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A Note on Bifurcation from the Essential Spectrum

Marino Badiale *

Dipartimento di Matematica
Università di Torino
Via Carlo Alberto 10, 10123 Torino, Italy
e-mail: badiale@dm.unito.it

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Abstract

In this paper we study a semilinear elliptic equation in all $\mathbb{R}^N$. This equation depends on a parameter λ, and we obtain, for small $\lambda < 0$, solutins which are small in $H^1(\mathbb{R}^N)$. In this sense we have solutions bifurcating from the origin and, as the differential operator involved is the laplacian, we say that we have solutions bifurcating from the bottom of the essential spectrum of the laplacian. By a change of variables we transform the original bifurcation problem into a perturbation one. We adopt a variational procedure, looking for critical points of a suitable functional. We apply a recently developped reduction method, which allows to reduce the original variational problem in $H^1(\mathbb{R}^N)$ to a variational problem in a finite-dimensional manifold, and then we solve this last problem. In this way we are also able to manage the presence of critical nonlinearities, in the sense of Sobolev embedding.

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1 Introduction and main results

We consider the following equation

$$\left\{ \begin{array}{rcl} -\Delta\psi - \lambda\psi & = & a(x)|\psi|^{p-1}\psi + b(x)|\psi|^{q-1}\psi, \quad x \in \mathbb{R}^N, \\ \lim_{|x|\to\infty} \psi(x) & = & 0, \end{array} \right. \tag{1}$$

where $N \geq 1$, $1 < p < q \leq \frac{N+2}{N-2}$ if $N \geq 3$ (and $q < +\infty$ if $N = 1, 2$), $p < 1 + 4/N$. λ is a negative parameter, and we look for families of solutions bifurcating from the origin in $H^1(\mathbb{R}^N)$, that is for couples (λ, ψ_λ) such that $\psi_\lambda \in H^1(\mathbb{R}^N)$, ψ_λ is a solution of (1) in the weak sense of $H^1(\mathbb{R}^N)$, $\lambda \in (\lambda_0, 0)$ for some $\lambda_0 < 0$ and $\psi_\lambda \to 0$ in $H^1(\mathbb{R}^N)$ as $\lambda \to 0$. Hence these families of solutions bifurcate not from an eigenvalue but from a point of the essential spectrum of $-\Delta$ in $\mathbb{R}^N$. This kind of bifurcation has been studied in several recent papers, see for example [14], [15], [16]. In this paper we continue the work initiated in [2] and [3], where problem (1) was studied in the case $N = 1$, and in [9], where many results of [2] where extended to any dimension N. In [2] and [9] it is assumed that the function a has, in some sense, a positive limit A at infinity, and through a change of variables the equation (1) is reduced to a perturbation problem (see below). The hypotheses used in [9] are, roughly speaking, of two different kinds: it is assumed either that the function $a(x) - A$ is integrable and $\int_{\mathbb{R}^N}(a(x) - A)dx \neq 0$, ot that it is asymptotic, at infinity, to $1/|x|^\gamma$ with $\gamma \in]0, N[$. So two interesting limit cases are left out: the case in which $a(x) - A$ is integrable but $\int_{\mathbb{R}^N}(a(x) - A)dx = 0$, and the case in which $a(x) - A$ is asymptotic to $1/|x|^N$. In this paper we will see that also in these cases, adding some more hypotheses, we obtain existence of solutions bifurcating from the origin in $H^1(\mathbb{R}^N)$. Furthemore, in the case $\int_{\mathbb{R}^N}(a(x) - A)dx = 0$ we obtain bifurcation of *two* families of solutions.

Now let us state precisely our results. For our first result (theorem 1.1 below) we will use the following hypotheses.

($\mathbf{h_1}$) a is continuous and bounded.

($\mathbf{h_2}$) There exists $A > 0$ such that the function $x \to (a(x) - A)|x|$ is in $L^1(\mathbb{R}^N)$.

($\mathbf{h_3}$) $\int_{\mathbb{R}^N}(a(x) - A)dx = 0$ and $\int_{\mathbb{R}^N}(a(x) - A)x_i dx \neq 0$ for some $i = 1, \ldots, N$.

($\mathbf{h_4}$) b is continuous and bounded.

