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Axiomatic/asymptotic evaluation of multilayered plate theories by using single and multi-points error criteria

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ABSTRACT

This paper deals with refined theories for multilayered composites plates. Layer-Wise (LW) and Equivalent Single Layer (ESL) theories are evaluated by means of axiomatic–asymptotic approach. Theories with forth order displacement fields in the thickness layer/plate direction *z* are implemented by referring to the Unified Formulation by Carrera. The effectiveness of each term of the made expansion is evaluated by comparing the related theories with a reference solution. As a result a reduced model is obtained which preserve the accuracy of the full-model (model that include the whole terms of the *z*-expansion) but it removes the not-significant terms in the same expansion (those terms that do no improve the results according to a given error criteria). Various single-point and multi-point error criteria have been analyzed and compared in order to establish such an effectiveness: error localized in an assigned point along *z*, error localized at each interface, error located at the *z*-value corresponding to the maximum value of the considered variables, etc. Applications are given in case of closed form solutions of orthotropic cross-ply, rectangular, simply supported plates loaded by bisinusoidal distribution of transverse pressure. Symmetrically and unsymmetrical laminated cases are considered along with sandwich plates. It is found the reduced model is strongly influenced by the used localized error and that in same case the reduced model which is obtained by of single point criteria can be very much improved by the use of multi-point criteria.

1. Introduction

The analysis of refined theories for anisotropic, multilayered composite structures is a topic of interest since decades. Classical theories based on the extension of developments originally made for homogenous one-layered beams (as Euler–Bernoulli [1–3]) or plates (Kirchhoof [4] and Reissner–Mindlin [5,6]) show severe limitation to analyze laminated composites structures. In particular these classical approaches are not able to describe so-called zig-zag field for the through-the-thickness distribution of displacement variables as well as interlaminar continuity of transverse both shear and normal stresses. These points were summarized in [7] as C_z^0-requirements, that is displacement and transverse stress field must be C^0 function along the *z*-thickness coordinate. Many theories are known that permit to overcome the limitation of classical theories, review papers on that topics are those by Ambartsumian [8,9], Librescu and Reddy [10], Kapania and Raciti [11], Noor and Burton [12,13], Librescu [14] and in particular the historical note by Carrera [15]. In general better description is obtained by referring to so-called Layer-Wise (LW) theories with respect to Equivalent Single Layer (ESL) ones. Each layer is considered as an independent plate in the case of LW theories while the number of the unknown variables remains independent by the number of the layers in the ESL approaches.

Among the many available techniques the present work refer to those based on Carrera Unified Formulation (CUF) which has been successfully introduced over the last decade. According to CUF it is possible to express the governing equations in terms of so-called fundamental nuclei whose form does not depend on either the expansion order, nor on the choices made for the base functions. More details on CUF can be found in [16–18]. ESL and LW approaches were successfully developed in the mainframe of CUF theory. In [19] ESL and LW models up to fourth *N*-order expansion in the thickness layer/plate direction were considered. The conclusion of that study was that LW approach could provide a realistic

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description of transverse stresses of laminate thick and thin plate while ESL approach accuracy depends on the laminate lay-out. Thermomechanical analysis of simply supported multilayered plates employing ESL and LW models was addressed in [20]. Applications to sandwich structures was given in [21]. Comparison of ESL and LW approaches can be found in [22] where the authors investigated the linearized buckling of laminated plates. Recent dynamic analysis have been provided in [23].

Due to its features CUF represents an interesting framework to compare and assess advanced theories. In particular it could be used to establish the accuracy of a given ESLM or LW theories with a given order of the expansion for the displacement variables, in case of displacement formulated theories. CUF as definition should be classified as axiomatic theories, that is the order of the expansion for the displacement variables is assumed ‘a priori’. It is well known that in contrast to axiomatic approach the asymptotic approach could be used. The latter expand the governing equations in terms of a perturbation parameter δ of the structures (e.g. the length-to-thickness ratio) by leading to class-of-problems related to set governing equations which contain the whole contribution with the same order of magnitude with respect to δ. Reviews and analysis on this approach with applications to plates and shells can be found in [24–30].

