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Gevrey regularity for a generalization of the Oleĭnik–Radkevič operator

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ABSTRACT

The Gevrey hypoellipticity of a class of models generalizing the Oleĭnik–Radkevič operator is studied. Some partial regularity result is also given. It is studied the partial and microlocal regularity of the operator

$$L(t,x;D_t,D_x)=D_t^2+\sum_{j=1}^{n}t^{2(r_j-1)}D_{x_j}^2$$

on Ω, open neighborhood of the origin in R^{n+1}, where the r_j's are positive integers such that $r_1<r_2<\cdots<r_n$.

1. Introduction

In [2] and [4] A. Bove, D.S. Tartakoff and M. Christ have proved that certain models of "sum of squares" operators of real vector fields with analytic coefficients have a Gevrey hypoellipticity threshold better than that one would have expected. In [2] moreover a detailed Gevrey partial regularity is obtained showing that the Gevrey regularity in some directions is better than in other directions.

More precisely in [2] the Oleĭnik–Radkevič operator, [8], is studied:

$$P=D_t^2+t^{2(p-1)}D_x^2+t^{2(q-1)}D_y^2,$$

where $p\leqslant q$ and $p,q\in\mathbb{N}$. It is proved that P is $G^{q/p}$-hypoelliptic and moreover that if u solves the problem $Pu=f$, f is analytic, then $u\in G^{(s_0,s_1,s_2)}$, where $s_0\geqslant 1-\frac{1}{q}+\frac{1}{p}$, $s_1\geqslant 1$ and $s_2\geqslant\frac{q}{p}$; their result is sharp.

In this paper we present a regularity result which concerns a more general class of Oleĭnik–Radkevič type of operators generalizing the result in [2] and [4]. Let us consider the "sum of squares" operator

$$L(t,x;D_t,D_x)=\sum_{j=0}^{n}X_j^2(t,x,D_t,D_x)=D_t^2+\sum_{j=1}^{n}t^{2(r_j-1)}D_{x_j}^2 \tag{1.1}$$

on Ω, open neighborhood of the origin in R^{n+1}, where the r_j's are positive integers such that $r_1<r_2<\cdots<r_n$.

By Hörmander's theorem [7] it is well known that these operators are C^∞-hypoelliptic. The classical results of Derridj and Zuily [5] and Rothschild and Stein [9] prove that for the operator L we have the following sub-elliptic a priori estimate with loss of $2(1-1/r_n)$ derivatives. We state it in the quadratic form version:

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$$\|u\|_{1/r_n}^2+\sum_{j=0}^{n}\|X_ju\|^2\leqslant C\big(\big|\langle Lu,u\rangle\big|+\|u\|^2\big). \tag{1.2}$$

Here $X_0=D_t$, $X_j=t^{r_j-1}D_j$, $\|\cdot\|_s$ denotes the H^s Sobolev norm and $\|\cdot\|=\|\cdot\|_0$ denotes the L^2 norm on a fixed open set Ω.

The results in [9] and [5] are actually applicable in a more general setting: for any operator being a sum of squares of real vector fields we have that if the fields and their brackets of length at most r span the tangent space then the sub-elliptic estimate with loss of $2(1-1/r)$ derivatives holds. Moreover the results obtained by Derridj and Zuily [5] say that the operator is hypoelliptic in all Gevrey classes G^s with $s\geqslant r$.

To understand in a clearer way the analytic (or Gevrey) hypoellipticity of sums of squares, F. Treves in [12] introduced the concept of Poisson stratification for such an operator.

We recall, without giving a definition, the main properties of the Poisson–Treves stratification for a "sum of squares":

Theorem 1.1. *([12], see also [3].) Let P be the operator $P(x;D)=\sum_1^k X_j^2(x;D)$, $X_j(x;D)$ vector fields with real analytic coefficients on an open neighborhood of the origin in $\mathbb{R}^n$. Let $X_j(x,\xi)$ be the symbol of the vector field X_j. Let $\Sigma=\operatorname{Char}(P)$ be the characteristic set of P that is*

$$\Sigma=\big\{(x,\xi)\in T^*\mathbb{R}^n\setminus\{0\}\colon\ X_j(x,\xi)=0\ \forall j\in\{1,\dots,k\}\big\}.$$

Then there is a stratification of Σ such that

1. *Each stratum is a real analytic manifold.*
2. *The symplectic form σ has constant rank on each stratum.*
3. *There is a sequence of integers, $\nu_1<\nu_2<\cdots<\nu_{p-1}<\nu_p=r$ (r denotes the maximum length of the Lie brackets involved in the Hörmander condition), and real analytic relatively open connected disjoint manifolds (strata) $\Sigma_{\nu_j,l}$, $l=1,\dots,l_j$, $j<p$. Here the index l counts the connected components at level ν_j. Moreover, all the Poisson brackets of the vector fields of length ν_j vanish on $\Sigma_{\nu_j,l}$, $l=1,\dots,l_j$, but there is at least one bracket of length ν_{j+1} which is non-identically zero.*

The length of a Poisson bracket of vector fields is just the number of vector fields forming the bracket; for example $X_j(x,\xi)$ is a bracket of length one while $\{X_1,\{X_1,X_2\}\}(x,\xi)$ is a bracket of length three.

Fore more details on the subject we refer to the papers [13] where it was first introduced and [3] for a different, constructive definition.

We recall the definition of depth of a point in Σ:

Definition 1.1. (See [1].) Let $\rho\in\Sigma$ be a characteristic point. Let ν_ρ be the smallest number such that there is a bracket of length ν_ρ which is non-zero at ρ. We define ν_ρ to be the depth of the point ρ and if $\rho\in\Sigma'$, where Σ' denotes a stratum in the stratification, ν_ρ will also be called the depth of the stratum Σ'.

We remark that the depth of a point is less or equal than the maximum length of the Lie brackets needed to generate the tangent space.

In the case of the operator L we have

$$\Sigma=\operatorname{Char}L=\big\{(t,x;\tau,\xi)\in T^*\mathbb{R}^{n+1}\setminus\{0\}\colon\ t=0,\ \tau=0\big\}$$

and the related stratification is given by

$$\begin{aligned}
&\Sigma_{r_1-1,\pm}=\big\{(t,x;\tau,\xi)\in T^*\mathbb{R}^{n+1}\setminus\{0\}\colon\ t=0,\ \tau=0,\ \xi_1\gtrless 0\big\},\\
&\Sigma_{r_2-1,\pm}=\big\{(t,x;\tau,\xi)\in T^*\mathbb{R}^{n+1}\setminus\{0\}\colon\ t=0,\ \tau=0,\ \xi_1=0,\ \xi_2\gtrless 0\big\},\\
&\quad\vdots\\
&\Sigma_{r_{j-1},\pm}=\big\{(t,x;\tau,\xi)\in T^*\mathbb{R}^{n+1}\setminus\{0\}\colon\ t=0,\ \tau=0,\ \xi_1=\cdots=\xi_{j-1}=0,\ \xi_j\gtrless 0\big\},\\
&\quad\vdots\\
&\Sigma_{r_{n-1}-1,\pm}=\big\{(t,x;\tau,\xi)\in T^*\mathbb{R}^{n+1}\setminus\{0\}\colon\ t=0,\ \tau=0,\ \xi_1=\cdots=\xi_{n-1}=0,\ \xi_n\gtrless 0\big\},\\
&\Sigma_{r_n}=\{\emptyset\}.
\end{aligned}$$

We recall the result in [1]:

Theorem 1.2. *(See [1].) Let P be a "sum of squares" operator. Let (x_0, ξ_0) be a point in the characteristic set Σ of P and $\nu_{(x_0,\xi_0)}$ its depth. Denote by Ω a neighborhood of (x_0, ξ_0). Then P is Gevrey-$\nu_{(x_0,\xi_0)}$ microlocally hypoelliptic at (x_0, ξ_0) i.e. if $WF_{\nu_{(x_0,\xi_0)}}(Pu) \cap \Omega = \varnothing$ then $(x_0, \xi_0) \notin WF_{\nu_{(x_0,\xi_0)}}(u)$.*

Here $u \in \mathscr{D}'(U)$, U is open subset in $\mathbb{R}^n$, and $WF_s(u)$ is the Gevrey-s wave front set of the distribution u.

In accordance with the above theorem we have that at the point $\rho_j = (0, 0, 0, e_j)$, with depth $\nu_{\rho_j} = r_j$, the operator L is Gevrey-r_j microlocally hypoelliptic at ρ_j i.e. $\rho_j \notin WF_{r_j}(u)$.

In virtue of the above results the operator L is G^{r_n}-hypoelliptic.

The non-isotropic Gevrey classes are defined as follows:

Definition 1.2. A smooth function $f(x_0, x_1, \ldots, x_n)$ belongs to the Gevrey space $G^{(\alpha_0,\alpha_1,\ldots,\alpha_n)}$ at the point x_0 provided that there exists a neighborhood, U, of x_0 and a constant C_f such that for all multi-indices β

$$\left|D^\beta f\right| \leqslant C_f^{|\beta|+1} \beta!^\alpha \quad \text{in } U,$$

where $\beta!^\alpha = \beta_0!^{\alpha_0}\beta_1!^{\alpha_1}\cdots\beta_n!^{\alpha_n}$.

Our result can be stated as follows:

Theorem 1.3. *The operator L in (1.1) is G^{r_n/r_1}-hypoelliptic and not better. More precisely we have that if u solves the equation $Lu = f$ and f is analytic then if $\rho_j \in \Sigma_{r_j-1}$ then $\rho_j \notin WF_{r_j/r_1}(u)$ and moreover $u \in G^{(s_0,s_1,\ldots,s_n)}$ where*

$$s_0 \geqslant r^*, \qquad s_j = \beta_j \geqslant \frac{r_n(r_j-1)}{r_n(r_1-1)+r_j-r_1} \quad \text{with } j = 1, \ldots, n;$$

where $r^* = \sup_j\{1 - \frac{1}{r_j} + \frac{\beta_j}{r_j}\}$, in particular $s_1 \geqslant 1$ and $s_n \geqslant r_n/r_1$.

Remark 1.1. The result stated above can be extended to the operators

$$L(t, x; D_t, D_x) = D_t^2 + \sum_{j=1}^{\nu} t^{2(r_j-1)}\Delta_{n_j}^2$$

where r_j are positive integers such that $r_1 < r_2 < \cdots < r_\nu$ and $\Delta_{n_j} = \sum_{j=n_{j-1}}^{n_j} D_j^2$ with $n_0 = 1$, $n_\nu = n+1$ and $n_0 < n_1 < \cdots < n_\nu$; in this case the points $(0, 0; 0, e_j)$ with $n_{j-1} + 1 \leqslant j \leqslant n_j$ are not in the $\frac{r_j}{r_1}$-Gevrey wave front set of u and $u \in G^{(s_0,s_1,\ldots,s_n)}$ where $s_0 \geqslant r^*$ and $s_j \geqslant (r_n(r_j-1))/(r_n(r_1-1)+r_j-r_1)$. This situation does not present additional difficulties compared to that we are going to handle.