($\mathbf{h_5}$) $b \in L^{\frac{2N}{N+2}}(\mathbb{R}^N)$. If $N \geq 2\frac{q-p}{p-1} - 1$ we also assume that there exists $\beta \in [1, \beta^*[$ such that $b \in L^\beta(\mathbb{R}^N)$, where

$$\beta^* = \frac{N(p-1)}{(N+1)(p-1) - 2(q-p)} \quad \text{if} \quad N > 2\frac{q-p}{p-1} - 1,$$
$$\beta^* = +\infty \quad \text{if} \quad N = 2\frac{q-p}{p-1} - 1.$$

For our next result (theorem 1.2 below) we will use the following hypotheses.

($\mathbf{h_6}$) a is continuous.

($\mathbf{h_7}$) There are $A > 0$, $A_1 \neq 0$ sucht that $|x|^N(a(x) - A) \to A_1$ as $|x| \to \infty$.

($\mathbf{h_8}$) The function $a(x) - A - \frac{A_1}{1+|x|^N}$ belongs to $L^1(\mathbb{R}^N)$.

($\mathbf{h_9}$) b is continuous and bounded.

($\mathbf{h_{10}}$) $b \in L^{\frac{2N}{N+2}}(\mathbb{R}^N)$. If $N > 2\frac{q-p}{p-1}$, we also assume that there exists $\beta \in [1, \beta^*]$ such that $b \in L^\beta(\mathbb{R}^N)$, where

$$\beta^* = \frac{N(p-1)}{N(p-1) - 2(q-p)}.$$

We can now state our main results.

Theorem 1.1 *Assume $1 < p < q \leq \frac{N+2}{N-2}$ if $N \geq 3$, and $q < +\infty$ if $N = 1, 2$. Assume $p < 1 + \frac{4}{N}$. Suppose that $(\mathbf{h_1}), \ldots, (\mathbf{h_5})$ hold. Then (1) has two families of solutions bifurcating from the origin in $H^1(\mathbb{R}^N)$.*

Theorem 1.2 *Assume $1 < p < q \leq \frac{N+2}{N-2}$ if $N \geq 3$, and $q < +\infty$ if $N = 1, 2$. Assume $p < 1 + \frac{4}{N}$. Suppose that $(\mathbf{h_6}), \ldots, (\mathbf{h_{10}})$ hold. Then (1) has a family of solutions bifurcating from the origin in $H^1(\mathbb{R}^N)$.*

Remark 1.3 When $p \geq 1 + 4/N$, the families of solutions that we find still bifurcate from the origin in $L^\infty(\mathbb{R}^N)$, but in $H^1(\mathbb{R}^N)$ they can bifurcate from infinity or can be bounded away both from zero and infinity.

To prove our results, we need a change of variables, which tranforms the original problem in a perturbation one. Let us set $u(x) = \varepsilon^{2/(1-p)}\psi(x/\varepsilon)$, $\lambda = -\varepsilon^2$, so that equation (1) changes to

$$-\Delta u + u = A|u|^{p-1}u + (a(x/\varepsilon) - A)|u|^{p-1}u + \varepsilon^{2\frac{q-p}{p-1}} b(x/\varepsilon)|u|^{q-1}u. \tag{2}$$

If $u_\varepsilon \in H^1(\mathbb{R}^N)$ is a family of solutions of (2), bounded as $\varepsilon \to 0$, then $\psi_\varepsilon(x) = \varepsilon^{2/(p-1)}u_\varepsilon(\varepsilon x)$ is a family of solutions of (1) and $\psi_\varepsilon(x) \to 0$ in $H^1(\mathbb{R}^N)$, as $\varepsilon \to 0$ (because $p < 1 + 4/N$). Hence, to find solutions of (1) bifurcating from the origin, we will look for bounded families of H^1-solutions of (2).

The paper is organized as follows: after the introduction (section 1) we give in section 2 a brief sketch of the critical point theory for perturbed functionals that we use to prove theorems 1.1 and 1.2. In section 3 we prove theorem 1.1 and in section 4 we prove theorem 1.2.

Notations

We collect below a list of the main notation used throughout the paper.

- By $B(x,r)$ we mean the open ball in $\mathbb{R}^N$ of center x and radius r.
- $x \cdot y$ is the usual scalar product of $x, y \in \mathbb{R}^N$.
- ω_N is the $(N-1)$-surface area of the sphere $\partial B(0,1)$.
- $2^* = \frac{2N}{N-2}$ is the critical exponent for the Sobolev embedding, when $N \geq 3$.
- We will use C to denote any positive constant, that can change from line to line.

