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Multiplicity of positive solutions for quasilinear elliptic equations involving critical nonlinearity

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Abstract: We are concerned with the following quasilinear elliptic equation

$$-\Delta u - \Delta(u^2)u = \mu|u|^{q-2}u + |u|^{2\cdot 2^*-2}u, u \in H_0^1(\Omega), \tag{QSE}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $N \geq 3$, $q_N < q < 2\cdot 2^*$, $2^* = 2N/(N-2)$, $q_N = 4$ for $N \geq 6$ and $q_N = \frac{2(N+2)}{N-2}$ for $N = 3, 4, 5$, and μ is a positive constant. By employing the Nehari manifold and the Lusternik-Schnirelman category theory, we prove that there exists $\mu^* > 0$ such that (QSE) admits at least $\mathrm{cat}_\Omega(\Omega)$ positive solutions when $\mu \in (0, \mu^*)$.

Keywords: Quasilinear elliptic equation; Positive solution; Nehari manifold

MSC: 35J20, 35J62, 35B33

1 Introduction and main result

1.1 Background

Consider the following quasilinear elliptic equations of the form

$$i\partial_t z = -\Delta z + W(x)z - h(|z|^2)z - \kappa\Delta(l(|z|^2))l'(|z|^2)z, \quad x \in \Omega, \tag{1.1}$$

where $\kappa \in \mathbb{R}^+$, $z : \mathbb{R} \times \Omega \to \mathbb{C}$, $W : \Omega \to \mathbb{R}$ is a given potential and l, h are real functions in $\mathbb{R}^+$. Of particular interest are solitary wave solutions of (1.1), i.e., $z(t, x) = \exp(-iEt)u(x)$, where $E \in \mathbb{R}$, u is a real function and satisfies the stationary quasilinear elliptic equation

$$-\Delta u - \kappa\Delta l(u^2)l'(u^2)u + V(x)u = h(u), \quad x \in \Omega \subset \mathbb{R}^N. \tag{1.2}$$

In particular, equation (1.2) is a special case of the following generalized quasilinear elliptic equations

$$-\mathrm{div}(\varphi^2(u)\nabla u) + \varphi(u)\varphi'(u)|\nabla u|^2 + V(x)u = h(u), \ x \in \Omega, \tag{1.3}$$

if ones take

$$\varphi^2(u) = 1 + \frac{([l(u^2)]')^2}{2}\kappa.$$

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Equation (1.3) corresponds to a large number of elliptic equations which appear in mathematical physics. In the literature, equation (1.3) has been derived as models of several physical phenomena corresponding to various types of $\varphi(s)$. If $\varphi(s) \equiv$ const, equation (1.3) is reduced to

$$-\Delta u + V(x)u = h(u),\ \ x \in \Omega,$$

which is the well-known elliptic equation in the quantum mechanic and also arises in biological models and propagation of laser beams(Ref. [21, 29]). If $\varphi(s) = \sqrt{1 + 2\kappa s^2}$, equation (1.3) can be rewritten as follows

$$-\Delta u - \kappa\Delta(u^2)u + V(x)u = h(u),\ x \in \Omega, \tag{1.4}$$

which is called the superfluid film equation in plasma physics and fluid mechanics(Ref.[28, 30]). If $\varphi^2(s) = 1 + \frac{\kappa s^2}{2(1+s^2)}$, then one can get the following equation of the form

$$-\Delta u - \kappa[\Delta(\sqrt{1 + u^2})]\frac{u}{2\sqrt{1 + u^2}} + V(x)u = h(u),\ x \in \Omega, \tag{1.5}$$

which models the self-channeling of a high-power ultrashort laser in matter(Ref. [30]). For the further physical background, we refer the readers to [15, 31, 33, 41, 45] and the reference therein.

1.2 Motivation

In the last decades, quasilinear Schrödinger equations have received a considerable attention by numerous researchers. To the best of our knowledge, the first existence results for quasilinear equations of the form of (1.4) with $\kappa \neq 0$ is due to [33, 41], in which, the main existence results are obtained, through a constrained minimization argument. Actually, in these papers, they obtained solutions in H_V of the problem with an unknown Lagrange multiplier λ:

$$-\Delta u - \frac{1}{2}\Delta(u^2)u + V(x)u = \lambda|u|^{p-1}u,\ x \in \mathbb{R}^N.$$

Here $H_V := \left\{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2 < \infty\right\}$. To investigate the case with any prescribed $\lambda > 0$ in the variational setting, one can formulate this problem as follows: consider the formal energy functional

$$\hat{J}(u) = \frac{1}{2}\int_{\mathbb{R}^N}\left[(1 + 2u^2)|\nabla u|^2 + V(x)u^2\right] - \frac{\lambda}{p+1}\int_{\mathbb{R}^N}|u|^{p+1}.$$

However, $\hat{J}$ is not well defined in H_V, except for $N = 1$. To overcome this difficulty, a change of variable $v = f^{-1}(u)$(see Section 2) was introduced in [31] and $\hat{J}$ can be rewritten in a new variable. Then this problem was resolved in an associated Orlicz space. Subsequently, a simpler and shorter proof of some results in [31] was given by M. Colin, L. Jeanjean [9]. Moreover, a dual approach was introduced in [9] so that problems of the form (1.4) can be dealt with in $H^1(\mathbb{R}^N)$ instead of the Orlicz space.

Initiated by M. Colin, L. Jeanjean, the dual approach introduced in [9] has been one of main tools in studying problem (1.4) by the variational approach and there have been the extensive results in the literature. By using such dual approach, J. M. do Ó, O. Miyagaki, S. M. Soares [12] considered problem (1.4) in $\mathbb{R}^2$ involving a critical growth of the Trudinger-Moser type(for instance see [10]). By using the mountain pass theorem and the concentration-compactness principle, a positive solution was obtained. For the semiclassical states of quasilinear problems, E. Gloss [23] considered the following problem in the subcritical case

$$-\varepsilon^2\Delta u - \varepsilon^2\Delta(u^2)u + V(x)u = h(u),\ x \in \mathbb{R}^N. \tag{1.6}$$

Under some sort of Berestycki and Lions conditions as in [5], in the framework of J. Byeon and L. Jeanjean[6], the author shows that (1.6) admits positive solutions. Moreover, there solutions exhibit a spike near local minimal points of the potential well V as $\varepsilon \to 0$. Later, through the same dual approach, Y. Wang and

W. Zou [52] considered the semiclassical states of the critical quasilinear Schrödinger equations (1.6). By the penalization argument by M. del Pino and P. Felmer[13], the authors proved the existence of positive bound states which concentrate around a local minimum point of V as $\varepsilon \to 0$. By the Nehari approach and the dual approach above, X. He, A. Qian and W. Zou [25] considered the semiclassical ground states of the critical quasilinear Schröinger equations (1.6). Moreover, the multiplicity was considered by the Ljusternik-Schnirelmann theory as well. In this aspect, we also would like to cite [7, 16, 26, 37, 39, 49, 51]. For the generalized quasilinear equation (1.3), by introducing a new variable replacement $v = G(u) = \int_0^u \varphi(s)\,\mathrm{d}s$, Y. Shen and Y. Wang [45] reduced (1.3) to a semi-linear elliptic equation

$$-\Delta v + V(x)\frac{G^{-1}(v)}{g(G^{-1}(v))} = \frac{h(G^{-1}(v))}{g(G^{-1}(v))}.$$

By virtue of the mountain pass theorem, positive solutions were obtained when the nonlinearity is *subcritical*. Subsequently, by adopting the same change of variable, Y. Deng, S. Peng and S. Yan [15] investigated the generalized quasilinear Schrödinger equations (1.3) involving critical growth. For more related results to quasilinear problems (1.3), we refer the readers to [3, 24] for uniqueness of solutions, [48] for non-degeneracy of solutions, [3, 35, 36, 38] for critical or supercritical exponent, [19, 42] for ground state solutions, [17, 20, 44] for multiple solutions, [46] for quasilinear p-Laplacian problems, [47] for asymptotical problems and [2] for the case $\kappa < 0$.

