y X = X \sqcup X/_\sim$, $q\colon X \sqcup X \to X \sqcup_Y X$ be the quotient map, and, for $k \in \{1, 2\}$,

$j_k = q \circ i_k$. If α is the action of M on X, we define an action β of M on $X \sqcup_Y X$ as follows: for every $j_k(x) \in X \sqcup_Y X$, $j_k(x) = j_k(xa)$, which is well-defined since $YM = Y$. Note that (j_1, id_M) and (j_2, id_M) are two homomorphisms from $X \curvearrowleft M$ to $X \sqcup_Y X \curvearrowleft M$. If Z is not a ceiling, then $j_1 \neq j_2$, although $j_1|_Y = j_2|_Y$ and so, in particular, $j_1|_Z = j_2|_Z$. □

Corollary 2.8. *Let M and N be two monoids, X be an M-set, Y be an N-set, and $(f_1, \overline{f})$ and $(f_2, \overline{f})$ be two homomorphisms from X to Y. If x is a top element of X, then $f_1 = f_2$ if and only if $f_1(x) = f_2(x)$.*

Corollary 2.9. *Let M and N be two monoids endowed with their right regular actions. Then the following properties are equivalent:*
(a) the pair $(f, \overline{f})$ is a homomorphism from M to N;
(b) there exists $a \in N$ such that $f = s_a^\lambda \circ \overline{f}$, where $s_a^\lambda \colon N \to N$ is the left shift by a*, defined as $s_a^\lambda(x) = ax$, for every $x \in N$.*

Proof. Let $\overline{f} \colon M \to N$ be a homomorphism. Then, for every $a \in N$, $(s_a^\lambda \circ \overline{f}, \overline{f})$ is a homomorphism from $M \curvearrowleft M$ to $N \curvearrowleft N$. Moreover, if $f \colon M \to N$ is another map such that $(f, \overline{f})$ is a homomorphism, then $f = s_{f(e_M)}^\lambda \circ \overline{f}$ according to Corollary 2.8 since e_M is a top element in M (Example 2.2). □

Proposition 2.10. *Let M and N be two monoids, X be an M-set, and Y be an N-set. Let us denote by α the action of N. Let $(f, \overline{f_1})$ and $(f, \overline{f_2})$ be two homomorphisms from X to Y.*
(a) For every $a \in M$, $\alpha \circ \overline{f_1}(a)|_{f(X)} = \alpha \circ \overline{f_2}(a)|_{f(X)}$.
(b) If either f is surjective or α is weakly free, then $\alpha \circ \overline{f_1} = \alpha \circ \overline{f_2}$.
(c) If α is free, then $\overline{f}_1 = \overline{f}_2$.

Proof. Item (a) can be easily deduced since, for every $a \in M$, and $x \in X$, $f(x)\overline{f_1}(a) = f(xa) = f(x)\overline{f_2}(a)$. Then items (b) and (c) trivially follow. □

Corollary 2.11. *Let M be a monoid, X be an M-set, α be the action of M on X, and $(f, \overline{f_1})$ and $(f, \overline{f_2})$ be two endomorphisms of X. Then:*
(a) for every $n \in \mathbb{N}$, $\alpha \circ \overline{f_1}^n = \alpha \circ \overline{f_2}^n$ provided that either f is surjective or α is weakly free;
(b) for every $n \in \mathbb{N}$, $\overline{f_1}^n = \overline{f_2}^n$ provided that α is free.

Proof. Both claims follow from Proposition 2.10 and Remark 2.4. □

The following result immediately descends from Corollary 2.11 and Example 2.2.

Corollary 2.12. *Let M be a left-cancellative monoid endowed with its right regular action, and $(f, \overline{f_1})$ and $(f, \overline{f_2})$ be two endomorphisms of M. Then, for every $n \in \mathbb{N}$, $\overline{f_1}^n = \overline{f_2}^n$.*

3 Algebraic entropy of endomorphisms

Let M be a monoid and X be an M-set. Then, for every $K \subseteq M$ and $Y \subseteq X$, define $YK = \{yk \in X \mid y \in Y, k \in K\}$. This notation will also be used if we consider M with its right regular action. For every set X, denote by $[X]^{<\omega}$ the family of finite subsets of X.

Let M be a monoid, X be an M-set, and $(f, \overline{f})$ be an endomorphism of $X \curvearrowleft M$. For every $n \in \mathbb{N} \setminus \{0\}$, $x \in X$ and $K \in [M]^{<\omega}$, define

$$\mathrm{T}_n(f, \overline{f}, x, K) = xK\overline{f}(K) \cdots \overline{f}^{n-1}(K),$$

which is called the *n-algebraic trajectory of $(f, \overline{f})$ with respect to x and K*. If K contains the neutral element e, then $\{\mathrm{T}_n(f, \overline{f}, x, K)\}_n$ is an increasing sequence of subsets. We now want to define the algebraic entropy of $(f, \overline{f})$.

Definition 3.1. Let M be a monoid, X be an M-set, and $(f,\overline{f})$ be an endomorphism of $X \curvearrowleft M$. For every $x \in X$ and $K \in [M]^{<\omega}$, we define

$$\mathrm{H}_{alg}(f,\overline{f},x,K) = \limsup_{n\to\infty} \frac{\log|\mathrm{T}_n(f,\overline{f},x,K)|}{n},$$

$$\mathrm{H}^{loc}_{alg}(f,\overline{f},x) = \sup\{\mathrm{H}_{alg}(f,\overline{f},x,K) \mid K \in [M]^{<\omega}\}, \quad \text{and} \quad \mathrm{h}_{alg}(f,\overline{f}) = \sup\{\mathrm{H}^{loc}_{alg}(f,\overline{f},x) \mid x \in X\},$$

and $\mathrm{h}_{alg}(f,\overline{f})$ is called the *algebraic entropy of* $(f,\overline{f})$.

In the notations of Definition 3.1, the limit superior in the definition of $\mathrm{H}_{alg}(f,\overline{f},x,K)$ is not a limit in general. We prove it in Remark 4.18.

Example 3.2. Let M be a monoid, X be an M-set, and $(f,\overline{f})$ be an endomorphism of $X \curvearrowleft M$.

(a) If $M = \{e\}$ (as in Example 2.5(a)), then $\mathrm{h}_{alg}(f,\overline{f}) = 0$ since, for every $n \in \mathbb{N} \setminus \{0\}$ and $x \in X$, $\mathrm{T}_n(f,\overline{f},x,\{e\}) = \{x\}$.

(b) If either M or X are finite, then it is easy to see that $\mathrm{h}_{alg}(f,\overline{f}) = 0$.

(c) Suppose now that the orbits of X are finite. Then, for every $x \in X$ and $K \in [M]^{<\omega}$, $\{\mathrm{T}_n(f,\overline{f},x,K)\}_{n\in\mathbb{N}}$ is bounded since $\mathrm{T}_n(f,\overline{f},x,K) \subseteq xM$, for every $n \in \mathbb{N} \setminus \{0\}$. Hence $\mathrm{h}_{alg}(f,\overline{f}) = 0$.

(d) If M is *locally finite* (i.e., every finite subset of M generates a finite submonoid) and $\overline{f} = id_M$, then $\mathrm{h}_{alg}(f, id_M) = 0$. In fact, for every $x \in X$, $K \in [M]^{<\omega}$, and $n \in \mathbb{N} \setminus \{0\}$,

$$|\mathrm{T}_n(f, id_M, x, K)| = |xK^n| \leq |\langle K\rangle| < \infty.$$

In the definition of the algebraic entropy, the morphism $f\colon X \to X$ does not play any explicit role. Hence, the following fact trivially holds.

Fact 3.3. If M is a monoid acting on a set X, and $(f_1,\overline{f})$ and $(f_2,\overline{f})$ are two endomorphisms of X, then $\mathrm{h}_{alg}(f_1,\overline{f}) = \mathrm{h}_{alg}(f_2,\overline{f})$.

The following results are devoted to understand the implications of what we obtained in §2.1, where we discussed the relationship between the components of a homomorphism.

Proposition 3.4. *Let M be a monoid, X be an M-set, and $(f,\overline{f_1})$ and $(f,\overline{f_2})$ be two endomorphisms of X. If, for every $n \in \mathbb{N}$, $\alpha \circ \overline{f_1}^n = \alpha \circ \overline{f_2}^n$, then $\mathrm{h}_{alg}(f,\overline{f_1}) = \mathrm{h}_{alg}(f,\overline{f_2})$.*

Proof. The claim follows once we show that, for every $n \in \mathbb{N} \setminus \{0\}$, $x \in X$, and $K \in [M]^{<\omega}$,

$$\mathrm{T}_n(f,\overline{f_1},x,K) = \mathrm{T}_n(f,\overline{f_2},x,K). \tag{2}$$

Let us prove the desired equality by induction. The case $n = 1$ is trivial. Suppose that (2) holds for a given n and we show that the equality holds also for $n+1$. In fact,

$$\begin{aligned}\mathrm{T}_{n+1}(f,\overline{f_1},x,K) &= x\overline{f_1}(K)\cdots\overline{f_1}^n(K) = \mathrm{T}_n(f,\overline{f_1},x,K)\overline{f_1}^n(K) = \bigcup_{k\in K} \alpha(\overline{f_1}^n(k))(\mathrm{T}_n(f,\overline{f_1},x,K)) = \\ &= \bigcup_{k\in K} \alpha(\overline{f_2}^n(k))(\mathrm{T}_n(f,\overline{f_1},x,K)) = \mathrm{T}_n(f,\overline{f_2},x,K)\overline{f_2}^n(K) = \mathrm{T}_{n+1}(f,\overline{f_2},x,K).\end{aligned}$$

□

Corollary 3.5. *Let M be a monoid, X be an M-set, α be the action of M on X, and $(f,\overline{f_1})$ and $(f,\overline{f_2})$ be two endomorphisms of X. Then $\mathrm{h}_{alg}(f,\overline{f_1}) = \mathrm{h}_{alg}(f,\overline{f_2})$ provided that either f is surjective or α is weakly free.*

Proof. The claim trivially follows from Proposition 3.4 and Corollary 2.11. □

3.1 Basic properties of the algebraic entropy

Let us now enlist some basic properties of the algebraic entropy.