By using axiomatic approach it has been shown that the introduction of high order terms in a model offers a benefit in terms of improved structural response analysis, as a drawback higher computational cost is requested. The possibility to obtain accurate high order theories with less computational cost could be offered by evaluating the importance/effectiveness of each term of the expansion in the solution process. With that information a decision could be taken and the corresponding term could be retained (if relevant) or discarded (if not significant). By doing that the so-called *asymptotic/axiomatic* technique is obtained: the effectiveness of each displacement variable of a model is compared to a reference solution and the terms which do not influence the response are discarded. This technique was proposed in the finite element framework in [31–33]. The genetic like algorithms were used in [34] to evaluate the importance of each displacement variables for FE plate models. The results of these works were presented in diagram. That diagram was stated as ‘Best Plate Theories curves’; it gives the minimum number of displacement variables versus the accuracy on a given stress or displacement parameter.

Closed form solution related to axiomatic/asymptotic technique have been proposed in [35]. The authors analyzed isotropic, orthotropic and composite plates considering different parameter (i.e. a/h ratios, orthotropic ratios and ply sequence) and the best models

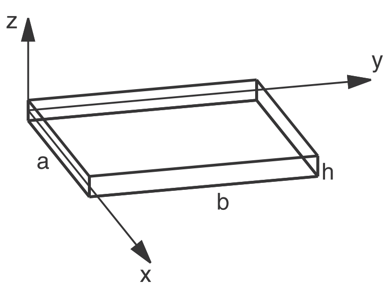


Fig. 1. Plate reference system.

Table 1
Symbols to indicate the status of a displacement variable.

Active term	Inactive term	Interface terms in LD4 (always active)
▲	△	■

Table 2
Example of representation of the reduced kinematics model with u_{z2} deactivated.

■	▲	▲	▲	■
■	▲	▲	▲	■
■	△	▲	▲	■

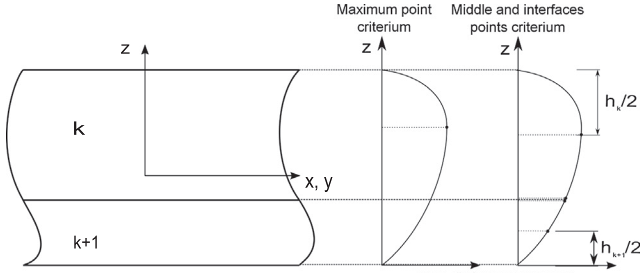


Fig. 2. Maximum point, interfaces and middle-layer points criterium.

Table 3
Comparison of ED4 and LD4 results with the exact solutions by Pagano [38]. Material data: $E_L/E_T = 25$, $(G_{LT}, G_{Lz})/E_T = 0.5$, $G_T T/E_T = 0.2$, $\nu_{LT} = \nu_{Lz}\nu_{TT} = 0.25$. Displacement $\bar{u}_z = u_z 100 E_T h^3/(p_0 a^4)$ evaluated at $z = 0$.

	3D	ED4	LD4
	$a/h = 4$		
Cylindrical bending cases			
$N_l = 3$	2.887	2.685	2.887
$N_l = 4$	4.181	3.830	4.180
	$a/h = 6$		
$N_l = 3$	1.635	1.514	1.635
$N_l = 4$	2.556	2.362	2.556
	$N_l = 3$		
	3D	ED4	LD4
Bisinusoidal loading cases, $b = 3a$			
$a/h = 4$	2.887	2.625	2.821
$a/h = 10$	0.919	0.867	0.919

Table 4
Composite square plate under bisinusoidal loadings. $\bar{u}_z = \frac{100\ u_z\ E_T\ h^3}{p_z\ a^4}$, $(\bar{\sigma}_{xz}, \bar{\sigma}_{yz}) = (\sigma_{xz}, \sigma_{yz})/(p_0\ a/h)$, $\bar{\sigma}_{xx} = \sigma_{xx}/(p_0\ (a/h)^2)$.