1.3 Our problem and main result

In the present paper, we mainly focus on the quasilinear elliptic equations with critical growth. Precisely, we investigate the problem

$$\begin{cases} -\Delta u - \Delta(u^2)u = \mu|u|^{q-2}u + |u|^{2\cdot 2^\star - 2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $N \geq 3$ and $2^\star = 2N/(N-2)$. In [32], it turns out that $p = 2 \cdot 2^\star$ behaves as a critical exponent for the quasilinear elliptic equations. So problem (1.7) can be regarded as the counterpart of the Brézis-Nirenberg problem in the quasilinear case. The first celebrated work is due to H. Brézis and L. Nirenberg [4]. They considered the well known Brézis-Nirenberg problem

$$\begin{cases} -\Delta u = \mu|u|^{q-2}u + |u|^{\cdot 2^\star - 2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.8}$$

In particular, they investigated the relation between the existence of positive solutions to (1.8) and μ, N, q. Precisely, they shows that problem (1.8) is solvable for any $q \in (2, 2^\star)$ and $\mu > 0$ if $N \geq 4$. In contrast, in dimension 3, the situation is much delicate. They shows that if $\Omega \subset \mathbb{R}^3$ is strictly starshaped about the origin, problem (1.8) with $q \in (2, 4]$ admits a positive solution if $\mu > 0$ large and no positive solution if $\mu > 0$ small. In [22], F. Gazzola and B. Ruf generalized some results in [4] to the semilinear critical elliptic problem with a wide class of lower order terms $-\Delta u = g(x, u) + |u|^{2^\star - 2}u$ in $\Omega \subset \mathbb{R}^N$. In particular, when $N = 3$, a similar hypothesis to [4] was imposed: there exists an open nonempty set $\Omega_0 \subset \Omega$ such that

$$\lim_{s \to +\infty} \frac{\int_0^s g(x, \tau)\,\mathrm{d}\tau}{s^4} = +\infty, \text{ uniformly for } x \in \Omega_0.$$

Since the pioneering work [4], there have been extensive works on semilinear elliptic equations with critical exponent. Compared to the semilinear case, the quasilinear equation becomes more complicated. In [40], a mountain-pass technique in a suitable Orlicz space is used to prove the existence of soliton solutions to quasilinear Schrödinger equations involving critical exponent in $\mathbb{R}^N$. In [11], a positive solution was obtained by using the concentration-compactness principle and the mountain pass theorem when $h(u)$ in (1.4) amounts to the sum of the two terms, $|u|^{q-1}u$ and $|u|^{p-1}u$, one of which is critical and the other subcritical.

In [34], for a class of quasilinear Schrödinger equations with critical exponent, X. Liu, J. Liu, Z.-Q. Wang established the existence of both one-sign and nodal ground states by the Nehari method. It is established in [43] the existence of solutions for a class of asymptotically periodic quasilinear elliptic equations in $\mathbb{R}^N$ with critical growth. For $h(u) = \lambda|u|^{q-2}u + |u|^{2\cdot 2^*-2}u$, $\lambda > 0$, $4 < q < 2\cdot 2^*$. Y. Deng, S. Peng, J. Wang [14], they proved the existence of the nodal solution for problem (1.4) by using Nehari technique. In [35], X. Liu, J. Liu, Z.-Q. Wang considered a kind of more general quasilinear elliptic equations. Via a perturbation method, they obtained positive solutions in the critical case.

In [1], C. O. Alves and Y. Deng considered the Brézis-Nirenberg problem involving the p-Laplacian operator. They were concerned with the p-Laplacian problem

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \mu|u|^{q-2}u + |u|^{p^*-2}u, \;\; u = 0 \text{ in } \Omega \subset \mathbb{R}^N,$$

where $p^* = pN/(N-p)$. Through the Lusternik-Schnirelman category theory, the authors obtained at least $\mathrm{cat}_\Omega(\Omega)$ positive solutions for $\mu > 0$ small and $N \geq p^2$. Motivated by [1], our main purpose of this paper is to investigate the multiplicity of positive solutions to quasilinear problem (1.7). Precisely, our main result reads as

Theorem 1.1. *Let $q_N < q < 2\cdot 2^*$, where $q_N = 4$ for $N \geq 6$ and $q_N = \frac{2(N+2)}{N-2}$ for $N = 3, 4, 5$. Then there is $\mu^* > 0$ such that for each $\mu \in (0, \mu^*)$, problem (1.7) has at least $cat_\Omega(\Omega)$ distinct solutions.*

Remark 1.1. *Here q_N is only used to guarantee that the least energy c_μ is below $\frac{1}{2N}S^{\frac{N}{2}}$ (see Lemma 3.1 below and also [35]). Obviously, $q_N > 4$ if $N \in \{3, 4, 5\}$. However, if Ω is strictly star-shaped about the origin and $N \in \{3, 4, 5\}$, problem (1.7) admits no solutions for some $q \in (4, q_N]$. To illustrate this difference between $N \geq 6$ and $N \in \{3, 4, 5\}$, in the following, assume by contradiction that for any $\mu > 0$ small, u_μ is a positive solution of (1.7) in the case $N = 3$ and $q = 6$. Then by the change of variable(see Section 2), $v = f^{-1}(u_\mu) \in H_0^1(\Omega)$ is a positive solution to $-\Delta v = \mu f^5(v)f'(v) + f^{11}(v)f'(v)$ in Ω. By Lemma 2.1, one can get that if $\mu > 0$ small enough, for some $C_1 > 0$(independent of v, μ), there holds that*

$$-\Delta v \geq \mu C_1 v^5 \;\; in \;\; \Omega. \tag{1.9}$$

Similar as in Theorem 2.4 in [4], by the Pohozaev's identity and Lemma 2.1, for some $C_2 > 0$(independent of v, μ), we have

$$\frac{\mu}{2}\int_\Omega f^6(v) \geq \frac{1}{2}\int_{\partial\Omega}(x\cdot\nu)(\frac{\partial v}{\partial \nu})^2 \geq C_2\left(\int_\Omega |\Delta v|\right)^2.$$

By the maximum principle, $v \leq c|\cdot|^{-1} \star |\Delta v|$ in Ω, where c is an universal constant. Since the L^3-weak norm

$$\||\cdot|^{-1}\|_{L^3_w} := \sup_{\lambda>0}\lambda\left[meas\{x\in\Omega : v(x) > \lambda\}\right]^{1/3} < \infty,$$

it follows from [8, Theorem 8.20] that for some $C_3 > 0$(independent of v, μ),

$$\|v\|_{L^3_w} \leq c\||\cdot|^{-1}\star|\Delta v|\|_{L^3_w} \leq C_3\|\Delta v\|_{L^1}.$$

Thanks to Lemma 2.1-(3) and (7), $f^6(t) \leq 2t^4$ for any t and then $\mu\int_\Omega v^4 \geq C_4\|v\|^2_{L^3_w}$, where $C_4 > 0$ is independent of v, μ. Using (1.9), we also have $\mu^{-1}\int_\Omega v^4 \geq C_1^2 C_2\|v\|_5^{10}$. By the interpolation inequality, $\|v\|_4^4 \leq K\|v\|_{3,w}^{\frac{3}{2}}\|v\|_5^{\frac{5}{2}}$, where $K > 0$(independent of v). Then

$$\int_\Omega v^4 \leq K\left(\mu C_4^{-1}\int_\Omega v^4\right)^{\frac{3}{4}}\left(\mu^{-1}C_1^{-2}C_2^{-1}\int_\Omega v^4\right)^{\frac{1}{4}} = K\mu^{\frac{1}{2}}C_1^{-\frac{1}{2}}C_2^{-\frac{1}{4}}C_4^{-\frac{3}{4}}\int_\Omega v^4,$$

which is a contradiction if $\mu < C_1 C_2^{\frac{1}{2}} C_4^{\frac{3}{2}} K^{-2}$.

