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Research Article

Strong Convergence Theorems for a Generalized Equilibrium Problem with a Relaxed Monotone Mapping and a Countable Family of Nonexpansive Mappings in a Hilbert Space

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We introduce a new iterative method for finding a common element of the set of solutions of a generalized equilibrium problem with a relaxed monotone mapping and the set of common fixed points of a countable family of nonexpansive mappings in a Hilbert space and then prove that the sequence converges strongly to a common element of the two sets. Using this result, we prove several new strong convergence theorems in fixed point problems, variational inequalities, and equilibrium problems.

1. Introduction

Throughout this paper, let $\mathbb{R}$ denote the set of all real numbers, let $\mathbb{N}$ denote the set of all positive integer numbers, let H be a real Hilbert space, and let C be a nonempty closed convex subset of H. Let $S : C \to C$ be a mapping. We call S nonexpansive if

$$\|Sx - Sy\| \le \|x - y\|, \quad \forall x, y \in C. \tag{1.1}$$

The set of fixed points of S is denoted by $\text{Fix}(S)$. We know that the set $\text{Fix}(S)$ is closed and convex. Let $\Phi : C \times C \to \mathbb{R}$ be a bifunction. The equilibrium problem for Φ is to find $z \in C$ such that

$$\Phi(z, y) \ge 0, \quad \forall y \in C. \tag{1.2}$$

The set of all solutions of the equilibrium problem is denoted by $\mathrm{EP}(\Phi)$, that is,

$$\mathrm{EP}(\Phi) = \{z \in C : \Phi(z, y) \geq 0,\ \forall y \in C\}. \tag{1.3}$$

Some iterative methods have been proposed to find an element of $\mathrm{EP}(\Phi) \cap \mathrm{Fix}(S)$; see [1, 2].

A mapping $A : C \to H$ is called inverse-strongly monotone if there exists $\lambda > 0$ such that

$$\langle x - y, Ax - Ay \rangle \geq \lambda \|Ax - Ay\|^2, \quad \forall x, y \in C. \tag{1.4}$$

Such a mapping A is also called λ-inverse-strongly monotone. It is known that each nonexpansive mapping is $1/2$-inverse-strongly monotone and each κ-strictly pseudocontraction is $(1-\kappa)/2$-inverse-strongly monotone; see [3, 4]. If there exists $u \in C$ such that

$$\langle v - u, Au \rangle \geq 0, \quad \forall v \in C, \tag{1.5}$$

then u is called a solution of the variational inequality. The set of all solutions of the variational inequality is denoted by $\mathrm{VI}(C, A)$. It is known that $\mathrm{VI}(C, A)$ is closed and convex. Recently Takahashi and Toyoda [5] introduced an iterative method for finding an element of $\mathrm{Fix}(S) \cap \mathrm{VI}(C, A)$; see also [6]. On the other hand, Plubtieng and Punpaeng [7] introduced an iterative method for finding an element of $\mathrm{Fix}(S) \cap \mathrm{EP}(\Phi) \cap \mathrm{VI}(C, A)$; see also [8].

Consider a general equilibrium problem:

$$\text{Find } z \in C \text{ such that } \Phi(z, y) + \langle Az, y - z \rangle \geq 0, \quad \forall y \in C. \tag{1.6}$$

The set of all solutions of the equilibrium problem is denoted by EP, that is,

$$\mathrm{EP} = \{z \in C : \Phi(z, y) + \langle Az, y - z \rangle \geq 0,\ \forall y \in C\}. \tag{1.7}$$

In the case of $A \equiv 0$, EP coincides with $\mathrm{EP}(\Phi)$. In the case $\Phi \equiv 0$, EP coincides with $\mathrm{VI}(C, A)$. Recently, S. Takahashi and W. Takahashi [9] introduced an iterative method to find an element of $\mathrm{EP} \cap \mathrm{Fix}(S)$. More precisely, they introduced the following iterative scheme: $u \in C$, $x_1 \in C$, and

$$\begin{gathered} \Phi(z_n, y) + \langle Ax_n, y - z_n \rangle + \lambda_n \langle y - z_n, z_n - x_n \rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n) S[\alpha_n u + (1 - \alpha_n) z_n], \quad n \in \mathbb{N}, \end{gathered} \tag{1.8}$$

where $\{\alpha_n\} \subset [0, 1]$, $\{\beta_n\} \subset [0, 1]$, and $\{\lambda_n\} \subset [0, 2\lambda]$ are three control sequences. They proved that $\{x_n\}$ converges strongly to $z = P_{\mathrm{Fix}(S) \cap \mathrm{EP}} u$.

A mapping $T : C \to H$ is said to be relaxed η-α monotone if there exist a mapping $\eta : C \times C \to H$ and a function $\alpha : H \to \mathbb{R}$ positively homogeneous of degree p, that is, $\alpha(tz) = t^p \alpha(z)$ for all $t > 0$ and $z \in H$ such that

$$\langle Tx - Ty, \eta(x, y) \rangle \geq \alpha(x - y), \quad \forall x, y \in C, \tag{1.9}$$

where $p > 1$ is a constant; see [10]. In the case of $\eta(x,y) = x - y$ for all $x, y \in C$, T is said to be relaxed α-monotone. In the case of $\eta(x,y) = x - y$ for all $x, y \in C$ and $\alpha(z) = k\|z\|^p$, where $p > 1$ and $k > 0$, T is said to be p-monotone; see [11–13]. In fact, in this case, if $p = 2$, then T is a k-strongly monotone mapping. Moreover, every monotone mapping is relaxed η-α monotone with $\eta(x,y) = x - y$ for all $x, y \in C$ and $\alpha \equiv 0$.

In this paper, we consider a new general equilibrium problem with a relaxed monotone mapping:

$$\text{Find } z \in C \text{ such that } \Phi(z,y) + \langle Tz, \eta(y,z)\rangle + \langle Az, y - z\rangle \geq 0, \quad \forall y \in C. \tag{1.10}$$

The set of all solutions of the equilibrium problem is denoted by GEP(Φ, T), that is,

$$\text{GEP}(\Phi,T) = \{z \in C : \Phi(z,y) + \langle Tz, \eta(y,z)\rangle + \langle Az, y - z\rangle \geq 0,\ \forall y \in C\}. \tag{1.11}$$

In the case of $A \equiv 0$, (1.10) is deduced to

$$\text{Find } z \in C \text{ such that } \Phi(z,y) + \langle Tz, \eta(y,z)\rangle \geq 0, \quad \forall y \in C. \tag{1.12}$$

The set of all solutions of (1.12) is denoted by EP(Φ, T), that is,

$$\text{EP}(\Phi,T) = \{z \in C : \Phi(z,y) + \langle Tz, \eta(y,z)\rangle \geq 0,\ \forall y \in C\}. \tag{1.13}$$

In the case of $T \equiv 0$, GEP(Φ, T) coincides with EP. In the case of $T \equiv 0$ and $A \equiv 0$, GEP(Φ, T) coincides with EP(Φ).

In this paper, we introduce a new iterative scheme for finding a common element of the set of solutions of a general equilibrium problem with a relaxed monotone mapping and the set of common fixed points of a countable family of nonexpansive mappings and then obtain a strong convergence theorem. More precisely, we introduce the following iterative scheme:

$$\begin{gathered}
x_1 \in C \text{ chosen arbitrarily,} \\
\Phi(u_n, y) + \langle Tu_n, \eta(y, u_n)\rangle + \langle Ax_n, y - u_n\rangle + \frac{1}{\lambda_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\
y_n = \alpha_n x_n + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\beta_n S_i x_n + (1 - \alpha_n)(1 - \beta_n)u_n, \\
C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\
D_n = \bigcap_{j=1}^{n} C_j, \\
x_{n+1} = P_{D_n} x_1, \quad n \geq 1,
\end{gathered} \tag{1.14}$$

where $T : C \to H$ is a relaxed η-α monotone mapping, $A : C \to H$ is a λ-inverse-strongly monotone mapping, and $\{S_n\}_{n=1}^{\infty} : C \to C$ is a countable family of nonexpansive mappings such that $F = \bigcap_{n=1}^{\infty} \mathrm{Fix}(S_n) \cap \mathrm{GEP}(\Phi, T) \neq \emptyset$, $\alpha_0 = 1$, and $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, and $\{\lambda_n\}_{n=1}^{\infty}$ are three control sequences. We prove that $\{x_n\}$ defined by (1.14) converges strongly to $x^* = P_F x_1$. Using the main result in this paper, we also prove several new strong convergence theorems for finding the elements of $\mathrm{Fix}(S) \cap \mathrm{EP}$, $\mathrm{Fix}(S) \cap \mathrm{EP}(\Phi)$, $\mathrm{Fix}(S) \cap \mathrm{EP}(\Phi, T)$, and $\mathrm{Fix}(S) \cap \mathrm{VI}(C, A)$, respectively, where $S : C \to C$ is a nonexpansive mapping.