2 Abstract theory for perturbed functionals

In the proof of our results we use a variational method to study critical points of perturbed functionals. The method has been developed in [4], [1] , [2] and then has been applied to many different problems, see [5], [6], [7], [8], [12]. We do not repeat here the main idea of the method. We just state the results that we need, referring to the quoted papers, in particular to [4], [1] , [2], for the proofs. Let us consider the Hilbert space $E = H^1(\mathbb{R}^N)$ with norm $||u||^2 = \int_{\mathbb{R}^N} \left(|\nabla u|^2 + u^2\right) dx$ and the family of functionals

$$f_\varepsilon(u) = \frac{1}{2}||u||^2 - F(u) + G(\varepsilon, u),$$

where

$$F(u) = \frac{A}{p+1} \int_{\mathbb{R}^N} |u|^{p+1} dx,$$

and $G = G_1 + G_2$ where

$$G_1(\varepsilon, u) = \begin{cases} \frac{-1}{p+1} \int_{\mathbb{R}^N} (a(x/\varepsilon) - A)|u|^{p+1} dx & \text{if} \quad \varepsilon \neq 0, \\ 0 & \text{if} \quad \varepsilon = 0 \end{cases}$$

and

$$G_2(\varepsilon, u) = \begin{cases} \frac{-1}{q+1} \varepsilon^{2\frac{q-p}{p-1}} \int_{\mathbb{R}^N} b(x/\varepsilon)|u|^{q+1} dx & \text{if} \quad \varepsilon \neq 0, \\ 0 & \text{if} \quad \varepsilon = 0 \end{cases}$$

We need the following properties:

(G_0) G is continuous in $(\varepsilon, u) \in \mathbb{R} \times E$ and $G(0, u) = 0$ for all $u \in E$;

(G_1) G is of class C^2 with respect to $u \in E$.

(G_2) The maps $(\varepsilon, u) \mapsto G'(\varepsilon, u)$, $(\varepsilon, u) \mapsto G''(\varepsilon, u)$ are continuous.

If $DF(u)$, $D_u G(\varepsilon, u)$ are the first differentials of F and G with respect to u, and $D^2 F(u)$, $D^2_{uu} G(\varepsilon, u)$ are the second differentials and $L(E, E)$ is the space of linear endomorphism of E, we define by $F'(u)$, respectively $G'(\varepsilon, u)$, the functions defined by setting

$$(F'(u)|v) = DF(u)[v], \quad \forall\, v \in E,$$

and, respectively,

$$(G'(\varepsilon,u)|v) = D_u G(\varepsilon,u)[v], \quad \forall\, v \in E,$$

where $(\cdot\,|\,\cdot)$ is the scalar product in E. Similarly, $F''(u)$, resp. $G''(\varepsilon,u)$, denotes the maps in $L(E,E)$ defined by

$$(F''(u)v|w) = D^2F(u)[v,w] \qquad (G''(\varepsilon,u)v|w) = D^2_{uu}G(\varepsilon,u)[v,w].$$

Let us recall that the equation $f_0'(u) = 0$ is the following nonlinear equation

$$\begin{cases} -\Delta u + u & = \; A|u|^{p-1}u, \quad x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N). \end{cases} \tag{3}$$

It is well known (see [10], [11], [13]) that there exists a unique positive radial solution z of (3), that z is strictly radial decreasing, has an exponential decay at infinity together with its derivatives, and that f_0 possesses a $N-$dimensional manifold of critical points

$$Z = \{z_\theta(x) = z(x+\theta)\,|\,\theta \in \mathbb{R}^N\}.$$

Of course $z_0 = z$. We make the following further assumptions on G.

($\mathbf{G_3}$) There exists a function $\varphi : (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}$, such that $\varphi(\varepsilon) \to 0$ as $\varepsilon \to 0$, and a continuous function $\Gamma : Z \to \mathbb{R}$ such that, for all $z_\theta \in Z$,

$$\Gamma(z_\theta) = \lim_{\varepsilon\to 0} \frac{G(\varepsilon, z_\theta)}{\varphi(\varepsilon)}$$

and

$$G'(\varepsilon, z) = o(|\varphi(\varepsilon)|^{1/2}).$$

With the same arguments of [2] (see also [1], [4]) we can prove the following theorem.