1.4 Main difficulties

In the following, we summarize some difficulties caused by the quasilinear term $\Delta(u^2)u$ and critical term $|u|^{2\cdot 2^*-2}u$ in seeking solutions. The main difficulties of the present paper are two-fold. First, due to the critical growth, the compactness does not hold in general. We adopt the Brézis-Nirenberg argument as in [4](see also [35]) to show that the least energy c_μ is below $\frac{1}{2N}S^{\frac{N}{2}}$ if $q > q_N$, which yields the compactness. Second, the term $\Delta(u^2)u$ results in the lack of smoothness to the formal energy functional of problem (1.7) in $H_0^1(\Omega)$. To overcome the difficulty, we use the dual approach introduced in [9] through a change of variable. But, due to the lack of homogeneity for the change of variable, the methods in [1] can not be applied in a direct way. So more delicate analyses and new tricks are needed.

1.5 Outline of this paper

This paper is organized as follows. In Section 2, the variational setting is set up and some preliminaries are given. Section 3 is devoted to proving Theorem 1.1 via the Nehari manifold and the Lusternik-Schnirelman category theory.

Notation. $C, C_1, C_2, \dots$ will denote different positive constants whose exact value is inessential. $|A|$ is the Lebesgue measure of a measurable set $A \subset \mathbb{R}^N$. $B_\rho(y) := \{x \in \mathbb{R}^N : |x-y| < \rho\}$. The usual norm in the Lebesgue space $L^p(\Omega)$ is denoted by $\|u\|_p$. E denotes the Sobolev space $H_0^1(\Omega)$ with the standard norm

$$\|u\| := \left(\int_\Omega |\nabla u|^2\right)^{1/2}.$$

2 Preliminary results

2.1 The dual approach

In this section we introduce a variational framework associated with problem (1.7). Formally (1.7) is the Euler-Lagrange equation associated to the natural energy functional

$$J_\mu(u) := \frac{1}{2}\int_\Omega (1+2u^2)|\nabla u|^2 - \frac{\mu}{q}\int_\Omega |u|^q - \frac{1}{2\cdot 2^*}\int_\Omega |u|^{2\cdot 2^*}. \tag{2.1}$$

However, it is not well defined in general in $H_0^1(\Omega)$. To overcome the difficulty, we use an argument developed in [9]. We make a change of variables $v := f^{-1}(u)$, where f is defined by

$$f'(t) = \frac{1}{(1+2f^2(t))^{1/2}} \text{ on } [0,+\infty) \quad \text{and} \quad f(t) = -f(-t) \text{ on } (-\infty, 0].$$

Let us collect some properties of f, which have been proved in [9, 52].

Lemma 2.1. *The function f satisfies the following properties:*

(1) *f is uniquely defined, C^∞ and invertible;*
(2) $|f'(t)| \le 1$ *for all* $t \in \mathbb{R}$;
(3) $|f(t)| \le |t|$ *for all* $t \in \mathbb{R}$;
(4) $f(t)/t \to 1$ *as* $t \to 0$;
(5) $f(t)/\sqrt{t} \to 2^{1/4}$ *as* $t \to +\infty$;
(6) $f(t)/2 \le tf'(t) \le f(t)$ *for all* $t > 0$;

(7) $|f(t)| \le 2^{1/4}|t|^{1/2}$ *for all* $t \in \mathbb{R}$;
(8) $|f(t)f'(t)| < 1/\sqrt{2}$ *for all* $t \in \mathbb{R}$;
(9) *the function* $f^p(t)f'(t)t^{-1}$ *is increasing for* $p \ge 3$ *and* $t > 0$.

It is easy to see from the proofs in [52] that (9) is strictly increasing.

Therefore, after the change of variables, we consider the following functional

$$I_\mu(v) := \frac{1}{2}\int_\Omega |\nabla v|^2 - \frac{\mu}{q}\int_\Omega f^q(v^+) - \frac{1}{2\cdot 2^*}\int_\Omega f^{2\cdot 2^*}(v^+), \tag{2.2}$$

which is well defined in E and belongs to C^1. Moreover,

$$\langle I'_\mu(v), w\rangle = \int_\Omega \nabla v \nabla w - \mu\int_\Omega f^{q-1}(v^+)f'(v^+)w - \int_\Omega f^{2\cdot 2^*-1}(v^+)f'(v^+)w \tag{2.3}$$

for all $v, w \in E$ and the critical points of I are the weak solutions of the Euler-Lagrange equation given by

$$-\Delta v = \mu f^{q-1}(v^+)f'(v^+) + f^{2\cdot 2^*-1}(v^+)f'(v^+), \quad v \in E.$$

Obviously, if $v \in E$ is a positive critical point of the functional I_μ, then $u = f(v) \in E$ is a solution of (1.7), see [9].

2.2 Nehari manifold

Let

$$\mathcal{M}_\mu := \{v \in E \setminus \{0\} : \langle I'_\mu(v), v\rangle = 0\} \tag{2.4}$$

is the Nehari manifold and $c_\mu := \inf_{\mathcal{M}_\mu} I_\mu$.

We denote by S the best Sobolev constant of the embedding $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$ given by $S := \inf\{\|u\|^2 : u \in H_0^1(\Omega), |u|_{2^*} = 1\}$. It is known that S is independent of Ω and is never achieved except when $\Omega = \mathbb{R}^N$ (see Proposition 1.43 in [50]).

Lemma 2.2. *The Nehari manifold $\mathcal{M}_\mu$ is a C^1-manifold, and it is a natural constraint.*

Proof. Let $g_s(u) = \int_\Omega f^{s-1}(u)f'(u)u$, where $s \ge 4$. Then using Lemma 2.1-(6),(8) and $f''(u) = -2f(u)(f'(u))^4$,

$$\begin{aligned}
\langle g'_s(u), u\rangle &= \int_\Omega (s-1)f^{s-2}(u)(f'(u))^2u^2 + f^{s-1}(u)f''(u)u^2 + f^{s-1}(u)f'(u)u \\
&= \int_\Omega (s-1)f^{s-2}(u)(f'(u))^2u^2 - 2f^s(u)(f'(u))^4u^2 + f^{s-1}(u)f'(u)u \\
&\ge \int_\Omega (s-1)f^{s-2}(u)(f'(u))^2u^2 - f^{s-2}(u)(f'(u))^2u^2 + f^{s-1}(u)f'(u)u \\
&= \int_\Omega (s-2)f^{s-2}(u)(f'(u))^2u^2 + f^{s-1}(u)f'(u)u \\
&\ge \frac{s}{2}\int_\Omega f^{s-1}(u)f'(u)u.
\end{aligned}$$

Let $J_\mu(u) = \int_\Omega |\nabla u|^2\,dx - \mu g_q(u^+) - g_{2\cdot 2^*}(u^+)$. For every $u \in \mathcal{M}_\mu$, we have by Lemma 2.1-(6),

$$\begin{aligned}\langle J'_\mu(u), u\rangle &= 2\int_\Omega |\nabla u|^2\,dx - \mu\langle g'_q(u^+), u\rangle - \langle g'_{2\cdot 2^*}(u^+), u\rangle \\ &\le 2\int_\Omega |\nabla u|^2 - \frac{q\mu}{2}\int_\Omega f^{q-1}(u^+)f'(u^+)u^+ - 2^*\int_\Omega f^{2\cdot 2^*-1}(u^+)f'(u^+)u^+ \\ &= \left(2-\frac{q}{2}\right)\int_\Omega |\nabla u|^2 + \left(\frac{q}{2}-2^*\right)\int_\Omega f^{2\cdot 2^*-1}(u^+)f'(u^+)u^+ \\ &\le \left(2-\frac{q}{2}\right)\int_\Omega |\nabla u|^2 + \left(\frac{q}{2}-2^*\right)\int_\Omega f^{2\cdot 2^*}(u) < 0.\end{aligned}$$

If u_μ is a critical point of I_μ on $\mathcal{M}_\mu$, there is $\theta \in \mathbb{R}$ such that $I'_\mu(u_\mu) = \theta J'_\mu(u_\mu)$. Since $\langle J'_\mu(u_\mu), u_\mu\rangle < 0$, we have $\theta = 0$. Hence $I'_\mu(u_\mu) = 0$ and then $\mathcal{M}_\mu$ is a natural constraint.