This is the plan of the paper: we will study the direction t and x_j using the same technique in [2], while the microlocal regularity will be obtained using the FBI technique (Fourier–Bros–Iagolnitzer (FBI) transform) and the microlocal version of the Rothschild–Stein estimate (1.2) obtained in [1]; for the sharpness of the result we will follow the ideas in [8] and [2].

We point out here that the Gevrey regularity r_n/r_1 is optimal, as is shown in the final section of the paper. We are also able to prove that the partial regularity w.r.t. t and x_n are also optimal. We have no claim of optimality for s_j, $1 \leqslant j \leqslant n-1$.

2. Microlocal regularity

We investigate the microlocal regularity of the operator L related with the *Poisson–Treves* stratification of $\Sigma = \mathrm{Char}(L)$ introduced above. The primary tools will be the Fourier–Bros–Iagolnitzer (FBI) transform (for more details on this we refer to [11,10,6]) and the microlocal version of the sub-elliptic a priori estimate obtained in [1] via FBI. We recall some basic definitions and results which will be useful for our purpose, see [11,6,1].

We consider the FBI transformation with the classical phase function

$$Tu(z, \lambda) = \int e^{-\frac{\lambda}{2}((z_0-t)^2+(z'-x)^2)} u(t, x)\, dt\, dx,$$

where $\lambda \gg 1$, $z = (z_0, z') \in \mathbb{C}^{1+n}$, $(t, x) \in \mathbb{R}^{1+n}$ and u is a compactly supported distribution.

Let us denote by Ω an open neighborhood of the point $z^0 = \pi \circ \mathcal{H}_T(z^0, \zeta^0)$ in $\mathbb{C}^{n+1}$; π denotes the space projection $\pi : \mathbb{C}_z^{n+1} \times \mathbb{C}_\zeta^{n+1} \to \mathbb{C}_z^{n+1}$ and $\mathcal{H}_T$ denotes the complex canonical transform associated to T. Let φ_0 be the weight function corresponding to the classical FBI transformation:

$$\varphi_0(z) = -\sup_{(t,x)\in\mathbb{R}^{n+1}} \operatorname{Im} -\frac{i}{2}\big((z_0-t)^2 + (z'-x)^2\big).$$

$\mathcal{H}_T$ maps $T^*\mathbb{R}^{n+1}$ into $\Lambda_{\varphi_0} = \{(z, \frac{2}{i}\partial_z\varphi_0(z))\}$. If we denote by $\tilde{L}$ our operator after the FBI we have that $\tilde{L}_{|\Lambda_{\varphi_0}} = L$. We recall briefly the characterization of the s-Gevrey wave front set in the FBI setting, see [11]: a point $(x_0, \xi_0) \in U$, U open subset of $T^*\mathbb{R}^{n+1} \setminus \{0\}$, does not belong to $WF_s(u)$ if and only if exists a neighborhood Ω of $x_0 - i\xi_0$ in $\mathbb{C}^{n+1}$ and positive constants C_1 and C_2 such that

$$\big|e^{-\varphi_0(z)} Tu(z,\lambda)\big| \leqslant C_1 e^{-\lambda^{\frac{1}{s}}/C_2}$$

for every $z \in \Omega$.

Since in the following we will work on the FBI side we will continue to denote with L the operator after the FBI transform. We recall the following result:

Theorem 2.1. *(See [1].) Let P be a "sum of squares" operator after the FBI transform and ν the depth of the point $(x_0, \xi_0) \in \operatorname{Char}(P)$. Let $\Omega_1 \Subset \Omega$, Ω is a neighborhood of the point $x_0 - i\xi_0$. Then*

$$\lambda^{2/\nu}\|u\|^2_{\varphi_0} + \sum_{j=1}^{k}\big\|X_j^{\Omega} u\big\|^2_{\varphi_0} \leqslant C\big(\langle P^{\Omega}u, u\rangle_{\varphi_0} + \lambda^{\alpha}\|u\|^2_{\varphi_0,\Omega\setminus\Omega_1}\big), \tag{2.1}$$

where α is a positive integer.

Here P^{Ω} is the Ω-realization of the operator, for more details see [6], and

$$\|u\|^2_{\varphi_0} = \int_{\Omega} e^{-2\lambda\varphi_0(z)}\big|u(z)\big|^2\, dz \wedge d\bar{z}.$$

In the special case of the operator L, (1.1), if we take $\rho_j \in \Sigma_{r_j-1}$, we can choose without loss of generality $\rho_j = (0, 0, 0, e_j)$, $j \geqslant 2$, then we have that the depth of ρ_j is r_j and applying the above theorem we have

$$\lambda^{2/r_j}\|u\|^2_{\varphi_0} + \sum_{j=0}^{n}\big\|X_j^{\Omega} u\big\|^2_{\varphi_0} \leqslant C\big(\langle L^{\Omega}u, u\rangle_{\varphi_0} + \lambda^{\alpha}\|u\|^2_{\varphi_0,\Omega\setminus\Omega_1}\big), \tag{2.2}$$

where Ω is a neighborhood of the point $0 - ie_j$, $\Omega_1 \Subset \Omega$ and L^{Ω} is the Ω-realization of the operator L.

We perturb canonically φ_0 using an analytic function $r(z, \zeta, \lambda)$ and solve, for small positive s, the Hamilton–Jacobi problem

$$\begin{cases} 2\dfrac{\partial\varphi}{\partial s}(s, z, \lambda) = r\Big(z, \dfrac{2}{i}\dfrac{\partial\varphi}{\partial z}(s, z, \lambda), \lambda\Big), \\ \varphi(0, z, \lambda) = \varphi_0(z). \end{cases}$$

Since $\mathbb{R}^{2(n+1)}$ and Λ_{φ_0} are isomorphic it is easier to construct the function r in $\mathbb{R}^{2(n+1)}$ near the characteristic point $\rho_j \in \Sigma_{r_j-1}$.

We choose

$$r(t, x, \tau, \xi) = \tau^2 + t^{2r_j} + \sum_{\substack{l=1\\ l\neq j}}^{n}\xi_j^2 + (\xi_j - 1)^2 + \sum_{i=1}^{j-1}\lambda^{-\theta_i}x_i^2 + \sum_{i=j}^{n}x_i^2$$

where $\theta_i = \frac{r_j - r_i}{r_j}$. We remark that $r(0, 0, 0, e_j, \lambda) = 0$ and $r_{|\Lambda_{\varphi_0}\setminus\{(ie_j, -e_j)\}}$ is strictly positive.

We write $\Lambda_{\varphi_s} = \exp(isH_r)(\Lambda_{\varphi_0})$. Our purpose is to use the estimate (2.2) with the new weight function φ_s. Consider the restriction to Λ_{φ_s} of the symbol of L, denoted by L^s. We have

$$L^s = L + s\sum_{j=0}^{n} X_j\{r, X_j\} + s^2\sum_{j=0}^{n}\{r, X_j\}^2 + \mathcal{O}\big(s^2\lambda^{\frac{2}{r_j}}\big).$$

We want to estimate

$$\big\|\{r, X_i\}^{\Omega} u\big\|^2_{\varphi_s} \quad \text{with } i = 0, \ldots, n.$$

We have

$$\begin{aligned}\{X_0, r(t,x,\tau,\xi,\lambda)\} &= \{\tau, r(t,x,\tau,\xi,\lambda)\} = 2r_j t^{2r_j-1} = a_0(t,\xi)X_j,\\ \{X_i, r(t,x,\tau,\xi,\lambda)\} &= \{t^{r_i-1}\xi_i, r(t,x,\tau,\xi,\lambda)\} = -2(r_i-1)t^{r_i-2}\xi_i\tau + 2\lambda^{-\theta_i}t^{r_i-1}x_i\\ &= a_{1i}(t,x,\xi)X_0 + a(x)\lambda^{-\theta_i}t^{r_i-1}, \quad i=1,\dots,j-1,\end{aligned}$$

and

$$\begin{aligned}\{X_i, r(t,x,\tau,\xi,\lambda)\} &= \{t^{r_i-1}\xi_i, r(t,x,\tau,\xi,\lambda)\} = -2(r_i-1)t^{r_i-2}\xi_i\tau + 2t^{r_i-1}x_i\\ &= a_{2i}(t,x,\xi)X_0 + a_{3i}(x,\xi)X_i, \quad i=j,\dots,n.\end{aligned}$$

The choice of θ_i allows us to take advantage of the inequality

$$\lambda^{2(1-\theta_i)}t^{2(r_i-1)} \leqslant \lambda^{\frac{2}{r_j}} + \lambda^2 t^{2(r_j-1)} = \lambda^{\frac{2}{r_j}} + a(\xi)\lambda^2 X_j^2.$$

In view of the above inequality and of Proposition (1.3) in [6] we have

$$\|\{r,X_i\}^\Omega u\|^2_{\varphi_s} \lesssim \|X_0^\Omega u\|^2_{\varphi_s} + \|X_j^\Omega u\|^2_{\varphi_s} + \lambda^{2/r_j}\|u\|^2_{\varphi_s}, \quad i=1,\dots,j-1,$$

and

$$\|\{r,X_i\}^\Omega u\|^2_{\varphi_s} \lesssim \|X_0^\Omega u\|^2_{\varphi_s} + \|X_i^\Omega u\|^2_{\varphi_s}, \quad i=j,\dots,n.$$

Rewriting (2.2) with the new weight φ_s

$$\lambda^{2/r_j}\|u\|^2_{\varphi_s} + \sum_{j=0}^{n}\|X_j^\Omega u\|^2_{\varphi_s} \leqslant C\big(\langle L^\Omega u, u\rangle_{\varphi_s} + \lambda^\alpha\|u\|^2_{\varphi_s,\Omega\setminus\Omega_1}\big).$$

We can conclude that there are a neighborhood Ω_0 of $0+ie_j$, a positive number δ and a positive integer α such that, for every $\Omega_1 \Subset \Omega_2 \Subset \Omega \subset \Omega_0$, there exists a constant $C>0$ such that, for $0<s<\delta$, we have

$$\lambda^{2/r_j}\|u\|_{\varphi_s,\Omega_1} \leqslant C\big(\|L^\Omega u\|_{\varphi_s,\Omega_2} + \lambda^\alpha\|u\|_{\varphi_s,\Omega\setminus\Omega_1}\big). \tag{2.3}$$

We now prove that if Lu is analytic in ρ_j then the point ρ_j does not belong to $WF_{r_j/r_1}(u)$.