Theorem 2.1 *Suppose that* ($\mathbf{G_0} - \mathbf{G_3}$) *hold and assume there exist an open bounded set $U \subset \mathbb{R}^N$ and $\theta^* \in U$ such that, setting $z^* = z_{\theta^*} \in Z$, we have*

$$\text{either} \;\; \min_{\theta\in U} \Gamma(z_\theta) > \Gamma(z^*), \quad \text{or} \;\; \max_{\theta\in U} \Gamma(z_\theta) \; < \Gamma(z^*). \tag{4}$$

Then, for ε small, f_ε has a critical point u_ε.

Notice that in [2] only the case $U = B(\theta^*, \delta)$ and $\varphi(\varepsilon) = \varepsilon^\alpha$ ($\alpha > 0$) is treated, but it is easy to see that the same arguments hold for general φ and U.

To apply teorem 2.1 we have to verify hypotheses ($\mathbf{G_0} - \mathbf{G_3}$). The verification of ($\mathbf{G_0} - \mathbf{G_2}$) is the same as in [9], so we state this fact, without proof, in the following lemma.

Lemma 2.2 *Assume that f_ε, F, G are defined as before. Assume that either the hypotheses of theorem 1.1 or those of theorem 1.2 are satisfied. Then also* ($\mathbf{G_0}$), ($\mathbf{G_1}$), ($\mathbf{G_2}$) *are satisfied.*

We are left with the verification of ($\mathbf{G_3}$), and this will be done in the next two sections.

3 Proof of theorem 1.1

In this section we prove theorem 1.1. As we pointed out above, we only have to verify that hypothesis $(\mathbf{G_3})$ holds.

Lemma 3.1 *Let us assume* $(\mathbf{h_1}),\ldots.(\mathbf{h_5})$. *Let us define* $\gamma = (\gamma_1,\ldots,\gamma_N) \in \mathbb{R}^N$ *by* $\gamma_i = \int_{\mathbb{R}^N}(a(x) - A)\, x_i dx$. *For* $\theta \in \mathbb{R}^N$ *we also define*

$$\Gamma(\theta) = -\frac{1}{p+1}\nabla z^{p+1}(\theta)\cdot\gamma. \tag{5}$$

Then

$$\lim_{\varepsilon\to 0}\frac{G(\varepsilon, z_\theta)}{\varepsilon^{N+1}} = \Gamma(\theta) \tag{6}$$

and

$$G'(\varepsilon, z_\theta) = O(\varepsilon^{\frac{N}{2}+1}). \tag{7}$$

Notice that in this case $\varphi(\varepsilon) = \varepsilon^{N+1}$.

Proof. The argument to prove (7) is the same as in [9] and we don't repeat it here. So let us prove (6). We will prove

$$\lim_{\varepsilon\to 0}\frac{G_1(\varepsilon, z_\theta)}{\varepsilon^{N+1}} = \Gamma(\theta) \tag{8}$$

$$\lim_{\varepsilon\to 0}\frac{G_2(\varepsilon, z_\theta)}{\varepsilon^{N+1}} = 0, \tag{9}$$

which of course imply (6). Let us begin by proving (8). By the change of variables $y = x/\varepsilon$ we have

$$\begin{aligned} -(p+1)G_1(\varepsilon, z_\theta) &= \int_{\mathbb{R}^N}\left(a\left(\frac{x}{\varepsilon}\right) - A\right)\, z^{p+1}(x+\theta)dx \\ &= \varepsilon^N\int_{\mathbb{R}^N}(a(y) - A)\, z^{p+1}(\varepsilon y+\theta)dy. \end{aligned}$$

By Lagrange's theorem, we get

$$z^{p+1}(\theta+\varepsilon y) = z^{p+1}(\theta) + \varepsilon\nabla z^{p+1}(\theta + h\varepsilon y)\cdot y,$$

where $h = h(y,\theta,\varepsilon) \in [0,1]$. Hence

$$\begin{aligned} -(p+1)G_1(\varepsilon, z_\theta) &= \varepsilon^N\int_{\mathbb{R}^N}(a(y) - A)z^{p+1}(\theta)dy + \\ &\quad \varepsilon^{N+1}\int_{\mathbb{R}^N}(a(y) - A)\nabla z^{p+1}(\theta+h\varepsilon y)\cdot y dy = \end{aligned}$$

$$\varepsilon^{N+1}\sum_{i=1}^{N}\int_{\mathbb{R}^N}(a(y)-A)D_i z^{p+1}(\theta+h\varepsilon y)y_i dy.$$