□

For $t > 0$, let

$$h(t) := I_\mu(tu) = \frac{t^2}{2}\int_\Omega |\nabla u|^2 - \frac{\mu}{q}\int_\Omega f^q(tu^+) - \frac{1}{2\cdot 2^*}\int_\Omega f^{2\cdot 2^*}(tu^+).$$

Lemma 2.3.
(1)For every $u^+ \neq 0$ there is a unique $t_u > 0$ such that $h'(t) > 0$ for $0 < t < t_u$ and $h'(t) < 0$ for $t > t_u$. Moreover, $tu \in \mathcal{M}_\mu$ if and only if $t = t_u$.
(2) There is $\rho > 0$ such that $c_\mu = \inf_{\mathcal{M}_\mu} I_\mu \ge \inf_{S_\rho} I_\mu > 0$, where $S_\rho := \{u \in E : \|u\| = \rho\}$. Moreover, $\|u\|^2 \ge 2c_\mu$ for all $u \in \mathcal{M}_\mu$.

Proof. (1) By Lemma 2.1-(7), we get

$$h(t) \ge \frac{t^2}{2}\int_\Omega |\nabla u|^2 - C_1 t^{q/2}\int_\Omega (u^+)^{q/2} - C_2 t^{2^*}\int_\Omega (u^+)^{2^*}.$$

It follows from $q > 4$ that $h(t) > 0$ whenever $t > 0$ is small enough. According to Lemma 2.1-(5), $h(t) \to -\infty$ as $t \to \infty$. Then $h(t)$ has a positive maximum. Note that $h'(t) = 0$ is equivalent to

$$\int_\Omega |\nabla u|^2 = \int_{u^+\neq 0}\left[\frac{\mu f^{q-1}(tu^+)f'(tu^+)}{tu^+} + \frac{f^{2\cdot 2^*-1}(tu^+)f'(tu^+)}{tu^+}\right](u^+)^2.$$

Using Lemma 2.1-(9), we finish the proof.

(2) By Lemma 2.1-(7) and the Sobolev inequality, $I_\mu(u) \ge \frac{1}{2}\|u\|^2 - C_1\|u\|^{q/2} - C_2\|u\|^{2^*}$. Then $\inf_{S_\rho} I_\mu > 0$ for sufficiently small ρ. The first inequality follows, since for each $u \in \mathcal{M}_\mu$, there exists $s > 0$ such that $su \in S_\rho$ and $I_\mu(t_u u) \ge I_\mu(su)$. Hence $c_\mu \le \frac{1}{2}\|u\|^2$ for every $u \in \mathcal{M}_\mu$. □

3 Proof of Theorem 1.1

3.1 Compactness

According to Lemma 2.3, it is standard to prove that the least energy value c_μ has a minimax characterization given by

$$c_\mu = \inf_{u\in E\setminus\{0\}} \sup_{t\ge 0} I_\mu(tu).$$

For $\mu_1 \geq \mu_2 \geq 0$, $I_{\mu_1}(u) = I_{\mu_2}(u) - \frac{\mu_1-\mu_2}{q}\int_\Omega f^q(u^+)$. Hence $\max_{t>0} I_{\mu_1}(tu) \leq \max_{t>0} I_{\mu_2}(tu)$ and therefore $c_{\mu_1} \leq c_{\mu_2}$. Moreover, for every $\lambda > 0$, we have by Lemma 2.3-(2),

$$c_\mu \geq \inf_{S_\rho} I_\mu \geq \inf_{S_\rho} I_\lambda > 0, \text{ for all } \mu \in [0, \lambda). \tag{3.1}$$

Denote by $I_{\mathcal{M}_\mu}$ the restriction of I_μ on $\mathcal{M}_\mu$.

Lemma 3.1.
(1) Assume $(u_n) \subset E$ satisfies that $I_\mu(u_n) \to c < \frac{1}{2N}S^{\frac{N}{2}}$ and $I'_\mu(u_n) \to 0$, then (u_n) has a convergent subsequence for $\mu > 0$.
(2) Assume $(u_n) \subset \mathcal{M}_\mu$ satisfies that $I_\mu(u_n) \to c < \frac{1}{2N}S^{\frac{N}{2}}$ and $I'_{\mathcal{M}_\mu}(u_n) \to 0$, then (u_n) has a convergent subsequence for $\mu > 0$.

Proof. First, we have $c_\mu < \frac{1}{2N}S^{\frac{N}{2}}$ for every $\mu > 0$. Using the similar argument as Lemma 3.3 in [35], it is easy to prove that. For the convenience of readers, we give the proof. Let $u_\varepsilon = w_\varepsilon\phi$, where ϕ is a smooth cut-off function such that $\phi(x) \equiv 1$ in some neighborhood of 0 and $w_\varepsilon(x) = \frac{(N(N-2)\varepsilon)^{\frac{N-2}{4}}}{(\varepsilon+|x|^2)^{\frac{N-2}{2}}}$. Following [35], we have $\int_\Omega |\nabla u_\varepsilon^2|^2 = S^{\frac{N}{2}} + O(\varepsilon^{\frac{N-2}{2}})$, $\int_\Omega |\nabla u_\varepsilon|^2 = O(\varepsilon^{\frac{N-2}{4}}|\ln\varepsilon|)$, $\int_\Omega |u_\varepsilon|^{2\cdot2^*} = S^{\frac{N}{2}} + O(\varepsilon^{\frac{N}{2}})$ and $\int_\Omega |u_\varepsilon|^q = O(\varepsilon^{\frac{N}{2}-\frac{1}{8}q(N-2)})$.

It is easy to see that there is $\varepsilon_0 > 0$, $0 < T_1 < T_2$ such that for $\varepsilon \leq \varepsilon_0$ the function $t \to J_\mu(tu_\varepsilon)$ assumes the maximum at some $t_0 \in [T_1, T_2]$. Hence for $q > q_N$,

$$\begin{aligned}\sup_{t\geq0} J_\mu(tu_\varepsilon) &= t_0^4\int_\Omega |\nabla u_\varepsilon^2|^2 + \frac{t_0^2}{2}\int_\Omega |\nabla u_\varepsilon|^2 - \frac{\mu t_0^q}{q}\int_\Omega u_\varepsilon^q - \frac{t_0^{2\cdot2^*}}{2\cdot2^*}\int_\Omega u_\varepsilon^{2\cdot2^*}\\ &\leq \frac{1}{4}t_0^4 S^{\frac{N}{2}} - \frac{t_0^{2\cdot2^*}}{2\cdot2^*}S^{\frac{N}{2}} + O(\varepsilon^{\frac{N-2}{4}}|\ln\varepsilon|) - C\varepsilon^{\frac{N}{2}-\frac{1}{8}q(N-2)}\\ &\leq \frac{1}{2N}S^{\frac{N}{2}} - C\varepsilon^{\frac{N}{2}-\frac{1}{8}q(N-2)} < \frac{1}{2N}S^{\frac{N}{2}}.\end{aligned}$$

