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Symmetric separation of variables for trigonometric integrable models

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Abstract

We study a problem of variable separation for the classical integrable hamiltonian systems possessing Lax matrices satisfying linear r-matrix algebra with skew-symmetric $sl(2) \otimes sl(2)$-valued trigonometric r-matrix. For all such the systems we produce new *symmetric* variables of separation. We show that the corresponding curve of separation differs from the spectral curve of the initial Lax matrix. The example of trigonometric Clebsch model is considered in details.

1. Introduction

The problem of the separation of variables is one of the most studied, but yet the most difficult and not completely resolved problems in the theory of integrable systems.

One of the most physically important classes of the integrable models are the models admitting Lax representation. For such the models one can advance in the solution of the problem of separation of variables [1]. This approach — sometimes called "the magic recipe" — is based on the poles of Baker-Akhiezer functions. In the classical case the idea of the approach can be traced back to the papers [2–7].

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The approach of [1] — despite being very effective for certain classes of examples — has two weak points. Although it permits to construct a set of variables that belong to a spectral curve of the Lax matrix that plays the role of the equations of separation, but it does not (in general) guaranty that the constructed variables are the canonical coordinates, i.e. that they are the variables of separation indeed.

Another problem arising in the framework of the "magic recipe" is that the number of variables produced by the method rarely coincides with the dimension of the corresponding phase space. Often it happens that the number of the coordinates constructed by the method of Sklyanin is less than the dimension of the phase space. Such the situation may occur even in the case when the corresponding variables are indeed the canonical ones. Although in some cases the problem can be resolved by certain tricks e.g. by complementing of the set of the obtained canonical variables by a linear integrals playing the role of the additional momenta of separation and by finding of the corresponding canonically conjugated variables [2,6,8] but, unfortunately, it is not always possible to do this.

That is why it is desirable to have a method producing at once a complete set of coordinates and momenta of separation. In order to do this we propose to re-consider a scheme of [1], changing its accents. The starting point of the scheme of [1] is a spectral curve of the Lax matrix. Contrary to this we propose to begin with a construction of a complete set of canonical coordinates and only after that start to check to what curve they belong, i.e. what equations of separation they satisfy. This changing of accents permits to obtain the separated variables for which the equations of separation does not coincide with a spectral curve of the Lax matrix. The change in the approach is suggested, in particular, by our recent result [13] on the separation of variables in the completely anisotropic Clebsch model, where it finally occurred that the canonical coordinates of separation satisfy two different equations of separation, coinciding with two different spectral curves of two different — two by two and four by four — Lax matrices.

In the present paper we consider examples of partially anisotropic models that correspond to the classical trigonometric r-matrix, i.e. the Poisson brackets of the corresponding Lax matrices are written in the form

$$\{L(u) \otimes 1, 1 \otimes L(v)\} = [r(u, v), L(u) \otimes 1 + 1 \otimes L(v)]. \tag{1}$$

Here $r(u, v)$ is a skew-symmetric trigonometric r-matrix in the following parametrization

$$r(u, v) = \frac{\sqrt{u}\sqrt{v}}{\sqrt{u-1} - \sqrt{v-1}}(\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2) + \frac{\sqrt{v-1}\sqrt{u-1} + 1}{\sqrt{u-1} - \sqrt{v-1}}\sigma_3 \otimes \sigma_3, \tag{2}$$

where σ_α, $\alpha \in \overline{1,3}$ are Pauli matrices. The change of variables $u = \dfrac{1}{\sin^2(\phi)}$, $v = \dfrac{1}{\sin^2(\psi)}$ brings the r-matrix (2) to the standard trigonometric form.

Although the mentioned above trick with the complement of the standard Sklyanin-type variables is valid in the trigonometric case [15,14,16], i.e. helps to produce a complete set of variables of separation, we have succeeded in a construction of a *new* family of variables of separation which is *complete at once* without any additional tricks. In order to do this we construct canonical coordinates using the technique of separating functions $B(u)$ and $A(u)$ such that zeros of the first function generate the Poisson-commuting coordinates and the values of the second one in these zeros generate the canonically conjugated momenta. The most important their property is a Poisson algebra to be satisfied by the functions $B(u)$ and $A(u)$ [20–22]

$$\{B(u), B(v)\} = b(u, v)B(u) - b(v, u)B(v),$$

$$\{A(u), B(v)\} = \alpha(u, v)B(u) - \beta(u, v)B(v),$$
$$\{A(u), A(v)\} = a(u, v)B(u) - a(v, u)B(v),$$

for some functions $a(u, v)$, $b(u, v)$, $\alpha(u, v)$, $\beta(u, v)$ such that the following limit holds true

$$\lim_{u \to v} (\alpha(u, v)B(u) - \beta(u, v)B(v)) = \partial_v B(v) + \gamma(v)B(v).$$

The corresponding restriction imposed on $B(u)$ and $A(u)$ permits their explicit construction without any appeal to Baker-Akhiezer function. We have found the explicit form of such the functions $B(u)$ and $A(u)$ in the trigonometric case

$$B(u) = \frac{i}{\sqrt{u}} L_1(u) - i\frac{\sqrt{u-1}}{\sqrt{u}} L_2(u) - L_3(u), \tag{3a}$$

$$A(u) = -\frac{u-1}{u} M_3 - i\frac{\sqrt{u-1}}{\sqrt{u}} L_1(u) + \frac{\sqrt{u-1}}{u} L_3(u), \tag{3b}$$

where $L_\alpha(u)$, $\alpha \in 1, 2, 3$ are the components of the Lax matrix in the above parametrization and M_3 is an additional linear integral satisfying the following Poisson-bracket relations with the components of the Lax matrix

$$\{M_3, L_\alpha(u)\} = \epsilon_{3\alpha\beta} L_\beta.$$

Although the technique of the separating functions is in the framework of the general scheme of Sklyanin [1], but contrary to Sklyanin we do not require that the corresponding coordinates and momenta belong to the standard spectral curve of the initial Lax matrix. Neither we require — as it is always done in [1] — that the function $A(u)$ is written in the closed form via the components of the Lax matrix only. In the final end it occurred that the constructed coordinates and momenta satisfy the spectral curve modified with the help of the linear integral M_3

$$\det\Big(L(x_k^{-2} + 1) + i\big((x_k^2 + 1) \cdot p_k + M_3\big) Id\Big) = 0.$$

This result would be impossible to obtain if one insisted from the beginning that the equations of separation coincide with the standard spectral curve of the initial Lax matrix.