As $(a(y)-A)y_i \in L^1(\mathbb{R}^N)$ and $D_i z^{p+1}$ is bounded, by dominated convergence, we obtain

$$\int_{\mathbb{R}^N}(a(y)-A)D_i z^{p+1}(\theta+h\varepsilon y)y_i dy \to \int_{\mathbb{R}^N}(a(y)-A)D_i z^{p+1}(\theta)y_i dy = D_i z^{p+1}(\theta)\gamma_i,$$

hence

$$\frac{-(p+1)G_1(\varepsilon, z_\theta)}{\varepsilon^{N+1}} \to \sum_{i=1}^{N} D_i z^{p+1}(\theta)\gamma_i = \nabla z^{p+1}(\theta)\cdot\gamma,$$

so (8) is proved. To prove (9), we distinguish two cases. Assume first $N < 2\frac{q-p}{p-1}-1$. In this case

$$\varepsilon^{-N-1}G_2(\varepsilon, z_\theta) = -\frac{1}{q+1}\,\varepsilon^{2\frac{q-p}{p-1}-N-1}\int_{\mathbb{R}^N} b\left(\frac{x}{\varepsilon}\right) z^{q+1}(x+\theta)dx,$$

and this expression of course vanishes as $\varepsilon \to 0$, uniformly with respect to θ, because the integral is bounded. Hence, let us assume $N \geq 2\frac{q-p}{p-1}-1$. We have

$$|\varepsilon^{-N-1}G_2(\varepsilon, z_\theta)| = \Big| -\frac{1}{q+1}\,\varepsilon^{2\frac{q-p}{p-1}-N-1}\int_{\mathbb{R}^N} b\left(\frac{x}{\varepsilon}\right) z_0^{q+1}(x+\theta)dx\Big| \leq$$

$$C\varepsilon^{2\frac{q-p}{p-1}-N-1}\left(\int_{\mathbb{R}^N}\left|b\left(\frac{x}{\varepsilon}\right)\right|^{\beta}dx\right)^{\frac{1}{\beta}}\left(\int_{\mathbb{R}^N} z_0^{\frac{\beta(q+1)}{\beta-1}}(x+\theta)\,dx\right)^{\frac{\beta-1}{\beta}} \leq$$

$$C\,\varepsilon^{2\frac{q-p}{p-1}-N-1+\frac{N}{\beta}}\left(\int_{\mathbb{R}^N}|b(y)|^{\beta}dy\right)^{\frac{1}{\beta}},$$

where β is given by $(\mathbf{h_5})$. This term goes to zero as, by $(\mathbf{h_5})$, $2\frac{q-p}{p-1}-N-1+\frac{N}{\beta} > 0$. ∎

We can then conclude the proof of theorem 1.1. We know that z is radial and strictly radial decreasing. Hence we can write $z_0(x) = \eta(|x|)$ where $\eta : \mathbb{R}^+ \to \mathbb{R}^+$, $\eta'(r) < 0$ if $r > 0$, $\eta'(0) = 0$ and $\eta'(r) \to 0$ as $r \to \infty$. This implies that $\zeta(r) = \frac{d}{dr}\eta^{p+1}(r)$ has a negative absolute minimum, that is, there are $r_0 > 0$ and $m_0 < 0$ such that $m_0 = \zeta(r_0) \leq \zeta(r) < 0$ for all $r > 0$. It is $\nabla z^{p+1}(\theta) = \zeta(|\theta|)\frac{1}{|\theta|}\theta$, hence

$$\Gamma(\theta) = -\frac{1}{p+1}\nabla z_0^{p+1}(\theta)\cdot\gamma = -\frac{1}{p+1}\zeta(|\theta|)\,\frac{1}{|\theta|}\,\theta\cdot\gamma.$$

It is then easy to see that $\theta^* = r_0 \frac{1}{|\gamma|}\gamma$ is an absolute maximum point for Γ, and $-\theta^*$ is an absolute minimum point. It is also possible to take $R > 0$ such that for all θ with $|\theta| \geq R$ or $|\theta \cdot \gamma| < \frac{1}{R}$ we have $|\Gamma(\theta)| < \frac{1}{2}|\Gamma(\theta^*)| = \frac{1}{2}|\Gamma(-\theta^*)|$ (recall that $\zeta(r)/r$ is bounded in $(0, +\infty)$). Let us define

$$U_1 = \{\theta \in \mathbb{R}^N \mid |\theta| < R,\ \theta \cdot \gamma > 1/R\}, \qquad U_2 = \{\theta \in \mathbb{R}^N \mid |\theta| < R,\ \theta \cdot \gamma < -1/R\}.$$