Note that $u_\varepsilon \in H_0^1(\Omega)\cap L^\infty(\Omega)$, then we have $\sup_{t\geq0} I_\mu(f^{-1}(tu_0)) = \sup_{t\geq0} J_\mu(tu_0)$. By [27], c_μ is also a mountain pass level. So

$$c_\mu < \frac{1}{2N}S^{\frac{N}{2}}, \text{ for every } \mu > 0. \tag{3.2}$$

(1) Using Lemma 2.1-(6), we have

$$\begin{aligned}C + C\|u_n\| &\geq I_\mu(u_n) - \frac{2}{q}\langle I'_\mu(u_n), u_n\rangle\\ &\geq (\frac{1}{2} - \frac{2}{q})\|u_n\|^2 + (\frac{1}{q} - \frac{1}{2\cdot2^*})\int_\Omega f^{2\cdot2^*}(u_n^+)\\ &\geq (\frac{1}{2} - \frac{2}{q})\|u_n\|^2,\end{aligned}$$

So (u_n) is bounded in E. Up to a subsequence, we can assume that $u_n \rightharpoonup u$ in E, $u_n \to u$ in $L^r(\Omega)$, $2 \leq r < 2^*$, $u_n \to u$ a.e. on $\mathbb{R}^N$. It is obvious that $I'_\mu(u) = 0$. Let $v_n := u_n - u$. Following a similar argument as Lemma 4.1 in [18], we have $I_\mu(v_n) = I_\mu(u_n) - I_\mu(u) + o(1)$, $I'_\mu(v_n) = I'_\mu(u_n) - I'_\mu(u) + o(1)$, where $o(1) \to 0$ as $n \to \infty$. Hence $I_\mu(v_n) \leq c$ and $I'_\mu(v_n) \to 0$. Since $\|v_n^-\|^2 = \langle I'_\mu(v_n), v_n^-\rangle \to 0$, we may assume $v_n \geq 0$ and

$$\int_\Omega |\nabla v_n|^2 \to l, \quad \int_\Omega f^{2\cdot2^*-1}(v_n)f'(v_n)v_n \to l.$$

We claim that $2^{\frac{2^*-2}{2}}\int_\Omega v_n^{2^*} \to l$. In fact, since

$$\frac{f^{2\cdot2^*-1}(t)f'(t)}{t^{2^*-1}} = \frac{f(t)}{\sqrt{1+2f^2(t)}}\cdot\left[\frac{f(t)}{\sqrt{t}}\right]^{2\cdot2^*-2} \to 2^{\frac{2^*-2}{2}}$$

by Lemma 2.1-(5), we have for every $\varepsilon > 0$, there is $R > 0$ enough large such that

$$\int_{|v_n|\le R} f^{2\cdot 2^*-1}(v_n)f'(v_n)v_n - 2^{\frac{2^*-2}{2}} v_n^{2^*} \le C_1 \int_{|v_n|\le R} v_n^{2^*}$$
$$\le C_1 R^{2^*-r} \int_{|v_n|\le R} v_n^r \le C_2\varepsilon$$

and

$$\int_{|v_n|>R} \left(\frac{f^{2\cdot 2^*-1}(v_n)f'(v_n)v_n}{v_n^{2^*}} - 2^{\frac{2^*-2}{2}} \right) v_n^{2^*} \le \varepsilon \int_{|v_n|>R} v_n^{2^*} \le C_3\varepsilon.$$

By the Sobolev inequality,

$$2^{\frac{2^*-2}{2}} \int_\Omega v_n^{2^*} \le 2^{\frac{2^*-2}{2}} S^{-\frac{2^*}{2}} \left(\int_\Omega |\nabla v_n|^2 \right)^{\frac{2^*}{2}}$$

and so $l \le 2^{\frac{2^*-2}{2}} S^{-\frac{2^*}{2}} l^{\frac{2^*}{2}}$.

Either $l = 0$ or $l \ge \frac{1}{2} S^{\frac{N}{2}}$. If $l = 0$, the conclusion follows. Assume $l \ge \frac{1}{2} S^{\frac{N}{2}}$. It follows from Lemma 2.1-(6) that

$$c \ge \frac{1}{2} l - \frac{1}{2^*} l = \frac{1}{N} l \ge \frac{1}{2N} S^{\frac{N}{2}} > c,$$

a contradiction.

(2) We take a similar argument as Lemma 4.2 in [1]. There exists a sequence $(\theta_n) \subset \mathbb{R}$ such that $I'_\mu(u_n) = \theta_n J'_\mu(u_n) + o(1)$ ($J_\mu(u)$ is in Lemma 2.2). By Lemma 2.2, we have $\langle J'_\mu(u), u\rangle < 0$ for all $u \in \mathcal{M}_\mu$.

If $\langle J'_\mu(u_n), u_n\rangle \to 0$, we obtain by the proof of Lemma 2.2,

$$2\int_\Omega |\nabla u_n|^2 + o(1) = \mu\langle g'_q(u_n), u_n\rangle + \langle g'_{2\cdot 2^*}(u_n), u_n\rangle$$
$$\ge \frac{q\mu}{2} \int_\Omega f^{q-1}(u_n)f'(u_n)u_n + 2^* \int_\Omega f^{2\cdot 2^*-1}(u_n)f'(u_n)u_n.$$

Note that $u_n \in \mathcal{M}_\mu$, then

$$\int_\Omega f^{q-1}(u_n)f'(u_n)u_n \to 0 \text{ and } \int_\Omega f^{2\cdot 2^*-1}(u_n)f'(u_n)u_n \to 0,$$

and therefore $\|u_n\| \to 0$, a contradiction to Lemma 2.3-(2).

Hence $\langle J'_\mu(u_n), u_n\rangle \to d < 0$. Then we get $\theta_n \to 0$ and $I'_\mu(u_n) \to 0$. Using the first conclusion, this completes the proof.

□

3.2 Asymptotic behavior of c_μ

Define

$$c_0 = \inf_{u\in E\setminus\{0\}} \sup_{t\ge 0} I_0(tu).$$

Lemma 3.2. $c_0 = \frac{1}{2N} S^{\frac{N}{2}}$.

Proof. Let $\psi \in C_0^\infty(\mathbb{R}^N)$ be a standard cut-off function satisfying $\psi \equiv 1$ on $B_{R_\varepsilon}(0)$ and $\psi \equiv 0$ on $\mathbb{R}^N \setminus B_{2R_\varepsilon}(0)$ with $R_\varepsilon = \varepsilon$. Up to a translation, we may assume that $B_{2R_\varepsilon}(0) \subset \Omega$ and set $u_\varepsilon(x) = \psi w_\varepsilon(x)$, where $w_\varepsilon(x) = \frac{c_N \varepsilon^{\frac{N-2}{4}}}{(\varepsilon+|x|^2)^{\frac{N-2}{2}}}$, $c_N = (N(N-2))^{\frac{N-2}{4}}$. It follows from [4] that

$$\int_\Omega |\nabla u_\varepsilon|^2 = S^{\frac{N}{2}} + O(\varepsilon^{\frac{N-2}{2}}) \text{ and } \int_\Omega |u_\varepsilon|^{2^*} = S^{\frac{N}{2}} + O(\varepsilon^{\frac{N}{2}}).$$

Set $v_\varepsilon = \frac{u_\varepsilon}{|u_\varepsilon|_{2^*}}$, then we have $\int_\Omega |\nabla v_\varepsilon|^2 = S + O(\varepsilon^{\frac{N-2}{2}})$. Consider

$$I_0(tv_\varepsilon) = \frac{t^2}{2}\int_\Omega |\nabla v_\varepsilon|^2 - \frac{1}{2^*}\int_\Omega 2^{\frac{2^*-2}{2}}(tv_\varepsilon)^{2^*} + \frac{1}{2^*}\int_\Omega 2^{\frac{2^*-2}{2}}(tv_\varepsilon)^{2^*} - \frac{1}{2}f^{2\cdot 2^*}(tv_\varepsilon).$$