Let X be a set. An *ideal* $\mathcal{I}$ of subsets of X is a family closed under taking subsets and finite unions. For example, for every set X, the family $[X]^{<\omega}$ is an ideal. A subfamily $\mathcal{F} \subseteq \mathcal{I}$ is *cofinal in* $\mathcal{I}$ if, for every $K \in \mathcal{I}$, there exists $F \in \mathcal{F}$ such that $K \subseteq F$.

Proposition 3.6. *Let M be a monoid, X be an M-set and $(f, \overline{f})$ be an endomorphism of X.*

(a) *If $K \subseteq F \in [M]^{<\omega}$, then $\mathrm{H}_{alg}(f, \overline{f}, x, K) \le \mathrm{H}_{alg}(f, \overline{f}, x, F)$, for every $x \in X$. Hence, if $\mathcal{F}$ is a cofinal family of $[M]^{<\omega}$, then*
$$\mathrm{H}_{alg}^{loc}(f, \overline{f}, x) = \sup\{\mathrm{H}_{alg}(f, \overline{f}, x, K) \mid K \in \mathcal{F}\}.$$
(b) *If $x, y \in X$ are two points such that $y \in xM$, then $\mathrm{H}_{alg}^{loc}(f, \overline{f}, x) \ge \mathrm{H}_{alg}^{loc}(f, \overline{f}, y)$.*

(c) *If Y is a ceiling of X, then $\mathrm{h}_{alg}(f, \overline{f}) = \sup_{y \in Y} \mathrm{H}_{alg}^{loc}(f, \overline{f}, y)$. In particular, if x is a top element of X, $\mathrm{h}_{alg}(f, \overline{f}) = \mathrm{H}_{alg}^{loc}(f, \overline{f}, x)$.*

Proof. Item (a) is trivial, and item (c) follows from (b).

(b) Let $x, y \in X$ and $a \in M$ be an element such that $y = xa$. Pick an arbitrary $K \in [M]^{<\omega}$ and define $K' = aK \cup K$. We claim that $\mathrm{H}_{alg}(f, \overline{f}, x, K') \ge \mathrm{H}_{alg}(f, \overline{f}, y, K)$ and thus $\mathrm{H}_{alg}^{loc}(f, \overline{f}, x) \ge \mathrm{H}_{alg}^{loc}(f, \overline{f}, y)$. In fact, for every $n \in \mathbb{N}$,
$$\mathrm{T}_n(f, \overline{f}, x, K') = xK'\overline{f}(K') \cdots \overline{f}^{n-1}(K') \supseteq xaK\overline{f}(K) \cdots \overline{f}^{n-1}(K) = \mathrm{T}_n(f, \overline{f}, y, K).$$
□

Proposition 3.7. *Let M be a monoid, X be an M-set, and $(f, \overline{f})$ be an automorphism of X. Suppose the following further properties:*

(a) *there exists a ceiling Y of X consisting of fixed points for the map f;*

(b) *M is either commutative or a group acting freely.*

Then $\mathrm{h}_{alg}(f^{-1}, \overline{f}^{-1}) = \mathrm{h}_{alg}(f, \overline{f})$.

Proof. According to Proposition 3.6(c), it is enough to evaluate the trajectories on points of the ceiling Y. Let us now notice that, for every $n \in \mathbb{N} \setminus \{0\}$, $x \in Y$, and $K \in [M]^{<\omega}$,
$$\begin{aligned} f^{n-1}(\mathrm{T}_n(f^{-1}, \overline{f}^{-1}, x, K)) &= f^{n-1}(xK\overline{f}^{-1}(K) \cdots \overline{f}^{-n+1}(K)) = f^{n-1}(x)\overline{f}^{n-1}(K) \cdots \overline{f}(K)K = \\ &= x\overline{f}^{n-1}(K) \cdots \overline{f}(K)K, \end{aligned}$$
because of the hypothesis (a). Hence, since f is bijective,
$$|\mathrm{T}_n(f^{-1}, \overline{f}^{-1}, x, K)| = |x\overline{f}^{n-1}(K) \cdots \overline{f}(K)K|. \tag{3}$$
If M is commutative, then (3) implies that $|\mathrm{T}_n(f^{-1}, \overline{f}^{-1}, x, K)| = |\mathrm{T}_n(f, \overline{f}, x, K)|$, which leads to the thesis. Suppose now that M is a group acting freely on X. According to Proposition 3.6(a), we can assume without loss of generality that $K = K^{-1}$. Hence, (3) and the fact that the action is free imply that
$$\begin{aligned} |\mathrm{T}_n(f^{-1}, \overline{f}^{-1}, x, K)| &= |x\overline{f}^{n-1}(K) \cdots \overline{f}(K)K| = |\overline{f}^{n-1}(K) \cdots \overline{f}(K)K| = |(\overline{f}^{n-1}(K) \cdots \overline{f}(K)K)^{-1}| = \\ &= |K\overline{f}(K) \cdots \overline{f}^{n-1}(K)| = |\mathrm{T}_n(f, \overline{f}, x, K)|, \end{aligned}$$
from which the claim descends. □

Proposition 3.8 (Weak logarithmic law)**.** *Let M be a monoid, X be an M-set, and $(f, \overline{f})$ be an endomorphism of X.*

(a) *For every $k \in \mathbb{N} \setminus \{0\}$, $\mathrm{h}_{alg}(f^k, \overline{f}^k) \le k\, \mathrm{h}_{alg}(f, \overline{f})$.*

(b) *Suppose that X has a ceiling consisting of fixed points for the map f, and M is either commutative or a group acting freely. Then, if* $(f, \overline{f})$ *is an automorphism, for every* $k \in \mathbb{Z} \setminus \{0\}$, $\mathrm{h}_{alg}(f^k, \overline{f}^k) \leq |k|\, \mathrm{h}_{alg}(f, \overline{f})$.

Proof. (a) Fix $k \in \mathbb{N} \setminus \{0\}$. Then, for every $n \in \mathbb{N} \setminus \{0\}$, $x \in X$ and $e \in K \in [M]^{<\omega}$,

$$\frac{\log|\mathrm{T}_n(f^k, \overline{f}^k, x, K)|}{n} = \frac{\log|xK\overline{f}^k(K)\cdots\overline{f}^{k(n-1)}(K)|}{n} \leq \frac{\log|\mathrm{T}_{kn-k+1}(f, \overline{f}, x, K)|}{n} = \\ = \frac{\log|\mathrm{T}_{kn-k+1}(f, \overline{f}, x, K)|}{kn-k+1} \cdot \frac{kn-k+1}{n}.$$

Hence $\mathrm{H}_{alg}(f^k, \overline{f}^k, x, K) \leq k\, \mathrm{H}_{alg}(f, \overline{f}, x, K)$, and so the claim follows in virtue of Proposition 3.6(a) and (c).

(b) It follows from item (a) and Proposition 3.7. In fact, if $k \in \mathbb{N} \setminus \{0\}$, then $\mathrm{h}_{alg}(f^k, \overline{f}^k) \leq k\, \mathrm{h}_{alg}(f, \overline{f})$ because of item (a), otherwise, if $k \in \mathbb{Z} \setminus \mathbb{N}$, then $\mathrm{h}_{alg}(f^k, \overline{f}^k) = \mathrm{h}_{alg}(f^{-k}, \overline{f}^{-k}) \leq -k\, \mathrm{h}_{alg}(f, \overline{f})$. □

Theorem 3.9. *Let* $(M, X, f, \overline{f})$ *and* $(N, Y, g, \overline{g})$ *be two objects of* **FlowMon-Set**, *and* $(h, \overline{h})$ *be a morphism of the category* **FlowMon-Set** *from* $(M, X, f, \overline{f})$ *and* $(N, Y, g, \overline{g})$.
(a) *If h is injective, then* $\mathrm{h}_{alg}(f, \overline{f}) \leq \mathrm{h}_{alg}(g, \overline{g})$.
(b) *If h and* $\overline{h}$ *are surjective, then* $\mathrm{h}_{alg}(f, \overline{f}) \geq \mathrm{h}_{alg}(g, \overline{g})$.
(c) *If h is bijective and* $\overline{h}$ *is surjective, then* $\mathrm{h}_{alg}(f, \overline{f}) = \mathrm{h}_{alg}(g, \overline{g})$.

Proof. Item (c) trivially follows from items (a) and (b). To prove them, let $n \in \mathbb{N} \setminus \{0\}$, $x \in X$, and $K \in [M]^{<\omega}$. Then

$$\begin{aligned} h(\mathrm{T}_n(f, \overline{f}, x, K)) &= h(xK\overline{f}(K)\cdots\overline{f}^{n-1}(K)) = h(x)\overline{h}(K\overline{f}(K)\cdots\overline{f}^{n-1}(K)) = \\ &= h(x)\overline{h}(K)\overline{h}(\overline{f}(K))\cdots\overline{h}(\overline{f}^{n-1}(K)) = h(x)\overline{h}(K)\overline{g}(\overline{h}(K))\cdots\overline{g}^{n-1}(\overline{h}(K)) = \\ &= \mathrm{T}_n(g, \overline{g}, h(x), \overline{h}(K)). \end{aligned} \tag{4}$$

Suppose that h is injective. Then (4) implies that $|\mathrm{T}_n(f, \overline{f}, x, K)| = |\mathrm{T}_n(g, \overline{g}, h(x), \overline{h}(K))|$, and thus

$$\mathrm{H}_{alg}(f, \overline{f}, x, K) = \mathrm{H}_{alg}(g, \overline{g}, h(x), \overline{h}(K)). \tag{5}$$

Since the inequality (5) holds for every $x \in X$ and $K \in [M]^{<\omega}$, $\mathrm{h}_{alg}(f, \overline{f}) \leq \mathrm{h}_{alg}(g, \overline{g})$.