It is then easy to see that

$$\Gamma(\theta^*) > \inf_{\theta \in U_1} \Gamma(\theta), \qquad \Gamma(-\theta^*) < \sup_{\theta \in U_2} \Gamma(\theta).$$

We can then apply theorem 2.1. We obtain two families $\{(\varepsilon, u_\varepsilon^1)\}$, $\{(\varepsilon, u_\varepsilon^2)\}$ of solutions of (2) such that $\{u_\varepsilon^i\}$ is a bounded set in $H^1(\mathbb{R}^N)$. To be precise, we have (see [2] and [9])

$$u_\varepsilon^i(x) = z_0(x + \theta_\varepsilon^i) + w(\varepsilon, \theta_\varepsilon^i)(x)$$

where $\theta_\varepsilon^i \in U_i$ and $w(\varepsilon, \theta_\varepsilon^i) \to 0$ as $\varepsilon \to 0$. Hence $\theta_\varepsilon^i \to \theta^i \in \overline{U}_i$, as $\varepsilon \to 0$, so $u_\varepsilon^i(x) \to z_0(x + \theta^i)$ and $\theta^1 \neq \theta^2$. This implies that $u_\varepsilon^1 \neq u_\varepsilon^2$, so the two families are distinct (at least for small ε's) and they give rise to two distinct families of solutions of (1) bifurcating from the origin.

4 Proof of theorem 1.2

To prove theorem 1.2 we have, as before, to prove only that hypothesis $(\mathbf{G_3})$ holds. Let us define $\varphi(\varepsilon) = \varepsilon^N |\log \varepsilon|$.

Lemma 4.1 *Let us assume* $(\mathbf{h_6}), \ldots . (\mathbf{h_{10}})$. *Define*

$$\Gamma(\theta) = -\frac{1}{p+1} A_1 \omega_N z_0^{p+1}(\theta). \tag{10}$$

Then we have

$$\lim_{\varepsilon \to 0} \frac{G(\varepsilon, z_\theta)}{\varphi(\varepsilon)} = \Gamma(\theta) \tag{11}$$

and

$$G'(\varepsilon, z_\theta) = o(|\varphi(\varepsilon)|^{1/2}). \tag{12}$$

Proof. The proof of (12) is the same as in [9], and we do not repeat it. To prove (11), we prove the following two statements.

$$\lim_{\varepsilon \to 0} \frac{G_1(\varepsilon, z_\theta)}{\varphi(\varepsilon)} = \Gamma(\theta) \tag{13}$$

$$\lim_{\varepsilon\to 0}\frac{G_2(\varepsilon,z_\theta)}{\varphi(\varepsilon)}=0. \tag{14}$$

Again, the proof of (14) is the same as in [9], so we only have to prove (13). For this, let us write

$$\begin{aligned} -(p+1)G_1(\varepsilon,z_\theta) &= \int_{\mathbb{R}^N}(a(x/\varepsilon)-A)\,z^{p+1}(x+\theta)dx \\ &= \int_{\mathbb{R}^N}\left(a(x/\varepsilon)-A-A_1(1+|x/\varepsilon|^N)^{-1}\right)z^{p+1}(x+\theta)dx + \\ &\quad \int_{\mathbb{R}^N}\left(A_1(1+|x/\varepsilon|^N)^{-1}\right)z^{p+1}(x+\theta)dx. \end{aligned} \tag{15}$$

By the change of variables $y=x/\varepsilon$, and using the hypothesis $a(x/\varepsilon)-A-A_1(1+|x/\varepsilon|^N)^{-1}\in L^1(\mathbb{R}^N)$, it is easy to get

$$\int_{\mathbb{R}^N}\left(a(x/\varepsilon)-A-A_1(1+|x/\varepsilon|^N)^{-1}\right)z^{p+1}(x+\theta)dx =$$

$$\varepsilon^N\int_{\mathbb{R}^N}\left(a(y)-A-A_1(1+|y|^N)^{-1}\right)z^{p+1}(\varepsilon y+\theta)dy =$$

$$\varepsilon^N\int_{\mathbb{R}^N}\left(a(y)-A-A_1(1+|y|^N)^{-1}\right)z^{p+1}(\theta)dy+o(\varepsilon^N)=o(\varphi(\varepsilon)).$$