Since Lu is real analytic the first term in (2.3) can be estimated by $Ce^{-\lambda/C}$ for a positive constant C.

On the other hand we have

$$\varphi_s(z) = \varphi_0(z) + \frac{s}{2}r\left(z, \frac{2}{i}\frac{\partial\varphi}{\partial z}(0,z), \lambda\right) + \mathcal{O}(s^2).$$

Hence

$$\varphi_s(z) - \varphi_0(z) \sim s\left(\sum_{i=1}^{j-1}\lambda^{-\theta_i}|z_i|^2 + \sum_{i=j+1}^{n}|z_i|^2 + |z_0|^2 + |z_j - i|^2\right).$$

Since on $\Omega\setminus\Omega_2$ $r > \beta + \sum\lambda^{-\theta_i}\beta_i$ we have

$$\varphi_s(z)_{|\Omega\setminus\Omega_2} \geqslant \varphi_0(z) + s\Big(\beta + \sum\lambda^{-\theta_i}\beta_i\Big).$$

The second term on the right hand side of (2.3) can be estimated by

$$\|u\|^2_{\varphi_s,\Omega\setminus\Omega_2} \leqslant e^{-\lambda C_1(s) - \sum\lambda^{1-\theta_i}C_i(s)}.$$

From (2.3) and the above argument we have

$$\|u\|^2_{\varphi_s,\Omega_1} \leqslant e^{-\lambda C_1(s) - \sum\lambda^{1-\theta_i}C_i(s)}.$$

Let Ω_3 be a sufficient small neighborhood of the point $0+ie_j$, then for a fixed positive s we obtain

$$\|u\|^2_{\varphi_0,\Omega_3} \leqslant \tilde{C}_s e^{-\epsilon\lambda^{1-\theta}},$$

where

$$1-\theta = \inf_j\{1-\theta_1,\dots,1-\theta_{j-1}\} = \inf_j\left\{\frac{r_1}{r_j},\dots,\frac{r_{j-1}}{r_j}\right\},$$

that is

$$\|u\|^2_{\varphi_0,\Omega_3} \leqslant \tilde{C}_s e^{-\epsilon\lambda^{r_1/r_j}}.$$

Proposition 2.1. *The operator L in (1.1) is $\frac{r_j}{r_1}$-Gevrey hypoelliptic microlocally at the stratum Σ_{r_j-1}, i.e. the point $\rho_j \in \Sigma_{r_j-1}$ is not in the $\frac{r_j}{r_1}$-Gevrey wave front set of u.*

3. Partial regularity

To study the partial regularity of the solutions we estimate the high order derivatives of the solutions in L^2 norm. As a matter of fact we estimate a suitable localization of a high derivative using (1.2). Actually we estimate $\varphi(t,x)D_j^\alpha u$, $j=1,\dots,n$, and $\varphi(t,x)D_t^\alpha u$. For $t\neq 0$ the operator L is elliptic and we shall not examine this region, elliptic operators are Gevrey hypoelliptic in any class G^s for $s\geqslant 1$.

3.1. Direction D_1

Let $\varphi(t,x)$ be a localizing function of Ehrenpreis–Hörmander type: for any Ω_1 and Ω, with Ω_1 compactly contained in Ω, there exists a constant C and a family of functions $\{\varphi_m\}\in C_0^\infty(\Omega)$ such that $\varphi_m\equiv 1$ on Ω_1 for every m and such that for $|\alpha|\leqslant 2r_n m$ we have $|D^\alpha\varphi_m|\leqslant C^{|\alpha|+1}m^{|\alpha|}$. We may assume that φ is independent of the variable t since every t-derivative landing on φ would leave a cut-off function supported where t is bounded away from zero, where the operator is elliptic. Let φ be a cut-off function of the type described above. We replace u by $\varphi(x)D_1^\alpha u$ in (1.2). We have

$$\|\varphi D_1^\alpha u\|^2_{1/r_n} + \sum_{j=0}^n \|X_j\varphi D_1^\alpha u\|^2 \leqslant C\big(\big|\langle L\varphi D_1^\alpha u, \varphi D_1^\alpha u\rangle\big| + \|\varphi D_1^\alpha u\|^2\big). \tag{3.1}$$

The scalar product in the right hand side leads to

$$\begin{aligned}\langle\varphi D_1^\alpha Lu, \varphi D_1^\alpha u\rangle + \sum_{j=1}^n\langle[X_j^2,\varphi D_1^\alpha]u, \varphi D_1^\alpha u\rangle &= 2\sum_{j=1}^n\langle[X_j,\varphi D_1^\alpha]u, X_j\varphi D_1^\alpha u\rangle + \sum_{j=1}^n\langle[[X_j,\varphi D_1^\alpha],X_j]u,\varphi D_1^\alpha u\rangle\\ &\quad+\langle\varphi D_1^\alpha Lu,\varphi D_1^\alpha u\rangle.\end{aligned}\tag{3.2}$$

The last term is trivial to estimate since Lu is analytic; we may assume, without loss of generality, that it is zero. For every j, $1\leqslant j\leqslant n$, we have

$$\begin{aligned}\big|\langle[X_j,\varphi D_1^\alpha]u, X_j\varphi D_1^\alpha u\rangle\big| &= \big|\langle t^{r_j-1}\varphi^{(1)}D_1^\alpha u, X_j\varphi D_1^\alpha u\rangle\big|\\ &\leqslant \big|\langle t^{r_j-1}D_1\varphi^{(1)}D_1^{\alpha-1}u, X_j\varphi D_1^\alpha u\rangle\big| + \big|\langle t^{r_j-1}\varphi^{(2)}D_1^{\alpha-1}u, X_j\varphi D_1^\alpha u\rangle\big|.\end{aligned}$$

We use Cauchy–Schwartz on the first scalar product and repeat the procedure on the second:

$$\begin{aligned}\big|\langle[X_j,\varphi D_1^\alpha]u, X_j\varphi D_1^\alpha u\rangle\big| &\leqslant C_1\|X_1\varphi^{(1)}D_1^{\alpha-1}u\|^2 + \frac{1}{C_1}\|X_j\varphi D_1^\alpha u\|^2 + \big|\langle t^{r_j-1}\varphi^{(2)}D_1^{\alpha-2}u, X_j\varphi D_1^\alpha u\rangle\big|\\ &\quad+\big|\langle t^{r_j-1}\varphi^{(3)}D_1^{\alpha-2}u, X_j\varphi D_1^\alpha u\rangle\big|.\end{aligned}$$

The latter can be handled as before with a constant C_2:

$$\begin{aligned}\big|\langle[X_j,\varphi D_1^\alpha]u, X_j\varphi D_1^\alpha u\rangle\big| &\leqslant C_1\|X_1\varphi^{(1)}D_1^{\alpha-1}u\|^2 + \frac{1}{C_1}\|X_j\varphi D_1^\alpha u\|^2 + C_2\|X_1\varphi^{(2)}D_1^{\alpha-2}u\|^2\frac{1}{C_2}\|X_j\varphi D_1^\alpha u\|^2\\ &\quad+\big|\langle t^{r_j-1}\varphi^{(3)}D_1^{\alpha-2}u, X_j\varphi D_1^\alpha u\rangle\big|.\end{aligned}$$

Iterating we obtain

$$\begin{aligned}\big|\langle[X_j,\varphi D_1^\alpha]u, X_j\varphi D_1^\alpha u\rangle\big| &\leqslant \sum_{\ell=1}^\alpha\left\{C_\ell\|X_1\phi^{(\ell)}D_1^{\alpha-\ell}u\|^2 + \frac{1}{C_\ell}\|X_j\varphi D_1^\alpha u\|^2\right\}\\ &\quad+C_{\alpha+1}\|\phi^{(\alpha+1)}u\|^2 + \frac{1}{C_{\alpha+1}}\|X_j\varphi D_1^\alpha u\|^2.\end{aligned}$$

Recalling that the constants C_ℓ are arbitrary we make the choice $C_\ell=\epsilon^{-1}2^\ell$, for a suitable fixed positive ϵ, we may absorb each second term in the above two lines on the left hand side of (3.1). Choosing $\varphi=\phi_m$, with $m\sim\alpha$, we have $C_{\alpha+1}\|\phi^{(\alpha+1)}u\|\leqslant C^{\alpha+1}\alpha!$. Finally to estimate the terms $C_\ell\|X_1\phi^{(\ell)}D_1^{\alpha-\ell}u\|^2$, we observe that for each of them there has been a shift of one or more x_1-derivatives from u to ϕ, but that they have the same form as $\|X_1\phi D_1^\alpha u\|^2$. We have to estimate the sum

$$\sum_{\ell=1}^{\alpha}\epsilon^{-1}2^\ell\big\|X_1\phi^{(\ell)}D_1^{\alpha-\ell}u\big\|^2=\epsilon^{-1}2\big\|X_1\phi^{(1)}D_1^{\alpha-1}u\big\|^2+\sum_{\ell=2}^{\alpha}\epsilon^{-1}2^\ell\big\|X_1\phi^{(\ell)}D_1^{\alpha-\ell}u\big\|^2. \tag{3.3}$$

We use the Rothschild–Stein sub-elliptic estimate replacing u with $\varphi^{(1)}D_1^{\alpha-1}u$ on the first term; repeating the above procedure we have

$$\big\|X_1\phi^{(1)}D_1^{\alpha-1}u\big\|^2\leqslant\sum_{\ell=1}^{\alpha-1}\left\{\epsilon^{-1}2^\ell\big\|X_1\phi^{(\ell+1)}D_1^{\alpha-\ell-1}u\big\|^2+\frac{\epsilon}{2^\ell}\big\|X_j\phi^{(1)}D_1^{\alpha-1}u\big\|^2\right\}$$

modulo terms which give analytic growth or which have the form $|\langle[[X_j,\varphi^{(1)}D_1^{\alpha-1}],X_j]u,\varphi^{(1)}D_1^{\alpha-1}u\rangle|$, we observe that for each of them there has been a shift of one x_1-derivative from u to ϕ, but that they have the same form as $|\langle[[X_j,\varphi D_1^\alpha],X_j]u,\varphi D_1^\alpha u\rangle|$ in (3.2), for the discussion of these terms see in the continuation of the proof. As above we may absorb the second term in the left hand side of the estimate. Therefore we have to estimate the sum

$$\sum_{\ell=2}^{\alpha}\frac{1}{\epsilon}\left(1+\frac{1}{\epsilon}\right)2^\ell\big\|X_1\phi^{(\ell)}D_1^{\alpha-\ell}u\big\|^2=\frac{1}{\epsilon}\left(1+\frac{1}{\epsilon}\right)2^2\big\|X_1\phi^{(2)}D_1^{\alpha-2}u\big\|^2+\sum_{\ell=3}^{\alpha}\frac{1}{\epsilon}\left(1+\frac{1}{\epsilon}\right)2^\ell\big\|X_1\phi^{(\ell)}D_1^{\alpha-\ell}u\big\|^2.$$

Repeating the above process p times we have

$$\sum_{\ell=1}^{\alpha}\epsilon^{-1}2^\ell\big\|X_1\phi^{(\ell)}D_1^{\alpha-\ell}u\big\|^2\leqslant\sum_{\ell=p}^{\alpha}\frac{1}{\epsilon}\left(1+\frac{1}{\epsilon}\right)^{p-1}2^\ell\big\|X_1\phi^{(\ell)}D_1^{\alpha-\ell}u\big\|^2$$

modulo terms which can be absorbed on the left side or which give analytic growth or which have the form $|\langle[[X_j,\varphi^{(\ell)}D_1^{\alpha-\ell}],X_j]u,\varphi^{(\ell)}D_1^{\alpha-\ell}u\rangle|$, $1\leqslant\ell\leqslant p-1$, we remark that they have the same form as $|\langle[[X_j,\varphi D_1^\alpha],X_j]u,\varphi D_1^\alpha u\rangle|$ in (3.2).