Since $\lim_{t\to 0} I_0(tv_\varepsilon) = 0$ and $\lim_{t\to\infty} I_0(tv_\varepsilon) = -\infty$ by Lemma 2.1-(5), we have $0 < t_1 < t_2 < \infty$ such that $I_0(t_\varepsilon v_\varepsilon) = \max_{t>0} I_0(tv_\varepsilon)$, $t_\varepsilon \in [t_1, t_2]$. A direct calculation implies that

$$\max_{t>0}\left[\frac{t^2}{2}\int_\Omega |\nabla v_\varepsilon|^2 - \frac{1}{2^*}\int_\Omega 2^{\frac{2^*-2}{2}}(tv_\varepsilon)^{2^*}\right] = \frac{1}{2N}S^{\frac{N}{2}} + O(\varepsilon^{\frac{N-2}{2}}).$$

For $x \in B_{R_\varepsilon}(0)$,

$$v_\varepsilon(x) \geq \frac{C_1\varepsilon^{\frac{N-2}{4}}}{(\varepsilon + R_\varepsilon^2)^{\frac{N-2}{2}}} \geq \frac{C_2}{\varepsilon^{\frac{N-2}{4}}} \to \infty, \quad \text{as } \varepsilon \to 0.$$

Note that $0 \leq 2^{\frac{2^*-2}{2}}s^{2^*} - \frac{1}{2}f^{2\cdot 2^*}(s) \to 0$, as $s \to \infty$, by Lemma 2.1-(5),(7). So

$$\int_{B_{R_\varepsilon}(0)} 2^{\frac{2^*-2}{2}}(tv_\varepsilon)^{2^*} - \frac{1}{2}f^{2\cdot 2^*}(tv_\varepsilon) \to 0, \quad \text{as } \varepsilon \to 0.$$

For $x \in B_{2R_\varepsilon}(0) \setminus B_{R_\varepsilon}(0)$, $v_\varepsilon(x) \leq C_3 w_\varepsilon(x) \leq \frac{C_4}{\varepsilon^{\frac{N-2}{4}}}$. So

$$\begin{aligned}&\int_{B_{2R_\varepsilon}(0)\setminus B_{R_\varepsilon}(0)} \left|2^{\frac{2^*-2}{2}}(tv_\varepsilon)^{2^*} - \frac{1}{2}f^{2\cdot 2^*}(tv_\varepsilon)\right| \\ &\leq C_5 \int_{B_{2R_\varepsilon}(0)\setminus B_{R_\varepsilon}(0)} |v_\varepsilon|^{2^*} \leq \frac{C_7\varepsilon^N}{\varepsilon^{\frac{N}{2}}} = C_7\varepsilon^{\frac{N}{2}} \to 0.\end{aligned}$$

Hence

$$\int_\Omega 2^{\frac{2^*-2}{2}}(tv_\varepsilon)^{2^*} - \frac{1}{2}f^{2\cdot 2^*}(tv_\varepsilon) \to 0, \quad \text{as } \varepsilon \to 0,$$

and therefore $c_0 \leq \frac{1}{2N}S^{\frac{N}{2}}$.

On the other hand, by the Ekeland variational principle, we can assume that $(u_n) \subset \mathcal{M}_0$ such that $I_0(u_n) \to c_0$, $I_0'(u_n) \to 0$. It is easy to prove that (u_n) is bounded in E. Similarly as Lemma 3.1, we may assume that $u_n \geq 0$. Obviously $\frac{f(u_n)}{f'(u_n)}$ is bounded in E. Then

$$\left\langle I_0'(u_n), \frac{f(u_n)}{f'(u_n)}\right\rangle = \int_\Omega \left[1 + 2f^2(u_n)(f'(u_n))^2\right]|\nabla u_n|^2 - \int_\Omega f^{2\cdot 2^*}(u_n) \to 0.$$

We may assume that

$$\int_\Omega \left[1 + 2f^2(u_n)(f'(u_n))^2\right]|\nabla u_n|^2 \to l, \quad \int_\Omega f^{2\cdot 2^*}(u_n) \to l.$$

We claim that $l > 0$. In fact, if $l = 0$, it follows from Lemma 2.1-(6) that $\int_\Omega f^{2\cdot 2^*-1}(u_n)f'(u_n)(u_n) \to 0$. Using $\langle I_0'(u_n), u_n\rangle \to 0$, we have $\|u_n\| \to 0$, a contradiction to Lemma 2.3-(2).

We obtain by Lemma 2.1-(8),

$$S \le \frac{\int_\Omega |\nabla (f^2(u_n))|^2}{\left(\int_\Omega f^{2\cdot 2^*}(u_n)\right)^{\frac{2}{2^*}}} \le \frac{l}{l^{\frac{2}{2^*}}} = l^{\frac{2}{N}},$$

and then

$$\begin{aligned} c_0 &= \frac{1}{2}\int_\Omega |\nabla u_n|^2 - \frac{1}{2\cdot 2^*}\int_\Omega f^{2\cdot 2^*}(u_n) + o(1) \\ &\ge \frac{1}{4}\int_\Omega \left[1 + 2f^2(u_n)(f'(u_n))^2\right] |\nabla u_n|^2 - \frac{1}{2\cdot 2^*}\int_\Omega f^{2\cdot 2^*}(u_n) + o(1) \\ &= \frac{1}{2N} S^{\frac{N}{2}} + o(1), \end{aligned}$$

where $o(1) \to 0$ as $n \to \infty$. The conclusion follows. □

Lemma 3.3. *$c_{\mu_n} \to c_0$ as $\mu_n \to 0$.*

Proof. We know that $c_{\mu_n} \le c_0$ for all $n \in \mathbb{N}$. Without loss of generality, we may assume that (μ_n) is nonincreasing. By Lemma 3.1 and (3.2), there exist nonnegative functions $u_n \in E$ such that $I_{\mu_n}(u_n) = c_{\mu_n}$ and $I'_{\mu_n}(u_n) = 0$. It is easy to obtain that (u_n) is bounded in E. Let $t_n u_n \in \mathcal{M}_0$. We have by Lemma 2.3,

$$c_0 \le I_0(t_n u_n) = I_{\mu_n}(t_n u_n) + \frac{\mu_n}{q}\int_\Omega f^q(t_n u_n) \le c_{\mu_n} + \frac{\mu_n}{q}\int_\Omega f^q(t_n u_n). \tag{3.3}$$

If $t_n \to \infty$, using (3.1) and Lemma 2.1-(6),(7),

$$0 < 2\inf_{S_\rho} I_{\mu_1} \le \int_\Omega |\nabla u_n|^2 = \frac{1}{t_n}\int_\Omega f^{2\cdot 2^*-1}(t_n u_n) f'(t_n u_n) u_n \le C t_n^{2^*-2}\int_\Omega |u_n|^{2^*}.$$

Hence $\int_\Omega |u_n|^{2^*} \to 0$. It follows from Lemma 2.1-(7) and the Hölder inequality that

$$\int_\Omega f^q(u_n) \le C\int_\Omega |u_n|^{\frac{q}{2}} \le C|u_n|_2^{\frac{q}{2}(1-\lambda)} \cdot |u_n|_{2^*}^{\frac{q}{2}\lambda} \to 0,$$

where $\lambda \in (0,1)$. Since $\langle I'_{\mu_n}(u_n), u_n\rangle = 0$, we have by Lemma 2.1-(6),(7),

$$\begin{aligned} \int_\Omega |\nabla u_n|^2 &= \mu_n \int_\Omega f^{q-1}(u_n) f'(u_n) u_n + \int_\Omega f^{2\cdot 2^*-1}(u_n) f'(u_n) u_n \\ &\le \mu_n \int_\Omega f^q(u_n) + C\int_\Omega |u_n|^{2^*} \to 0, \end{aligned}$$

a contradiction. So (t_n) is bounded and moreover by (3.3),

$$c_0 \le \liminf_{n\to\infty} c_{\mu_n} \le \limsup_{n\to\infty} c_{\mu_n} \le c_0.$$

□

3.3 The barycenter map

On $\mathcal{M}_\mu$, we define the map

$$\beta(u) := \frac{1}{2^{\frac{N}{2-N}} S^{\frac{N}{2}}}\int_\Omega (u^+)^{2^*} x\, dx.$$

Since Ω is a smooth bounded domain of $\mathbb{R}^N$, we choose $r > 0$ small enough that

$$\Omega_r^+ := \{x \in \mathbb{R}^N : \operatorname{dist}(x, \Omega) < r\}$$

and

$$\Omega_r^- := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > r\}$$

are homotopically equivalent to Ω. Moreover we can assume that $B_r := B_r(0) \subset \Omega$.