The functions $B(u)$ and $A(u)$ given by the formulas (3) produce the complete set of the coordinates of separation for any integrable system possessing Lax matrix satisfying the Poisson brackets (1) with the r-matrix (2). In order to be more concrete we consider a class of examples of the Lax matrices possessing the poles in the point $u = \infty$. The special attention devoted to this class of the Lax matrices is explained by the fact that they are connected with the finite-gap sectors of partially anisotropic Landau-Lifshitz and chiral field equations [10,11]. The example of the first non-trivial from the separation of variables point of view model, namely, trigonometric Clebsch model is considered in details. We explicitly write the corresponding coordinates and momenta of separation, the reconstruction formulae and the Abel-type equations. Note, that the reconstruction formulae that are explicitly obtained for the Clebsch model, and may be also explicitly obtained also for other integrable models associated with trigonometric r-matrix, are *symmetric* in terms of the coordinates on the separation curve. That is why we call the obtained separated variables to be *symmetric*.

In the end of the Introduction we would like to answer a natural question that may arise in the context of SoV for trigonometric integrable models. This question is the following: is not it possible to obtain a complete set of separated variables for trigonometric integrable models using trigonometric degeneration of the complete set of separated variables for the elliptic integrable models? In particular, is not it possible to obtain a complete set of separated variables for the "trigonometric" Clebsch model using "symmetric" separated variable for the "elliptic" Clebsch model from [23] or "asymmetric" separated variables for the same model from [13]? The answer to this question is negative. After trigonometric degeneration of the "elliptic" separated variables from [23] or [13] one of the separated coordinates becomes constant or goes to infinity and the completeness of the obtained set of separated coordinates is lost.

The structure of the present letter is the following: in the Section 2 we remind the general scheme of the variable separation based on the method of separating functions. In the Section 3 we describe the trigonometric integrable models and separation of variables for them. In the Section 4 we consider an example of the trigonometric Clebsch model. At last in the Section 5 we briefly conclude and describe the on-going problems.

2. Separation of variables: general scheme

2.1. Definitions

Let us recall the definitions of Liouville integrability and separation of variables in the general theory of Hamiltonian systems [1]. An integrable Hamiltonian system with D degrees of freedom is determined on a $2D$-dimensional symplectic manifold $\mathcal{M}$ — symplectic leaf in the Poisson manifold $(\mathcal{P}, \{\ ,\ \})$ by D independent functions (first integrals) I_j commuting with respect to the Poisson bracket

$$\{I_i, I_j\} = 0, \qquad i, j \in \overline{1, D}.$$

For the Hamiltonian H of the system may be taken any first integral I_j.

To find separated variables means to find (at least locally) a set of coordinates $x_i, p_j, i, j \in \overline{1, D}$ such that there exist D relations — equations of separation

$$\Phi_i(x_i, p_i, I_1, ..., I_D) = 0, \qquad i \in \overline{1, D}, \tag{4}$$

where the coordinates $x_i, p_j, i, j \in \overline{1, D}$ are canonical, i.e.

$$\{x_i, p_j\} = \delta_{ij}, \quad \{x_i, x_j\} = 0, \ \{p_i, p_j\} = 0, \quad \forall i, j \in \overline{1, D}.$$

The separated variables provide a way to a construction of the action-angle coordinates from the Liouville theorem and half a way to explicit integration of the equations of motion.

Unfortunately, in the general case no algorithm is known to construct a set of separated variables for any given integrable system. One of the possible methods of their construction is the so-called method of separating functions permitting one to construct a set of canonical coordinates.

2.2. Separating functions and canonical coordinates

Let us now remind a method of construction of canonical coordinates using separating functions. Generally speaking this method can be considered independently of separation of variables. That is why in this subsection we do not assume any special properties of the Poisson manifold $\mathcal{P}$ or Poisson structure $\{\ ,\ \}$. Neither we assume integrability or existence of the Lax representation.

Let $B(u)$ and $A(u)$ be some functions of the dynamical variables and an auxiliary parameter u, which is constant with respect to the bracket $\{\ ,\ \}$. Let the points $x_i, i \in \overline{1, D}$ be zeros of the function $B(u)$ and $p_i, i \in \overline{1, D}$ be the values of $A(u)$ in these points, i.e.

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$$B(u)|_{u=x_i} = 0, \quad p_i = A(u)|_{u=x_i}.$$

We wish to construct Poisson brackets among these new coordinates using the Poisson brackets between $B(u)$ and $A(u)$. The following Proposition holds true:

Proposition 2.1. *Let $B(x_i) = 0$, $p_j = A(x_j)$. Then:*

$$(i)\ \{x_i, x_j\} = \Big(\frac{\{B(u), B(v)\}}{\partial_u B(u)\partial_v B(v)}\Big)|_{u=x_i, v=x_j}, \text{ where } i \neq j,$$

$$(ii)\ \{x_j, p_i\} = \Big(\frac{\{A(u), B(v)\}}{\partial_v B(v)}\Big)|_{u=x_i, v=x_j} + \{x_i, x_j\}(\partial_u A(u))|_{u=x_i}, \text{ where } i \neq j,$$

$$\begin{aligned}(iii)\ \{p_i, p_j\} = &\Big(\{A(u), A(v)\}\Big)|_{u=x_i, v=x_j} + \{p_i, x_j\}(\partial_v A(v))|_{v=x_j} \\ &+ \{x_i, p_j\}(\partial_u A(u))|_{u=x_i} - \\ &- \{x_i, x_j\}(\partial_u A(u)\partial_v A(v))|_{u=x_i, v=x_j}, \text{ where } i \neq j.\end{aligned}$$

Sketch of the Proof. The equalities (i)-(iii) are obtained by the decomposition of $B(u)$, $A(u)$, $B(v)$, $A(v)$ in Taylor power series in the neighborhood of the points $u = x_i$, $v = x_j$ in the expressions $\{B(u), B(v)\}$, $\{A(u), B(v)\}$, $\{A(u), A(v)\}$ and by considering the limits $u \to x_i$, $v \to x_j$ after the calculation of the Poisson brackets.