2. Preliminaries

Let $A : C \to H$ be a λ-inverse-strongly monotone mapping and let I denote the identity mapping of H. For all $x, y \in C$ and $k > 0$, one has [6]

$$\|(I - kA)x - (I - kA)y\|^2 \leq \|x - y\|^2 + k(k - 2\lambda)\|Ax - Ay\|^2. \tag{2.1}$$

Hence, if $k \in (0, 2\lambda]$, then $I - kA$ is a nonexpansive mapping of C into H.

For each point $x \in H$, there exists a unique nearest point of C, denoted by $P_C x$, such that $\|x - P_C x\| \leq \|x - y\|$ for all $y \in C$. Such a P_C is called the metric projection from H onto C. The well-known Browder's characterization of P_C ensures that P_C is a firmly nonexpansive mapping from H onto C, that is,

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle, \quad x, y \in H. \tag{2.2}$$

Further, we know that for any $x \in H$ and $z \in C$, $z = P_C x$ if and only if

$$\langle x - z, z - y \rangle \geq 0, \quad \forall y \in C. \tag{2.3}$$

Let S be a nonexpansive mapping of C into itself such that $\mathrm{Fix}(S) \neq \emptyset$. Then we have

$$\hat{x} \in \mathrm{Fix}(S) \Longleftrightarrow \|Sx - x\|^2 \leq 2\langle x - Sx, x - \hat{x} \rangle, \quad \forall x \in C, \tag{2.4}$$

which is obtained directly from the following:

$$\begin{aligned} \|x - \hat{x}\|^2 \geq \|Sx - S\hat{x}\|^2 = \|Sx - \hat{x}\|^2 = \|Sx - x + (x - \hat{x})\|^2 \\ = \|Sx - x\|^2 + \|x - \hat{x}\|^2 + 2\langle Sx - x, x - \hat{x} \rangle. \end{aligned} \tag{2.5}$$

This inequality is a very useful characterization of $\mathrm{Fix}(S)$. Observe what is more that it immediately yields that $\mathrm{Fix}(S)$ is a convex closed set.

Let Φ be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying the following conditions:

(A1) $\Phi(x, x) = 0$ for all $x \in C$;

(A2) Φ is monotone, that is, $\Phi(x, y) + \Phi(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$, $\lim_{t \downarrow 0} \Phi(tz + (1 - t)x, y) \leq \Phi(x, y)$;

(A4) for each $x \in C$, $y \mapsto \Phi(x, y)$ is convex and lower semicontinuous.

Definition 2.1 (see [10]). Let E be a Banach space with the dual space E^* and let K be a nonempty subset of E. Let $T : K \to E^*$ and $\eta : K \times K \to E$ be two mappings. The mapping $T : K \to E^*$ is said to be η-hemicontinuous if, for any fixed $x, y \in K$, the function $f : [0,1] \to (-\infty, \infty)$ defined by $f(t) = \langle T((1-t)x + ty), \eta(x,y)\rangle$ is continuous at 0^+.

Lemma 2.2. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $T : C \to H$ be an η-hemicontinuous and relaxed η- α monotone mapping. Let Φ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1) and (A4). Let $r > 0$ and $z \in C$. Assume that*

(i) *$\eta(x,x) = 0$ for all $x \in C$;*

(ii) *for any fixed $u, v \in C$, the mapping $x \mapsto \langle Tv, \eta(x,u)\rangle$ is convex.*

Then the following problems (2.6) *and* (2.7) *are equivalent:*

$$\text{Find } x \in C \text{ such that } \Phi(x,y) + \langle Tx, \eta(y,x)\rangle + \frac{1}{r}\langle y - x, x - z\rangle \geq 0, \quad \forall y \in C; \tag{2.6}$$

$$\text{Find } x \in C \text{ such that } \Phi(x,y) + \langle Ty, \eta(y,x)\rangle + \frac{1}{r}\langle y - x, x - z\rangle \geq \alpha(y - x), \quad \forall y \in C. \tag{2.7}$$

Proof. Let $x \in C$ be a solution of the problem (2.6). Since T is relaxed η-α monotone, we have

$$\begin{aligned} &\Phi(x,y) + \langle Ty, \eta(y,x)\rangle + \frac{1}{r}\langle y - x, x - z\rangle \\ &\quad\geq \Phi(x,y) + \alpha(y-x) + \frac{1}{r}\langle y - x, x - z\rangle + \langle Tx, \eta(y,x)\rangle \\ &\quad\geq \alpha(y-x), \quad \forall y \in C. \end{aligned} \tag{2.8}$$

Thus $x \in C$ is a solution of the problem (2.7).

Conversely, let $x \in C$ be a solution of the problem (2.7). Letting

$$y_t = (1-t)x + ty, \quad \forall t \in (0,1), \tag{2.9}$$

then $y_t \in C$. Since $x \in C$ is a solution of the problem (2.7), it follows that

$$\Phi(x,y_t) + \langle Ty_t, \eta(y_t,x)\rangle + \frac{1}{r}\langle y_t - x, x - z\rangle \geq \alpha(y_t - x) = t^p\alpha(y - x). \tag{2.10}$$

The conditions (i), (ii), (A1), and (A4) imply that

$$\begin{aligned} \langle Ty_t, \eta(y_t,x)\rangle &\leq (1-t)\langle Ty_t, \eta(x,x)\rangle + t\langle Ty_t, \eta(y,x)\rangle \\ &= t\langle T(x + t(y-x)), \eta(y,x)\rangle, \\ \Phi(x,y_t) &\leq (1-t)\Phi(x,x) + t\Phi(x,y) = t\Phi(x,y). \end{aligned} \tag{2.11}$$

It follows from (2.10)-(2.11) that

$$\Phi(x,y)+\langle T(x+t(y-x)),\eta(y,x)\rangle+\frac{1}{r}\langle y-x,x-z\rangle\geq t^{p-1}\alpha(y-x),\quad\forall y\in C. \tag{2.12}$$

Since T is η-hemicontinuous and $p>1$, letting $t\to 0$ in (2.12), we get

$$\Phi(x,y)+\langle Tx,\eta(y,x)\rangle+\frac{1}{r}\langle y-x,x-z\rangle\geq 0 \tag{2.13}$$

for all $y\in C$. Therefore, $x\in C$ is also a solution of the problem (2.6). This completes the proof. □

Definition 2.3 (see [14]). Let E be a Banach space with the dual space E^* and let K be a nonempty subset of E. A mapping $F: K\to 2^E$ is called a KKM mapping if, for any $\{x_1,\ldots,x_n\}\subset K$, $\overline{\text{co}}\{x_1,\ldots,x_n\}\subset\bigcup_{i=1}^n F(x_i)$, where 2^E denotes the family of all the nonempty subsets of E.

Lemma 2.4 (see [14]). *Let M be a nonempty subset of a Hausdorff topological vector space X and let $F: M\to 2^X$ be a KKM mapping. If $F(x)$ is closed in X for all $x\in X$ in K and compact for some $x\in K$, then $\bigcap_{x\in M}F(x)\neq\emptyset$.*

Next we use the concept of KKM mapping to prove two basic lemmas for our main result. The idea of the proof of the next lemma is contained in the paper of Fang and Huang [10].