Let $H^1_{0,rad}(B_r) := \{u \in H^1_0(B_r) : u \text{ is radial}\}$ and

$$m(\mu) := \inf\{I_{\mu,B_r}(u) : u \in \mathcal{M}_{\mu,B_r}\},$$

where

$$I_{\mu,B_r}(u) := \frac{1}{2}\int\limits_{B_r} |\nabla u|^2 - \frac{\mu}{q}\int\limits_{B_r} f^q(u^+) - \frac{1}{2\cdot 2^*}\int\limits_{B_r} f^{2\cdot 2^*}(u^+)$$

and

$$\mathcal{M}_{\mu,B_r} := \{u \in H^1_{0,rad}(B_r)\setminus\{0\} : \langle I'_{\mu,B_r}(u), u\rangle = 0\}.$$

Obviously, $m(\mu)$ is nonincreasing in μ. From the above lemmas, it is easy to have

Lemma 3.4.
(1) I_{μ,B_r} satisfies the $(PS)_c$ condition for all $c \in (0, \frac{1}{2N}S^{\frac{N}{2}})$ on $H^1_{0,rad}(B_r)$, and moreover,

$$m(\mu) \in (0, \tfrac{1}{2N}S^{\frac{N}{2}}) \text{ for } \mu > 0.$$

(2)

$$m(\mu) \to \tfrac{1}{2N}S^{\frac{N}{2}} \text{ as } \mu \to 0.$$

Let $c_{0,\mathbb{R}^N} := \inf\{I_{0,\mathbb{R}^N}(u) : u \in \mathcal{M}_{0,\mathbb{R}^N}\}$, where

$$I_{0,\mathbb{R}^N}(u) := \frac{1}{2}\int\limits_{\mathbb{R}^N} |\nabla u|^2 - \frac{1}{2\cdot 2^*}\int\limits_{\mathbb{R}^N} f^{2\cdot 2^*}(u^+)$$

and

$$\mathcal{M}_{0,\mathbb{R}^N} := \{u \in H^1(\mathbb{R}^N)\setminus\{0\} : \langle I'_{0,\mathbb{R}^N}(u), u\rangle = 0\}.$$

Taking the same argument as Proposition 1.43 in [50], we have $c_0 = c_{0,\mathbb{R}^N} = \frac{1}{2N}S^{\frac{N}{2}}$.

Lemma 3.5. *There is $\mu^\star > 0$ such that if $\mu \in (0, \mu^\star)$ and $u \in \mathcal{M}_\mu$ with $I_\mu(u) \le m(\mu)$, then $\beta(u) \in \Omega_r^+$.*

Before proving Lemma 3.5, we start with the following lemma (see [50]).

Lemma 3.6. *Let $(u_n) \subset H^1_0(\Omega)$ be a nonnegative function sequence with $|u_n|_{2^*} = 1$ and $\|u_n\|^2 \to S$. Then there exists a sequence $(y_n, \lambda_n) \in \mathbb{R}^N \times \mathbb{R}^+$ such that $v_n(x) := \lambda_n^{\frac{N-2}{2}} u_n(\lambda_n x + y_n)$ contains a convergent subsequence denoted again by (v_n) such that $v_n \to v$ in $D^{1,2}(\mathbb{R}^N)$ with $v(x) > 0$ in $\mathbb{R}^N$. Moreover, we have $\lambda_n \to 0$ and $y_n \to y \in \overline{\Omega}$.*

Proof of Lemma 3.5. Arguing by contradiction, if there exist $\mu_n \to 0$, $(u_n) \subset \mathcal{M}_{\mu_n}$ and $c_{\mu_n} \le I_{\mu_n}(u_n) \le m(\mu_n)$ such that $\beta(u_n) \notin \Omega_r^+$, then (u_n) is bounded. By Lemma 2.1-(6), we know $C_1, C_2 > 0$ such that, without loss of generality,

$$0 < C_1 \le \int\limits_{\Omega} f^{2\cdot 2^*-1}(u_n^+) f'(u_n^+) u_n^+ \le C_2 < \infty, \text{ for each } n.$$

Let $g(t) = \langle I_0'(tu_n), tu_n\rangle$, $t \geq 0$. Obviously, $g(0) = 0$, $g(1) = \int_\Omega |\nabla u_n|^2 - \int_\Omega f^{2\cdot 2^*-1}(u_n^+)f'(u_n^+)u_n^+ > 0$ and $g(t) \to -\infty$, as $t \to \infty$. Using Lemma 2.1-(9), there is a unique $t_n > 1$ such that $g(t_n) = 0$, that is, $t_n u_n \in \mathcal{M}_0$. Since $\langle I_{\mu_n}'(u_n), u_n\rangle = 0$ and $\langle I_0'(t_n u_n), t_n u_n\rangle = 0$, we know

$$\frac{1}{t_n}\int_\Omega f^{2\cdot 2^*-1}(t_n u_n^+)f'(t_n u_n^+)u_n^+ = \mu_n \int_\Omega f^{q-1}(u_n^+)f'(u_n^+)u_n^+ + \int_\Omega f^{2\cdot 2^*-1}(u_n^+)f'(u_n^+)u_n^+.$$

Case 1. $u_n \rightharpoonup u \neq 0$, as $n \to \infty$. Then (t_n) is bounded. Otherwise, if $t_n \to \infty$, we have by Lemma 2.1-(5) and the Fatou's Lemma,

$$\liminf_{n\to\infty} \frac{1}{t_n}\int_\Omega f^{2\cdot 2^*-1}(t_n u_n^+)f'(t_n u_n^+)u_n^+ = +\infty,$$

a contradiction. Assume $t_n \to t_0 \geq 1$, as $n \to \infty$, then we show $t_0 = 1$.

If $t_0 > 1$, then noting $\mu_n \to 0$, we get by Lemma 2.1-(9) and the Fatou's Lemma,

$$\begin{aligned} 0 &= \lim_{n\to\infty}\left[\frac{1}{t_n}\int_\Omega f^{2\cdot 2^*-1}(t_n u_n^+)f'(t_n u_n^+)u_n^+ - \int_\Omega f^{2\cdot 2^*-1}(u_n^+)f'(u_n^+)u_n^+\right] \\ &\geq \left[\frac{1}{t_0}\int_\Omega f^{2\cdot 2^*-1}(t_0 u^+)f'(t_0 u^+)u^+ - \int_\Omega f^{2\cdot 2^*-1}(u^+)f'(u^+)u^+\right] > 0, \end{aligned}$$

a contradiction.