	$\bar{u}_z(z = h/2)$	$\bar{\sigma}_{xx}(z = h/2)$	$\bar{\sigma}_{xz}(z = 0)$	$\bar{\sigma}_{yz}(z = 0)$	$\bar{\sigma}_{zz}(z = h/2)$
	$a/h = 100$, 0°/90°/0°				
CLT	0.2836	0.5637	0.3280	0.0239	0.8448
FSDT	0.2852	0.5634	0.3279	0.0240	0.8465
ED4	0.2854	0.5639	0.4542	0.0449	0.0100
LD4	0.2854	0.5639	0.4054	0.0727	0.0100
	$a/h = 100$, 0°/90°/0°/90°				
CLT	0.3374	0.0239	0.1736	0.1447	0.9798
FSDT	0.3383	0.0239	0.1736	0.1447	0.980
ED4	0.3387	0.0240	0.2683	0.2235	0.0349
LD4	0.3387	0.0239	0.2848	0.2848	0.0100
	$a/h = 2$, 0°/90°/0°				
CLT	0.2828	0.5625	0.3277	0.0238	0.0169
FSDT	3.1598	0.3536	0.2255	0.1040	0.0456
ED4	5.3692	1.2074	0.2687	0.1632	0.5091
LD4	5.5094	1.4089	0.2541	0.2000	0.5025
	$a/h = 2$, 0°/90°/0°/90°				
CLT	0.3374	0.0239	0.1736	0.1447	0.0196
FSDT	2.6402	0.0239	0.1736	0.1447	0.0196
ED4	5.2633	0.1753	0.2325	0.2029	0.5105
LD4	5.3511	0.1786	0.2065	0.2650	0.5012

Table 5
Reduced LD4 model for symmetric laminated plate.