Lemma 2.5. *Let H be a real Hilbert space and C be a nonempty bounded closed convex subset of H. Let $T: C\to H$ be an η-hemicontinuous and relaxed η-α monotone mapping, and let Φ be a bifunction from $C\times C$ to $\mathbb{R}$ satisfying (A1) and (A4). Let $r>0$. Assume that*

(i) *$\eta(x,y)+\eta(y,x)=0$ for all $x,y\in C$;*

(ii) *for any fixed $u,v\in C$, the mapping $x\mapsto\langle Tv,\eta(x,u)\rangle$ is convex and lower semicontinuous;*

(iii) *$\alpha: H\to\mathbb{R}$ is weakly lower semicontinuous; that is, for any net $\{x_\beta\}$, x_β converges to x in $\sigma(H,H)$ which implies that $\alpha(x)\leq\liminf\alpha(x_\beta)$.*

Then problem (2.6) is solvable.

Proof. Let $z\in C$. Define two set-valued mappings $F_z, G_z: C\to 2^H$ as follows:

$$\begin{aligned}F_z(y)&=\left\{x\in C:\Phi(x,y)+\langle Tx,\eta(y,x)\rangle+\frac{1}{r}\langle y-x,x-z\rangle\geq 0\right\},\quad\forall y\in C,\\ G_z(y)&=\left\{x\in C:\Phi(x,y)+\langle Ty,\eta(y,x)\rangle+\frac{1}{r}\langle y-x,x-z\rangle\geq\alpha(y-x)\right\},\quad\forall y\in C.\end{aligned} \tag{2.14}$$

We claim that F_z is a KKM mapping. If F_z is not a KKM mapping, then there exist $\{y_1, \ldots, y_n\} \subset C$ and $t_i > 0$, $i = 1, \ldots, n$, such that

$$\sum_{i=1}^{n} t_i = 1, \qquad y = \sum_{i=1}^{n} t_i y_i \notin \bigcup_{i=1}^{n} F_z(y_i). \tag{2.15}$$

By the definition of F, we have

$$\Phi(y, y_i) + \langle Ty, \eta(y_i, y)\rangle + \frac{1}{r}\langle y_i - y, y - z\rangle < 0, \quad \forall i = 1, \ldots, n. \tag{2.16}$$

It follows from (A1), (A4), and (ii) that

$$\begin{aligned} 0 &= \Phi(y, y) \\ &= \Phi\left(y, \sum_{i=1}^{n} t_i y_i\right) + \left\langle Ty, \eta\left(\sum_{i=1}^{n} t_i y_i, y\right)\right\rangle \\ &\le \sum_{i=1}^{n} t_i \Phi(y, y_i) + \sum_{i=1}^{n} t_i \langle Ty, \eta(y_i, y)\rangle \\ &< \sum_{i=1}^{n} t_i \frac{1}{r}\langle y - y_i, y - z\rangle \\ &= 0, \end{aligned} \tag{2.17}$$

which is a contradiction. This implies that F_z is a KKM mapping.

Now, we prove that

$$F_z(y) \subset G_z(y), \quad \forall y \in C. \tag{2.18}$$

For any given $y \in C$, taking $x \in F_z(y)$, then

$$\Phi(x, y) + \langle Tx, \eta(y, x)\rangle + \frac{1}{r}\langle y - x, x - z\rangle \ge 0. \tag{2.19}$$

Since T is relaxed η-α monotone, we have

$$\begin{aligned} &\Phi(x, y) + \langle Ty, \eta(y, x)\rangle + \frac{1}{r}\langle y - x, x - z\rangle \\ &\quad \ge \Phi(x, y) + \langle Tx, \eta(y, x)\rangle + \alpha(y - x) + \frac{1}{r}\langle y - x, x - z\rangle \\ &\quad \ge \alpha(y - x). \end{aligned} \tag{2.20}$$

It follows that $x \in G_z(y)$ and so

$$F_z(y) \subset G_z(y), \quad \forall y \in C. \tag{2.21}$$

This implies that G_z is also a KKM mapping. Now, since $x \mapsto \langle Ty, \eta(x,y) \rangle$ is a convex lower-semicontinuous function, we know that it is weakly lower semicontinuous. Thus from the definition of G_z and the weak lower semicontinuity of α, it follows that $G_z(y)$ is weakly closed for all $y \in C$. Since C is bounded closed and convex, we know that C is weakly compact, and so $G_z(y)$ is weakly compact in C for each $y \in C$. It follows from Lemmas 2.2 and 2.4 that

$$\bigcap_{y \in C} F_z(y) = \bigcap_{y \in C} G_z(y) \neq \emptyset. \tag{2.22}$$

Hence there exists $x \in C$ such that

$$\Phi(x,y) + \langle Tx, \eta(y,x) \rangle + \frac{1}{r}\langle y - x, x - z \rangle \geq 0, \quad \forall y \in C. \tag{2.23}$$

This completes the proof. □

Lemma 2.6. *Let H be a real Hilbert space and let C be a nonempty bounded closed convex subset of H. Let $T : C \to H$ be an η-hemicontinuous and relaxed η-α monotone mapping and let Φ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1), (A2), and (A4). Let $r > 0$ and define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \left\{ z \in C : \Phi(z,y) + \langle Tz, \eta(y,z) \rangle + \frac{1}{r}\langle y - z, z - x \rangle \geq 0,\ \forall y \in C \right\} \tag{2.24}$$

for all $x \in H$. Assume that

(i) $\eta(x,y) + \eta(y,x) = 0$, *for all $x, y \in C$;*

(ii) *for any fixed $u, v \in C$, the mapping $x \mapsto \langle Tv, \eta(x,u) \rangle$ is convex and lower semicontinuous and the mapping $x \mapsto \langle Tu, \eta(v,x) \rangle$ is lower semicontinuous;*

(iii) $\alpha : H \to \mathbb{R}$ *is weakly lower semicontinuous;*

(iv) *for any $x, y \in C$, $\alpha(x - y) + \alpha(y - x) \geq 0$.*

Then, the following holds:

(1) *T_r is single-valued;*

(2) *T_r is a firmly nonexpansive mapping, that is, for all $x, y \in H$,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle; \tag{2.25}$$

(3) *$F(T_r) = EP(\Phi, T)$;*

(4) *$EP(\Phi, T)$ is closed and convex.*

Proof. The fact that T_r is nonempty is exactly the thesis of the previous lemma. We claim that T_r is single-valued. Indeed, for $x \in H$ and $r > 0$, let $z_1, z_2 \in T_r x$. Then,

$$\begin{aligned} &\Phi(z_1, z_2) + \langle T z_1, \eta(z_2, z_1)\rangle + \frac{1}{r}\langle z_2 - z_1, z_1 - x\rangle \geq 0, \\ &\Phi(z_2, z_1) + \langle T z_2, \eta(z_1, z_2)\rangle + \frac{1}{r}\langle z_1 - z_2, z_2 - x\rangle \geq 0. \end{aligned} \tag{2.26}$$

Adding the two inequalities, from (i) we have

$$\Phi(z_1, z_2) + \Phi(z_2, z_1) + \langle T z_1 - T z_2, \eta(z_2, z_1)\rangle + \frac{1}{r}\langle z_1 - z_2, z_2 - z_1\rangle \geq 0. \tag{2.27}$$

From (A2), we have

$$\langle T z_1 - T z_2, \eta(z_2, z_1)\rangle + \frac{1}{r}\langle z_1 - z_2, z_2 - z_1\rangle \geq 0, \tag{2.28}$$

that is,

$$\frac{1}{r}\langle z_1 - z_2, z_2 - z_1\rangle \geq \langle T z_2 - T z_1, \eta(z_2, z_1)\rangle. \tag{2.29}$$

Since T is relaxed η-α monotone and $r > 0$, one has

$$\langle z_1 - z_2, z_2 - z_1\rangle \geq r\alpha(z_2 - z_1). \tag{2.30}$$

In (2.29) exchanging the position of z_1 and z_2, we get

$$\frac{1}{r}\langle z_2 - z_1, z_1 - z_2\rangle \geq \langle T z_1 - T z_2, \eta(z_1, z_2)\rangle \geq \alpha(z_1 - z_2), \tag{2.31}$$

that is,

$$\langle z_1 - z_2, z_2 - z_1\rangle \geq r\alpha(z_1 - z_2). \tag{2.32}$$

Now, adding the inequalities (2.30) and (2.32), by using (iv) we have

$$-2\|z_1 - z_2\|^2 = 2\langle z_1 - z_2, z_2 - z_1\rangle \geq 0. \tag{2.33}$$

Hence, $z_1 = z_2$.