Since $I_{\mu_n}(u_n) = c_0 + o(1)$, then $I_0(t_n u_n) = I_{\mu_n}(t_n u_n) + o(1) = c_0 + o(1)$, where $o(1) \to 0$, as $n \to \infty$. Moreover, $I_{0,\mathbb{R}^N}(t_n u_n) = I_0(t_n u_n) \to c_0$, as $n \to \infty$, and $t_n u_n \in \mathcal{M}_{0,\mathbb{R}^N}$. By the definition of c_0, there exist λ_n such that $I_{0,\mathbb{R}^N}'(t_n u_n) + \lambda_n K'(t_n u_n) = o(1)$, where $K(u) := \langle I_{0,\mathbb{R}^N}'(u), u\rangle$. Taking a similar argument as the proof in Lemma 2.2, we have

$$\begin{aligned} \langle K'(t_n u_n), t_n u_n\rangle &\leq 2\int_{\mathbb{R}^N} t_n^2|\nabla u_n|^2 - 2^*\int_{\mathbb{R}^N} f^{2\cdot 2^*-1}(t_n u_n^+)f'(t_n u_n^+)t_n u_n^+ \\ &= (2-2^*)\int_{\mathbb{R}^N} t_n^2|\nabla u_n|^2 < 0, \end{aligned}$$

then $\lambda_n = o(1)$, that is, $I_{0,\mathbb{R}^N}'(t_n u_n) = o(1)$. Hence $I_{0,\mathbb{R}^N}'(u_n) = o(1)$. It follows from the weak convergence that $I_{0,\mathbb{R}^N}'(u) = 0$, i.e.

$$-\Delta u = f^{2\cdot 2^*-1}(u^+)f'(u^+)u^+ \quad \text{in } \mathbb{R}^N.$$

According to Proposition 1.43 in [50], we have $u \equiv 0$, a contradiction.

Case 2. $u_n \rightharpoonup 0$, as $n \to \infty$. Using a similar argument as Lemmas 3.1 and 3.2 (note that $c_{\mu_n} \leq I_{\mu_n}(u_n) \leq m(\mu_n)$), we have

$$I_{\mu_n}(u_n) = \frac{1}{2}\int_\Omega |\nabla u_n|^2 - \frac{2^{\frac{2^*-2}{2}}}{2^*}\int_\Omega (u_n^+)^{2^*} + o(1) = \frac{1}{2N}S^{\frac{N}{2}} + o(1)$$

and

$$\int_\Omega |\nabla u_n|^2 - 2^{\frac{2^*-2}{2}}\int_\Omega (u_n^+)^{2^*} = o(1).$$

Taking a similar argument as Lemma 3.3 in [1],

$$\|w_n^+\|_{2^*} = 1 \text{ and } \|w_n\|^2 \to S,$$

where $w_n := u_n/\|u_n^+\|_{2^*}$. Then the function $\tilde{w}_n(x) := w_n^+(x)$ satisfies

$$\|\tilde{w}_n\|_{2^*} = 1 \text{ and } \|\tilde{w}_n\|^2 \to S.$$

By Lemma 3.6, there exists a sequence $(y_n, \lambda_n) \in \mathbb{R}^N \times \mathbb{R}^+$ such that $v_n(x) := \lambda_n^{\frac{N-2}{2}} \tilde{w}_n(\lambda_n x + y_n)$ converges strongly to $v \in D^{1,2}(\mathbb{R}^N)$. Hence

$$\beta(u_n) = \frac{1}{2^{\frac{N}{2-N}} S^{\frac{N}{2}}} \int_\Omega (u_n^+)^{2^*} x\, dx = \frac{\|u_n^+\|_{2^*}^{2^*}}{2^{\frac{N}{2-N}} S^{\frac{N}{2}}} \int_\Omega (\tilde{w}_n)^{2^*} x\, dx.$$

For $\varphi \in C_0^\infty(\mathbb{R}^N)$ with $\varphi(x) = x$ for all $x \in \overline{\Omega}$, it follows from the Lebesgue theorem and $\|u_n^+\|_{2^*}^{2^*} \to 2^{\frac{N}{2-N}} S^{\frac{N}{2}}$ that

$$\beta(u_n) = \frac{\|u_n^+\|_{2^*}^{2^*}}{2^{\frac{N}{2-N}} S^{\frac{N}{2}}} \int_{\mathbb{R}^N} \varphi(\lambda_n x + y_n) v_n^{2^*}\, dx \to y \in \overline{\Omega}.$$

□.

3.4 Proof of Theorem 1.1

Let $I_\mu^{m(\mu)} := \{u \in H_0^1(\Omega) : I_\mu(u) \le m(\mu)\}$. Using Lemma 3.4, we choose a nonnegative radial function $v_\mu \in \mathcal{M}_{\mu,B_r}$ such that $I_\mu(v_\mu) = I_{\mu,B_r}(v_\mu) = m(\mu)$ and define $y : \Omega_r^- \to I_\mu^{m(\mu)}$ by

$$y(y) = \begin{cases} v_\mu(x-y), & x \in B_r(y), \\ 0, & x \notin B_r(y). \end{cases}$$

For each $y \in \Omega_r^-$, we have

$$(\beta \circ y)(y) = \frac{1}{2^{\frac{N}{2-N}} S^{\frac{N}{2}}} \int_\Omega v_\mu(x-y)^{2^*} x\, dx = \frac{1}{2^{\frac{N}{2-N}} S^{\frac{N}{2}}} \int_\Omega v_\mu(z)^{2^*} (z+y)\, dz,$$

so

$$(\beta \circ y)(y) = \frac{1}{2^{\frac{N}{2-N}} S^{\frac{N}{2}}} \int_\Omega v_\mu(z)^{2^*} y\, dz = \alpha(\mu) y,$$

where $\alpha(\mu) = \frac{1}{2^{\frac{N}{2-N}} S^{\frac{N}{2}}} \int_\Omega v_\mu(z)^{2^*} dz$. Taking the same argument in Lemma 3.5, we have the follow lemma (that is, $\|v_\mu\|_{2^*}^{2^*} \to 2^{\frac{N}{2-N}} S^{\frac{N}{2}}$).

Lemma 3.7. *If $\mu \to 0$, then $\alpha(\mu) \to 1$.*

Consider the homotopy

$$\psi_\mu(t, x) = (1-t)x + t\beta \circ y(x),$$

where $t \in [0, 1]$ and $x \in \Omega_r^-$. According to Lemma 3.7, it is easy to prove that, without loss of generality, there exists $\mu^\star > 0$ such that for $\mu \in (0, \mu^\star)$

$$\psi_\mu(t, x) \in \Omega_r^+,$$

for all $x \in \Omega_r^-$ and $t \in [0, 1]$.

Lemma 3.8. *If $N \ge 3$ and $\mu \in (0, \mu^\star)$ then $cat_{I_{\mathcal{M}_\mu}^{m(\mu)}}(I_{\mathcal{M}_\mu}^{m(\mu)}) \ge cat_\Omega(\Omega)$.*

Proof. Assume that

$$I_{\mathcal{M}_\mu}^{m(\mu)} = A_1 \cup \cdots \cup A_n,$$

where A_j, $j = 1, \cdots, n$, is closed and contractible in $I_{\mathcal{M}_\mu}^{m(\mu)}$, i.e. there is $h_j \in C([0, 1] \times A_j, I_{\mathcal{M}_\mu}^{m(\mu)})$ such that, for every $u, v \in A_j$,

$$h_j(0, u) = u, \quad h_j(1, u) = h_j(1, v).$$

Consider $B_j := y^{-1}(A_j)$, $1 \le j \le n$. The sets B_j are closed and

$$\Omega_r^- = B_1 \cup \cdots \cup B_n.$$

Using the deformation $g_j : [0,1] \times B_j \to \Omega_r^+$ by

$$g_j(t,x) = \begin{cases} \psi_\mu(2t,x), & 0 \le t \le \frac{1}{2}, \\ \beta \circ h_j(2t-1, y(x)), & \frac{1}{2} \le t \le 1, \end{cases}$$

the sets B_j are contractible in Ω_r^+. It follows that

$$\mathrm{cat}_\Omega(\Omega) = \mathrm{cat}_{\Omega_r^+}(\Omega_r^-) \le n.$$

□

Completion of Proof of Theorem 1.1. A standard argument as Proof of Theorem 1 in [1] (see also Theorem 5.26 in [50]) implies that I_μ has at least $\mathrm{cat}_\Omega(\Omega)$ critical points. The proof is complete. □

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