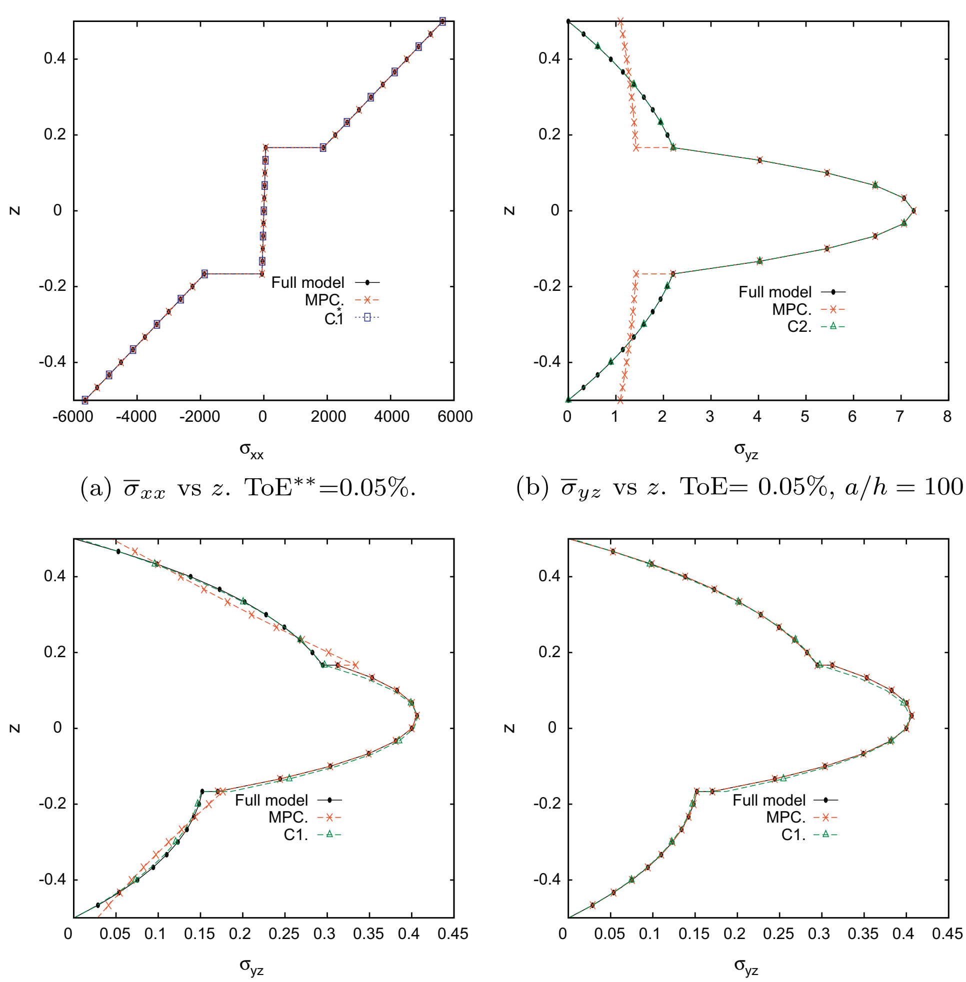
Tolerance on error: 0.05%

Case													
$a/h = 100$													
MPC $M_e : 20/39$	■	△	△	△	■	△	▲	△	■	▲	△	△	■
	■	△	△	△	■	△	▲	△	■	△	△	△	■
	■	▲	▲	▲	■	▲	△	△	■	▲	△	△	■
C1* $M_e : 15/39$	■	▲	△	△	■	△	△	△	■	▲	△	△	■
	■	△	△	△	■	△	▲	△	■	△	△	△	■
	■	△	△	△	■	△	△	△	■	△	△	△	■
C2 $M_e : 26/39$	■	▲	▲	△	■	△	△	△	■	▲	▲	△	■
	■	▲	▲	△	■	△	▲	△	■	▲	▲	△	■
	■	▲	▲	△	■	▲	△	△	■	▲	▲	△	■
$a/h = 2$													
MPC $M_e : 29/39$	■	▲	▲	▲	■	▲	△	△	■	▲	▲	▲	■
	■	▲	△	△	■	▲	▲	▲	■	▲	△	△	■
	■	▲	▲	▲	■	▲	△	△	■	▲	△	△	■
C2 $M_e : 25/39$	■	▲	▲	▲	■	▲	△	△	■	▲	▲	▲	■
	■	▲	▲	△	■	△	▲	△	■	▲	▲	△	■
	■	△	△	△	■	△	△	△	■	△	▲	△	■

Tolerance= 0.01%

Case													
$a/h = 100$													
MPC $M_e : 21/39$	■	▲	△	△	■	△	▲	△	■	▲	△	△	■
	■	△	△	△	■	△	▲	△	■	△	△	△	■
	■	▲	▲	▲	■	▲	△	△	■	▲	△	△	■
C1* $M_e : 15/39$	■	▲	△	△	■	△	△	△	■	▲	△	△	■
	■	△	△	△	■	△	▲	△	■	△	△	△	■
	■	△	△	△	■	△	△	△	■	△	△	△	■
C2 $M_e : 28/39$	■	▲	▲	△	■	△	△	△	■	▲	▲	△	■
	■	▲	▲	△	■	△	▲	△	■	▲	▲	△	■
	■	▲	▲	△	■	▲	△	▲	■	▲	▲	▲	■
$a/h = 2$													
MPC $M_e : 35/39$	■	▲	▲	▲	■	▲	▲	△	■	▲	▲	▲	■
	■	▲	▲	▲	■	▲	▲	▲	■	▲	▲	△	■
	■	▲	▲	▲	■	▲	△	▲	■	▲	▲	△	■
C2 $M_e : 29/39$	■	▲	▲	▲	■	▲	▲	△	■	▲	▲	▲	■
	■	▲	▲	△	■	△	▲	△	■	▲	▲	△	■
	■	△	▲	△	■	▲	△	△	■	▲	▲	△	■