Next we show that T_r is firmly nonexpansive. Indeed, for $x, y \in H$, we have

$$\begin{aligned}&\Phi(T_r x, T_r y) + \langle TT_r x, \eta(T_r y, T_r x)\rangle + \frac{1}{r}\langle T_r y - T_r x, T_r x - x\rangle \geq 0,\\&\Phi(T_r y, T_r x) + \langle TT_r y, \eta(T_r x, T_r y)\rangle + \frac{1}{r}\langle T_r x - T_r y, T_r y - y\rangle \geq 0.\end{aligned} \tag{2.34}$$

Adding the two inequalities and by (i) and (A2), we get

$$\langle TT_r x - TT_r y, \eta(T_r y, T_r x)\rangle + \frac{1}{r}\langle T_r y - T_r x, T_r x - T_r y - x + y\rangle \geq 0, \tag{2.35}$$

that is,

$$\begin{aligned}\frac{1}{r}\langle T_r y - T_r x, T_r x - T_r y - x + y\rangle &\geq \langle TT_r y - TT_r x, \eta(T_r y, T_r x)\rangle\\&\geq \alpha(T_r y - T_r x).\end{aligned} \tag{2.36}$$

In (2.36) exchanging the position of $T_r x$ and $T_r y$, we get

$$\frac{1}{r}\langle T_r x - T_r y, T_r y - T_r x - y + x\rangle \geq \alpha(T_r x - T_r y). \tag{2.37}$$

Adding the inequalities (2.36) and (2.37), we have

$$2\langle T_r x - T_r y, T_r y - T_r x - y + x\rangle \geq r(\alpha(T_r x - T_r y) + \alpha(T_r y - T_r x)). \tag{2.38}$$

It follows from (iv) that

$$\langle T_r x - T_r y, T_r y - T_r x - y + x\rangle \geq 0, \tag{2.39}$$

that is,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle. \tag{2.40}$$

This shows that T_r is firmly nonexpansive.

Next, we claim that $F(T_r) = \mathrm{EP}(\Phi, T)$. Indeed, we have the following:

$$\begin{aligned} u \in F(T_r) &\Longleftrightarrow u = T_r u \\ &\Longleftrightarrow \Phi(u,y) + \langle Tu, \eta(y,u)\rangle + \frac{1}{r}\langle y-u, u-u\rangle \ge 0, \quad \forall y \in C \\ &\Longleftrightarrow \Phi(u,y) + \langle Tu, \eta(y,u)\rangle \ge 0, \quad \forall y \in C \\ &\Longleftrightarrow u \in \mathrm{EP}(\Phi, T). \end{aligned} \tag{2.41}$$

Finally, we prove that $\mathrm{EP}(\Phi, T)$ is closed and convex. Indeed, Since every firm nonexpansive mapping is nonexpansive, we see that T_r is nonexpansive from (2). On the other hand, since the set of fixed points of every nonexpansive mapping is closed and convex, we have that $\mathrm{EP}(\Phi, T)$ is closed and convex from (2) and (3). This completes the proof. □

3. Main Results

In this section, we prove a strong convergence theorem which is our main result.

Theorem 3.1. *Let C be a nonempty bounded closed convex subset of a real Hilbert space H and let $\Phi : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1), (A2), (A3), and (A4). Let $T : C \to H$ be an η-hemicontinuous and relaxed η-α monotone mapping, let $A : C \to H$ be a λ-inverse-strongly monotone mapping, and let $\{S_n\}_{n=1}^{\infty} : C \to C$ be a countable family of nonexpansive mappings such that $F = \bigcap_{n=1}^{\infty} \mathrm{Fix}(S_n) \cap GEP(\Phi, T) \neq \emptyset$. Assume that the conditions (i)–(iv) of Lemma 2.6 are satisfied. Let $\alpha_0 = 1$ and assume that $\{\alpha_n\}_{n=1}^{\infty} \subset (0,1)$ is a strictly decreasing sequence. Assume that $\{\beta_n\}_{n=1}^{\infty} \subset (c,d)$ with some $c, d \in (0,1)$ and $\{\lambda_n\}_{n=1}^{\infty} \subset [a,b]$ with some $a, b \in (0, 2\lambda)$. Then, for any $x_1 \in C$, the sequence $\{x_n\}$ generated by* (1.14) *converges strongly to $x^* = P_F x_1$. In particular, if C contains the origin 0, taking $x_1 = 0$, then the sequence $\{x_n\}$ generated by* (1.14) *converges strongly to the minimum norm element in F.*

Proof. We split the proof into following steps.

Step 1. F is closed and convex, the sequence $\{x_n\}$ generated by (1.14) is well defined, and $F \subset D_n$ for all $n \ge 1$.

First, we prove that F is closed and convex. It suffices to prove that $\mathrm{GEP}(\Phi, T)$ is closed and convex. Indeed, it is easy to prove the conclusion by the following fact:

$$\begin{aligned} \forall p \in \mathrm{GEP}(\Phi, T) &\Longleftrightarrow \Phi(p,y) + \langle Tp, \eta(y,p)\rangle + \frac{1}{\lambda_n}\langle y-p, \lambda_n Ap\rangle \ge 0, \quad \forall y \in C \\ &\Longleftrightarrow \Phi(p,y) + \langle Tp, \eta(y,p)\rangle + \frac{1}{\lambda_n}\langle y-p, p-(p-\lambda_n Ap)\rangle \ge 0, \quad \forall y \in C \\ &\Longleftrightarrow p = T_{\lambda_n}(I-\lambda_n A)p. \end{aligned} \tag{3.1}$$

This implies that $\mathrm{GEP}(\Phi, T) = \mathrm{Fix}[T_{\lambda_n}(I - \lambda_n A)]$. Noting that $T_{\lambda_n}(I - \lambda_n A)$ is a nonexpansive mapping for $\lambda_n < 2\lambda$ and the set of fixed points of a nonexpansive mapping is closed and convex, we have that $\mathrm{GEP}(\Phi, T)$ is closed and convex.

Next we prove that the sequence $\{x_n\}$ generated by (1.14) is well defined and $F \subset D_n$ for all $n \geq 1$. It is easy to see that C_n is closed and convex for all $n \in \mathbb{N}$ from the construction of C_n. Hence, D_n is closed and convex for all $n \in \mathbb{N}$. For any $p \in F$, since $u_n = T_{\lambda_n}(x_n - \lambda_n A x_n)$ and $I - \lambda_n A$ is nonexpansive, we have (note that $\{\alpha_n\}$ is strictly decreasing)

$$\begin{aligned}
\|y_n - p\| &= \left\| \alpha_n(x_n - p) + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\beta_n(S_i x_n - p) + (1-\alpha_n)(1-\beta_n)(u_n - p) \right\| \\
&\leq \alpha_n\|x_n - p\| + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\beta_n\|S_i x_n - p\| + (1-\alpha_n)(1-\beta_n)\|u_n - p\| \\
&\leq \alpha_n\|x_n - p\| + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\beta_n\|x_n - p\| \\
&\quad + (1-\alpha_n)(1-\beta_n)\|T_{\lambda_n}(x_n - \lambda_n A x_n) - T_{\lambda_n}(p - \lambda_n A p)\| \\
&\leq \alpha_n\|x_n - p\| + (1-\alpha_n)\beta_n\|x_n - p\| \\
&\quad + (1-\alpha_n)(1-\beta_n)\|(x_n - \lambda_n A x_n) - (p - \lambda_n A p)\| \\
&\leq \alpha_n\|x_n - p\| + (1-\alpha_n)\beta_n\|x_n - p\| + (1-\alpha_n)(1-\beta_n)\|x_n - p\| \\
&= \|x_n - p\|.
\end{aligned} \tag{3.2}$$

So, $F \subset C_n$ for all $n \in \mathbb{N}$. Hence $F \subset \bigcap_{j=1}^{n} C_j$, that is, $F \subset D_n$ for all $n \in \mathbb{N}$. Since D_n is closed, convex, and nonempty, the sequence $\{x_n\}$ is well defined.