Now we are ready to formulate the following important Lemma:

Lemma 2.1. *Let the coordinates x_i and p_j, $i, j \in \overline{1, D}$ be defined as above. Let the functions $A(u)$, $B(u)$ satisfy the following Poisson algebra*

$$(i)\ \{B(u), B(v)\} = b(u, v)B(u) - b(v, u)B(v), \tag{5a}$$

$$(ii)\ \{A(u), B(v)\} = \alpha(u, v)B(u) - \beta(u, v)B(v), \tag{5b}$$

$$(iii)\ \{A(u), A(v)\} = a(u, v)B(u) - a(v, u)B(v). \tag{5c}$$

Then the Poisson bracket among the functions x_i and p_j, $\forall i, j \in \overline{1, D}$, $i \neq j$ are trivial

$$(i)\ \{x_i, x_j\} = 0,$$

$$(ii)\ \{x_j, p_i\} = 0,$$

$$(iii)\ \{p_i, p_j\} = 0.$$

If, moreover holds also the condition

$$\lim_{u \to v} (\alpha(u, v)B(u) - \beta(u, v)B(v)) = \partial_v B(v) + \gamma(v)B(v) \tag{6}$$

then the corresponding Poisson brackets are canonical, i.e.: $\{x_i, p_i\} = 1, \forall i \in \overline{1, D}$.

Remark 1. Observe, that the method of the separating functions $A(u)$ and $B(u)$ does not, generally speaking, guarantee that the constructed canonical coordinates satisfy the equations of separation (4) for some integrable Hamiltonian system defined by the Poisson-commuting Hamiltonians $\{I_i, i \in \overline{1, D}\}$. Nevertheless it is often the case and it is necessary only to find the explicit form of the corresponding functions $\Phi_i(x_i, p_i, I_1, ..., I_D)$. This phenomenon is explained by the fact that both the functions $A(u)$, $B(u)$ and the generating functions of integrals

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of motions are constructed using the r-matrix technique with the same classical r-matrix. That is why it is quite natural (although not guaranteed) that the constructed in such a way canonical coordinates and integrals of motion are connected by some equations of separation. In the next sections we will illustrate this by a set of new examples.

3. Trigonometric models and separation of variables

3.1. Trigonometric r-matrix

Let us consider the following trigonometric r-matrix [12]

$$r(\phi-\psi)=\frac{1}{sin(\phi-\psi)}(\sigma_1\otimes\sigma_1+\sigma_2\otimes\sigma_2)+\frac{cos(\phi-\psi)}{sin(\phi-\psi)}\sigma_3\otimes\sigma_3. \tag{7}$$

It satisfies usual classical Yang-Baxter equation [12]

$$[r^{12}(\phi-\psi),r^{13}(\phi-\chi)]=[r^{23}(\psi-\chi),r^{12}(\phi-\psi)+r^{13}(\phi-\chi)]$$

and coincides with the trigonometric degeneration of the elliptic r-matrix of Sklyanin [9].

Using the addition formulas for the trigonometric functions and re-parametrization $u=\frac{1}{sin^2(\phi)}$, $v=\frac{1}{sin^2(\psi)}$ it is easy to show that the trigonometric r-matrix (7) is re-written in the following irrational form

$$r(u,v)=\frac{\sqrt{u}\sqrt{v}}{\sqrt{u-1}-\sqrt{v-1}}(\sigma_1\otimes\sigma_1+\sigma_2\otimes\sigma_2)+\frac{\sqrt{v-1}\sqrt{u-1}+1}{\sqrt{u-1}-\sqrt{v-1}}\sigma_3\otimes\sigma_3, \tag{8}$$

where σ_α, $\alpha\in\overline{1,3}$ are the Pauli matrices.

Hereafter in our paper we will use the trigonometric r-matrix only in the form (8).

3.2. Trigonometric Lax algebra

Let us consider on the space of $sl(2)$-valued spectral-parameter-dependent matrices the following Poisson bracket

$$\{L(u)\otimes 1,1\otimes L(v)\}=[r(u,v),L(u)\otimes 1+1\otimes L(v)], \tag{9}$$

where $r(u,v)$ is given by the formula (8) and $L(u)=\sum_{\alpha=1}^{3}L_\alpha(u)\sigma_\alpha$, $L(v)=\sum_{\alpha=1}^{3}L_\alpha(v)\sigma_\alpha$.

The type dependence of $L_\alpha(u)$ on the spectral parameter u and dynamical variables is determined by the parenthesis (9) the r-matrix (8) and the number and type of poles of the Lax matrix [17,18]. In the present paper we will consider mainly the examples of the Lax matrices with the poles in the point $u=\infty$.

The following proposition holds true:

Proposition 3.1. *The Lax matrices $L(u)$ satisfying (9) and having the poles in the point $u=\infty$ are written as follows*

$$\begin{aligned}L(u)=&\sqrt{u}(S_1(u)\sigma_1+S_2(u)\sigma_2)+\sqrt{u-1}S_3(u)\sigma_3+\sqrt{u}\sqrt{u-1}(T_1(u)\sigma_1+T_2(u)\sigma_2)\\&+uT_3(u)\sigma_3,\end{aligned} \tag{10}$$

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where

$$S_\alpha(u) = \sum_{m=0}^{\infty} S_\alpha^{(m)} u^m, \quad T_\alpha(u) = \sum_{m=0}^{\infty} T_\alpha^{(m)} u^m, \tag{11}$$