Suppose otherwise that h and $\overline{h}$ are surjective. Then, for every $y \in Y$ and every $F \in [N]^{<\omega}$, there exists $x \in X$ and $K \in [M]^{<\omega}$ such that $h(x) = y$ and $\overline{h}(K) = F$. Then, according to (4),

$$|\mathrm{T}_n(g, \overline{g}, y, F)| = |\mathrm{T}_n(g, \overline{g}, h(x), \overline{h}(K))| = |h(\mathrm{T}_n(f, \overline{f}, x, K))| \leq |\mathrm{T}_n(f, \overline{f}, x, K)|.$$

Thus $\mathrm{H}_{alg}(g, \overline{g}, y, F) \leq \mathrm{H}_{alg}(f, \overline{f}, x, K)$, and so, since $y \in Y$ and $F \in [N]^{<\omega}$ can be taken arbitrarily, $\mathrm{h}_{alg}(g, \overline{g}) \leq \mathrm{h}_{alg}(f, \overline{f})$. □

Remark 3.10. In the notation of Theorem 3.9, let us note that the injectivity of h implies that $\overline{h}$ satisfies a mild version of injectivity. More precisely, we claim that, for every $a, b \in M$, if $\beta(\overline{h}(a)) = \beta(\overline{h}(b))$, then $\alpha(a) = \alpha(b)$. In fact, $\beta(\overline{h}(a)) = \beta(\overline{h}(b))$ implies that, for every $x \in X$, $h(xa) = h(x)\overline{h}(a) = h(x)\overline{h}(b) = h(xb)$, and so the claim follows since h is injective.

The following consequences of Theorem 3.9 can be deduced.

Corollary 3.11 (Invariance under conjugation)**.** *The algebraic entropy is invariant along isomorphisms of the category* **FlowMon-Set**.

Corollary 3.12 (Monotonicity for subspaces)**.** *Let M be a monoid, X be an M-set, and* $(f, \overline{f})$ *be a endomorphism of X. Suppose that Y is a subset of X and N is a submonoid of M satisfying the following properties:*
(a) $YN = Y$;
(b) $f(Y) \subseteq Y$ *(i.e., Y is f-invariant);*

(c) $\overline{f}(N) \leq N$ *(i.e., N is* $\overline{f}$*-invariant).*

Then $(f|_Y, \overline{f}|_N)$ *is an endomorphism of* $Y \curvearrowleft N$*, and* $\mathrm{h}_{alg}(f|_Y, \overline{f}|_N) \leq \mathrm{h}_{alg}(f, \overline{f})$.

Proof. If we denote by h the inclusion of Y in X and by $\overline{h}$ the inclusion of N in M, we can apply Theorem 3.9 to obtain the desired result. □

Let $q\colon X \to Y$ be a map. For every $E \subseteq X \times X$, we define $(q \times q)(E) = \{(q(x), q(y)) \in Y \times Y \mid (x, y) \in E\}$ and $R_q = \{(x, y) \in X \times X \mid q(x) = q(y)\}$.

Corollary 3.13 (Monotonicity for quotients)**.** *Let M and N be two monoids, X and Y be an M-set and an N-set, respectively,* $(f, \overline{f})$ *be an endomorphism of* $X \curvearrowleft M$*,* $q\colon X \to Y$ *be a surjective map and* $p\colon M \to N$ *be a surjective homomorphism. Suppose that the following properties hold:*

(a) $(f \times f)(R_q) \subseteq R_q$;

(b) $(\overline{f} \times \overline{f})(R_p) \subseteq R_p$;

(c) the pair (q, p) *is a homomorphism from* $X \curvearrowleft M$ *to* $Y \curvearrowleft N$.

Then there exists an endomorphism $(g, \overline{g})$ *of* $Y \curvearrowleft N$ *making the pair* (q, p) *a morphism of the category* **FlowMon-Set** *from* $(M, X, f, \overline{f})$ *to* $(N, Y, g, \overline{g})$*. Moreover,* $\mathrm{h}_{alg}(g, \overline{g}) \leq \mathrm{h}_{alg}(f, \overline{f})$.

Proof. Let $y \in Y$ and $b \in N$. Since q and p are surjective, then there exist $x \in X$ and $a \in M$ such that $q(x) = y$ and $p(a) = b$. Then we define $g(y) = g(q(x)) = q(f(x))$ and $\overline{g}(b) = \overline{g}(p(a)) = p(\overline{f}(a))$, and these two maps are well-defined because of the properties (a) and (b), respectively. Moreover, $\overline{g}$ is an endomorphism of N since both p and $\overline{f}$ are homomorphisms. Then it is easy to check that (q, p) has the desired properties. The last claim follows from Theorem 3.9(b). □

Let us specify Corollary 3.13 in some particular situations, in order to get a better understanding of the hypotheses (a)–(c).

Corollary 3.14. *Let* $q\colon M \to N$ *be a surjective homomorphism of monoids, and f be an endomorphism of M. Moreover, suppose that* $(f \times f)(R_q) \subseteq R_q$*. Then there exists an endomorphism g of N such that* (q, q) *is a morphism of* **FlowMon-Set** *from* (M, M, ρ, f, f) *to* (N, N, ρ, g, g)*. Moreover,* $\mathrm{h}_{alg}(g, g) \leq \mathrm{h}_{alg}(f, f)$.

Proof. In order to apply Corollary 3.13, it is enough to check that the properties (a)–(c) hold. Since $(f \times f)(R_q) \subseteq R_q$, both items (a) and (b) are fulfilled. Moreover, item (c) follows from Example 2.5(b). □

Note that, in the notation of Corollary 3.14, if M and N are groups, the request that $(f \times f)(R_q) \subseteq R_q$ is equivalent to asking that $\ker q$ is f-invariant.

Corollary 3.15. *Let M be a monoid, X be an M-set,* $q\colon X \to Y$ *be a surjective map between sets, and* $(f, \overline{f})$ *be an endomorphism of X. Suppose that the following properties hold:*

(a) $(f \times f)(R_q) \subseteq R_q$;

(b) for every $a \in M$, $(a \times a)(R_q) \subseteq R_q$.

Then there exists an action of M on Y and a map $g\colon Y \to Y$ *such that* $(g, \overline{f})$ *is an endomorphism of* $Y \curvearrowleft M$*, and* (q, id_M) *is a morphism of* **FlowMon-Set** *from* $(M, X, f, \overline{f})$ *to* $(M, Y, g, \overline{f})$*. Moreover,* $\mathrm{h}_{alg}(g, \overline{f}) \leq \mathrm{h}_{alg}(f, \overline{f})$.

Proof. First of all, because of item (b), we can define an action of M on Y as follows: if y is a point of Y and $x \in X$ satisfies $q(x) = y$, then $ya = q(x)a = q(xa)$, for every $a \in M$. Because of the definition, it is easy to check that (q, id_M) is actually a homomorphism from $X \curvearrowleft M$ to $Y \curvearrowleft M$. Thus items (a) and (c) of Corollary 3.13 are fulfilled. Moreover, item (b) is trivial, and thus the claim follows from Corollary 3.13 since the map $\overline{g}$ defined in the proof coincides with $\overline{f}$. □

For every set X, denote by Δ_X the *diagonal of* X, i.e., the family $\Delta_X = \{(x, x) \in X \times X \mid x \in X\}$.

Corollary 3.16. *Let $p: M \to N$ be a surjective homomorphism of monoids, α be an action of M on X, and $(f, \overline{f})$ be an endomorphism of $X \curvearrowleft M$. Suppose that the following properties hold:*
(a) $(\overline{f} \times \overline{f})(R_p) \subseteq R_p$;
(b) $(\alpha \times \alpha)(R_p) \subseteq \Delta_{X^X}$.

Then there exists an action β of N on X and an endomorphism $\overline{g}$ of N such that $(f, \overline{g})$ is an endomorphism of $X \curvearrowleft N$, and (id_X, p) is a morphism of **FlowMon-Set** *from $(M, X, \alpha, f, \overline{f})$ to $(N, X, \beta, f, \overline{g})$. Moreover, $\mathrm{h}_{alg}(f, \overline{g}) \leq \mathrm{h}_{alg}(f, \overline{f})$.*

Proof. Let us define the action β. Let $b \in N$ and $a \in M$ be an element such that $p(a) = b$. Then we put $\beta(b) = \beta(p(a)) = \alpha(a)$, which is well-defined because of item (b). Because of the definition of β, (id_X, p) is a homomorphism from $X \curvearrowleft M$ to $X \curvearrowleft N$. Thus the properties (b) and (c) of Corollary 3.13 are satisfied, while item (a) is trivial. Then the claim follows once we notice that the map g defined in the proof of Corollary 3.13 coincides with f. □

Let us consider an application of Corollary 3.16.

Remark 3.17. Let M be a monoid with a weakly free action α on a set X, and $(f, \overline{f})$ be an endomorphism of $X \curvearrowleft M$. We can introduce an equivalence relation $\sim_X$ on M as follows: for every $a, b \in M$, $a \sim_X b$ if and only if $\alpha(a) = \alpha(b)$. It is not hard to check that $\sim_X$ is actually a *congruence* (i.e., an equivalence relation such that, for every $a, b, c, d \in M$, if $a \sim_X b$ and $c \sim_X d$, then $ac \sim_X bd$), and thus we can consider the quotient monoid $M/_{\sim_X}$. Let $p: M \to M/_{\sim}$ denote the quotient map. It is easy to check that the hypothesis of Corollary 3.16 is fulfilled. Thus there exist an action β of $M/_{\sim_X}$ on X and an endomorphism $\overline{g}$ of $M/_{\sim_X}$ such that (id_X, p) is a morphism from $(M, X, \alpha, f, \overline{f})$ to $(M/_{\sim_X}, X, \beta, f, \overline{g})$. Moreover, note that β is free according to its definition in the proof of Corollary 3.16.