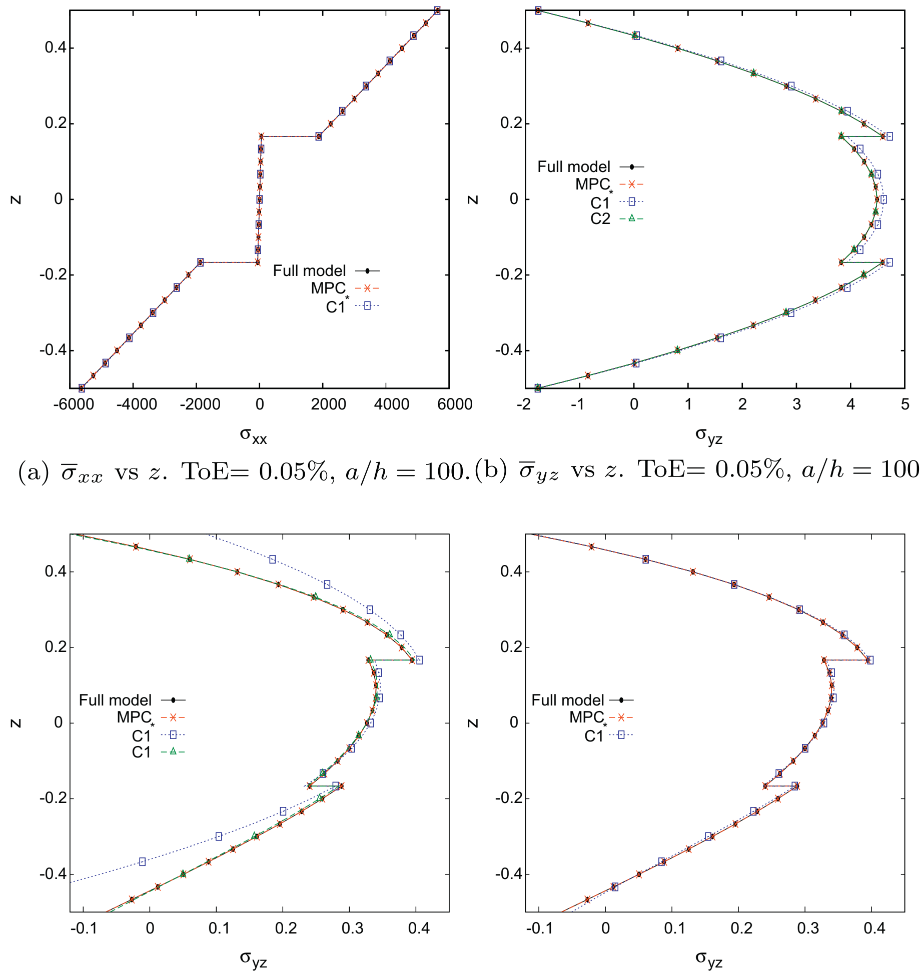
∗: In this case only the middle point of the middle layer has been considered.



(a) $\overline{\sigma}_{xx}$ vs z. ToE**=0.05%. (b) $\overline{\sigma}_{yz}$ vs z. ToE= 0.05%, $a/h = 100$.

(c) $\overline{\sigma}_{yz}$ vs z. ToE= 0.05%, $a/h = 2$. (d) $\overline{\sigma}_{yz}$ vs z. ToE= 0.01%, $a/h = 2$.

Fig. 3. $\bar{\sigma}_{yz}$ and vs z for LD4 model. Composite plate, ply sequence: 0°/90°/0°. *: only the middle point of the middle layer has been considered. **: ToE denotes the used tolerance for the adopted criteria.



(a) $\overline{\sigma}_{xx}$ vs z. ToE= 0.05%, $a/h = 100$. (b) $\overline{\sigma}_{yz}$ vs z. ToE= 0.05%, $a/h = 100$.

(c) $\overline{\sigma}_{yx}$ vs z. ToE= 0.05%, $a/h = 2$. (d) $\overline{\sigma}_{yx}$ vs z. ToE= 0.01%, $a/h = 2$.