Step 2. $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$ and there exists $x^* \in C$ such that $x_n \to x^*$ as $n \to \infty$.

From the definition of D_n, we see that $D_{n+1} \subset D_n$ for all $n \in \mathbb{N}$ and hence

$$x_{n+2} = P_{D_{n+1}} x_1 \in D_{n+1} \subset D_n. \tag{3.3}$$

Noting that $x_{n+1} = P_{D_n} x_1$, we get

$$\|x_{n+1} - x_1\| \leq \|x_{n+2} - x_1\| \tag{3.4}$$

for all $n \geq 1$. This shows that $\{\|x_n - x_1\|\}$ is increasing. Since C is bounded, $\{\|x_n - x_1\|\}$ is bounded. So, we have that $\lim_{n\to\infty}\|x_n - x_1\|$ exists.

Noting that $x_{n+1} = P_{D_n}x_1$ and $x_{m+1} = P_{D_m}x_1 \in D_m \subset D_n$ for all $m \geq n$, we have

$$\langle x_{n+1} - x_1, x_{m+1} - x_{n+1} \rangle \geq 0. \tag{3.5}$$

It follows from (3.5) that

$$\begin{aligned}
&\|x_{m+1} - x_{n+1}\|^2 \\
&\quad = \|x_{m+1} - x_1 - (x_{n+1} - x_1)\|^2 \\
&\quad = \|x_{m+1} - x_1\|^2 + \|x_{n+1} - x_1\|^2 - 2\langle x_{m+1} - x_1, x_{n+1} - x_1 \rangle \\
&\quad = \|x_{m+1} - x_1\|^2 + \|x_{n+1} - x_1\|^2 - 2\langle x_{n+1} - x_1, x_{m+1} - x_{n+1} + x_{n+1} - x_1 \rangle \\
&\quad = \|x_{m+1} - x_1\|^2 - \|x_{n+1} - x_1\|^2 - 2\langle x_{n+1} - x_1, x_{m+1} - x_{n+1} \rangle \\
&\quad \leq \|x_{m+1} - x_1\|^2 - \|x_{n+1} - x_1\|^2.
\end{aligned} \tag{3.6}$$

By taking $m = n + 1$ in (3.6), we get

$$\|x_{n+2} - x_{n+1}\| \leq \|x_{n+2} - x_1\|^2 - \|x_{n+1} - x_1\|^2. \tag{3.7}$$

Since the limits of $\|x_n - x_1\|$ exists we get

$$\lim_{n \to \infty} \|x_{n+2} - x_{n+1}\| = 0, \tag{3.8}$$

that is, $x_{n+1} - x_n \to 0$ as $n \to \infty$. Moreover, from (3.6) we also have

$$\lim_{m,n \to \infty} \|x_{m+1} - x_{n+1}\| = 0. \tag{3.9}$$

This shows that $\{x_n\}$ is a Cauchy sequence. Hence, there exists $x^* \in C$ such that

$$x_n \longrightarrow x^* \in C, \quad \text{as } n \longrightarrow \infty. \tag{3.10}$$

Step 3. $\lim_{n\to\infty}\|x_n - u_n\| = 0$.
Since $x_{n+1} \in C_n$ and $x_n - x_{n+1} \to 0$ as $n \to \infty$, we have

$$\|y_n - x_{n+1}\| \leq \|x_n - x_{n+1}\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \tag{3.11}$$

and hence

$$\|y_n - x_n\| \leq \|y_n - x_{n+1}\| + \|x_n - x_{n+1}\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.12}$$

Note that u_n can be rewritten as $u_n = T_{\lambda_n}(x_n - \lambda_n A x_n)$ for all $n \geq 1$. Take $p \in F$. Since $p = T_{\lambda_n}(p - \lambda_n Ap)$, A is λ-inverse-strongly monotone, and $0 < \lambda_n < 2\lambda$, we know that, for all $n \in \mathbb{N}$,

$$\begin{aligned}
\|u_n - p\|^2 &= \|T_{\lambda_n}(x_n - \lambda_n A x_n) - T_{\lambda_n}(p - \lambda_n Ap)\|^2 \\
&\leq \|x_n - \lambda_n A x_n - p - \lambda_n Ap\|^2 \\
&= \|(x_n - p) - \lambda_n(Ax_n - Ap)\|^2 \\
&= \|x_n - p\|^2 - 2\lambda_n\langle x_n - p, Ax_n - Ap\rangle + \lambda_n^2\|Ax_n - Ap\|^2 \\
&\leq \|x_n - p\|^2 - 2\lambda_n\lambda\|Ax_n - Ap\|^2 + \lambda_n^2\|Ax_n - Ap\|^2 \\
&= \|x_n - p\|^2 + \lambda_n(\lambda_n - 2\lambda)\|Ax_n - Ap\|^2 \\
&\leq \|x_n - p\|^2.
\end{aligned} \tag{3.13}$$

Using (1.14) and (3.13), we have (note that $\{\alpha_n\}$ is strictly decreasing)

$$\begin{aligned}
\|y_n - p\|^2 &= \left\|\alpha_n(x_n - p) + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\beta_n(S_i x_n - p) + (1 - \alpha_n)(1 - \beta_n)(u_n - p)\right\|^2 \\
&\leq \alpha_n\|x_n - p\|^2 + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\beta_n\|S_i x_n - p\|^2 + (1 - \alpha_n)(1 - \beta_n)\|u_n - p\|^2 \\
&\leq \alpha_n\|x_n - p\|^2 + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\beta_n\|x_n - p\|^2 \\
&\quad + (1 - \alpha_n)(1 - \beta_n)\left(\|x_n - p\|^2 + \lambda_n(\lambda_n - 2\lambda)\|Ax_n - Ap\|^2\right) \\
&= \|x_n - p\|^2 + (1 - \alpha_n)(1 - \beta_n)\lambda_n(\lambda_n - 2\lambda)\|Ax_n - Ap\|^2,
\end{aligned} \tag{3.14}$$

and hence

$$\begin{aligned}
(1 - \alpha_n)(1 - d)a(2\lambda - b)\|Ax_n - Ap\|^2 &\leq (1 - \alpha_n)(1 - \beta_n)\lambda_n(2\lambda - \lambda_n)\|Ax_n - Ap\|^2 \\
&\leq \|x_n - p\|^2 - \|y_n - p\|^2 \\
&\leq \|x_n - y_n\|(\|x_n - p\| + \|y_n - p\|).
\end{aligned} \tag{3.15}$$

Since $\{x_n\}$ and $\{y_n\}$ are both bounded, $\alpha_n \to 0$, and $x_n - y_n \to 0$, we have

$$Ax_n - Ap \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.16}$$

Using Lemma 2.6, we get

$$\begin{aligned}\|u_n-p\|^2 &= \|T_{\lambda_n}(x_n-\lambda_nAx_n)-T_{\lambda_n}(p-\lambda_nAp)\|^2\\ &\le \langle x_n-\lambda_nAx_n-(p-\lambda_nAp), u_n-p\rangle\\ &= \frac{1}{2}\Big(\|x_n-\lambda_nAx_n-(p-\lambda_nAp)\|^2+\|u_n-p\|^2\\ &\quad -\|x_n-\lambda_nAx_n-(p-\lambda_nAp)-(u_n-p)\|^2\Big)\\ &\le \frac{1}{2}\Big(\|x_n-p\|^2+\|u_n-p\|^2-\|x_n-u_n-\lambda_n(Ax_n-Ap)\|^2\Big)\\ &= \frac{1}{2}\Big(\|x_n-p\|^2+\|u_n-p\|^2-\|x_n-u_n\|^2\\ &\quad +2\lambda_n\langle x_n-u_n, Ax_n-Ap\rangle-\lambda_n^2\|Ax_n-Ap\|^2\Big).\end{aligned} \tag{3.17}$$