Keeping on with the same procedure, after $\alpha-1$ iterates, we obtain a term of the form

$$\frac{1}{\epsilon}\left(1+\frac{1}{\epsilon}\right)^{\alpha-1}2^\alpha\big\|X_1\varphi^{(\alpha)}u\big\|^2.$$

Choosing $\varphi=\varphi_m$ with $m\sim\alpha$ we have $\epsilon^{-\alpha}(1+\epsilon)^{\alpha-1}2^\alpha\|X_1\varphi^{(\alpha)}u\|\leqslant C^{\alpha+1}\alpha!$.

To conclude this part of the estimate we thus need to bound the term with the double commutator.

We turn our attention to the second sum on the right hand side of (3.2). For every j we have

$$\begin{aligned}\big|\big\langle\big[\big[X_j,\varphi D_1^\alpha\big],X_j\big]u,\varphi D_1^\alpha u\big\rangle\big|&=\big|\big\langle t^{2(r_j-1)}\varphi^{(2)}D_1^\alpha u,\varphi D_1^\alpha u\big\rangle\big|\\&\leqslant\big|\big\langle t^{r_j-1}D_1\varphi^{(2)}D_1^{\alpha-1}u,t^{r_j-1}D_1\varphi D_1^{\alpha-1}u\big\rangle\big|+\big|\big\langle t^{r_j-1}D_1\varphi^{(2)}D_1^{\alpha-1}u,t^{r_j-1}\varphi^{(1)}D_1^{\alpha-1}u\big\rangle\big|\\&\quad+\big|\big\langle t^{r_j-1}\varphi^{(3)}D_1^{\alpha-1}u,t^{r_j-1}D_1\varphi D_{x_1}^{\alpha-1}u\big\rangle\big|+\big|\big\langle t^{r_j-1}\varphi^{(3)}D_1^{\alpha-1}u,t^{r_j-1}\varphi^{(1)}D_1^{\alpha-1}u\big\rangle\big|\\&=I_1+I_2+I_3+I_4.\end{aligned}$$

We study each single term. Term I_1:

$$I_1\leqslant2\left\|\frac{1}{m}X_1\varphi^{(2)}D_1^{\alpha-1}u\right\|^2+\frac{1}{2}\big\|mX_1\varphi D_1^{\alpha-1}u\big\|^2.$$

We have introduced the "weight" m to balance the number of x_1-derivatives on u with the number of derivatives on φ; In other words we take the factor m like a derivative on φ and $m^{-1}\varphi^{(2)}$ as $\varphi^{(1)}$. Hence the terms on the right hand side have the same form as $\|X_1\varphi D_1^\alpha u\|^2$.

The term I_2:

$$\begin{aligned}I_2&=\big|\big\langle t^{r_j-1}D_1\varphi^{(2)}D_1^{\alpha-1}u,t^{r_j-1}\varphi^{(1)}D_1^{\alpha-1}u\big\rangle\big|\\&\leqslant\epsilon_1\big\|X_1\varphi^{(2)}D_1^{\alpha-2}u\big\|^2+\frac{1}{\epsilon_1}\big\|X_1\varphi^{(1)}D_1^{\alpha-1}u\big\|^2+\big|\big\langle t^{r_j-1}\varphi^{(3)}D_1^{\alpha-2}u,t^{r_j-1}D_1\varphi^{(1)}D_1^{\alpha-1}u\big\rangle\big|\\&\leqslant\cdots\leqslant\sum_{j=1}^{\alpha}\epsilon_j\big\|X_1\varphi^{(j)}D_1^{\alpha-j}u\big\|^2+C_\epsilon\big\|X_1\varphi^{(1)}D_1^{\alpha-1}u\big\|^2+\epsilon_\alpha\big\|\varphi^{(\alpha+1)}u\big\|^2.\end{aligned}$$

The above sum can be handled with the same process used to estimate the sum (3.3).

The term I_3:

$$\begin{aligned}
I_3 &= \left|\left\langle t^{r_j-1}\varphi^{(3)}D_1^{\alpha-1}u, t^{r_j-1}D_1\varphi D_1^{\alpha-1}u\right\rangle\right| \\
&\leqslant C_1\left\|\frac{1}{m^2}X_1\varphi^{(3)}D_1^{\alpha-1}u\right\|^2+\frac{1}{C_1}\left\|m^2X_1\varphi D_1^{\alpha-2}u\right\|^2+C_2\left\|\frac{1}{m}X_1\varphi^{(3)}D_1^{\alpha-2}u\right\|^2+\frac{1}{C_2}\left\|mX_1\varphi^{(1)}D_1^{\alpha-2}u\right\|^2 \\
&\quad+\left|\left\langle t^{r_j-1}\varphi^{(4)}D_1^{\alpha-2}u, t^{r_j-1}D_1\varphi^{(1)}D_1^{\alpha-2}u\right\rangle\right| \\
&\leqslant\cdots\leqslant\sum_{k=1}^{\alpha}C_1\left\|\frac{1}{m^2}X_1\varphi^{(k+2)}D_1^{\alpha-k}u\right\|^2+\sum_{k=1}^{\alpha-1}\frac{1}{C_1}\left\|m^2X_1\varphi^{(k-1)}D_1^{\alpha-(k+1)}u\right\|^2 \\
&\quad+\sum_{k=1}^{\alpha}\left\{C_2\left\|\frac{1}{m}X_1\varphi^{(k+2)}D_1^{\alpha-(k+1)}u\right\|^2+\frac{1}{C_2}\left\|mX_1\varphi^{(k)}D_1^{\alpha-(k+1)}u\right\|^2\right\}+\left|\left\langle t^{r_j-1}\varphi^{(\alpha+2)}u, t^{r_j-1}D_1\varphi^{(\alpha-1)}u\right\rangle\right|.
\end{aligned}$$

Choosing $\varphi=\phi_m$, with $m\sim\alpha$, we have $|\langle t^{r_j-1}\varphi^{(\alpha+2)}u, t^{r_j-1}D_1\varphi^{(\alpha-1)}u\rangle|\leqslant C^{\alpha+1}\alpha!$. To estimate the terms in the sums, we observe that with the help of the weight m we have essentially, on each of them, shifted one or more x_1-derivatives from u to ϕ, but that they have the same form as $\|X_1\phi D_1^{\alpha}u\|^2$.

The term I_4:

$$\begin{aligned}
I_4 &= \left|\left\langle t^{r_j-1}\varphi^{(3)}D_1^{\alpha-1}u, t^{r_j-1}\varphi^{(1)}D_1^{\alpha-1}u\right\rangle\right| \\
&\leqslant\left|\left\langle t^{r_j-1}D_1\varphi^{(3)}D_1^{\alpha-2}u, t^{r_j-1}D_1\varphi^{(1)}D_1^{\alpha-2}u\right\rangle\right|+\left|\left\langle t^{r_j-1}D_1\varphi^{(3)}D_1^{\alpha-2}u, t^{r_j-1}D_1\varphi^{(1)}D_1^{\alpha-2}u\right\rangle\right| \\
&\quad+\left|\left\langle t^{r_j-1}\varphi^{(4)}D_1^{\alpha-2}u, t^{r_j-1}D_1\varphi^{(1)}D_1^{\alpha-2}u\right\rangle\right|+\left|\left\langle t^{r_j-1}\varphi^{(4)}D_1^{\alpha-2}u, t^{r_j-1}\varphi^{(2)}D_1^{\alpha-2}u\right\rangle\right|.
\end{aligned}$$

Iterating we obtain

$$\begin{aligned}
I_4 &\leqslant\sum_{k=1}\left|\left\langle t^{r_j-1}D_1\varphi^{(k+2)}D_1^{\alpha-(k+1)}u, t^{r_j-1}D_1\varphi^{(k)}D_1^{\alpha-(k+1)}u\right\rangle\right|+\sum_{k=1}\left|\left\langle t^{r_j-1}\varphi^{(k+2)}D_1^{\alpha-(k+1)}u, t^{r_j-1}\varphi^{(k+1)}D_1^{\alpha-(k+1)}u\right\rangle\right| \\
&\quad+\sum_{k=1}\left|\left\langle t^{r_j-1}\varphi^{(k+3)}D_1^{\alpha-(k+1)}u, t^{r_j-1}D_1\varphi^{(k)}D_1^{\alpha-(k+1)}u\right\rangle\right|+\left|\left\langle t^{r_j-1}\varphi^{(\alpha+2)}u, t^{r_j-1}\varphi^{(\alpha)}u\right\rangle\right|.
\end{aligned}$$

Observing that the terms in the first sum have the same form as I_1, the terms in the second sum have the same form as I_2 and those in the third sum have the same form as I_3 we can handle each of them as above. Finally, the last term, for $\varphi=\varphi_m$ with $m\sim\alpha$ can be estimated by $C^{\alpha+1}\alpha!$.

Using the estimate (1.2) with u replaced by $m^k\varphi^{(j)}D_1^{\alpha-(j+k)}u$ or $m^{-k}\varphi^{(j+k)}D_1^{\alpha-j}u$ and applying recursively the same strategy followed above we are able to shift all free derivatives on φ. Hence we have

$$\left\|\varphi D_1^{\alpha}u\right\|_{1/r_n}^2+\sum_{j=0}^{n}\left\|X_j\varphi D_1^{\alpha}u\right\|^2\leqslant C^{\alpha+1}m^{\alpha}.$$

Choosing $\varphi=\varphi_m$, with $m\sim\alpha$, we have the analytic growth in the direction x_1.