Fig. 4. $\bar{\sigma}_{xx}$ and $\bar{\sigma}_{yz}$ and vs z for ED4 model. Composite plate, ply sequence: 0°/90°/0°, a/h = 100. ∗: only the middle point of the middle layer has been considered.

were derived from ESL approach. Navier closed-form solution were considered, and the analysis was limited to simply supported plates. In a more recent paper [36], LW and ESL reduced models had been analyzed and for isotopic, laminated and sandwich plates. The influence of geometric, lay-up and mechanical parameters was taken into account. The conclusion of that study showed that the reduced Layer-Wise models can save up to 50% of the degrees-of-freedom of the full models with the same accuracy on stress/displacement results.

In all the previous works the axiomatic/asymptotic method was applied by considering stress/displacement variables evaluated in fixed points over the plate domain (in-plane and through-the-thickness positions). In particular is has been found that although stress/dispalcement component values are always correct when evaluated at the given points, their distribution along the thickness may show a significant deviation from the reference solution. To overcome what above the present work explores and compares the use of different criteria to establish the accuracy of a given terms of the ESLM and LW theories for multilayered plates analysis. That is the errors localized at various points along the thickness are used to evaluate the effectiveness of displacement variables. The number and the position of the points depend on the case studied. In addition the influence of the tolerance on error is considered on the number of terms that have to be considered for a given case. As in [35] this work is based on closed-form solutions and only consider displacement formulations based on Principle of Virtual Displacement, PVD, applications. Unsymmetrical and symmetrical laminated plates and sandwich plates are analyzed: the influence of length-to thickness ratio, stacking-ply-sequence and face-to-core ratio are considered. This paper is organized as follows: a description of the adopted formulation is provided in Section 2; the asymptotic–axiomatic method is presented in Section 3 while the adopted error criteria are described in Section 4. Results and discussion are given in Section 5. Conclusions are outlined in Section 6.

2. Refined theories based on Carrera Unified Formulation

Plate geometry and notations are given in Fig. 1. According to CUF [7] the displacement of a plate structure can be described as:

$$\mathbf{u} = F_\tau \cdot \mathbf{u}_\tau \qquad \tau = 1, 2, \ldots, N+1 \tag{1}$$

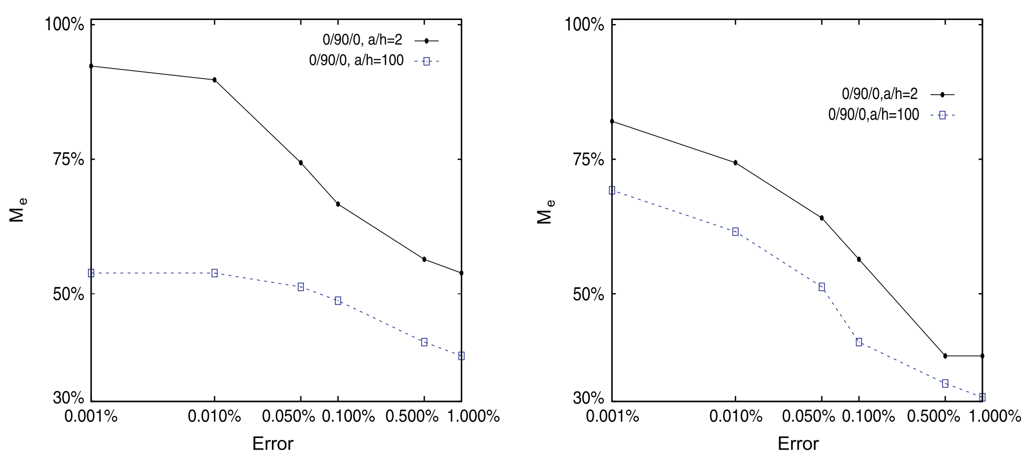
where $\mathbf{u}$ is the displacement vector and N is the expansion order. If Equivalent Single Layer approach is employed F_τ functions can be Mc-Laurin functions of z defined as $F_\tau = z^{\tau-1}$. In the following the ESL models are synthetically indicated as EDN, where N is the expansion order. An example of an ED4 displacement field is reported