So, we have

$$\|u_n-p\|^2\le\|x_n-p\|^2-\|x_n-u_n\|^2+2\lambda_n\langle x_n-u_n, Ax_n-Ap\rangle-\lambda_n^2\|Ax_n-Ap\|^2. \tag{3.18}$$

From (3.18), we have

$$\begin{aligned}&\|y_n-p\|^2\\ &\quad=\left\|\alpha_n(x_n-p)+\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\beta_n(S_ix_n-p)+(1-\alpha_n)(1-\beta_n)(u_n-p)\right\|^2\\ &\quad\le\alpha_n\|x_n-p\|^2+\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\beta_n\|x_n-p\|^2+(1-\alpha_n)(1-\beta_n)\|u_n-p\|^2\\ &\quad\le\alpha_n\|x_n-p\|^2+(1-\alpha_n)\beta_n\|x_n-p\|^2+(1-\alpha_n)(1-\beta_n)\\ &\qquad\times\Big(\|x_n-p\|^2-\|x_n-u_n\|^2+2\lambda_n\langle x_n-u_n, Ax_n-Ap\rangle-\lambda_n^2\|Ax_n-Ap\|^2\Big)\\ &\quad\le\|x_n-p\|^2-(1-\alpha_n)(1-\beta_n)\|x_n-u_n\|^2+2(1-\alpha_n)(1-\beta_n)\lambda_n\langle x_n-u_n, Ax_n-Ap\rangle,\end{aligned} \tag{3.19}$$

and hence

$$\begin{aligned}(1-d)(1-\alpha_n)\|x_n-u_n\|^2&\le(1-\beta_n)(1-\alpha_n)\|x_n-u_n\|^2\\ &\le\|x_n-y_n\|(\|x_n-p\|+\|y_n-p\|)\\ &\quad+2(1-\alpha_n)(1-\beta_n)\lambda_n\|x_n-u_n\|\|Ax_n-Ap\|.\end{aligned} \tag{3.20}$$

By using $\|x_n - y_n\| \to 0$ and (3.16), we have

$$\|x_n - u_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.21}$$

Step 4. $\lim_{n\to\infty}\|x_n - S_i x_n\| = 0$, for all $i = 0, 1, \ldots$.

It follows from the definition of scheme (1.14) that

$$y_n + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\beta_n(x_n - S_i x_n) - (1-\alpha_n)\beta_n x_n = \alpha_n x_n + (1-\alpha_n)(1-\beta_n)u_n, \tag{3.22}$$

that is,

$$\begin{aligned}\sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\beta_n(x_n - S_i x_n) &= x_n - y_n - x_n + \alpha_n x_n + (1-\alpha_n)\beta_n x_n + (1-\alpha_n)(1-\beta_n)u_n \\ &= x_n - y_n + (1-\alpha_n)(\beta_n - 1)x_n + (1-\alpha_n)(1-\beta_n)u_n \\ &= x_n - y_n + (1-\alpha_n)(1-\beta_n)(u_n - x_n).\end{aligned} \tag{3.23}$$

Hence, for any $p \in F$, one has

$$\sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\beta_n\langle x_n - S_i x_n, x_n - p\rangle = (1-\alpha_n)(1-\beta_n)\langle u_n - x_n, x_n - p\rangle + \langle x_n - y_n, x_n - p\rangle. \tag{3.24}$$

Since each S_i is nonexpansive, by (2.4) we have

$$\|S_i x_n - x_n\|^2 \le 2\langle x_n - S_i x_n, x_n - p\rangle. \tag{3.25}$$

Hence, combining this inequality with (3.24), we get

$$\frac{1}{2}\sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\beta_n\|S_i x_n - x_n\|^2 \le (1-\alpha_n)(1-\beta_n)\langle u_n - x_n, x_n - p\rangle + \langle x_n - y_n, x_n - p\rangle, \tag{3.26}$$

that is (noting that $\{\alpha_n\}$ is strictly decreasing),

$$\begin{aligned}\|S_i x_n - x_n\|^2 &\le \frac{2(1-\alpha_n)(1-\beta_n)}{(\alpha_{i-1} - \alpha_i)\beta_n}\langle u_n - x_n, x_n - p\rangle + \frac{2}{(\alpha_{i-1} - \alpha_i)\beta_n}\langle x_n - y_n, x_n - p\rangle \\ &\le \frac{2(1-\alpha_n)(1-\beta_n)}{(\alpha_{i-1} - \alpha_i)\beta_n}\|u_n - x_n\|\|x_n - p\| + \frac{2}{(\alpha_{i-1} - \alpha_i)\beta_n}\|x_n - y_n\|\|x_n - p\|.\end{aligned} \tag{3.27}$$

Since $\|u_n - x_n\| \to 0$ and $\|x_n - y_n\| \to 0$, we have

$$\lim_{n\to\infty}\|S_i x_n - x_n\| = 0, \quad \forall i = 1, 2, \ldots. \tag{3.28}$$

Step 5. $x_n \to x^* = P_F x_1$.

First we prove $x^* \in \bigcap_{i=1}^{\infty} \mathrm{Fix}(S_i)$. Indeed, since $x_n \to x^*$ and $S_i x_n - x_n \to 0$, we have $x^* \in \mathrm{Fix}(S_i)$ for each $i = 1, 2, \ldots$. Hence, $x^* \in \bigcap_{i=1}^{\infty} \mathrm{Fix}(S_i)$.

Next, we show that $x^* \in \mathrm{GEP}(\Phi, T)$. Noting that $u_n = T_{\lambda_n}(x_n - \lambda_n A x_n)$, one obtains

$$\Phi(u_n, y) + \langle T u_n, \eta(y, u_n) \rangle + \langle A x_n, y - u_n \rangle + \frac{1}{\lambda_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C. \tag{3.29}$$

Put $u_t = ty + (1-t)x^*$ for all $t \in (0,1)$ and $y \in C$. Then, we have $u_t \in C$. So, from (A2), (i), and (3.29) we have

$$\begin{aligned}
\langle u_t - u_n, A u_t \rangle &\geq \langle u_t - u_n, A u_t \rangle - \langle u_t - u_n, A x_n \rangle - \left\langle u_t - u_n, \frac{u_n - x_n}{\lambda_n} \right\rangle \\
&\quad + \Phi(u_t, u_n) + \langle T u_n, \eta(u_n, u_t) \rangle \\
&= \langle u_t - u_n, A u_t - A u_n \rangle + \langle u_t - u_n, A u_n - A x_n \rangle - \left\langle u_t - u_n, \frac{u_n - x_n}{\lambda_n} \right\rangle \\
&\quad + \Phi(u_t, u_n) + \langle T u_n, \eta(u_n, u_t) \rangle.
\end{aligned} \tag{3.30}$$

Since $x_n - u_n \to 0$, we have $\|A u_n - A x_n\| \to 0$. Further, from monotonicity of A, we have $\langle u_t - u_n, A u_t - A u_n \rangle \geq 0$. So, from (A4), (ii), and η-hemicontinuity of T we have

$$\langle u_t - x^*, A u_t \rangle \geq \Phi(u_t, x^*) + \langle T x^*, \eta(x^*, u_t) \rangle. \tag{3.31}$$