σ_α are the Pauli Matrices and the Poisson brackets among $S_\alpha^{(m)}$ and $T_\alpha^{(m)}$ have the form

$$\{S_1^{(m)}, S_2^{(n)}\} = S_3^{(m+n)} - S_3^{(m+n+1)}, \quad \{S_1^{(m)}, S_3^{(n)}\} = -S_2^{(m+n)}, \quad \{S_2^{(m)}, S_3^{(n)}\} = S_1^{(m+n)}. \tag{12a}$$

$$\begin{aligned} &\{S_1^{(m)}, T_2^{(n)}\} = T_3^{(m+n)}, \quad \{S_1^{(m)}, T_3^{(n)}\} = -T_2^{(m+n)} + T_2^{(m+n+1)}, \\ &\{S_2^{(m)}, T_3^{(n)}\} = T_1^{(m+n)} - T_1^{(m+n+1)}, \end{aligned} \tag{12b}$$

$$\begin{aligned} &\{T_1^{(m)}, S_2^{(n)}\} = T_3^{(m+n)}, \quad \{T_1^{(m)}, S_3^{(n)}\} = -T_2^{(m+n)} + T_2^{(m+n+1)}, \\ &\{T_2^{(m)}, S_3^{(n)}\} = T_1^{(m+n)} - T_1^{(m+n+1)}, \end{aligned} \tag{12c}$$

$$\{T_1^{(m)}, T_2^{(n)}\} = S_3^{(m+n+1)}, \quad \{T_1^{(m)}, T_3^{(n)}\} = -S_2^{(m+n+1)}, \quad \{T_2^{(m)}, T_3^{(n)}\} = S_1^{(m+n+1)}. \tag{12d}$$

Sketch of the Proof. The proof follows from the results of [10,11] on the elliptic algebra based on the elliptic r-matrix of Sklyanin in the trigonometric degeneration $j_1 = j_2 = 0$, $j_3 = 1$, where j_α, $\alpha = 1, 2, 3$ are the branching points of the elliptic curve.

Remark 2. The division of the trigonometric Lax matrix $L(u)$ into two parts: $L(u) = S(u) + T(u)$ corresponds to the decomposition of trigonometric function into odd and even parts. In order to see this it is necessary to make a change of variables $u = \dfrac{1}{\sin^2(\phi)}$, $v = \dfrac{1}{\sin^2(\psi)}$. This division leads to the additional Z_2-grading in the Lax algebra (12).

3.3. Lax matrices of finite-dimensional integrable systems

One of the main advantages of the proposed basis in the algebra of Lax operators consists in the fact that it gives a convenient way to describe finite dimensional Hamiltonian systems and "finite" Lax operators that correspond to them.

To construct the corresponding Lax operators it is necessary to consider the subspaces

$$\mathcal{J}_{2M+2} = \mathrm{Span}_{\mathbb{C}}\{S_\alpha^{(m)}, T_\beta^{(m)} m \geq M, \alpha, \beta \in \overline{1,3}\},$$

$$\mathcal{J}_{2M+1} = \mathrm{Span}_{\mathbb{C}}\{S_\alpha^{(m)}, m > M, \alpha \in \overline{1,3}; T_\alpha^{(n)}, n \geq M, \beta \in \overline{1,3}\}$$

The following Proposition holds true:

Proposition 3.2. *The spaces $\mathcal{J}_{2M+2}$ and $\mathcal{J}_{2M+1}$, $M \geq 0$ are ideals in the Poisson algebra given by Poisson brackets (12).*

Sketch of the Proof. It follows from the explicit form of the Poisson brackets (12) and is checked by direct verifications.

Using the above Proposition and the fact that projection onto the quotient space over the ideal is a homomorphism one proves the following Proposition:

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Proposition 3.3. *The Lax matrices of the form (10) with*

$$S_\alpha(u) = \sum_{m=0}^{M-1} S_\alpha^{(m)} u^m, \quad T_\alpha(u) = \sum_{m=0}^{M-1} T_\alpha^{(m)} u^m, \tag{13}$$

and

$$S_\alpha(u) = \sum_{m=0}^{M} S_\alpha^{(m)} u^m, \quad T_\alpha(u) = \sum_{m=0}^{M-1} T_\alpha^{(m)} u^m, \tag{14}$$

where $M \in \mathbb{Z}_+$ satisfy linear tensor bracket (9).

Remark 3. The Poisson bracket between the coordinate functions $S_\alpha^{(k)}$, $T_\beta^{(k)}$ have the form (12) where in the right-hand-side it is necessary to put $S_\alpha^{(k)} = 0$, $T_\beta^{(l)} = 0$ if $k, l \geq M$ in the case of the Lax matrices (13) or to put $S_\alpha^{(k)} = 0$, $T_\beta^{(l)} = 0$ if $k > M$, $l \geq M$ in the case of the Lax matrices (14).

3.4. Integrals of motion

3.4.1. General case

By the virtue of the r-matrix form of the Poisson brackets we have that the function

$$I^{(2)}(u) = \sum_{\alpha=1}^{3} L_\alpha^2(u) \tag{15}$$

generate Poisson commuting integrals.

Let us now specify the form of the function $I^{(2)}(u)$ using the explicit form of the components of the Lax matrix $L_\alpha(u)$ in terms of the functions $T_\alpha(u)$ and $S_\alpha(u)$. By the direct calculation one shows that

$$I^{(2)}(u) = I^{2,+}(u) + 2u\sqrt{u-1}\, I^{2,-}(u),$$

where

$$I^{2,+}(u) = u(S_1^2(u) + S_2^2(u)) + (u-1)S_3^2(u) + u(u-1)(T_1^2(u)) + T_2^2(u)) + u^2 T_3^2(u),$$

$$I^{2,-}(u) = \sum_{\alpha=1}^{3} T_\alpha(u) S_\alpha(u).$$

It is possible to show, that due to the $SO(2)$-symmetry of the r-matrix there exists also additional linear integral M_3 such that

$$\{M_3, L_\alpha(u)\} = \epsilon_{3\alpha\beta} L_\beta.$$

By the virtue of this property we have that $\{M_3, I^{(2)}(u)\} = 0$.