3.2. Direction D_n

Let $\varphi(x)$ be a cut-off function of Ehrenpreis–Hörmander type described above. We replace u by $\varphi D_n^{\alpha}u$ in (1.2). We have

$$\left\|\varphi D_n^{\alpha}u\right\|_{1/r_n}^2+\sum_{j=0}^{n}\left\|X_j\varphi D_n^{\alpha}u\right\|^2\leqslant C\left(\left|\left\langle L\varphi D_n^{\alpha}u, \varphi D_n^{\alpha}u\right\rangle\right|+\left\|\varphi D_n^{\alpha}u\right\|^2\right). \tag{3.4}$$

The scalar product in the right hand side leads to

$$2\sum_{j=2}^{n}\left\langle X_j\left[X_j,\varphi D_n^{\alpha}\right]u, \varphi D_n^{\alpha}u\right\rangle+\sum_{j=2}^{n}\left\langle\left[X_j,\left[X_j,\varphi D_n^{\alpha}\right]\right]u, \varphi D_n^{\alpha}u\right\rangle+\left\langle\varphi D_n^{\alpha}Lu, \varphi D_n^{\alpha}u\right\rangle.$$

The last term has a trivial estimate since Lu is analytic. For the sake of simplicity we assume that it is zero. In the case $j=n$ we have

$$\begin{aligned}
&\left\langle X_j\left[X_j,\varphi D_n^{\alpha}\right]u, \varphi D_n^{\alpha}u\right\rangle+\left\langle\left[X_j,\left[X_j,\varphi D_n^{\alpha}\right]\right]u, \varphi D_n^{\alpha}u\right\rangle \\
&\quad\leqslant C\left\|t^{r_n-1}\varphi^{(1)}D_n^{\alpha}u\right\|^2+\frac{1}{C}\left\|X_n\varphi D_n^{\alpha}u\right\|^2+\left\langle t^{2(r_n-1)}\varphi^{(2)}D_n^{\alpha}u, \varphi D_n^{\alpha}u\right\rangle.
\end{aligned}$$

Thus

$$\begin{aligned}\|t^{r_n-1}\varphi^{(1)}D_n^\alpha u\|^2 &= \langle t^{r_n-1}\varphi^{(1)}D_n^\alpha u, t^{r_n-1}\varphi^{(1)}D_n^\alpha u\rangle \\ &\leqslant C_1\|X_n\varphi^{(1)}D_n^{\alpha-1}u\|^2 + \frac{1}{C_1}\|t^{r_n-1}\varphi^{(2)}D_n^{\alpha-1}u\|^2 \leqslant \cdots \\ &\leqslant C\sum_{k=1}^{s}\|X_n\varphi^{(j)}D_n^{\alpha-j}u\|^2 + \frac{1}{C}\|t^{r_n-1}\varphi^{(s+1)}D_n^{\alpha-s}u\|^2. \end{aligned} \tag{3.5}$$

Choosing $s+1=\alpha$ we have

$$\|t^{r_n-1}\varphi^{(s+1)}D_n^{\alpha-s}u\| = \|t^{r_n-1}\varphi^{(\alpha)}D_n u\| \leqslant |\varphi^{(\alpha)}|\|X_n u\|.$$

We obtain analytic growth choosing $\varphi=\varphi_m$ with $m\sim\alpha$. We further observe that the terms in the sum have the same form as $\|X_n\varphi D_n^\alpha u\|^2$ where one or more x_n-derivatives have been shifted from u to φ. In addition, we have

$$\begin{aligned}\langle t^{2(r_n-1)}\varphi^{(2)}D_n^\alpha u, \varphi D_n^\alpha u\rangle &\leqslant 2\|mX_n\varphi D_n^{\alpha-1}u\|^2 + \left\|\frac{1}{m}X_n\varphi^{(2)}D_n^{\alpha-1}u\right\|^2 + \|X_n\varphi^{(1)}D_n^{\alpha-1}u\|^2 + \|t^{r_n-1}\varphi^{(2)}D_n^{\alpha-1}u\|^2 \\ &\quad + 2\left\|\frac{1}{m}t^{r_n-1}\varphi^{(3)}D_n^{\alpha-1}u\right\|^2 + \|mt^{r_n-1}\varphi^{(1)}D_n^{\alpha-1}u\|^2.\end{aligned}$$

The role of the weight m—a derivative on φ—allows us to estimate the last three terms as well the first three as in (3.5) i.e. like terms of the type $\|X_n\varphi^{(l)}D_n^{\alpha-l}u\|^2$ for which we use estimate (3.4) with u replaced by $\varphi^{(j)}D_n^{\alpha-j}$, $m^k\varphi^{(j)}D_n^{\alpha-j-k}$ or $m^{-k}\varphi^{(j+k)}D_n^{\alpha-j}$. Therefore we can conclude that these terms will give an analytic growth.

The case $1\leqslant j\leqslant n-1$. We have

$$\begin{aligned}|\langle X_j[X_j,\varphi D_n^\alpha]u, \varphi D_n^\alpha u\rangle| &= |\langle t^{r_j-1}\varphi^{(1)}D_n^\alpha u, X_j\varphi D_n^\alpha u\rangle| \\ &\leqslant C_1\|t^{r_j-1}\varphi^{(1)}D_n^\alpha u\|^2 + \frac{1}{C_1}\|X_j\varphi D_n^\alpha u\|^2\end{aligned}$$

and

$$\begin{aligned}|\langle [X_j,[X_j,\varphi D_n^\alpha]]u, \varphi D_n^\alpha u\rangle| &= |\langle t^{2(r_j-1)}\varphi^{(2)}D_n^\alpha u, \varphi D_n^\alpha u\rangle| \\ &\leqslant \left\|\frac{1}{m}t^{r_j-1}\varphi^{(2)}D_n^\alpha u\right\|^2 + \|mt^{r_j-1}\varphi D_n^\alpha u\|^2.\end{aligned}$$

The last term in the first inequality will be absorbed on the right hand side of (3.4), if C_1^{-1} is chosen small enough. Since the first term does not have sufficient powers of t to take maximal advantage of the a priori estimate (1.2), we will use the sub-ellipticity. We will analyze in detail the first term in the first inequality. The same strategy can be used to analyze the terms on the right hand side of the second inequality. We work with the elliptic pseudo-differential operator Λ, whose symbol is $(1+|\xi_n|^2)^{1/2}$. We write

$$\|t^{r_j-1}\varphi^{(1)}D_n^\alpha u\|^2 = \|t^{r_j-1}\varphi^{(1)}D_n^{\alpha-\frac{1}{r_n}}u\|_{1/r_n}^2 + \|t^{r_j-1}[\varphi^{(1)},\Lambda^{\frac{1}{r_n}}]D_n^{\alpha-\frac{1}{r_n}}u\|^2$$

where

$$[\varphi^{(1)},\Lambda^{\frac{1}{r_n}}] = \sum_{k\geqslant 1}\frac{1}{k!}D_n^k(\varphi^{(1)})\partial_{\xi_n}^k(\Lambda^{\frac{1}{r_n}}).$$

Remark that the bracket has a whole asymptotic expansion in decreasing powers of Λ and increasing number of derivatives on φ. The second term will produce terms of the form $(k!)^{-1}\|t^{r_j-1}\varphi^{(k+1)}D_n^{\alpha-k-\frac{1}{r_n}}u\|_{1/r_n}^2$ which can be handled as the first term. To estimate the first term we use the sub-elliptic estimate (1.2) replacing u with $t^{r_j-1}\varphi^{(1)}D_n^{\alpha-\frac{1}{r_n}}u$. We have

$$\begin{aligned}&\|t^{r_j-1}\varphi^{(1)}D_n^{\alpha-\frac{1}{r_n}}u\|_{1/r_n}^2 + \sum_{l\geqslant 0}\|X_l t^{r_j-1}\varphi^{(1)}D_n^{\alpha-\frac{1}{r_n}}u\|^2 \\ &\quad\leqslant 2\sum_{l\geqslant 0}|\langle X_l[X_l,t^{r_j-1}\varphi^{(1)}]D_n^{\alpha-\frac{1}{r_n}}u, t^{r_j-1}\varphi^{(1)}D_n^{\alpha-\frac{1}{r_n}}u\rangle| + \|t^{r_j-1}\varphi^{(1)}D_n^{\alpha-\frac{1}{r_n}}u\|^2\end{aligned}$$

$$
\begin{aligned}
&+\sum_{l\geqslant 0}\left|\left\langle\left[X_l,\left[X_l,t^{r_j-1}\varphi^{(1)}\right]\right]D_n^{\alpha-\frac{1}{r_n}}u,t^{r_j-1}\varphi^{(1)}D_n^{\alpha-\frac{1}{r_n}}u\right\rangle\right|+\left\|t^{r_j-1}\varphi^{(1)}D_n^{\alpha-\frac{1}{r_n}}Lu\right\|^2\\
&\leqslant 2(r_j-1)\left|\left\langle t^{r_j-2}\varphi^{(1)}D_n^{\alpha-\frac{1}{r_n}},X_0t^{r_j-1}\varphi^{(1)}D_n^{\alpha-\frac{1}{r_n}}\right\rangle\right|+\left\|t^{r_j-1}\varphi^{(1)}D_n^{\alpha-\frac{1}{r_n}}u\right\|^2\\
&+(r_j-1)(r_j-2)\left|\left\langle t^{r_j-3}\varphi^{(1)}D_n^{\alpha-\frac{1}{r_n}},X_0t^{r_j-1}\varphi^{(1)}D_n^{\alpha-\frac{1}{r_n}}\right\rangle\right|\\
&+2\sum_{l\geqslant 1}2\left|\left\langle t^{r_j+r_l-2}\varphi^{(2)}D_n^{\alpha-\frac{1}{r_n}},X_lt^{r_j-1}\varphi^{(1)}D_n^{\alpha-\frac{1}{r_n}}\right\rangle\right|+\left\|t^{r_j-1}\varphi^{(1)}D_n^{\alpha-\frac{1}{r_n}}Lu\right\|^2\\
&+\sum_{l\geqslant 1}\left|\left\langle t^{r_j-1+2(r_l-1)}\varphi^{(3)}D_n^{\alpha-\frac{1}{r_n}},t^{r_j-1}\varphi^{(1)}D_n^{\alpha-\frac{1}{r_n}}\right\rangle\right|\\
&\leqslant 2C(r_j-1)\left\|t^{r_j-2}\varphi^{(1)}D_n^{\alpha-\frac{1}{r_n}}\right\|^2+(r_j-1)(r_j-2)\left\|t^{r_j-3}\varphi^{(1)}D_n^{\alpha-\frac{1}{r_n}}\right\|^2\\
&+2C\sum_{l\geqslant 1}\left\|t^{r_l+r_j-2}\varphi^{(2)}D_n^{\alpha-\frac{1}{r_n}}\right\|^2+\frac{1}{m}\left\|t^{r_l+r_j-2}\varphi^{(3)}D_n^{\alpha-\frac{1}{r_n}}\right\|^2+m\left\|t^{r_l+r_j-2}\varphi^{(1)}D_n^{\alpha-\frac{1}{r_n}}\right\|^2
\end{aligned}
$$

modulo terms which can be absorbed on the left or which give analytic growth. The only way to handle the first term on the right is to capitalize on sub-ellipticity again. We have

$$\left\|t^{r_j-2}\varphi^{(1)}D_n^{\alpha-\frac{1}{r_n}}u\right\|^2\sim\left\|t^{r_j-2}\varphi^{(1)}D_n^{\alpha-\frac{2}{r_n}}u\right\|^2_{1/r_n}$$

modulo terms involving the bracket of Λ^{1/r_n} with $\varphi^{(1)}$. Iterating the above strategy at the k-th step we obtain a term of the form $\|t^{r_j-(k+1)}\varphi^{(1)}D_n^{\alpha-\frac{k+1}{r_n}}u\|^2_{1/r_n}$. When $k=r_j-1$ we have $\|\varphi^{(1)}D_n^{\alpha-\frac{r_j}{r_n}}u\|^2_{1/r_n}$. Iterating this cycle s times we obtain, if we focus our analysis only on the field X_j, j fixed, a term of the form

$$C^s\left\|\varphi^{(s)}D_n^{\alpha-s\frac{r_j}{r_n}}u\right\|^2_{1/r_n}. \tag{3.6}$$