From (A1), (A4), (ii), and (3.31) we also have

$$\begin{aligned}
0 &= \Phi(u_t, u_t) + \langle T x^*, \eta(u_t, u_t) \rangle \\
&\leq t \left[\Phi(u_t, y) + \langle T x^*, \eta(y, u_t) \rangle \right] \\
&\quad + (1-t) \left[\Phi(u_t, x^*) + \langle T x^*, \eta(x^*, u_t) \rangle \right] \\
&\leq t \left[\Phi(u_t, y) + \langle T x^*, \eta(y, u_t) \rangle \right] + (1-t) \langle u_t - x^*, A u_t \rangle \\
&= t \left[\Phi(u_t, y) + \langle T x^*, \eta(y, u_t) \right] + (1-t) t \langle y - x^*, A u_t \rangle,
\end{aligned} \tag{3.32}$$

and hence

$$0 \leq \Phi(u_t, y) + \langle T x^*, \eta(y, u_t) + (1-t) \langle y - x^*, A u_t \rangle. \tag{3.33}$$

Letting $t \to 0$, from (A3) and (ii) we have, for each $y \in C$,

$$0 \leq \Phi(x^*, y) + \langle T x^*, \eta(y, x^*) + \langle y - x^*, A x^* \rangle. \tag{3.34}$$

This implies that $x^* \in \mathrm{GEP}(\Phi, T)$. Hence, we get $x^* \in F = \bigcap_{n=1}^{\infty} \mathrm{Fix}(S_n) \cap \mathrm{GEP}(\Phi, T)$.

Finally, we show that $x^* = P_F x$. Indeed, from $x_{n+1} = P_{D_n} x$ and $F \subset D_n$, we have

$$\langle x - x_{n+1}, x_{n+1} - v \rangle \geq 0, \quad \forall v \in F. \tag{3.35}$$

Taking the limit in (3.35) and noting that $x_n \to x^*$ as $n \to \infty$, we get

$$\langle x - x^*, x^* - v \rangle \geq 0, \quad \forall v \in F. \tag{3.36}$$

In view of (2.3), one sees that $x^* = P_F x$. This completes the proof. □

Corollary 3.2. *Let C be a nonempty bounded closed convex subset of a Hilbert space H and let $\Phi : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1), (A2), (A3), and (A4). Let $T : C \to H$ be an η-hemicontinuous and relaxed η-α monotone mapping and let $S : C \to C$ be a nonexpansive mapping such that* $\operatorname{Fix}(S) \cap EP(\Phi, T) \neq \emptyset$. *Assume that the conditions (i)–(iv) of Lemma 2.6 are satisfied. Assume that $\{\alpha_n\}_{n=1}^{\infty} \subset (0,1)$ with* $\limsup_{n\to\infty} \alpha_n < 1$, *$\{\beta_n\}_{n=1}^{\infty} \subset (c,d)$ with some $c, d \in (0,1)$ and $\{\lambda_n\}_{n=1}^{\infty} \subset (a,\infty)$ with $a \in (0,\infty)$. Let $x_1 \in C$ and let $\{x_n\}$ be generated by*

$$\begin{gathered}
\Phi(u_n, y) + \langle Tu_n, \eta(y, u_n) \rangle + \frac{1}{\lambda_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C, \\
y_n = \alpha_n x_n + (1-\alpha_n)\beta_n S x_n + (1-\alpha_n)(1-\beta_n) u_n, \\
C_n = \{ z \in C : \| y_n - z \| \leq \| x_n - z \| \}, \\
D_n = \bigcap_{j=1}^{n} C_j, \\
x_{n+1} = P_{D_n} x_1, \quad n \geq 1.
\end{gathered} \tag{3.37}$$

Then the sequence $\{x_n\}$ converges strongly to $x^* = P_{\operatorname{Fix}(S)\cap EP(\Phi,T)} x_1$. In particular, if C contains the origin 0, taking $x_1 = 0$, the sequence $\{x_n\}$ converges strongly to the minimum norm element in $\operatorname{Fix}(S) \cap EP(\Phi, T)$.

Proof. In Theorem 3.1, put $A \equiv 0$, $S_1 = \cdots = S_n = \cdots = S$. Then, we have

$$\begin{aligned}
y_n &= \alpha_n x_n + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) \beta_n S_i x_n + (1-\alpha_n)(1-\beta_n) u_n \\
&= \alpha_n x_n + (1-\alpha_n)\beta_n S x_n + (1-\alpha_n)(1-\beta_n) u_n, \\
\| S x_n - x_n \|^2 &\leq \frac{2(1-\alpha_n)(1-\beta_n)}{(1-\alpha_n)\beta_n} \| u_n - x_n \| \| x_n - p \| + \frac{2}{(1-\alpha_n)\beta_n} \| x_n - y_n \| \| x_n - p \|.
\end{aligned} \tag{3.38}$$

On the other hand, for all $\lambda \in (0,\infty)$, we have that

$$\langle x - y, Ax - Ay \rangle \geq \lambda \| Ax - Ay \|^2, \quad \forall x, y \in C. \tag{3.39}$$

So, taking $a, b \in (0, \infty)$ with $0 < a \le b < \infty$ and choosing a sequence $\{\lambda_n\}$ of real numbers with $a \le \lambda_n \le b$, we obtain the desired result by Theorem 3.1. □

Corollary 3.3. *Let C be a nonempty bounded closed convex subset of a Hilbert space H and let $\Phi : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1), (A2), (A3), and (A4). Let $T : C \to H$ be a monotone mapping and let $S : C \to C$ be a nonexpansive mapping such that* $\mathrm{Fix}(S) \cap EP(\Phi, T) \neq \emptyset$*. Assume that $\{\alpha_n\}_{n=1}^{\infty} \subset (0,1)$ with $\limsup_{n\to\infty} \alpha_n < 1$, $\{\beta_n\}_{n=1}^{\infty} \subset (c,d)$ with some $c, d \in (0,1)$ and $\{\lambda_n\}_{n=1}^{\infty} \subset (a,\infty)$ with $a \in (0,\infty)$. Let $x_1 \in C$ and let $\{x_n\}$ be generated by*

$$
\begin{gathered}
\Phi(u_n, y) + \langle Tu_n, y - u_n \rangle + \frac{1}{\lambda_n} \langle y - u_n, u_n - x_n \rangle \ge 0, \quad \forall y \in C, \\
y_n = \alpha_n x_n + (1-\alpha_n)\beta_n S x_n + (1-\alpha_n)(1-\beta_n) u_n, \\
C_n = \{ z \in C : \| y_n - z \| \le \| x_n - z \| \}, \\
D_n = \bigcap_{j=1}^{n} C_j, \\
x_{n+1} = P_{D_n} x, \quad n \ge 1.
\end{gathered} \tag{3.40}
$$

Then the sequence $\{x_n\}$ converges strongly to $x^* = P_{\mathrm{Fix}(S)\cap EP(\Phi,T)} x_1$. In particular, if C contains the origin 0, taking $x_1 = 0$, the sequence $\{x_n\}$ converges strongly to the minimum norm element in $\mathrm{Fix}(S) \cap EP(\Phi, T)$.

Proof. In Corollary 3.2, put $\alpha \equiv 0$ and $\eta(x,y) \equiv x - y$ for all $x, y \in C$. Then $T : C \to H$ is a monotone mapping and we obtain the desired result by Theorem 3.1. □

Corollary 3.4. *Let C be a closed convex subset of a Hilbert space H and let $\Phi : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1), (A2), (A3), and (A4). Let $A : C \to H$ be a λ-inverse-strongly monotone mapping and let $S : C \to C$ be a nonexpansive mapping such that* $\mathrm{Fix}(S) \cap EP \neq \emptyset$*. Assume that $\{\alpha_n\}_{n=1}^{\infty} \subset (0,1)$ with $\limsup_{n\to\infty} \alpha_n < 1$, $\{\beta_n\}_{n=1}^{\infty} \subset (c,d)$ with some $c, d \in (0,1)$ and $\{\lambda_n\}_{n=1}^{\infty} \subset (a,b)$ with $0 < a, b < 2\lambda$. Let $x_1 \in C$ and let $\{x_n\}$ be generated by*

$$
\begin{gathered}
\Phi(u_n, y) + \langle Au_n, y - u_n \rangle + \frac{1}{\lambda_n} \langle y - u_n, u_n - x_n \rangle \ge 0, \quad \forall y \in C, \\
y_n = \alpha_n x_n + (1-\alpha_n)\beta_n S x_n + (1-\alpha_n)(1-\beta_n) u_n, \\
C_n = \{ z \in C : \| y_n - z \| \le \| x_n - z \| \}, \\
D_n = \bigcap_{j=1}^{n} C_j, \\
x_{n+1} = P_{D_n} x, \quad n \ge 1.
\end{gathered} \tag{3.41}
$$

Then the sequence $\{x_n\}$ converges strongly to $x^* = P_{\mathrm{Fix}(S)\cap EP} x_1$. In particular, if C contains the origin 0, taking $x_1 = 0$, the sequence $\{x_n\}$ converges strongly to the minimum norm element in $\mathrm{Fix}(S) \cap EP$.