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3.4.2. Case of one-poled Lax matrices

In the case of the one-poled Lax matrices with the poles in the point $u=\infty$ we obtain that

$$I^{2,+}(u)=\sum_{m=0}^{\infty} I^{2,+}_m u^m,$$
$$I^{2,-}(u)=\sum_{m=0}^{\infty} I^{2,-}_m u^m,$$

where $I^{2,\pm}_m$ are polynomials in the coordinate functions given by the following explicit formulae

$$I^{2,+}_m=\sum_{k=0}^{m-2}\sum_{\alpha=1}^{3} T^{(k)}_\alpha T^{(m-2-k)}_\alpha+\sum_{k=0}^{m-1}(\sum_{\alpha=1}^{3} S^{(k)}_\alpha S^{(m-1-k)}_\alpha - T^{(k)}_1 T^{(m-1-k)}_1 - T^{(k)}_2 T^{(m-1-k)}_2) - \sum_{k=0}^{m} S^{(k)}_3 S^{(m-k)}_3, \tag{16a}$$

$$I^{2,-}_m=\sum_{\alpha=1}^{3}\sum_{k=0}^{m} T^{(k)}_\alpha S^{(m-k)}_\alpha. \tag{16b}$$

The additional linear integral M_3 is written in this case simply as follows

$$M_3=S^{(0)}_3.$$

Remark 4. After the restriction to the finite-dimensional quotients some of the functions $I^{2,+}_m$ and $I^{2,-}_n$ become Casimir functions and some of the functions $I^{2,+}_m$, $I^{2,-}_n$ turn zero. The restrictions onto the quotients is made by putting in the functions (16) $S^{(m)}_\alpha=0$, $T^{(m)}_\beta=0$ if $m\geq M$ (in the case of the Lax matrices (13)) or $S^{(m)}_\alpha=0$, $T^{(n)}_\beta=0$ if $m>M$, $n\geq M$ (in the case of the Lax matrices (14)).

3.5. The separating functions

Let us consider the following linear in the matrix elements of the Lax matrix functions

$$B(u)=\frac{i}{\sqrt{u}}L_1(u)-i\frac{\sqrt{u-1}}{\sqrt{u}}L_2(u)-L_3(u), \tag{17a}$$

$$A(u)=-\frac{u-1}{u}M_3-i\frac{\sqrt{u-1}}{\sqrt{u}}L_1(u)+\frac{\sqrt{u-1}}{u}L_3(u), \tag{17b}$$

where M_3 is such a function on the Lax algebra that

$$\{M_3, L_\alpha(u)\}=\epsilon_{3\alpha\beta}L_\beta. \tag{18}$$

The following Proposition is true:

Proposition 3.4. *Let the components of the Lax matrix $L_\alpha(u)$ satisfy the Poisson brackets (9). Let, moreover, function M_3 be such a function on the phase space that the Poisson relation (18) holds. Then the above defined functions $A(u)$ and $B(u)$ possess the following Poisson brackets*

$$(i)\{B(u), B(v)\} = B(u) - B(v), \tag{19a}$$

$$(ii)\ \{A(u), B(v)\} = \frac{\sqrt{u-1}\sqrt{v-1}}{\sqrt{u-1} - \sqrt{v-1}}(B(u) - B(v)), \tag{19b}$$

$$(iii)\ \{A(u), A(v)\} = 0. \tag{19c}$$

Sketch of the Proof. The Proposition is proven by the direct calculation using the explicit form of the functions $A(u)$, $B(v)$, the Poisson brackets (9) and the relation (18).

It is easy to see from the relations (19) that the relations (5) are satisfied. In order to satisfy the relation (6) it is necessary to make the change of variables

$$u = x^{-2} + 1,\ v = y^{-2} + 1.$$

In terms of these new spectral parameters the polynomial $B(x)$ will have the correct order and the function $A(x)$ will produce canonically conjugated momenta, i.e. the relation (6) is also satisfied.[1]

Hence we have constructed a set of canonical coordinates associated with the trigonometric r-matrix. It is left to show that they satisfy the equations of separation. For this purpose we define the following linear function $C(u)$ on the algebra of the Lax matrices

$$C(u) = \frac{i}{\sqrt{u}} L_1(u) + i\frac{\sqrt{u-1}}{\sqrt{u}} L_2(u) - L_3(u). \tag{20}$$

Now we can formulate the next Proposition:

Proposition 3.5. *The functions $A(u)$, $B(u)$, $C(u)$ satisfy the following algebraic relation*

$$\left(\frac{u}{(u-1)} A(u) + M_3\right)^2 - \frac{u}{(u-1)} B(u)C(u) + \sum_{\alpha=1}^{3} L_\alpha(u)^2 = 0. \tag{21}$$

Sketch of the Proof. The Proposition is proven by the direct calculation using the explicit form of the functions $A(u)$, $B(v)$, $C(u)$.

Taking into account that by the very definition $B(x_j) = 0$, $A(x_j) = p_j$ we obtain that the equations (21) are the equations of separation for the trigonometric models which can be written explicitly as follows

$$\sum_{\alpha=1}^{3} L_\alpha^2(x_k^{-2} + 1) + ((x_k^2 + 1) \cdot p_k + M_3)^2) = 0.$$

Now, if we choose the Pauli matrices normalized as follows

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix},\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

then the equations of separation are written in the following determinant form

$$\det(L(x_k^{-2} + 1) + i((1 + x_k^2) \cdot p_k + M_3)Id) = 0.$$

[1] Hereafter for technical reason we chose the following convention for a sign of the square root: $\sqrt{x^2} = -x$.

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In the end of this section let us show the completeness of the constructed separated variables. The following Proposition holds true:

Proposition 3.6. *Let $L(u)$ be "finite" Lax operator with the poles in the point $u = \infty$. Then the number of zeros of the polynomial $B(x)$ coincide with the half of the dimension of the non-degenerate symplectic leaf in the space of the "finite" Lax matrices.*

Proof. Let us check the order of the polynomial $B(x)$ in x^{-1}. The direct calculation gives

$$B(x) = S_1(u) - S_2(u)\cdot x^{-1} + iS_3(u)\cdot x^{-1} + T_1(u)\cdot x^{-1} - T_2(u)\cdot x^{-2} + iT_3(u)\cdot(1+x^{-2}). \tag{22}$$