The mentioned corollary also implies that $\mathrm{h}_{alg}(f, \overline{g}) \leq \mathrm{h}_{alg}(f, \overline{f})$. We claim that, in this setting, also the opposite inequality holds, and thus $\mathrm{h}_{alg}(f, \overline{g}) = \mathrm{h}_{alg}(f, \overline{f})$. Let $K \in [M]^{<\omega}$ and $x \in X$. Then, since (id_X, p) is a morphism from $(M, X, \alpha, f, \overline{f})$ to $(M/_{\sim_X}, X, \beta, f, \overline{g})$, for every $n \in \mathbb{N} \setminus \{0\}$, we have that

$$\begin{aligned} \mathrm{T}_n(f, \overline{g}, x, p(K)) &= xp(K)\overline{g}(p(K)) \cdots (\overline{g})^{n-1}(p(K)) = xp(K)p(\overline{f}(K)) \cdots p(\overline{f}^{n-1}(K)) = \\ &= id_X(x)p(K\overline{f}(K) \cdots \overline{f}^{n-1}(K)) = id_X(xK\overline{f}(K) \cdots \overline{f}^{n-1}(K)) = \mathrm{T}_n(f, \overline{f}, x, K). \end{aligned}$$

The previous chain implies that $\mathrm{H}_{alg}(f, \overline{f}, x, K) = \mathrm{H}_{alg}(f, \overline{g}, x, p(K))$, and thus the claim follows since $x \in X$ and $K \in [M]^{<\omega}$ can be arbitrarily taken.

3.2 Basic properties of some categorical constructions

Let M and N be two monoids, and X and Y be an M-set and an N-set respectively. Then we define the *product action* of $M \times N$ on $X \times Y$ as follows: for every $(x, y) \in X \times Y$ and $(a, b) \in M \times N$, $(x, y)(a, b) = (xa, yb)$. Moreover, if $(f, \overline{f})$ and $(g, \overline{g})$ are two endomorphisms of $X \curvearrowleft M$ and $Y \curvearrowleft N$, respectively, then the pair $(f \times g, \overline{f} \times \overline{g})$ is an endomorphism of $X \times Y \curvearrowleft M \times N$. In fact, for every $(x, y) \in X \times Y$ and $(a, b) \in M \times N$,

$$(f \times g)((x, y)(a, b)) = (f \times g)(xa, yb) = (f(xa), g(yb)) = (f(x)\overline{f}(a), g(y)\overline{g}(b)) = ((f \times g)(x, y))((\overline{f} \times \overline{g})(a, b)).$$

Theorem 3.18 (Weak addition theorem)**.** *Let M and N be two monoids, X be an M-set, Y be an N-set, and $(f, \overline{f})$ and $(g, \overline{g})$ be endomorphisms of X and Y, respectively. Then $\mathrm{h}_{alg}(f \times g, \overline{f} \times \overline{g}) = \mathrm{h}_{alg}(f, \overline{f}) + \mathrm{h}_{alg}(g, \overline{g})$.*

Proof. For every $n \in \mathbb{N} \setminus \{0\}$, $(x, y) \in X \times Y$, and $F \times K \in [M \times N]^{<\omega}$, where $F \in [M]^{<\omega}$ and $K \in [N]^{<\omega}$, it is easy to check that

$$\mathrm{T}_n(f \times g, \overline{f} \times \overline{g}, (x, y), F \times K) = \mathrm{T}_n(f, \overline{f}, x, F) \times \mathrm{T}_n(g, \overline{g}, y, K),$$

and thus $\mathrm{H}_{alg}(f \times g, \overline{f} \times \overline{g}, (x, y), F \times K) = \mathrm{H}_{alg}(f, \overline{f}, x, F) + \mathrm{H}_{alg}(g, \overline{g}, y, K)$. Then, according to Proposition 3.6(a), $\mathrm{H}^{loc}_{alg}(f \times g, \overline{f} \times \overline{g}, (x, y)) = \mathrm{H}^{loc}_{alg}(f, \overline{f}, x) + \mathrm{H}^{loc}_{alg}(g, \overline{g}, y)$, and so the claim follows. □

Let $\{M_k\}_{k\in I}$ be a family of monoids and, for every $k \in I$, X_k be an M_k-set. For every $k \in I$, denote by $i_k\colon X_k \to \bigsqcup_k X_k$ the canonical inclusion of X_k in the disjoint union. We define the *coproduct action* of $\Pi_k M_k$ on $\bigsqcup_k X_k$ as follows: for every $i_j(x) \in \bigsqcup_k X_k$ and every $(a_k)_k \in \Pi_k M_k$, $i_j(x)(a_k)_k = i_j(xa_j)$. Moreover, if, for every $k \in I$, $(f_k, \overline{f_k})$ is an endomorphism of $X_k \curvearrowleft M_k$, we define the map $\bigoplus_{k\in I} f_k\colon \bigsqcup_k X_k \to \bigsqcup_k X_k$ by the law $(\bigoplus_{k\in I} f_k)(i_j(x)) = i_j(f_j(x))$, for every $i_j(x) \in \bigsqcup_k X_k$. Then the pair $(\bigoplus_k f_k, \Pi_k\overline{f_k})$ is an endomorphism of $\bigsqcup_k X_k \curvearrowleft \Pi_k M_k$. In fact, for every $i_j(x) \in \bigsqcup_k X_k$ and $(a_k)_k \in \Pi_k M_k$,

$$\begin{aligned}\bigoplus_{k\in I} f_k(i_j(x)(a_k)_k) &= \bigoplus_{k\in I} f_k(i_j(xa_j)) = i_j(f_j(xa_j)) = i_j(f_j(x)\overline{f_j}(a_j)) = i_j(f_j(x))(\overline{f_k}(a_k))_k = \\ &= \left(\bigoplus_{k\in I} f_k(i_j(x))\right)\left(\left(\prod_{k\in I}\overline{f_k}\right)((a_k)_k)\right).\end{aligned}$$

Theorem 3.19. *Let $\{M_k\}_{k\in I}$ be a family of monoids, and, for every $k \in I$, X_k and $(f_k, \overline{f}_k)$ be an M_k-set and an endomorphism of X_k, respectively. Then* $\mathrm{h}_{alg}(\bigoplus_k f_k, \Pi_k\overline{f_k}) = \sup\{\mathrm{h}_{alg}(f_k, \overline{f_k}) \mid k \in I\}$.

Proof. For every $k \in I$, denote by $q_k\colon \Pi_k M_k \to M_k$ the canonical projection. It is easy to check that, because of the definition of the coproduct action, for every $K \in [\Pi_k M_k]^{<\omega}$ and every $i_j(x) \in \bigsqcup_k X_k$, $i_j(x)K = i_j(xq_j(K))$. Then, for every $n \in \mathbb{N}\setminus\{0\}$, $i_j(x) \in \bigsqcup_k X_k$ and $K \in [\Pi_k M_k]^{<\omega}$,

$$\begin{aligned}\mathrm{T}_n\left(\bigoplus_{k\in I} f_k, \prod_{k\in I}\overline{f_k}, i_j(x), K\right) &= i_j\left(xq_j(K\left(\prod_{k\in I}\overline{f_k}(K)\right)\cdots\left(\prod_{k\in I}\overline{f_k}^{\,n-1}(K)\right))\right) = \\ &= i_j\left(xq_j(K)q_j\left(\prod_{k\in I}\overline{f_k}(K)\right)\cdots q_j\left(\prod_{k\in I}\overline{f_k}^{\,n-1}(K)\right)\right) = i_j(\mathrm{T}_n(f_j, \overline{f_j}, x, q_j(K))),\end{aligned}$$

which implies that $\mathrm{H}_{alg}(\bigoplus_k f_k, \Pi_k\overline{f_k}, i_j(x), K) = \mathrm{H}_{alg}(f_j, \overline{f_j}, x, q_j(K))$. Hence, we obtain the inequality $\mathrm{H}^{loc}_{alg}(\bigoplus_k f_k, \Pi_k\overline{f_k}, i_j(x)) \le \mathrm{H}^{loc}_{alg}(f_j, \overline{f_j}, x)$. Moreover, since q_j is surjective, every finite subset of M_j is the image of a finite subset of $\Pi_k X_k$, and so $\mathrm{H}^{loc}_{alg}(\bigoplus_k f_k, \Pi_k\overline{f_k}, i_j(x)) = \mathrm{H}^{loc}_{alg}(f_j, \overline{f_j}, x)$. Finally, note that

$$\begin{aligned}\mathrm{h}_{alg}\left(\bigoplus_{k\in I} f_k, \prod_{k\in I}\overline{f}_k\right) &= \sup_{i_j(x)\in\bigsqcup_k X_k} \mathrm{H}^{log}_{alg}\left(\bigoplus_{k\in I} f_k, \prod_{k\in I}\overline{f}_k, i_j(x)\right) = \sup_{j\in I}\sup_{x\in X_k} \mathrm{H}^{loc}_{alg}\left(\bigoplus_{k\in I} f_k, \prod_{k\in I}\overline{f}_k, i_j(x)\right) = \\ &= \sup_{j\in I}\sup_{x\in X_k} \mathrm{H}^{loc}_{alg}(f_j, \overline{f}_j, x) = \sup_{j\in I}\mathrm{h}_{alg}(f_j, \overline{f}_j).\end{aligned}$$

□

4 Relationship with other entropies

Let us compare the algebraic entropy introduced in the previous section with other known entropies.

4.1 Relationship with the algebraic entropy of group endomorphisms

Let M be a monoid. Let $f\colon M \to M$ be an endomorphism of M. Fix a finite subset $K \in [M]^{<\omega}$, and $n \in \mathbb{N}\setminus\{0\}$. Then we define the subset

$$\mathrm{T}^{alg}_n(f, K) = Kf(K)\cdots f^{n-1}(K) \subseteq M.$$

Definition 4.1 ([8], for group endomorphisms)**.** Let M be a monoid, and $f\colon M \to M$ be an endomorphism of M. Then the *algebraic entropy of f with respect to K* is defined as

$$\mathrm{H}_{alg}(f, K) = \lim_{n\to\infty}\frac{\log|\mathrm{T}^{alg}_n(f, K)|}{n}. \tag{6}$$

Finally, the *algebraic entropy of f* is $\mathrm{h}_{alg}(f) = \sup\{\mathrm{H}_{alg}(f, K) \mid K \in [M]^{<\omega}\}$.