Proof. In Theorem 3.1, put $T \equiv 0$, $\eta \equiv 0$, $\alpha \equiv 0$, and $S_1 = \cdots = S_n = \cdots = S$. We obtain the desired result by Theorem 3.1. □

Corollary 3.5. *Let C be a closed convex subset of a Hilbert space H and let $\Phi : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1), (A2), (A3), and (A4). Let $S : C \to C$ be a nonexpansive mapping such that $\mathrm{Fix}(S) \cap EP(\Phi) \neq \emptyset$. Assume that $\{\alpha_n\}_{n=1}^{\infty} \subset (0,1)$ with $\limsup_{n\to\infty}\alpha_n < 1$, $\{\beta_n\}_{n=1}^{\infty} \subset (c,d)$ with some $c, d \in (0,1)$, and $\{\lambda_n\}_{n=1}^{\infty} \subset (a,\infty)$ with $0 < a < \infty$. Let $x_1 \in C$ and let $\{x_n\}$ be generated by*

$$\begin{aligned}
&\Phi(u_n, y) + \frac{1}{\lambda_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C,\\
&y_n = \alpha_n x_n + (1-\alpha_n)\beta_n S x_n + (1-\alpha_n)(1-\beta_n)u_n,\\
&C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\},\\
&D_n = \bigcap_{j=1}^{n} C_j,\\
&x_{n+1} = P_{D_n}x, \quad n \geq 1.
\end{aligned} \tag{3.42}$$

Then the sequence $\{x_n\}$ converges strongly to $x^* = P_{\mathrm{Fix}(S)\cap EP(\Phi)}x_1$. In particular, if C contains the origin 0, taking $x_1 = 0$, the sequence $\{x_n\}$ converges strongly to the minimum norm element in $\mathrm{Fix}(S) \cap EP(\Phi)$.

Proof. In Corollary 3.4, by putting $A \equiv 0$ we obtain the desired result. □

Corollary 3.6. *Let C be a closed convex subset of a Hilbert space H and let $A : C \to H$ be a λ-inverse-strongly monotone mapping. Let $S : C \to C$ be a nonexpansive mapping such that $\mathrm{Fix}(S) \cap VI(C,A) \neq \emptyset$. Assume that $\{\alpha_n\}_{n=1}^{\infty} \subset (0,1)$ with $\limsup_{n\to\infty}\alpha_n < 1$, $\{\beta_n\}_{n=1}^{\infty} \subset (c,d)$ with some $c, d \in (0,1)$, and $\{\lambda_n\}_{n=1}^{\infty} \subset (a,b)$ with $0 < a, b < 2\lambda$. Let $x_1 \in C$ and let $\{x_n\}$ be generated by*

$$\begin{aligned}
&u_n = P_C(x_n - \lambda_n A x_n),\\
&y_n = \alpha_n x_n + (1-\alpha_n)\beta_n S x_n + (1-\alpha_n)(1-\beta_n)u_n,\\
&C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\},\\
&D_n = \bigcap_{j=1}^{n} C_j,\\
&x_{n+1} = P_{D_n}x, \quad n \geq 1.
\end{aligned} \tag{3.43}$$

Then the sequence $\{x_n\}$ converges strongly to $x^* = P_{\mathrm{Fix}(S)\cap VI(C,A)}x_1$. In particular, if C contains the origin 0, taking $x_1 = 0$, the sequence $\{x_n\}$ converges strongly to the minimum norm element in $\mathrm{Fix}(S) \cap VI(C,A)$.

Proof. In Theorem 3.1, put $\Phi \equiv 0$, $T \equiv 0$, $\eta \equiv 0$, $\alpha \equiv 0$, and $S_1 = \cdots = S_n = \cdots = S$. Then, we have

$$u_n = P_C(x_n - \lambda_n A x_n), \quad \forall x \geq 1. \tag{3.44}$$

Then, we obtain the desired result by Theorem 3.1. □

Remark 3.7. The novelty of this paper lies in the following aspects.

(i) A new general equilibrium problem with a relaxed monotone mapping is considered.

(ii) The definition of D_n is of independent interest.

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References

[1] A. Tada and W. Takahashi, "Weak and strong convergence theorems for a nonexpansive mapping and an equilibrium problem," *Journal of Optimization Theory and Applications*, vol. 133, no. 3, pp. 359–370, 2007.

[2] S. Takahashi and W. Takahashi, "Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces," *Journal of Mathematical Analysis and Applications*, vol. 331, no. 1, pp. 506–515, 2007.

[3] W. Takahashi, *Nonlinear Functional Analysis*, Fixed Point Theory and Its Applications, Yokohama Publishers, Yokohama, Japan, 2000.

[4] F. E. Browder and W. V. Petryshyn, "Construction of fixed points of nonlinear mappings in Hilbert space," *Journal of Mathematical Analysis and Applications*, vol. 20, pp. 197–228, 1967.

[5] W. Takahashi and M. Toyoda, "Weak convergence theorems for nonexpansive mappings and monotone mappings," *Journal of Optimization Theory and Applications*, vol. 118, no. 2, pp. 417–428, 2003.

[6] H. Iiduka and W. Takahashi, "Strong convergence theorems for nonexpansive mappings and inverse-strongly monotone mappings," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 61, no. 3, pp. 341–350, 2005.

[7] S. Plubtieng and R. Punpaeng, "A new iterative method for equilibrium problems and fixed point problems of nonexpansive mappings and monotone mappings," *Applied Mathematics and Computation*, vol. 197, no. 2, pp. 548–558, 2008.

[8] S. Wang, H. Zhou, and J. Song, "Viscosity approximation methods for equilibrium problems and fixed point problems of nonexpansive mappings and inverse-strongly monotone mappings," *Methods and Applications of Analysis*, vol. 14, no. 4, pp. 405–419, 2007.

[9] S. Takahashi and W. Takahashi, "Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mapping in a Hilbert space," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 3, pp. 1025–1033, 2008.

[10] Y. P. Fang and N. J. Huang, "Variational-like inequalities with generalized monotone mappings in Banach spaces," *Journal of Optimization Theory and Applications*, vol. 118, no. 2, pp. 327–338, 2003.

[11] D. Goeleven and D. Motreanu, "Eigenvalue and dynamic problems for variational and hemivariational inequalities," *Communications on Applied Nonlinear Analysis*, vol. 3, no. 4, pp. 1–21, 1996.

[12] A. H. Siddiqi, Q. H. Ansari, and K. R. Kazmi, "On nonlinear variational inequalities," *Indian Journal of Pure and Applied Mathematics*, vol. 25, no. 9, pp. 969–973, 1994.

[13] R. U. Verma, “Nonlinear variational inequalities on convex subsets of Banach spaces,” *Applied Mathematics Letters*, vol. 10, no. 4, pp. 25–27, 1997.

[14] K. K. Marzukiewicz, “Ein beweis des fixpuntsatzen fur n-dimensionale simplexe,” *Fundamenta Mathematicae*, vol. 14, pp. 132–137, 1929.