Let us study the second integral on the right-hand side of (15). We have

$$\begin{aligned} &\int_{\mathbb{R}^N}(1+|x/\varepsilon|^N)^{-1}z^{p+1}(x+\theta)dx \\ =\;& \varepsilon^N\int_{\mathbb{R}^N}\frac{1}{\varepsilon^N+|x|^N}\,z^{p+1}(x+\theta)dx \\ =\;& \varepsilon^N\int_0^{+\infty}\left[\int_{\partial B(0,\rho)}\frac{1}{\varepsilon^N+|x|^N}z^{p+1}(x+\theta)d\sigma_x\right]d\rho \\ =\;& \varepsilon^N\int_0^{+\infty}\frac{1}{\varepsilon^N+\rho^N}\left[\int_{\partial B(0,\rho)}z^{p+1}(x+\theta)d\sigma_x\right]d\rho \\ =\;& \varepsilon^N\int_0^{+\infty}\frac{\rho^{N-1}}{\varepsilon^N+\rho^N}\left[\int_{\partial B(0,1)}z^{p+1}(\rho y+\theta)d\sigma_y\right]d\rho. \end{aligned}$$

Here by $d\sigma_x$ or $d\sigma_y$ we mean the surface measures of $\partial B(0,\rho)$ or $\partial B(0,1)$. By an integration by parts, this is equal to

$$\varepsilon^N\left[\frac{1}{N}\log(\rho^N+\varepsilon^N)\left(\int_{|y|=1}z^{p+1}(\theta+\rho y)d\sigma_y\right)\Bigg|_{\rho=0}^{\rho=+\infty}\right]- \tag{16}$$

$$\varepsilon^N \int_0^{+\infty} \frac{1}{N} \log(\rho^N + \varepsilon^N) \left(\int_{|y|=1} \nabla z^{p+1}(\theta + \rho y) \cdot y d\sigma_y \right) d\rho.$$

Let us now study these two terms. We know that, for some $K, \delta > 0$, it is $z^{p+1}(x) \le Ke^{-\delta|x|}$. Hence we easily get

$$\left| \int_{|y|=1} z^{p+1}(\theta + \rho y) d\sigma_y \right| \le K_1 e^{-\delta_1 \rho},$$

for suitable $K_1, \delta_1 > 0$, independent from θ when θ is in a fixed bounded set. From this we get easily

$$\log(\rho^N + \varepsilon^N) \int_{|y|=1} z^{p+1}(\theta + \rho y) d\sigma_y \to 0 \quad \text{as} \quad \rho \to +\infty,$$

and hence, for small ϵ's,

$$\varepsilon^N \left[\frac{1}{N} \log(\rho^N + \varepsilon^N) \left(\int_{|y|=1} z^{p+1}(\theta + \rho y) d\sigma_y \right) \Big|_{\rho=0}^{\rho=+\infty} \right] =$$

$$-\varepsilon^N \log\varepsilon \int_{|y|=1} z^{p+1}(\theta) d\sigma_y = \varepsilon^N |\log\varepsilon| z^{p+1}(\theta) \omega_N = \omega_N \varphi(\varepsilon) z^{p+1}(\theta).$$

We estimate the other integral in (16) in the following way: for small ε's,

$$\left| -\varepsilon^N \int_0^{+\infty} \frac{1}{N} \log(\rho^N + \varepsilon^N) \left(\int_{|y|=1} \nabla z^{p+1}(\theta + \rho y) \cdot y d\sigma_y \right) d\rho \right| \le$$

$$\varepsilon^N \frac{1}{N} \int_0^{+\infty} |\log(\rho^N + \varepsilon^N)| \left(\int_{|y|=1} |\nabla z^{p+1}(\theta + \rho y)| d\sigma_y \right) d\rho \le$$

$$\varepsilon^N \frac{1}{N} \int_0^{+\infty} (\log(\rho^N + 1) + |\log(\rho^N)|) \left(\int_{|y|=1} |\nabla z^{p+1}(\theta + \rho y)| d\sigma_y \right) d\rho.$$