A standard approach to prove that the limit in (6) exists is by using Fekete's Lemma (see, for example, [8]). We refer to [8] for a comprehensive survey on the algebraic entropy of monoid and group endomorphisms.

Theorem 4.2. *Let M be a monoid, X be an M-set, and $(f,\overline{f})$ be an endomorphism of X. Then*

$$\mathrm{h}_{alg}(f,\overline{f}) \leq \mathrm{h}_{alg}(\overline{f}).$$

Moreover, if the action of M on X is free,

$$\mathrm{h}_{alg}(f,\overline{f}) = \mathrm{h}_{alg}(\overline{f}).$$

Proof. Let $x \in X$ and $K \in [M]^{<\omega}$. Then, for every $n \in \mathbb{N} \setminus \{0\}$,

$$|\mathrm{T}_n(f,\overline{f},x,K)| = |xK\overline{f}(K)\cdots\overline{f}^{n-1}(K)| \leq |K\overline{f}(K)\cdots\overline{f}^{n-1}(K)| = |\mathrm{T}_n^{alg}(\overline{f},K)|, \tag{7}$$

which implies that $\mathrm{H}_{alg}(f,\overline{f},x,K) \leq \mathrm{H}_{alg}(\overline{f},K)$. Since x and K can be taken arbitrarily, $\mathrm{h}_{alg}(f,\overline{f}) \leq \mathrm{h}_{alg}(\overline{f})$.

If the action of M is free, for every $x \in X$, $K \in [M]^{<\omega}$, and $n \in \mathbb{N} \setminus \{0\}$, (7) becomes a chain of equalities and thus the desired claim can be deduced. □

Let us specialise the previous result for endomorphisms of monoids equipped with their right regular actions.

Theorem 4.3. *Let M be a monoid and f be an endomorphism of M. If we endow M with its right regular action, then*

$$\mathrm{h}_{alg}(f,f) = \mathrm{h}_{alg}(f).$$

Proof. Since $e \in M$ is a top element, $\mathrm{h}_{alg}(f,f) = \mathrm{H}_{alg}^{loc}(f,f,e)$ according to Proposition 3.6(c). Then the conclusion follows from the observation that, for every $K \in [M]^{<\omega}$ and every $n \in \mathbb{N} \setminus \{0\}$, $\mathrm{T}_n(f,f,e,K) = \mathrm{T}_n^{alg}(f,K)$. □

The previous corollary proves that the algebraic entropy of endomorphisms of sets endowed with monoid actions extends the usual algebraic entropy of monoid endomorphisms. Hence, we can see the results proved in §3 as generalisations of results known for the usual algebraic entropy ([8, 10]).

Example 4.4. (a) Let M be a left-cancellative monoid endowed with its right regular action. According to Corollary 2.9, every morphism $(f,\overline{f})$ of M is of the form $(s_a^\lambda \circ \overline{f},\overline{f})$, for some $a \in M$. Then Theorem 4.2 implies that $\mathrm{h}_{alg}(s_a^\lambda \circ \overline{f},\overline{f}) = \mathrm{h}_{alg}(\overline{f})$. Hence, as one may expect, $\mathrm{h}_{alg}(s_a^\lambda, id_M) = \mathrm{h}_{alg}(id_M)$ which is 0 if, for example, M is an abelian group ([9]). Moreover, for every $k \in \mathbb{N} \setminus \{0\}$ and every $n \in \mathbb{Z}$, the map $f_{k,n}\colon \mathbb{Z} \to \mathbb{Z}$ defined as $f_{k,n}(x) = kx + n$, for every $x \in \mathbb{Z}$, and the endomorphism $f_k = f_{k,0}\colon \mathbb{Z} \to \mathbb{Z}$ satisfy $\mathrm{h}_{alg}(f_{k,n}, f_k) = \mathrm{h}_{alg}(f_k) = \log k$ ([9, Example 3.1]).

(b) Consider the action $\mathbb{Z} \curvearrowleft \mathbb{N}$, defined as in Example 2.5(c), i.e., for every $n \in \mathbb{N}$, $n\colon x \mapsto x + n$, for every $x \in \mathbb{Z}$. Let $k \in \mathbb{N}$ and consider the endomorphism $f_k\colon x \mapsto kx$, where $x \in \mathbb{Z}$, of $\mathbb{Z}$. Consistently with Example 2.5(c), define $\overline{f_k}\colon \mathbb{N} \to \mathbb{N}$ such that $\overline{f_k}(n) = kn$, for every $n \in \mathbb{N}$. Then, since the action of $\mathbb{N}$ is free, $\mathrm{h}_{alg}(f_k,\overline{f_k}) = \mathrm{h}_{alg}(\overline{f_k})$. Moreover, by easily adapting the classical proof showing that $\mathrm{h}_{alg}(f_k) = \log k$ (see, for example [9, Example 3.1]), it is possible to prove that also $\mathrm{h}_{alg}(\overline{f_k}) = \log k$. Hence, $\mathrm{h}_{alg}(f) = \mathrm{h}_{alg}(\overline{f_k}) = \mathrm{h}_{alg}(f_k,\overline{f_k})$.

Example 4.4(b) inspires the following question.

Question 4.5. *Let M be a left-cancellative commutative monoid, $G(M)$ be its associated group, f be an endomorphism of $G(M)$ such that M is f-invariant, and $\overline{f}\colon M \to M$ the induced endomorphism defined as in Example 2.5(c). Is it true that $\mathrm{h}_{alg}(f) = \mathrm{h}_{alg}(\overline{f}) = \mathrm{h}_{alg}(f,\overline{f})$?*

4.2 Relationship with the coarse entropy

Let X be a set, $U, V \subseteq X \times X$, $x \in X$, and $A \subseteq X$. Then we define

$$U \circ V = \{(x,z) \in X \times X \mid \exists y \in X : (x,y) \in U, (y,z) \in V\}, \quad U^{-1} = \{(y,x) \in X \times X \mid (x,y) \in U\},$$
$$U[x] = \{y \in X \mid (x,y) \in U\}, \quad \text{and} \quad U[A] = \bigcup \{U[a] \mid a \in A\}.$$

Moreover, note that $(U \circ V)[A] = V[U[A]]$.

Definition 4.6 ([31, 36]). Let X be a set. A family $\mathcal{E} \subseteq \mathcal{P}(X)$ is a *quasi-coarse structure* if it satisfies the following properties:
(C_1) $\mathcal{E}$ is an ideal;
(C_2) $\Delta_X \in \mathcal{E}$;
(C_3) for every $E, F \in \mathcal{E}$, $E \circ F \in \mathcal{E}$.

If, moreover, $\mathcal{E}$ satisfies
(C_4) $E^{-1} \in \mathcal{E}$, for every $E \in \mathcal{E}$,

then $\mathcal{E}$ is a *coarse structure*. The pair $(X, \mathcal{E})$ is a *quasi-coarse space* (*coarse space*) if $\mathcal{E}$ is a quasi-coarse structure (coarse structure, respectively).

Let $(X, \mathcal{E})$ be a quasi-coarse space. Then $(X, \mathcal{E})$ is *locally finite* if, for every $x \in X$ and every $E \in \mathcal{E}$, $E[x]$ is finite. Moreover, $(X, \mathcal{E})$ has *bounded geometry* if, for every $E \in \mathcal{E}$, there exists N_E such that $|E[x]| \leq N_E$, for every $x \in X$. A quasi-coarse space with bounded geometry is, in particular, locally finite.

A map $f: (X, \mathcal{E}_X) \to (Y, \mathcal{E}_Y)$ between quasi-coarse spaces is called
- *bornologous* if, for every $E \in \mathcal{E}_X$, $(f \times f)(E) \in \mathcal{E}_Y$;
- an *asymorphism* if it is bijective and both f and f^{-1} are bornologous.

Example 4.7. (a) Let (X, d) be a *quasi-metric space*, i.e., a set X endowed with a *pseudo-quasi-metric* (for the sake of simplicity, we refer to it as *quasi-metric* in the sequel) d, which is a map $d: X \times X \to \mathbb{R}$ such that
   - $d(x, x) = 0$, for every $x \in X$;
   - $d(x, z) \leq d(x, y) + d(y, z)$, for every $x, y, z \in X$.

   Then d induces a quasi-coarse structure $\mathcal{E}_d$ on X, called *metric-quasi-coarse structure*, as follows. We define, for every $R \geq 0$, $E_R = \bigcup_{x \in X}(\{x\} \times B(x, R))$, where $B(x, R)$ denotes the closed ball centred in x with radius R, and then $\mathcal{E}_d = \{E \subseteq X \times X \mid \exists R \geq 0 : E \subseteq E_R\}$. If d is a metric, then $\mathcal{E}_d$ is a coarse structure.
   A quasi-coarse space $(X, \mathcal{E})$ is *quasi-metrisable* (a coarse space $(X, \mathcal{E})$ is *metrisable*) if there exists a quasi-metric d (a metric d, respectively) on X such that $\mathcal{E} = \mathcal{E}_d$. Moreover, a quasi-coarse space $(X, \mathcal{E})$ is quasi-metrisable if and only if $\mathcal{E}$ contains a countable cofinal family ([36]). Similarly, a coarse space $(X, \mathcal{E})$ is metrisable if and only if $\mathcal{E}$ contains a countable cofinal family ([30, 31]).