Taking into account that $u = x^{-2} + 1$ we obtain that order of the polynomial $B(x)$ in x^{-1} is $N = 2M$ in the case of the Lax operator (13) and $N = 2M + 1$ in the case of the Lax operator (14). On the other hand "finite" Lax operators (13) and (14) are defined on the spaces $\mathfrak{g}^{(M)}$ of the dimensions $3 \times 2M$ and $3 \times (2M + 1)$ correspondingly. Taking into account the explicit form of the Poisson brackets it is easy to show that these spaces coincide with dual spaces to the sequences of semidirect extensions of the Lie algebra $so(3)$ with three dimensional commutative algebras over $\mathbb{C}$: $\mathfrak{g}^{(M)} \simeq (so(3) + \mathbb{C}^3) + \mathbb{C}^3) + ...+) + \mathbb{C}^3$. It is well-known that the number of the Casimir functions of such the algebras is equal to the number of three dimensional spaces in this extension (including the initial algebra $so(3)$), i.e. is equal to $2M$ in case of the Lax operator (13) or $(2M + 1)$ in case of the Lax operator (14). Hence the dimensions of the corresponding symplectic leafs is $2 \times 2M$ and $2 \times (2M + 1)$ correspondingly, which is exactly two times the order of the polynomial $B(x)$.

Proposition is proven.

4. Separation of variables in trigonometric Clebsch model

4.1. The trigonometric Clebsch model

The trigonometric Clebsch model is the first non-trivial from the separation of variables point of view models obtained in our scheme. It corresponds to the Lax operator (13) and the case $M = 1$

$$L(u) = \sqrt{u}(S_1\sigma_1 + S_2\sigma_2) + \sqrt{u-1}S_3\sigma_3 + \sqrt{u}\sqrt{u-1}(T_1\sigma_1 + T_2\sigma_2) + uT_3\sigma_3, \tag{23}$$

where $S_\alpha \equiv S_\alpha^{(0)}$, $T_\alpha \equiv T_\alpha^{(0)}$, $\alpha \in 1, 2, 3$.

The corresponding Poisson brackets coincide with the Lie-Poisson brackets on $e^*(3)$

$$\{S_\alpha, S_\beta\} = \epsilon_{\alpha\beta\gamma}S_\gamma, \quad \{S_\alpha, T_\beta\} = \epsilon_{\alpha\beta\gamma}T_\gamma, \quad \{T_\alpha, T_\alpha\} = 0.$$

The generating function $I^{(2)}(u) = \sum\limits_{\alpha=1}^{3} L_\alpha^2(u)$ of the integrals of motion is written as follows

$$I^{(2)}(u) = I^{2,+}(u) + 2u\sqrt{u-1}I^{2,-}(u),$$
$$\text{where } I^{2,+}(u) = u^2 I_2^{2,+} + uI_1^{2,+} + I_0^{2,+},\ \ I^{2,-}(u) = I_0^{2,-},$$

$$I_2^{2,+} = \sum_{\alpha=1}^{3} T_\alpha^2,$$

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$$I_1^{2,+}=\sum_{\alpha=1}^{3}S_\alpha^2-(T_1^2+T_2^2),$$
$$I_0^{2,+}=-S_3^2,$$
$$I_0^{2,-}=\sum_{\alpha=1}^{3}T_\alpha S_\alpha.$$

The functions $C_1\equiv I_0^{2,-}$ and $C_2\equiv I_2^{2,+}$ are Casimir functions. The functions $H\equiv I_1^{2,+}$ and $K\equiv I_0^{2,+}$ are the Hamiltonian and integral of motion

$$\{H,K\}=0.$$

The "additional" linear integral is $M_3=S_3$. It is functionally dependent on K: $K=-M_3^2$.

4.2. The separated variables

Let us now apply the results of the previous section to the trigonometric Clebsch model. For this purpose it is enough to specify the functions $A(u)$ and $B(u)$ in the new notations

$$B(x)=x^{-2}\big(i(S_1+iT_3)x^2-i(iS_3-S_2+T_1)x+i(iT_3-T_2)\big), \tag{24a}$$
$$A(x)=-\frac{-iS_1x+iT_1+T_3x}{x^2}, \tag{24b}$$

The separated coordinates x_1, x_2 are the solutions of the equation $B(x)=0$ and have the following explicit form

$$x_1=\frac{T_1-S_2+iS_3-\sqrt{T_1^2-2T_1S_2+2iT_1S_3+S_2^2-2iS_2S_3-S_3^2+4S_1T_2-4iS_1T_3+4iT_3T_2+4T_3^2}}{2(S_1+iT_3)},$$
$$x_2=\frac{T_1-S_2+iS_3+\sqrt{T_1^2-2T_1S_2+2iT_1S_3+S_2^2-2iS_2S_3-S_3^2+4S_1T_2-4iS_1T_3+4iT_3T_2+4T_3^2}}{2(S_1+iT_3)}.$$

The canonically conjugated momenta are

$$p_1=A(x_1)=-\frac{-iS_1x_1+iT_1+T_3x_1}{x_1^2},$$
$$p_2=A(x_2)=-\frac{-iS_1x_2+iT_1+T_3x_2}{x_2^2}.$$

The Poisson commutation relations are the canonical ones

$$\{x_i,p_j\}=\delta_{ij},\quad \{x_i,x_j\}=0,\quad \{p_i,p_j\}=0,\qquad i,j\in\overline{1,2}.$$

By the virtue of the results of the previous section the canonical coordinates p_k, x_k satisfy the following equations of separation

$$\frac{(1+x_k^2)^2}{x_k^4}C_2+\frac{(1+x_k^2)}{x_k^2}H-M_3^2-\frac{2(1+x_k^2)}{x_k^3}C_1+\big((1+x_k^2)p_k+M_3\big)^2=0,\qquad k\in 1,2. \tag{25}$$

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4.3. The reconstruction formulae