Using up all x_n-derivatives we estimate the left hand side of (3.4) with $C\tilde{C}^\alpha m^{\alpha(r_n/r_j)}$, choosing $\varphi=\varphi_m$ with $m\sim\alpha$ we have a growth corresponding to G^{r_n/r_j}. More in general we obtain terms of the type $\|\varphi^{(\beta)}D_n^{\alpha-\sum\beta_l(r_{q_l}/r_n)}u\|^2_{1/r_n}$, $\beta=\sum\beta_l$ and using all x_n-derivatives we can estimate the left hand side of (3.4) with a term of the type $C^{\alpha+1}m^{\alpha(r_n/r_k)}$, $r_k=\inf_l\{r_{q_l}\}$, that is we have $\frac{r_n}{r_j}$-Gevrey growth. We can conclude that these terms give a growth corresponding to G^s where $s=\sup_j\{\frac{r_n}{r_j}\}=\frac{r_n}{r_1}$.

On the other side we have to estimate terms of the form $\|t^{r_j+r_l-2}\varphi^{(2)}D_n^{\alpha-\frac{1}{r_n}}u\|^2$ and $m^{-1}\|t^{r_j+r_l-2}\varphi^{(3)}D_n^{\alpha-\frac{1}{r_n}}u\|^2$. If $r_j+r_l-2\geqslant r_n-1$ we take maximal advantage of the a priori estimate reducing α by one and obtaining terms of the type $\|X_nt^{r_j+r_l-r_n-1}\varphi^{(2)}D_n^{\alpha-1-\frac{1}{r_n}}u\|^2$ and $m^{-1}\|X_nt^{r_j+r_l-r_n-1}\varphi^{(3)}D_n^{\alpha-1-\frac{1}{r_n}}u\|^2$ on which we use the estimate (1.2) replacing u by $t^{r_j+r_l-r_n-1}\varphi^{(2)}D_n^{\alpha-1-\frac{1}{r_n}}u$ and by $t^{r_j+r_l-r_n-1}\varphi^{(3)}D_n^{\alpha-1-\frac{1}{r_n}}u$ restarting the cycle. If $r_j+r_l-2<r_n-1$ we use the sub-ellipticity, as we did above. Iterating these processes many times, removing powers of t with D_n and taking profit from the sub-ellipticity, we obtain terms of the form

$$m^{-k}\left\|t^{\sum c_j(r_{l_j-1})-c_n(r_n-1)-p}\varphi^{(c+k)}D_n^{\alpha-c_n-\frac{p}{r_n}-\frac{c}{r_n}+\frac{c_n}{r_n}}u\right\|^2$$

where $c=\sum c_j$. We remark that the quantity c_n/r_n in the exponent of D_n is due to the fact that when we take maximal advantage of the a priori estimate (1.2) reducing by one the number of x_n-derivatives we do not take profit from the sub-ellipticity, hence we must add it. We remark that in the particular case in which $c_n=0$, $c=c_j=s$ and $p=s(r_j-1)-1$ we have (3.6).

Iterating the procedure until all the x_n-derivatives are used up, we come to estimate the left hand side of (3.4) with terms of the form

$$C^\alpha m^{-k}\left|\varphi^{(c+k)}\right|\left\|t^{\sum c_j(r_{l_j-1})-c_n(r_n-1)-p}u\right\|^2$$

where $0<\sum c_j(r_{l_j-1})-c_n(r_n-1)-p\leqslant r_n-1$ and $\alpha-c_n-\frac{p}{r_n}-\frac{c-c_n}{r_n}=0$.

We obtained that

$$\left\|\varphi D_n^\alpha u\right\|^2_{1/r_n}+\sum_{j=0}^n\left\|X_j\varphi D_n^\alpha u\right\|^2\leqslant C_1^{\alpha+1}m^c+C_2^{\alpha+1}m^{\alpha\frac{r_n}{r_1}}$$

where c denotes the maximum constant. Since $0 < \sum c_j(r_{l_j-1}) - c_n(r_n - 1) - p < r_n - 1$ and $p \geqslant 1$ we have that $c < \alpha \frac{r_n}{r_1}$ and we can conclude that

$$\left\|\varphi D_n^\alpha u\right\|_{1/r_n}^2 + \sum_{j=0}^n \left\|X_j \varphi D_n^\alpha u\right\|^2 \leqslant C^{\alpha+1} m^{\alpha \frac{r_n}{r_1}}.$$

Choosing φ with $m \sim \alpha$ we have a growth corresponding to G^{r_n/r_1} in the direction x_n.

3.3. The direction D_k, $2 \leqslant k \leqslant n-1$

Let $\varphi(x)$ be a cut-off function of Ehrenpreis–Hörmander type described above. With k fixed, $2 \leqslant k \leqslant n-1$, we replace u by $\varphi(x) D_k^{2\alpha} u$ in (1.2). We write

$$\left\|\varphi D_k^{2\alpha} u\right\|_{1/r_n}^2 + \sum_{j=0}^n \left\|X_j \varphi D_k^{2\alpha} u\right\|^2 \leqslant C\left(\left|\left\langle L\varphi D_k^{2\alpha} u, \varphi D_k^{2\alpha} u\right\rangle\right| + \left\|\varphi D_k^{2\alpha} u\right\|^2\right). \tag{3.7}$$

Without loss of generality we may consider an even number of free derivatives. We stress that if u is a solution of the problem $Lu = 0$ then $D_k u$ is again solution of the same problem, since D_k and L commute.

The scalar product in the right hand side leads to

$$2\sum_{j=1}^{k-1} \left\langle X_j\left[X_j, \varphi D_k^{2\alpha}\right]u, \varphi D_k^{2\alpha} u\right\rangle + 2\sum_{j=k}^{n} \left\langle X_j\left[X_j, \varphi D_k^{2\alpha}\right]u, \varphi D_k^{2\alpha} u\right\rangle + \sum_{j=1}^{k-1} \left\langle \left[X_j, \left[X_j, \varphi D_k^{2\alpha}\right]\right]u, \varphi D_k^{2\alpha} u\right\rangle$$
$$+ \sum_{j=k}^{n} \left\langle \left[X_j, \left[X_j, \varphi D_k^{2\alpha}\right]\right]u, \varphi D_k^{2\alpha} u\right\rangle + \left\langle \varphi D_k^{2\alpha} Lu, \varphi D_k^{2\alpha} u\right\rangle.$$

The last term has a trivial estimate since Lu is analytic. We assume that it is zero.

For the terms with $j \geqslant k$, that is when $r_j \geqslant r_k$, we have

$$\left\langle X_j\left[X_j, \varphi D_k^{2\alpha}\right]u, \varphi D_k^{2\alpha} u\right\rangle \leqslant C\left\|t^{r_j+1}\varphi^{(1)} D_k^{2\alpha} u\right\|^2 + \frac{1}{C}\left\|X_j \varphi D_k^{2\alpha}\right\|^2.$$

While the last term is absorbed on the right hand side of (3.7), for the first we have

$$\left\|t^{r_j+1}\varphi^{(1)} D_k^{2\alpha} u\right\|^2 \leqslant \left\|\frac{1}{2m} X_k \varphi^{(1)} D_k^{2\alpha} u\right\|^2 + \left\|4m X_k \varphi^{(1)} D_k^{2\alpha-2} u\right\|^2 + \left\|\frac{1}{2m} X_k \varphi^{(3)} D_k^{2\alpha-2} u\right\|^2 + \left\|t^{r_j+1}\varphi^{(3)} D_k^{2\alpha-2} u\right\|^2.$$

With s steps we get

$$\left\|t^{r_j+1}\varphi^{(1)} D_k^{2\alpha} u\right\|^2 \leqslant C\sum_{l=1}^{s} \left\|(2m)^{-1} X_k \varphi^{(2l+1)} D_k^{2\alpha-2l} u\right\|^2 + \frac{1}{C}\sum_{l=1}^{s} \left\|2m X_k \varphi^{(2l+1)} D_k^{2\alpha-2(l+1)} u\right\|^2$$
$$+ \left\|t^{r_j+1}\varphi^{(2s+1)} D_k^{2\alpha-2s} u\right\|^2.$$

We observe that the terms in the sums have the same form as $\|X_k \varphi D_k^{2\alpha} u\|^2$. We use the a priori estimate (1.2) where u is replaced by $(2m)^{-1}\varphi^{(2s+1)} D_k^{2\alpha-2s} u$ and by $2m\varphi^{(2s+1)} D_k^{2\alpha-2(s+1)} u$ to restart the process. For $2s+1 = 2\alpha-1$ the last term can be estimated by

$$\left\|t^{r_j+1}\varphi^{(2s+1)} D_k^{2\alpha-2s} u\right\|^2 \leqslant \frac{1}{4m^2}\left\|X_k \varphi^{(2\alpha-1)} D_k^2 u\right\|^2 + C\left|\varphi^{(2\alpha)}\right|^2 + C_1(2m)^2\left|\varphi^{(2\alpha-1)}\right|^2.$$

Choosing φ with $m \sim \alpha$ the last two terms give analytic growth.