(b) Let M be a monoid and X be an M-set. Then X can be endowed with a canonical quasi-coarse structure $\mathcal{E}_M$ induced by M as follows: we define, for every $K \in [M]^{<\omega}$,

$$E_K = \bigcup_{x \in X}(\{x\} \times xK), \text{ and } \mathcal{E}_M = \{E \subseteq X \times X \mid \exists K \in [M]^{<\omega} : E \subseteq E_K\}.$$

   The quasi-coarse structure $\mathcal{E}_M$ is called *action-quasi-coarse structure*. In order to see that it is actually a quasi-coarse structure it is enough to prove property (C_3) of Definition 4.6 while the other properties are easy to check. The desired property follows from the observation that, for every $F, K \in [M]^{<\omega}$, $E_F \circ E_K = E_{FK}$ and $FK \in [M]^{<\omega}$. Moreover, if M is a group, $\mathcal{E}_M$ is a coarse structure, called *action-coarse structure*. These coarse structures were deeply investigated in [27] in terms of balleans. Furthermore, $(X, \mathcal{E}_M)$ has bounded geometry, in fact, for every $E \in \mathcal{E}_M$, there exists $K \in [M]^{<\omega}$ such that $E \subseteq E_K$, and thus $|E[x]| \leq |E_K[x]| \leq |K|$, for every $x \in X$.
   In the previous notation, $(X, \mathcal{E}_M)$ is quasi-metrisable if M is countable, and it is metrisable if M is a countable group.

Note that, if we consider a group endowed with its right regular action, then the action-coarse structure coincides with the finitary-group coarse structure ([12]).

Let M and N be two monoids, and X and Y be an M-set and an N-set, respectively. Suppose that $(f, \bar{f})$ is a homomorphism from X to Y. We claim that $f\colon (X, \mathcal{E}_M) \to (Y, \mathcal{E}_N)$ is bornologous. In order to prove it, we show that, for every $K \in [M]^{<\omega}$,

$$(f \times f)(E_K) = E_{\bar{f}(K)} \cap (f(X) \times f(X)). \tag{8}$$

In fact, if (8) holds, for every $E \in \mathcal{E}_M$, there exists $K \in [M]^{<\omega}$ such that $E \subseteq E_K$, and thus $(f \times f)(E) \subseteq E_{\bar{f}(K)} \in \mathcal{E}_N$, which implies that f is bornologous. Let $K \in [M]^{<\omega}$. Then, for every $(x, y) \in E_K$, there exists $k \in K$ such that $y = xk$, and thus

$$(f \times f)(x, y) = (f(x), f(xk)) = (f(x), f(x)\bar{f}(k)) \in E_{\bar{f}(K)} \cap (f(X) \times f(X)).$$

As for the opposite inclusion, if $(f(x), f(y)) \in E_{\bar{f}(K)} \cap (f(X) \times f(X))$, then there exists $k \in K$ such that $f(y) = f(x)\bar{f}(k) = f(xk)$, and so $(f(x), f(y)) = (f \times f)(x, xk)$, where $(x, xk) \in E_K$.

Let us add one more result concerning the action-quasi-coarse structure.

Proposition 4.8. *Let M be a monoid and X be an M-set. Then, for every $a \in U(M) = \{x \in M \mid \exists y \in M : xy = yx = e\}$, $a\colon (X, \mathcal{E}_M) \to (X, \mathcal{E}_M)$ is an asymorphism.*

Proof. It is enough to prove that, for every $a \in U(M)$, $a\colon (X, \mathcal{E}_M) \to (X, \mathcal{E}_M)$ is bornologous. In fact, once the claim is proved, we can note that a and $a^{-1}\colon X \to X$, which is the inverse of a, are bornologous, and so a is an asymorphism. Let us now fix $a \in U(M)$. For every $K \in [M]^{<\omega}$ and every point $(x, xk) \in E_K$, where $k \in K$,

$$(a \times a)(x, xk) = (xa, xka) = (xa, xaa^{-1}ka) \in E_{a^{-1}Ka},$$

and so $(a \times a)(E_K) \subseteq E_{a^{-1}Ka}$, where $a^{-1}Ka \in [M]^{<\omega}$, which proves that a is bornologous. □

Let $(X, \mathcal{E})$ be a quasi-coarse space and $f\colon X \to X$ be a bornologous self-map. For every $n \in \mathbb{N} \setminus \{0\}$, $x \in X$, and $E \in \mathcal{E}$, we define

$$\begin{aligned} \mathrm{T}^c_n(f, x, E) &= (E \circ (f \times f)(E) \circ \cdots \circ (f^{n-1} \times f^{n-1})(E))[x] = \\ &= (f^{n-1} \times f^{n-1})(E)[(f^{n-2} \times f^{n-2})(E)[\cdots [(f \times f)(E)[E[x]]] \cdots]], \end{aligned} \tag{9}$$

which is called the *n-coarse trajectory of f with respect to x and E*. Note that, if X is locally finite, every trajectory of a bornologous self-map is finite.

Let us define the coarse entropy.

Definition 4.9 ([37])**.** Let $(X, \mathcal{E})$ be a locally finite quasi-coarse space and $f\colon X \to X$ be a bornologous self-map. If $x \in X$ and $E \in \mathcal{E}$, we define

$$\mathrm{H}_c(f, x, E) = \limsup_{n \to \infty} \frac{\log|\mathrm{T}^c_n(f, x, E)|}{n},$$
$$\mathrm{H}^{loc}_c(f, x) = \sup_{E \in \mathcal{E}} \mathrm{H}_c(f, x, E), \quad \text{and,} \quad \mathrm{h}_c(f) = \sup_{x \in X} \mathrm{H}^{loc}_c(f, x).$$

The value $\mathrm{h}_c(f)$ is called the *coarse entropy of f*.

Proposition 4.10. *Let M be a monoid, X be an M-set, and $f\colon (X, \mathcal{E}_M) \to (X, \mathcal{E}_M)$ be a bornologous self-map. Then, for every $x \in X$, $\mathrm{H}^{loc}_c(f, x) = \sup\{\mathrm{H}_c(f, x, E_K) \mid K \in [M]^{<\omega}\}$.*

Proof. The proof follows from a more general result, [37, Proposition 2.2], stating that it is enough to take a cofinal subfamily of $\mathcal{E}$ in order to compute $\mathrm{H}^{loc}_c(f, x)$. □

For a monoid M, an M-set X, and an endomorphism $(f,\overline{f})$ of X, we want to compare the algebraic entropy of $(f,\overline{f})$ with the coarse entropy of $f\colon (X,\mathcal{E}_M)\to(X,\mathcal{E}_M)$.

Lemma 4.11. *Let M be a monoid, X be an M-set, and $(f,\overline{f})$ be an endomorphism of X. For every $n\in\mathbb{N}\setminus\{0\}$, $x\in X$, and $K\in[M]^{<\omega}$, $|\mathrm{T}_n^c(f,x,E_K)|\leq|\mathrm{T}_n(f,\overline{f},x,K)|$. Moreover, if f is surjective, $|\mathrm{T}_n^c(f,x,E_K)|=|\mathrm{T}_n(f,\overline{f},x,K)|$.*

Proof. Let $n\in\mathbb{N}\setminus\{0\}$, $x\in X$, and $K\in[M]^{<\omega}$. Then, according to (8),

$$\begin{aligned}\mathrm{T}_n^c(f,x,E_K)&=(f^{n-1}\times f^{n-1})(E_K)[\cdots[(f\times f)(E_K)[E_K[x]]]\cdots]\subseteq\\&\subseteq E_{\overline{f}^{n-1}(K)}[\cdots[E_{\overline{f}(K)}[E_K[x]]]\cdots]=xK\overline{f}(K)\cdots\overline{f}^{n-1}(K)=\mathrm{T}_n(f,\overline{f},x,K).\end{aligned}\tag{10}$$

Thus the desired conclusion holds. Moreover, if f is surjective, again according to (8), the inclusion in (10) is an equality, and thus $|\mathrm{T}_n^c(f,x,E_K)|=|\mathrm{T}_n(f,\overline{f},x,K)|$. □

Theorem 4.12. *Let M be a monoid, X be an M-set, and $(f,\overline{f})$ be an endomorphism of X. Then*

$$\mathrm{h}_c(f)\leq\mathrm{h}_{alg}(f,\overline{f}).$$

Moreover, if f is surjective, then

$$\mathrm{h}_c(f)=\mathrm{h}_{alg}(f,\overline{f}).$$

Proof. Let $x\in X$. Then, according to Lemma 4.11, we have that, for every $K\in[M]^{<\omega}$, $\mathrm{H}_c(f,x,E_K)\leq\mathrm{H}_{alg}(f,\overline{f},x,K)$. Thus, by Proposition 4.10,

$$\mathrm{H}_c^{loc}(f,x)=\sup_{K\in[M]^{<\omega}}\mathrm{H}_c(f,x,E_K)\leq\sup_{K\in[M]^{<\omega}}\mathrm{H}_{alg}(f,\overline{f},x,K)=\mathrm{H}_{alg}^{loc}(f,\overline{f},x),$$

which implies that $\mathrm{h}_c(f)\leq\mathrm{h}_{alg}(f,\overline{f})$. If f is surjective, the equality $\mathrm{h}_c(f)=\mathrm{h}_{alg}(f,\overline{f})$ can be similarly shown. □

Corollary 4.13. *Let M be a monoid and X be an M-set. Then* $\mathrm{h}_{alg}(id_X,id_M)\in\{0,\infty\}$.

Proof. Since the identity map is surjective, Theorem 4.12 implies that $\mathrm{h}_{alg}(id_X,id_M)=\mathrm{h}_c(id_X)$, and the conclusion follows since $\mathrm{h}_c(id_X)\in\{0,\infty\}$ ([37, Theorem 4.4]). □

Remark 4.14. Let us discuss one more consequence of Theorem 4.12. A coarse structure $\mathcal{E}$ on X is said to be *monogenic* ([31]) if there exists $E\in\mathcal{E}$ such that the countable family $\{E^n\subseteq X\times X\mid n\in\mathbb{N}\}$, where

$$E^n=\underbrace{E\circ\cdots\circ E,}_{n\text{ times}}$$

is cofinal in $\mathcal{E}$. Let G be a group acting on a set X. Since, for every $F,K\in[G]^{<\omega}$, $E_F\circ E_K=E_{FK}$, $\mathcal{E}_G$ is monogenic if and only if G is finitely generated.