Let us now reconstruct the dynamical variables S_α, T_β, $\alpha, \beta \in 1, 2, 3$ using the variables of separation p_i, x_j and the values of the Casimir functions. For this purpose it is necessary to solve the system of six linear-quadratic equations on six variables S_α, T_β

$$(x_1 + x_2) = \frac{(T_1 - S_2 + iS_3)}{(S_1 + iT_3)}, \tag{26a}$$

$$x_1 x_2 = \frac{(-T_2 + iT_3)}{(S_1 + iT_3)}, \tag{26b}$$

$$p_1 = -\frac{-iS_1 x_1 + iT_1 + T_3 x_1}{x_1^2}, \tag{26c}$$

$$p_2 = -\frac{-iS_1 x_2 + iT_1 + T_3 x_2}{x_2^2}, \tag{26d}$$

$$C_1 = T_1 S_1 + T_2 S_2 + T_3 S_3, \tag{26e}$$

$$C_2 = T_1^2 + T_2^2 + T_3^2. \tag{26f}$$

The following Proposition holds true:

Proposition 4.1. *(i) The system of equations (26) is solved as follows*

$$S_1 = \frac{i(x_1 - x_2)^2 C_2}{2x_1 x_2((x_1 - x_2)x_1^2 p_1 - x_2^2 p_2(x_1 - x_2))} + \\ + \frac{i(x_1^4 x_2(x_2 - 2x_1 + x_1^2 x_2)p_1^2 - 2x_1^3 x_2^3(-1 + x_1 x_2)p_2 p_1 + x_1 x_2^4(x_1 x_2^2 - 2x_2 + x_1)p_2^2)}{2x_1 x_2((x_1 - x_2)x_1^2 p_1 - x_2^2 p_2(x_1 - x_2))}, \tag{27a}$$

$$S_2 = -\frac{i(x_1 - x_2)C_1}{x_1 x_2(p_1 x_1^2 - x_2^2 p_2)} + \frac{i(x_2 + x_1)(x_1 - x_2)C_2}{2x_2^2 x_1^2(p_1 x_1^2 - x_2^2 p_2)} - \\ - \frac{i(-x_2^2 x_1^4(x_1^3 + x_1^2 x_2 - x_1 + x_2)p_1^2 + 2x_1^4 x_2^4(x_2 + x_1)p_2 p_1 - x_1^2 x_2^4(x_2^3 + x_1 x_2^2 - x_2 + x_1)p_2^2)}{2x_2^2(x_1 - x_2)x_1^2(p_1 x_1^2 - x_2^2 p_2)}, \tag{27b}$$

$$S_3 = -\frac{(x_1 - x_2)C_1}{x_1 x_2(p_1 x_1^2 - x_2^2 p_2)} + \frac{((x_2 + x_1)(x_1 - x_2)C_2}{2x_2^2 x_1^2(p_1 x_1^2 - x_2^2 p_2)} \\ - \frac{((x_2^2 x_1^4(1 + x_1^2)p_1^2 - x_1^2 x_2^4(1 + x_2^2)p_2^2)}{2x_2^2 x_1^2(p_1 x_1^2 - x_2^2 p_2)}, \tag{27c}$$

$$T_1 = -\frac{i x_2 x_1^2 p_1}{(x_1 - x_2)} + \frac{i x_1 p_2 x_2^2}{(x_1 - x_2)}, \tag{27d}$$

$$T_2 = -\frac{i(x_1 - x_2)^2 C_2}{2x_1 x_2((x_1 - x_2)x_1^2 p_1 - x_2^2 p_2(x_1 - x_2))} - \\ - \frac{i(-x_2^2 x_1^4(x_1 - 1)(x_1 + 1)p_1^2 + 2x_1^3 x_2^3(-1 + x_1 x_2)p_2 p_1 - x_1^2 x_2^4(x_2 - 1)(x_2 + 1)p_2^2)}{2x_1 x_2((x_1 - x_2)x_1^2 p_1 - x_2^2 p_2(x_1 - x_2))}, \tag{27e}$$

$$T_3 = -\frac{(x_1 - x_2)C_2}{2x_1 x_2(p_1 x_1^2 - x_2^2 p_2)}$$

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$$-\frac{(x_2^2x_1^4(1+x_1^2)p_1^2-2x_1^3x_2^3(x_1x_2+1)p_2p_1+x_1^2x_2^4(1+x_2^2)p_2^2)}{2(x_1-x_2)(p_1x_1^2-x_2^2p_2)x_1x_2}. \tag{27f}$$

(ii) If the Poisson commutation relations among p_i, x_j are the canonical ones

$$\{x_i,p_j\}=\delta_{ij},\quad \{x_i,x_j\}=0,\quad \{p_i,p_j\}=0,\qquad i,j\in\overline{1,2}$$

then the Poisson brackets between the variables S_α, T_β repeat standard Lie brackets of $e(3)$

$$\{S_\alpha,S_\beta\}=\epsilon_{\alpha\beta\gamma}S_\gamma,\quad \{S_\alpha,T_\beta\}=\epsilon_{\alpha\beta\gamma}T_\gamma,\quad \{T_\alpha,T_\alpha\}=0.$$

4.4. The Abel-type equations

Using either the equations of separation or the reconstruction formulas it is possible to show that the Hamiltonians are expressed via the coordinates of separation and values of the Casimir functions as follows[2]

$$H=\frac{(2x_1^4x_2p_1-2x_1x_2^4p_2)}{x_1^2x_2^2(p_1x_1^2-p_2x_2^2)}C_1+\frac{(-x_1^4(1+x_2^2)p_1+x_2^4(1+x_1^2)p_2}{(x_1^2x_2^2(p_1x_1^2-p_2x_2^2))}C_2+ \\ +\frac{(x_2^4x_1^4(1+x_1^2)p_2p_1^2-x_2^4x_1^4(1+x_2^2)p_2^2p_1)}{x_1^2x_2^2(p_1x_1^2-p_2x_2^2)},$$

$$M_3=-\frac{(x_1-x_2)C_1}{x_1x_2(p_1x_1^2-x_2^2p_2)}+\frac{((x_2+x_1)(x_1-x_2)C_2}{2x_2^2x_1^2(p_1x_1^2-x_2^2p_2)} \\ -\frac{((x_2^2x_1^4(1+x_1^2)p_1^2-x_1^2x_2^4(1+x_2^2)p_2^2)}{2x_2^2x_1^2(p_1x_1^2-x_2^2p_2)}.$$