On the other hand we have

$$\begin{aligned}
\left|\left\langle \left[X_j, \left[X_j, \varphi D_k^{2\alpha}\right]\right]u, \varphi D_k^{2\alpha} u\right\rangle\right| &= \left|\left\langle t^{2(r_j-1)}\varphi^{(2)} D_k^{2\alpha} u, \varphi D_k^{2\alpha} u\right\rangle\right| \\
&\leqslant \left|\left\langle t^{2(r_j-1)} D_k^2 \varphi^{(2)} D_k^{2\alpha} u, \varphi D_k^{2\alpha} u\right\rangle\right| + \left|\left\langle t^{2(r_j-1)}\varphi^{(2)} D_k^{2\alpha} u, \varphi^{(2)} D_k^{2\alpha} u\right\rangle\right| \\
&\leqslant C\left\|\frac{1}{4m^2} X_k \varphi^{(2)} D_k^{2\alpha} u\right\|^2 + \frac{1}{C}\left\|(2m)^2 X_k \varphi D_k^{2\alpha-2} u\right\|^2 + \left\|X_k \varphi^2 D_k^{2\alpha-2}\right\|^2 \\
&\quad + \left|\left\langle t^{2(r_j-1)}\varphi^{(4)} D_k^{2\alpha-2} u, \varphi^{(2)} D_k^{2\alpha-2} u\right\rangle\right| \leqslant \cdots
\end{aligned}$$

$$\leqslant C\sum_{f=0}^{s}\left\|\frac{1}{4m^2}X_k\varphi^{(2(f+1))}D_k^{2\alpha-2f}u\right\|^2+\frac{1}{C}\sum_{f=0}^{s}\left\|(2m)^2X_k\varphi^{(2f)}D_k^{2\alpha-2(f+1)}u\right\|^2$$
$$+\left|\left\langle t^{2(r_j-1)}\varphi^{(2(s+1))}D_k^{2\alpha-2s}u,\varphi^{(2(s+1))}D_k^{2\alpha-2(s+1)}u\right\rangle\right|.$$

We restart the process using the a priori estimate (1.2), replacing u by $4m^2\varphi^{(2(f+1))}D_k^{2\alpha-2f}u$ and by $(2m)^{-2}\varphi^{(2f)}\times D_k^{2\alpha-2(f+1)}u$. We remark that at the step $2(s+1)=2\alpha$ we can estimate the last term in the right hand side with $|\varphi^{(2\alpha)}|^2(\|u\|_2^2+\|u\|^2)$ which gives analytic growth.

The case $j\leqslant k-1$. We have

$$\left|\left\langle X_j\left[X_j,\varphi D_k^{2\alpha}\right]u,\varphi D_k^{2\alpha}u\right\rangle\right|\leqslant C\left\|t^{r_j-1}\varphi^{(1)}D_k^{2\alpha}u\right\|^2+\frac{1}{C}\left\|X_j\varphi D_k^{2\alpha}u\right\|^2.$$

The last term is absorbed on the right hand side of (3.7). Since the first term does not have sufficient powers of t to take maximal advantage of the a priori estimate we write

$$\left\|t^{r_j-1}\varphi^{(1)}D_k^{2\alpha}u\right\|^2=\left\langle t^{2(r_j-1)}\varphi^{(1)}D_k^{2\alpha}u,\varphi^{(1)}D_k^{2\alpha}u\right\rangle.$$

Let $\Lambda=(1+|\xi_k|^2)^{1/2}$ and let β_j be a positive parameter which will be chosen later. We write

$$\begin{aligned}\left\|t^{r_j-1}\varphi^{(1)}D_k^{2\alpha}u\right\|^2&\leqslant\left|\left\langle t^{2(r_j-1)}\Lambda^{2\beta_j}\varphi^{(1)}D_k^{2\alpha-\beta_j}u,\varphi^{(1)}D_k^{2\alpha-\beta_j}u\right\rangle\right|+\left|\left\langle t^{2(r_j-1)}\Lambda^{2\beta_j}\varphi^{(1)}D_k^{2\alpha-\beta_j}u,\left[\varphi^{(1)},\Lambda^{\beta_j}\right]D_k^{2\alpha}u\right\rangle\right|\\&\quad+\left|\left\langle t^{2(r_j-1)}\left[\varphi^{(1)},\Lambda^{-\beta_j}\right]D_k^{2\alpha}u,\Lambda^{2\beta_j}\varphi^{(1)}D_k^{2\alpha-\beta_j}u\right\rangle\right|\\&\quad+\left|\left\langle t^{2(r_j-1)}\Lambda^{\beta_j}\left[\varphi^{(1)},\Lambda^{-\beta_j}\right]D_k^{2\alpha}u,\Lambda^{-\beta_j}\left[\varphi^{(1)},\Lambda^{-\beta_j}\right]D_k^{2\alpha}u\right\rangle\right|\\&=H_1+H_2+H_3+H_4.\end{aligned}$$

Set

$$\beta_j=\frac{r_n(r_j-1)+r_k-r_j}{r_n(r_k-1)}.$$

Because of the inequality

$$t^{2(r_j-1)}\Lambda^{2\beta_j}\leqslant t^{2(r_k-1)}\Lambda^2+\Lambda^{\frac{2}{r_n}},\tag{3.8}$$

we obtain

$$\begin{aligned}H_1&\leqslant\left|\left\langle\left(t^{2(r_k-1)}\Lambda^2+\Lambda^{\frac{2}{r_n}}\right)\varphi^{(1)}D_k^{2\alpha-\beta_j}u,\varphi^{(1)}D_k^{2\alpha-\beta_j}u\right\rangle\right|\\&\leqslant\left\|X_k\varphi^{(1)}D_k^{2\alpha-\beta_j}u\right\|^2+\left\|\varphi^{(1)}D_k^{2\alpha-\beta_j}u\right\|_{1/r_n}^2+\left\|t^{r_k-1}\varphi^{(1)}D_k^{2\alpha-\beta_j}u\right\|^2.\end{aligned}\tag{3.9}$$

On the first two terms we restart the process using the sub-elliptic estimate (1.2) with u replaced by $\varphi^{(1)}D_k^{2\alpha-\beta_j}u$. Iterating, if we focus our analysis on the field X_j, j fixed, at the step s we obtain terms of the form

$$\left\|X_k\varphi^{(s)}D_k^{2\alpha-s\beta_j}u\right\|^2+\left\|\varphi^{(s)}D_k^{2\alpha-s\beta_j}u\right\|_{1/r_n}^2.$$

Using up all x_k-derivatives we estimate the left hand side of (3.7) with

$$\left|\varphi^{(s+1)}\right|^2\left(\|X_ku\|^2+\|u\|_{1/r_n}^2\right)\leqslant C_1C^{2\alpha}(2m)^{s+1}$$

where $s+1=2\alpha/\beta_j$. Choosing $\varphi=\varphi_m$ with $m\sim\alpha$ we have a growth corresponding to G^{1/β_j}. More in general we obtain terms of the type

$$\left\|X_k\varphi^{(s)}D_k^{2\alpha-\sum s_j\beta_j}u\right\|^2+\left\|\varphi^{(s)}D_k^{2\alpha-\sum s_j\beta_j}u\right\|_{1/r_n}^2$$

where $s=\sum s_j$. Using all x_n-derivatives we can estimate the right hand side of (3.7) with $C^{2\alpha+1}(2m)^{2\alpha/\beta}$ where $\beta=\inf_j\{\beta_j\}$. These terms give the growth G^s where

$$s=\sup_{1\leqslant j\leqslant k-1}\left\{\frac{1}{\beta_j}\right\}=\frac{r_n(r_k-1)}{r_n(r_1-1)+r_k-r_1}.$$

We turn our attention to the last term in (3.9). We have

$$\begin{aligned}
\left|\left\langle t^{2(r_j-1)}\varphi^{(1)}D_k^{2\alpha-\beta_j}u,\varphi^{(1)}D_k^{2\alpha-\beta_j}u\right\rangle\right| &\leqslant \frac{2}{C}\left\|2mX_k\varphi^{(1)}D_k^{2\alpha-2-\beta_j}u\right\|^2+C\left\|\frac{1}{2m}X_k\varphi^{(1)}D_k^{2\alpha-\beta_j}u\right\|^2\\
&\quad+\frac{1}{2m}\left\|X_k\varphi^{(3)}D_k^{2\alpha-2-\beta_j}u\right\|^2+\left\|t^{r_k-1}\varphi^{(3)}D_k^{2\alpha-\beta_j-2}u\right\|^2\leqslant\cdots\\
&\leqslant\frac{2}{C}\sum_{f=1}^{s}\left\|2mX_k\varphi^{(2f-1)}D_k^{2\alpha-\beta_j-2f}u\right\|^2+C\sum_{f=1}^{s}\left\|\frac{1}{2m}X_k\varphi^{(2f+1)}D_k^{2\alpha-\beta_j-2f}u\right\|^2\\
&\quad+\left\|t^{r_k-1}\varphi^{(2s+1)}D_k^{2\alpha-\beta_j-2s}u\right\|^2.
\end{aligned}$$

When $1\leqslant 2\alpha-\beta-2s<2$ the last term can be estimated by $|\varphi^{(2s)+1}|\|u\|_1^2$, where $2s+1\leqslant 2\alpha-\beta_j<2\alpha/\beta_j$.

The term H_2:

$$\left|\left\langle t^{2(r_j-1)}\Lambda^{2\beta_j}\varphi^{(1)}D_k^{2\alpha-\beta_j}u,\left[\varphi^{(1)},\Lambda^{-\beta_j}\right]D_k^{2\alpha-\beta}u\right\rangle\right|$$

where

$$\left[\varphi^{(1)},\Lambda^{-\beta_j}\right]=\sum_{p\geqslant 1}(-1)^p\binom{\beta_j}{p}\varphi^{(p+1)}\Lambda^{-p-\beta_j}=\sum C_{\beta_j,p}\varphi^{(p+1)}\Lambda^{-p-\beta_j}.$$

We examine a generic term of the sum and obtain

$$\begin{aligned}
\left\langle t^{2(r_j-1)}\Lambda^{2\beta}\varphi^{(1)}D_k^{2\alpha-\beta}u,\varphi^{(p+1)}D_k^{2\alpha-\beta-p}u\right\rangle &\leqslant\left\langle\left(X_k^2+t^{2(r_k-1)}+\Lambda^{\frac{2}{r_n}}\right)\varphi^{(1)}D_k^{2\alpha-\beta}u,\varphi^{(p+1)}D_k^{2\alpha-\beta-p}u\right\rangle\\
&\leqslant 2\left(\left\|X_k\varphi^{(1)}D_k^{2\alpha-\beta_j}u\right\|^2+\left\|\varphi^{(1)}D_k^{2\alpha-\beta_j}u\right\|_{1/r_n}^2\right)\\
&\quad+\frac{1}{2}\left(\left\|X_k\varphi^{(p+1)}D_k^{2\alpha-\beta_j-p}u\right\|^2+\left\|\varphi^{(p+1)}D_k^{2\alpha-\beta_j-p}u\right\|_{1/r_n}^2\right)\\
&\quad+2\left\|t^{r_k-1}\varphi^{(1)}D_k^{2\alpha-\beta_j}u\right\|^2+\frac{1}{2}\left\|t^{r_k-1}\varphi^{(p+1)}D_k^{2\alpha-\beta_j-p}u\right\|^2.
\end{aligned}$$

The last two terms can be handled as the last term in (3.9) while the first is taken care of using the sub-elliptic a priori inequality. The terms H_3 and H_4 can be treated like H_2.