Let G be a finitely generated group acting on a set X. Since $\mathcal{E}_G$ is monogenic, in particular, $(X,\mathcal{E}_G)$ is metrisable, and thus there exists a metric d on X such that $\mathcal{E}_G=\mathcal{E}_d$. Suppose that G acts transitively on X, which is equivalent to the requirement $\bigcup\mathcal{E}_G=X\times X$ (i.e., $(X,\mathcal{E}_G)$ is *connected*). For a point $x\in X$, we consider the sequence $\gamma(n,x)=|B(x,n)|$, for every $n\in\mathbb{N}$. We define the *growth rate of X* as the growth type of the sequence $\{\gamma(n,x)\}_{n\in\mathbb{N}}$, which does not depend on the point ([2]). Then, applying [37, Theorem 4.9], we obtain the following properties:
- X has polynomial growth type if $\mathrm{h}_{alg}(id_X,id_G)=\mathrm{h}_c(id_X)=0$;
- X has sub-exponential growth type if and only if $\mathrm{h}_{alg}(id_X,id_G)=\mathrm{h}_c(id_X)=0$;
- X has exponential growth type if and only if $\mathrm{h}_{alg}(id_X,id_G)=\mathrm{h}_c(id_X)=\infty$.

Taking into account Theorem 4.2, this result extends the known relation between the growth type of finitely generated groups (see [17]) and the algebraic entropy of their identity maps ([8]).

We have seen how an endomorphism of a set endowed with a monoid action induces a bornologous self-map. In Theorem 4.16 we discuss the opposite implication in some special cases.

Notation 4.15. In order to keep the notation uniform, in the sequel we consider the group S_X of permutation of a set X acting on the right on X, while it usually acts on the left.

Theorem 4.16. *Let $f\colon (X,\mathcal{E})\to(X,\mathcal{E})$ be an injective bornologous self-map of a coarse space X with bounded geometry. Then there exists a subgroup G of S_X acting on X and an endomorphism $\overline{f}$ of G such that $\mathcal{E}=\mathcal{E}_G$ and $(f,\overline{f})$ is an endomorphism of $X\curvearrowleft G$.*

Proof. Fix an entourage $E\in\mathcal{E}$ such that $\Delta_X\subseteq E=E^{-1}$. Let us consider a non-directed graph $\Gamma_X(E)$ that has X and $(E\circ E)\setminus\Delta_X$ as set of vertices and edges, respectively. Since $(X,\mathcal{E})$ has bounded geometry, the degree of $\Gamma_X(E)$ is bounded by $N_{E\circ E}$ because $\sup_x\{|E\circ E[x]|\}\le N_{E\circ E}$. Then, by [16, Corollary 12.2], there exists a partition

$$X=X_0^E\sqcup\cdots\sqcup X_{N_{E\circ E}}^E$$

of X satisfying the following property: for every $i\in\{0,\dots,N_{E\circ E}\}$ and every $x,y\in X_i^E$, $(x,y)\in E\circ E$ if and only if $x=y$. The existence of such a partition easily implies that, for every $x\in X$ and every $i\in\{0,\dots,N_{E\circ E}\}$, $|E[x]\cap X_i^E|\le 1$. Thus, in particular, we can enumerate

$$E[x]=\{a_0^E(x),\dots,a_{N_x^E}^E(x)\},\tag{11}$$

for some $N_x^E\le N_{E\circ E}$, where, without loss of generality, $a_0^E(x)=x$.

Let us now fix two indices $i,j\in\{0,\dots,N_{E\circ E}\}$ and $n\in\mathbb{N}$, and define a permutation

$$\sigma_{i,j}^{E,n}=\prod_{\substack{x\in X_i^E:\\ j\le N_x^E}}(f^n(x),f^n(a_j^E(x))),\tag{12}$$

where $f^0=id_X$ and $(x,a_j^E(x))$ denotes the permutation of X that swaps x and $a_j^E(x)$ leaving untouched the remaining points. Note that (12), for $n=0$, is well-defined since, if there exists $x,y\in X_i^E$ such that $a_j^E(x)=a_j^E(y)$, then $(x,y)\in E\circ E$ and so $x=y$. Moreover, the injectivity of f implies that, for every $n\in\mathbb{N}$, the single swaps in (12) are disjoint, and so (12) is well-defined. For every $n\in\mathbb{N}$, set

$$S(E,n)=\{id_X\}\cup\{\sigma_{i,j}^{E,n}\mid i,j\in\{0,\dots,N_{E\circ E}\}\}.$$

We claim that, for every $n\in\mathbb{N}$,

$$(f^n\times f^n)(E)\cup\Delta_X=E_{S(E,n)}=\{(x,y)\in X\times X\mid y\in xS(E,n)\}.\tag{13}$$

Let $(x,y)\in(f^n\times f^n)(E)\cup\Delta_X$ and $i\in\{0,\dots,N_{E\circ E}\}$ such that $x\in X_i^E$. If $x=y$, there is nothing to prove since $id_X\in S(E,n)$. Otherwise, there exists $(z,w)\in E$ such that $f^n(z)=x$ and $f^n(w)=y$. Then, according to (11), there exists $j\in\{0,\dots,N_{E\circ E}\}$ such that $a_j(z)=w$, and so

$$y=f^n(w)=f^n(z)\sigma_{i,j}^{E,n}=x\sigma_{i,j}^{E,n}\in xS(E,n).$$

Conversely, let $x\in X$ and $y\in xS(E,n)$, for some $n\in\mathbb{N}$. Then either $x=y$ or there exist $i,j\in\{0,\dots,N_{E\circ E}\}$, $z\in X_i^E\cap f^{-n}(x)$ and $w=a_j^E(z)$ such that $f^n(w)=y$. In both cases, $y\in((f^n\times f^n)(E)\cup\Delta_X)[x]$. Thus the claim is proved.

Let G be the subgroup of S_X generated by the family $\bigcup\{S(E,n)\mid\Delta_X\subseteq E^{-1}=E\in\mathcal{E},n\in\mathbb{N}\}$, which is its closure under composition since, for every $\Delta_X\subseteq E^{-1}=E\in\mathcal{E}$, $n\in\mathbb{N}$, and $i,j\in\{0,\dots,N_{E\circ E}\}$, $\sigma_{i,j}^{E,n}\circ\sigma_{i,j}^{E,n}=id_X$. Note that G trivially acts on X as a subgroup of S_X. We claim that $\mathcal{E}=\mathcal{E}_G$. Equation (13) implies that $\mathcal{E}\subseteq\mathcal{E}_G$ since f is bornologous. In order to prove the opposite inclusion, let us consider an arbitrary element $\rho\in G$. Then $\rho=\sigma_1\cdots\sigma_m$, for some $m\in\mathbb{N}$, where, for every $k\in\{1,\dots,m\}$, there exist $\Delta_X\subseteq(E_k)^{-1}=E_k\in\mathcal{E}$ and $n_k\in\mathbb{N}$ such that $\sigma_k\in S(E_k,n_k)$. Thus, according to (13), for every $x\in X$ and $y\in x\rho=x\sigma_1\cdots\sigma_m$,

$$\begin{aligned}(x,y)&=(x,x\sigma_1)\circ(x\sigma_1,x\sigma_1\sigma_2)\circ\cdots\circ(x\sigma_1\cdots\sigma_{m-1},x\sigma_1\cdots\sigma_m)\in\\ &\in((f^{n_1}\times f^{n_1})(E_1)\cup\Delta_X)\circ((f^{n_2}\times f^{n_2})(E_2)\cup\Delta_X)\circ\cdots\circ((f^{n_m}\times f^{n_m})(E_m)\cup\Delta_X)\in\mathcal{E},\end{aligned}$$

which shows that $E_{\{\rho\}} \in \mathcal{E}$. Then, since $\mathcal{E}$ is closed under finite unions, for every $F \in [G]^{<\omega}$, $E_F \in \mathcal{E}$.

Let now $\overline{f}\colon G \to G$ be the map defined on the generators of G as $\overline{f}(\sigma_{i,j}^{E,n}) = \sigma_{i,j}^{E,n+1}$, for every $\Delta_X \subseteq E^{-1} = E \in \mathcal{E}$, $i, j \in \{0, \dots, N_{E\circ E}\}$ and $n \in \mathbb{N}$, and then extended to a homomorphism of G. Then $\overline{f}$ is well-defined, as $\sigma_{i,j}^{E,n+1} = \sigma_{k,l}^{F,m+1}$ if $\sigma_{i,j}^{E,n} = \sigma_{k,l}^{F,m}$, for every $\Delta_X \subseteq E^{-1} = E, \Delta_X \subseteq F^{-1} = F \in \mathcal{E}$, $i, j \in \{0, \dots, N_{E\circ E}\}$, $k, l \in \{0, \dots, N_{F\circ F}\}$, and $n, m \in \mathbb{N}$. Moreover, $id_X = \sigma_{0,0}^{\Delta_X,n}$, for every $n \in \mathbb{N}$, and so $\overline{f}(id_X) = id_X$, which shows that $\overline{f}$ is a endomorphism of G. It remains to prove that $(f, \overline{f})$ is an endomorphism of $X \curvearrowleft G$. Let $\rho \in G$. Then,

$$\rho = \sigma_{i_1,j_1}^{E_1,n_1} \cdots \sigma_{i_m,j_m}^{E_m,n_m},$$

for some $m, n_1, \dots, n_m \in \mathbb{N}$, $\Delta_X \subseteq E_1^{-1} = E_1, \dots, \Delta_X \subseteq E_m^{-1} = E_m \in \mathcal{E}$, and $i_1, j_1 \in \{0, \dots, N_{E_1\circ E_1}\}, \dots, i_m, j_m \in \{0, \dots, N_{E_m\circ E_m}\}$. Finally, it is easy to check that, for every $x \in X$,

$$f(x\rho) = f(x)\overline{f}(\sigma_{i_1,j_1}^{E_1,n_1}) \cdots \overline{f}(\sigma_{i_m,j_m}^{E_m,n_m}) = f(x)\overline{f}(\rho).$$

□

Theorem 4.16 extends [27, Theorem 1], to which the provided proof is inspired. The mentioned theorem states that, for every coarse space $(X, \mathcal{E})$ with bounded geometry, there exists a group $G \leq S_X$ satisfying $\mathcal{E} = \mathcal{E}_G$.