Using this representation we obtain the following equations of motion for the separated coordinates

$$\frac{dx_1}{dt_1}=-\frac{2p_2x_1x_2(x_1-x_2)C_1}{(p_1x_1^2-p_2x_2^2)^2}+\frac{p_2(x_1-x_2)(x_2+x_1)C_2}{(p_1x_1^2-p_2x_2^2)^2-p_2(-x_2^2x_1^4(1+x_1^2)p_1^2}+ \\ +\frac{p_2(-x_2^2x_1^4(1+x_1^2)p_1^2+2x_1^2x_2^4(1+x_1^2)p_2p_1-x_2^4x_1^2(1+x_2^2)p_2^2)}{(p_1x_1^2-p_2x_2^2)^2}, \tag{28a}$$

$$\frac{dx_2}{dt_1}=\frac{2p_1x_1x_2(x_1-x_2)C_1}{(p_1x_1^2-p_2x_2^2)^2}-\frac{p_1(x_1-x_2)(x_2+x_1)C_2}{(p_1x_1^2-p_2x_2^2)^2}+ \\ +\frac{p_1(x_2^2x_1^4(1+x_1^2)p_1^2-2x_2^2x_1^4(1+x_2^2)p_2p_1+x_2^4x_1^2(1+x_2^2)p_2^2)}{(p_1x_1^2-p_2x_2^2)^2}, \tag{28b}$$

$$\frac{dx_1}{dt_2}=\frac{x_1(x_1-x_2)C_1}{x_2(p_1x_1^2-p_2x_2^2)^2}-\frac{(x_2+x_1)(x_1-x_2)C_2}{2x_2^2(p_1x_1^2-p_2x_2^2)^2}+ \\ +\frac{(-x_2^2x_1^4(1+x_1^2)p_1^2+2x_1^2x_2^4(1+x_1^2)p_2p_1-x_2^4x_1^2(1+x_2^2)p_2^2)}{2x_2^2(p_1x_1^2-p_2x_2^2)^2}, \tag{28c}$$

[2] We hereafter prefer to use integral $M_3=S_3$ instead of the integral $K=-(M_3)^2$.

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$$\frac{dx_2}{dt_2} = -\frac{x_2(x_1-x_2)C_1}{x_1(p_1x_1^2-p_2x_2^2)^2} + \frac{(x_2+x_1)(x_1-x_2)C_2}{2x_1^2(p_1x_1^2-p_2x_2^2)^2} - \frac{(x_2^2x_1^4(1+x_1^2)p_1^2 - 2x_2^2x_1^4(1+x_2^2)p_2p_1 + x_2^4x_1^2(1+x_2^2)p_2^2)}{2x_1^2(p_1x_1^2-p_2x_2^2)^2}, \tag{28d}$$

where $\frac{dx_i}{dt_1} = \{x_i, H\}$, $\frac{dx_j}{dt_2} = \{x_j, M_3\}$.

With the help of these equations of motion we derive the Abel-type equations

$$\frac{dx_1}{2x_1^2(p_1x_1^2+M_3+p_1)} + \frac{dx_2}{2x_2^2(p_2x_2^2+M_3+p_2)} = -dt_1,$$
$$\frac{p_1dx_1}{(p_1x_1^2+M_3+p_1)} + \frac{p_2dx_2}{(p_2x_2^2+M_3+p_2)} = -dt_2.$$

The Abel-type equations are integrated using the fact that momenta satisfy the equations of separation (25) and all the integrals are constants along the trajectories of the both flows.

Let us now transform the equation of the spectral curve (25) and the above Abel-type equations to more standard from the point of view of the algebraic geometry form. For this purpose we perform the following change of variables

$$P_i = (p_i(x_i^2+1) + M_3), \quad X_i = x_i^{-1} \qquad i \in 1, 2.$$

Under this transformation the spectral curve (25) becomes of second order in P_i and fourth order in X_i, i.e. coincides with the elliptic curve

$$P_i^2 = -(X_i^2+1)^2C_2 - (X_i^2+1)H + M_3^2 + 2(X_i^2+1)X_iC_1, \qquad i \in 1, 2. \tag{29}$$

The corresponding Abel-type equations are re-written as follows

$$\frac{dX_1}{2P_1} + \frac{dX_2}{2P_2} = dt_1, \tag{30a}$$

$$\frac{(P_1-M_3)}{X_1^2+1}\frac{dX_1}{P_1} + \frac{(P_2-M_3)}{X_2^2+1}\frac{dX_2}{P_2} = dt_2. \tag{30b}$$

As one can check, the first equation in the quadratures (30) contains a holomorphic differential on the elliptic curve (29), whereas the second one includes a meromorphic differential of third kind. Hence equations (30) written in integral form give rise to the so called generalized Abel mapping of third kind. The problem of its inversion has been posed in [25] and solved in terms of theta-functions of one variable in [26].

5. Conclusion and discussion

In the present paper we have constructed new *symmetric* variables of separation for the classical integrable Hamiltonian systems governed by skew-symmetric trigonometric r-matrix. The important feature of the constructed models is that the corresponding curve of separation is a *shifted* spectral curve of the initial Lax matrix.

In the present paper we have considered in details the example of trigonometric Clebsch model and have explicitly found the corresponding separated coordinates and momenta, the reconstruction formulas and Abel-type equations. We plan to perform a similar detailed study of the variables of separation for the other integrable models associated with a classical trigonometric

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r-matrix. Among such the models there are physically important ones. They are: trigonometric Gaudin models [24] and Jyaynes-Cummings-Dicke-type models [19].

Note that in the case of the standard skew-symmetric rational r-matrix a complete symmetric set of separated variables is obtained by another method (see e.g. [22] and references therein) and the corresponding curve of the separation coincides with a standard spectral curve of the Lax matrix. The similar statement holds true for the case of elliptic integrable models, where the separation curve is a standard spectral curve of the initial Lax matrix (see [23] for the example of the "elliptic" Clebsch model). Nevertheless, this does not mean that the presented example of SoV is completely isolated. The method of construction of separated variables using shifted spectral curves is generalized onto certain classes of *non-skew-symmetric* classical r-matrices. The work over this problem is now in progress.

Another very interesting problem is to prolong the results of the present article onto the quantum case. This problem is open.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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