This last integral converges, because the integrand function

$$f(\rho) = (\log(\rho^N + 1) + |\log(\rho^N)|) \left(\int_{|y|=1} |\nabla z^{p+1}(\theta + \rho y)| d\sigma_y \right)$$

is continuous in $(0, +\infty)$, in $\rho = 0$ is dominated by the integrable function $|\log(\rho)|$ and has an exponential decay at infinity, uniformly in θ when θ is in a bounded set. Hence we can write

$$\left| \varepsilon^N \int_0^{+\infty} \frac{1}{N} \log(\rho^N + \varepsilon^N) \left(\int_{|y|=1} \nabla z^{p+1}(\theta + \rho y) \cdot y d\sigma_y \right) d\rho \right| \le C\varepsilon^N = o(\varphi(\varepsilon)).$$

Collecting all these results we obtain

$$G_1(\varepsilon, z_\theta) = \Gamma(\theta)\varphi(\varepsilon) + o(\varphi(\varepsilon)),$$

hence (11) is proved. Let us now prove (12). We give the arguments in the case $N \geq 3$; the cases $N = 1, 2$ are similar and in fact easier. We have

$$||G_1'(\varepsilon, z_\theta)|| = \sup_{||v|| \leq 1} \left|(G_1'(\varepsilon, z_\theta) \,|\, v)\right| = \sup_{||v|| \leq 1} \left| \int_{\mathbb{R}^N} (a(x/\varepsilon) - A) z^p(x+\theta) v dx \right| \leq$$

$$\sup_{||v|| \leq 1} \left(\int_{\mathbb{R}^N} |a(x/\varepsilon) - A|^{\frac{2N}{N+2}} |z(x+\theta)|^{p\frac{2N}{N+2}} dx \right)^{\frac{N+2}{2N}} \left(\int_{\mathbb{R}^N} |v|^{\frac{2N}{N-2}} dx \right)^{\frac{N-2}{2N}} \leq$$

$$C \left(\int_{\mathbb{R}^N} |a(x/\varepsilon) - A|^{\frac{2N}{N+2}} dx \right)^{\frac{N+2}{2N}}.$$

By hypothesis ($\mathbf{h_7}$), the function $|a(y) - A|^{\frac{2N}{N+2}}$ is asymptotic, at infinity, to the functions $|y|^{-\frac{2N^2}{N+2}}$, so of course it belongs to $L^1(\mathbb{R}^N)$. Hence, by the change of variables $x/\varepsilon = y$, we obtain

$$||G_1'(\varepsilon, z_\theta)|| \leq C\varepsilon^{1+N/2}$$

so (12) is proved. ∎

To conclude the proof of theorem 1.2 we notice that $z(\theta)$ has a strict maximum at $\theta = 0$, so the same holds for $z^{p+1}(\theta)$. Hence we can apply theorem 2.1, taking $U = B(0,1)$ and $\theta^* = 0$. We obtain a family $\{u_\varepsilon\}$ of solutions of (2), from which we come back, in the usual way, to a family (λ, ψ_λ) of solutions of (1) bifurcating from the origin.

Remark 4.2 Let us try to explain the meaning of hypothesis ($\mathbf{h_8}$). From ($\mathbf{h_7}$) we deduce that $g(x) = |x|^N(a(x) - A) - A_1 \to 0$ as $|x| \to \infty$. We can write

$$a(x) - A - \frac{A_1}{|x|^N} = \frac{g(x)}{|x|^N}$$

and

$$a(x) - A - \frac{A_1}{1+|x|^N} = A_1 \left(\frac{1}{|x|^N} - \frac{1}{1+|x|^N} \right) + \frac{g(x)}{|x|^N}.$$

The function $a(x) - A - \frac{A_1}{1+|x|^N}$ is of course locally integrable, so hypothesis ($\mathbf{h_8}$) deals with its behavior at infinity. It is obvious that $\frac{1}{|x|^N} - \frac{1}{1+|x|^N}$ is integrable at infinity, so ($\mathbf{h_8}$) is equivalent to the hypothesis that $\frac{g(x)}{|x|^N}$ is integrable at infinity. We know that $g(x) \to 0$, so what we need is that $g(x)$ decays fast enough at infinity: for example if $g(x)$ is asymptotic to $1/|x|^\alpha$, for any $\alpha > 0$, or to $1/(\log|x|)^\alpha$, for any $\alpha > 1$, then ($\mathbf{h_8}$) is satisfied, while it is not satisfied if $g(x)$ is asymptotic to $1/\log|x|$.

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