Corollary 4.17. *Let $f\colon (X, \mathcal{E}) \to (X, \mathcal{E})$ be a bornologous bijective self-map of a coarse space $(X, \mathcal{E})$ with bounded geometry. Then there exists a group G acting on X and an endomorphism $\overline{f}$ of G such that*
(a) $\mathcal{E} = \mathcal{E}_M$,
(b) $(f, \overline{f})$ is a endomorphism of $X \curvearrowleft G$, and
(c) $\mathrm{h}_c(f) = \mathrm{h}_{alg}(f, \overline{f})$.

Proof. The desired claims follow from Theorems 4.16 and 4.12. □

Remark 4.18. Right after Definition 3.1 we claimed that the limit superior in the definition of $\mathrm{H}_{alg}(f, \overline{f}, x, K)$ is not a limit in general. We use Theorem 4.16 to prove this statement. In [37, Example 2.3(c)] the author provides a metric space (X, d) with bounded geometry with a point $x \in X$ such that the sequence $\{(\log|\mathrm{T}_n^c(id_X, x, E_1)|)/n\}_{n\in\mathbb{N}}$ (in the notation of Example 4.7(a)) has no limit. Let G be the subgroup of S_X such that $\mathcal{E}_G = \mathcal{E}_d$ whose existence is guaranteed by Theorem 4.16. Note that (id_X, id_G) is trivially an automorphism of $X \curvearrowleft G$. Moreover, according to (13), there exists $K \in [G]^{<\omega}$ such that $E_F = E_1$ since $\Delta_X \subseteq E_1 = (E_1)^{-1}$. Since id_X is surjective, Lemma 4.11 implies that, for every $n \in \mathbb{N} \setminus \{0\}$, $|\mathrm{T}_n^c(id_X, x, E_1)| = |\mathrm{T}_n(id_X, id_G, x, K)|$, and so $\{(\log|\mathrm{T}_n(id_X, id_G, x, K)|)/n\}_{n\in\mathbb{N}}$ does not have a limit.

Question 4.19. *In the notation of Theorem 4.16, can we loose the injectivity request on the map f?*

Question 4.20. *Let f be a bornologous self-map of a quasi-coarse space $(X, \mathcal{E})$ with bounded geometry. Do there exist a monoid M acting on X and an endomorphism $\overline{f}$ of M such that $\mathcal{E} = \mathcal{E}_M$ and $(f, \overline{f})$ is an endomorphism of $X \curvearrowleft M$?*

Question 4.21. *Let f be a bornologous self-map of a quasi-coarse space $(X, \mathcal{E})$ with bounded geometry. Let M and N be two monoids acting on X such that $\mathcal{E} = \mathcal{E}_M = \mathcal{E}_N$, and $(f, \overline{f_M})$ and $(f, \overline{f_N})$ be endomorphisms of $X \curvearrowleft M$ and $X \curvearrowleft N$, respectively. Is it true that $\mathrm{h}_{alg}(f, \overline{f_M}) = \mathrm{h}_{alg}(f, \overline{f_N})$? What happens if X is a coarse space and M and N are groups?*

Let us spend a few words on the importance of Questions 4.19 and 4.21. Suppose that they were true. Then the algebraic entropy of endomorphisms of G-sets would induce as a new entropy notion h_c' of bornologous self-maps of coarse spaces with bounded geometry in the following way: in the notation of Theorem 4.16, let us define $\mathrm{h}_c'(f) = \mathrm{h}_{alg}(f, \overline{f})$. This new entropy would coincide with the usual coarse entropy h_c on surjective maps. Outside this realm, h_c often takes value 0 ([37]), while h_c' would be more meaningful. Furthermore, h_{alg} would extend both the algebraic entropy of group endomorphisms and this new notion h_c'.

References

[1] R. L. Adler, A. G. Konheim, M. H. McAndrew, *Topological entropy*, Trans. Amer. Math. 114 (1965) 309–319.

[2] J. A. Álvarez López, A. Candel, *Generic coarse geometry of leaves*, Lecture Notes in Mathematics, Springer International Publishing (2018).

[3] R. Bowen, *Entropy for group endomorphisms and homogeneous spaces*, Trans. Amer. Math. Soc. 153 (1971) 401–414.

[4] T. Ceccherini-Silberstein, M. Coornaert, F. Krieger, *An analogue of Fekete's lemma for subadditive functions on cancellative amenable semigroups*, J. Analyse Math. 124 (2014) 59–81.

[5] A. H. Clifford, G. B. Preston, *The Algebraic Theory of Semigroups*, volume 1, American Mathematical Society (1961).

[6] A. H. Clifford, G. B. Preston, *The Algebraic Theory of Semigroups*, volume 2, American Mathematical Society (1967).

[7] D. Dikranjan, A. Fornasiero, A. Giordano Bruno, *Algebraic entropy for amenable semigroup actions*, J. Algebra, 556 (2020), 467–546.

[8] D. Dikranjan, A. Giordano Bruno, *Topological entropy and algebraic entropy for group endomorphisms*, Proceedings ICTA2011 Islamabad, Pakistan July 4–10 2011 Cambridge Scientific Publishers 133–214.

[9] D. Dikranjan, A. Giordano Bruno, *Entropy on abelian groups*, Advances in Mathematics 298 (2016), 612–653.

[10] D. Dikranjan, A. Giordano Bruno, *Entropy on normed semigroups (Towards a unifying approach to entropy)*, Dissertationes Math. 542 (2019), 1–90.

[11] D. Dikranjan, B. Goldsmith, L. Salce, P. Zanardo, *Algebraic entropy for abelian groups*, Trans. Amer. Math. Soc. 361 (2009), 3401–3434.

[12] D. Dikranjan, N. Zava, *Categories of coarse groups: quasi-homomorphisms and functorial coarse structures*, Topology Appl., 273 (2020), 106980.

[13] J. Dydak, C. S. Hoffland, *An alternative definition of coarse structures*, Topology Appl. 155 (2008), no. 9, 1013–1021.

[14] W. Geller, M. Misiurewicz, *Coarse entropy*, Fund. Math. 255 (2021), no. 1, 91–109.

[15] A. Giordano Bruno, P. Spiga, *Some properties of the growth and of the algebraic entropy of group endomorphisms*, Journal of Group Theory 20 (2017), 763–774.

[16] F. Harary, *Graph Theory*, Addison-Wesley Publ. Comp., 1969.

[17] P. de la Harpe, *Topics in geometric group theory*. Chicago Lectures in Math., the University of Chicago Press, Chicago (2000).

[18] K.H. Hofmann, L. Stoyanov, *Topological entropy of group and semigroup actions*, Adv. Math. 115 (1995) no. 1, 54–98.

[19] N. Jacobson, *Basic Algebra I*, volume I, second edition, W. H. Freeman, 1985.

[20] M. Kilp, U. Knauer, A. V. Mikhalev, *Monoids, Acts and Categories: with Applications to Wreath Products and Graphs*, Expositions in Mathematics 29, Walter de Gruyter, Berlin (2000).

[21] A. N. Kolmogorov, *New metric invariants of transitive dynamical systems and automorphisms of Lebesgue spaces*, Doklady Akad. Nauk. SSSR 119 (1958) 861–864.

[22] P. W. Nowak, G. Yu, *Large Scale Geometry*, European Mathematical Society, 2012.

[23] J. Peters, *Entropy on discrete abelian groups*, Adv. Math. 33 (1979), 1–13.

[24] J. Peters, *Entropy of automorphisms on L.C.A. groups*, Pacific J. Math. 96 (2) (1981) 475–488.

[25] O. V. Petrenko, I. V. Protasov, *Balleans and G-spaces*, Ukr. Math. J. 64, 387–393 (2012).

[26] I. V. Protasov, *Asymptotic proximities*, Applied Gen. Top., 9 (2008), no. 2, 189–195.

[27] I. V. Protasov, *Balleans of bounded geometry and G-spaces*, Algebra Discrete Math., 2008, Issue 2, 101–108.

[28] I. V. Protasov, T. Banakh, *Ball Structures and Colorings of Groups and Graphs*, Math. Stud. Monogr. Ser., vol. 11, VNTL, Lviv, 2003.

[29] I. V. Protasov, O. I. Protasova, *Sketch of group balleans*, Matematychni Studii, 22 (2004), no. 1, 10–20.

[30] I. V. Protasov, M. Zarichnyi, *General Asymptology*, 2007 VNTL Publishers, Lviv, Ukraine.

[31] J. Roe, *Lectures on Coarse Geometry*, Univ. Lecture Ser., vol. 31, American Mathematical Society, Providence RI, 2003.

[32] C. E. Shannon, *A Mathematical Theory of Communication*, Bell System Technical Journal, 27 (3) (1948) 379–423.

[33] Y. G. Sinai, *On the concept of entropy of a dynamical system*, Doklady Akad. Nauk. SSSR 124 (1959) 789–781 (in Russian).

[34] S. Virili, *Entropy for endomorphisms of LCA groups*, Top. Appl. 159 (9) (2012), 2546–2556.

[35] M. D. Weiss, *Algebraic and other entropies of group endomorphisms*, Math. Systems Theory 8 (3) (1974/75) 243–248.

[36] N. Zava, *Generalisations of coarse spaces*, Topology Appl., 263 (2019), 230–256.

[37] N. Zava, *On a notion of entropy in coarse geometry*, Topol. Algebra Appl., 7 (2019), 48–68.