Using the same strategy we have

$$\begin{aligned}
\left|\left\langle\left[X_j,\left[X_j,\varphi D_k^{2\alpha}\right]\right]u,\varphi D_k^{2\alpha}u\right\rangle\right| &=\left\langle t^{2(r_j-1)}\varphi^{(2)}D_k^{2\alpha}u,\varphi D_k^{2\alpha}u\right\rangle\\
&\leqslant\left|\left\langle t^{2(r_j-1)}\Lambda^{2\beta_j}\varphi^{(1)}D_k^{2\alpha-\beta_j}u,\left[\varphi^{(1)},\Lambda^{\beta_j},\right]D_k^{2\alpha}u\right\rangle\right|\\
&\quad+\left|\left\langle t^{2(r_j-1)}\left[\varphi^{(1)},\Lambda^{-\beta_j}\right]D_k^{2\alpha}u,\Lambda^{2\beta_j}\varphi^{(1)}D_k^{2\alpha-\beta_j}u\right\rangle\right|\\
&\quad+\left|\left\langle t^{2(r_j-1)}\Lambda^{\beta_j}\left[\varphi^{(1)},\Lambda^{-\beta_j}\right]D_k^{2\alpha}u,\Lambda^{-\beta_j}\left[\varphi^{(1)},\Lambda^{-\beta_j}\right]D_k^{2\alpha}u\right\rangle\right|\\
&=R_1+R_2+R_3+R_4.
\end{aligned}$$

We have

$$\begin{aligned}
R_1 &\leqslant\left(\frac{C}{2m}\right)^2\left(\left\|X_k\varphi^{(2)}D_k^{2\alpha-\beta_j}u\right\|^2+\left\|\varphi^{(2)}D_k^{2\alpha-\beta_j}u\right\|_{1/r_n}^2+\left\|t^{r_k-1}\varphi^{(2)}D_k^{2\alpha-\beta_j}u\right\|^2\right)\\
&\quad+\left(\frac{2m}{C}\right)^2\left(\left\|X_k\varphi D_k^{2\alpha-\beta_j}u\right\|^2+\left\|\varphi D_k^{2\alpha-\beta_j}u\right\|_{1/r_n}^2+\left\|t^{r_k-1}\varphi D_k^{2\alpha-\beta_j}u\right\|^2\right).
\end{aligned}$$

These terms can be studied with the same method used for H_1, (3.9). The terms R_2, R_3 and R_4 can be treaded like the terms H_2, H_3 and H_4.

Iterating this process, using (3.8) the sub-elliptic estimate, we may estimate the left hand side of (3.7) with terms of the form

$$(2m)^{-2p}\left(\left\|X_k\varphi^{(p+c+s)}D_k^{2\alpha-\sum s_j\beta_j-c}u\right\|^2+\left\|\varphi^{(p+c+s)}D_k^{2\alpha-\sum s_j\beta_j-c}u\right\|_{1/r_n}^2\right)$$

where $s=\sum s_j$. Iterating until all the x_k-derivatives are used up, that is until $2\alpha-\sum s_j\beta_j-c\sim 0$, since for $c\geqslant 1$ we have that $c+s\leqslant 2\alpha/\beta_k$, $\beta_k=\inf_j\{\beta_j\}$, we can conclude that

$$\left\|\varphi D_k^{2\alpha}u\right\|_{1/r_n}^2+\sum_{j=0}^{n}\left\|X_j\varphi D_k^{2\alpha}u\right\|^2\leqslant C^{2\alpha+1}(2m)^{2\alpha/\beta_k}$$

where C depends on u and β_j. Choosing $\varphi = \varphi_m$ with $m \sim \alpha$ we have a growth corresponding to G^{1/β_k} in the direction x_k where

$$\beta_k = \frac{r_n(r_1 - 1) + r_k - r_1}{r_n(r_k - 1)}.$$

We remark that $\beta_k^{-1} < r_n/r_1$ for all $k = 2, \ldots, n-1$.

3.4. Direction D_t

Once again our primary tool will be the sub-elliptic estimate (1.2). As in the case of direction x_j, we replace u by $\varphi(x)D_t^\alpha u$ in (1.2); φ is a cut-off function. We recall that φ does not depend on t, indeed every t-derivative landing on φ would leave a cut-off function supported in a region where t is bounded away from zero, where the operator is elliptic. We have

$$\left\|\varphi D_t^\alpha u\right\|_{1/r_n}^2 + \sum_{j=0}^{n} \left\|X_j \varphi D_t^\alpha u\right\|^2 \leqslant C\left(\left|\langle L\varphi D_t^\alpha u, \varphi D_1^\alpha u\rangle\right| + \left\|\varphi D_t^\alpha u\right\|^2\right). \tag{3.10}$$

We consider the scalar product in the right hand side of the above inequality. We must study terms of the type

$$\left|\left\langle \left[X_j, \varphi D_t^\alpha\right]u, X_j \varphi D_t^\alpha u\right\rangle\right|$$

where $j = 1, \ldots, n$. We have

$$\begin{aligned}
\left[X_j, \varphi D_t^\alpha\right]u &= \left[t^{r_j-1}D_j, \varphi D_t^\alpha\right] = t^{r_j-1}\varphi^{(1)}D_t^\alpha u - \varphi\left[D_t^\alpha, t^{r_j-1}\right]D_j u \\
&= t^{r_j-1}\varphi^{(1)}D_t^\alpha u - \varphi \sum_{l=1}^{r_j-1} \frac{\alpha!(r_j-1)!}{(i)^l l!(\alpha-l)!(r_j-1-l)!} t^{r_j-1-l} D_t^{\alpha-l} D_j u.
\end{aligned}$$

Without loss of generality we analyze one of the terms. A similar method can be used for the other terms.

Consider $\alpha(r_j-1)D_t t^{r_j-2}\varphi D_t^{\alpha-2}D_j u$, that is we have to estimate $\alpha(r_j-1)\|X_0 t^{r_j-2}\varphi D_t^{\alpha-2}D_j u\|$. Applying the sub-elliptic estimate with $u = \alpha(r_j-1)t^{r_j-2}\varphi D_t^{\alpha-2}D_j u$ and arguing as above, we study the first term coming from the commutator with X_j. We obtain the term $\alpha(\alpha-2)C_{r_j}t^{2r_j-4}\varphi D_t^{\alpha-3}D_j u$. We stress that another vector field does not produce additional difficulties. We have to estimate $C_{r_j}\alpha(\alpha-2)\|X_j t^{r_j-3}\varphi D_t^{\alpha-3}D_j u\|$. Hence after two steps we have

$$\left\|X_j \varphi D_t^\alpha u\right\| \to C_{r_j}\alpha(\alpha-2)\left\|X_j t^{r_j-3}\varphi D_t^{\alpha-3}D_j u\right\|.$$

Repeating the process p times, we have

$$\left\|X_j \varphi D_t^\alpha u\right\|^2 \to \cdots \to C_{r_j}\frac{\alpha!}{(\alpha-1)(\alpha-(p+1))!}\left\|X_j t^{r_j-(p+1)}\varphi D_t^{\alpha-(p+1)}D_j u\right\|^2.$$

In this way after $r_j - 1$ iterates we have to analyze a term of the form $C_{r_j}(\alpha!)/((\alpha-1)(\alpha-r_j)!)D_t^{\alpha-r_j}D_j u$. Arguing in the same way after l steps we have

$$\left\|X_j \varphi D_t^\alpha u\right\|^2 \to \cdots \to \frac{C_{r_j}\alpha!}{(\alpha-1)\cdots(\alpha-1-(l-1)r_j)(\alpha-r_j l)!}\left\|X_j \varphi D_t^{\alpha-lr_j}D_j^l u\right\|^2.$$

Iterating this cycle α/r_j times we use up all free derivatives in t and we are left with

$$\alpha!\left((\alpha/r_j)!\right)^{-1}\left\|X_j \varphi D_j^{\alpha/r_j}u\right\|.$$

Since in the direction x_j we have a growth as G^{β_j} where $\beta_j = (r_n(r_j-1))/(r_n(r_1-1)+r_j-r_1)$ we can estimate this term with $C^{|\alpha|+1}(\alpha!)^{1-\frac{1}{r_j}+\frac{\beta_j}{r_j}}$. Since $j \in \{1, \ldots, n\}$ we must take the $\sup_j\{1 - \frac{1}{r_j} + \frac{\beta_j}{r_j}\} = s_0$. We have obtained a growth corresponding to G^{s_0} in the direction t.

3.5. Sharpness

By Lemma 1 in [8] we know that there exists a real number z such that the ordinary differential equation

$$u'' = \sum_{j=2}^{n} t^{2(r_j-1)}u - zt^{r_1-1}u$$

has a non-trivial solution u defined on the whole real line, rapidly decreasing at infinity and such that $u(0) \neq 1$.

We define

$$v(t,x) = \int_0^{+\infty} e^{i\rho x_n + \sum_{j=2}^{n-1} i\rho^{r_j/r_n} x_j + \sqrt{z}\rho^{r_1/r_n}} e^{-\rho^{r_1/r_n}} u\big(\rho^{1/r_n} t\big)\, d\rho.$$

v is a solution to the problem $Lu = 0$ and we have

$$\big|\partial_{x_n}^{\alpha} v(0)\big| \sim C \int_0^{\infty} \rho^{\alpha} e^{-\rho^{\frac{r_1}{r_n}}}\, d\rho \sim C^{\alpha+1} \alpha!^{r_n/r_1}$$

and

$$\big|\partial_{x_1}^{\alpha} u(0)\big| \sim C \int_0^{\infty} \rho^{\alpha \frac{r_1}{r_n}} e^{-\rho^{\frac{r_1}{r_n}}}\, d\rho \sim C^{\alpha+1} \alpha!.$$

We can conclude

Proposition 3.1. *The Gevrey regularity r_n/r_1 is optimal. Moreover the operator L, in (1.1), is analytic in the direction x_1 and r_n/r_1 in the x_n-direction. The latter values are optimal.*

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