$$\begin{aligned} u_x &= u_{x_1} + z\, u_{x_2} + z^2\, u_{x_3} + z^3\, u_{x_4} + z^4\, u_{x_5} \\ u_y &= u_{y_1} + z\, u_{y_2} + z^2\, u_{y_3} + z^3\, u_{y_4} + z^4\, u_{y_5} \\ u_z &= u_{z_1} + z\, u_{z_2} + z^2\, u_{z_3} + z^3\, u_{z_4} + z^4\, u_{z_5} \end{aligned} \tag{2}$$

Table 6
Reduced ED4 models for symmetrical laminated plates.

	Tolerance= 0.05%	Tolerance= 0.01%
	$a/h = 100$	
MPC	$M_e : 8/15$ △ ▲ △ ▲ △ △ ▲ △ ▲ △ ▲ ▲ ▲ △ ▲	$M_e : 13/15$ ▲ ▲ △ ▲ △ ▲ ▲ ▲ ▲ ▲ ▲ ▲ ▲ ▲ ▲
C1*	$M_e : 6/15$ △ ▲ △ ▲ △ △ ▲ △ ▲ △ ▲ ▲ △ △ △	$M_e : 8/15$ ▲ ▲ △ ▲ △ ▲ ▲ △ ▲ △ ▲ ▲ △ △ △
C2	$M_e : 8/15$ △ ▲ △ ▲ △ △ ▲ △ ▲ △ ▲ ▲ ▲ △ ▲	$M_e : 8/15$ △ ▲ △ ▲ △ △ ▲ △ ▲ △ ▲ ▲ ▲ △ ▲
	$a/h = 2$	
MPC	$M_e : 15/15$ ▲ ▲ ▲ ▲ ▲ ▲ ▲ ▲ ▲ ▲ ▲ ▲ ▲ ▲ ▲	$M_e : 15/15$ ▲ ▲ ▲ ▲ ▲ ▲ ▲ ▲ ▲ ▲ ▲ ▲ ▲ ▲ ▲
C1*	$M_e : 9/15$ ▲ ▲ ▲ ▲ ▲ △ ▲ △ ▲ △ ▲ ▲ △ △ △	$M_e : 13/15$ ▲ ▲ ▲ ▲ ▲ ▲ ▲ △ ▲ ▲ ▲ ▲ ▲ ▲ △
C1	$M_e : 14/15$ ▲ ▲ ▲ ▲ ▲ ▲ ▲ △ ▲ ▲ ▲ ▲ ▲ ▲ ▲	$M_e : 15/15$ ▲ ▲ ▲ ▲ ▲ ▲ ▲ ▲ ▲ ▲ ▲ ▲ ▲ ▲ ▲

∗: In this case only the middle point of the middle layer has been considered.



(a) MPC. (b) C1.

Fig. 5. Number of required displacement variables (M_e) vs error and a/h. LD4 approach.

As mentioned in [35], classical models such as CLT and FSDT can be considered as special cases of full linear expansion (ED1 case).

Layer Wise theories can be conveniently build by using Legendre's polynomials expansion in each layer. The displacement field is described as

$$\mathbf{u}^k = F_t \cdot \mathbf{u}_t^k + F_b \cdot \mathbf{u}_b^k + F_r \cdot \mathbf{u}_r^k = F_\tau \mathbf{u}_\tau^k$$
$$\tau = t, b, r \quad r = 2, 3, \ldots, N \qquad k = 1, 2, \ldots, N_l \tag{3}$$

where k is the generic k-layer of a plate and N_l is the number of the layers. Subscripts t and b correspond to the top and the bottom of a layer. Functions F_τ depend on a coordinate ζ_k: $-1 \leqslant \zeta_k \leqslant 1$. Functions F_τ consist of combination of Legendre's polynomials P.

$$F_t = \frac{P_0 + P_1}{2} \qquad F_b = \frac{P_0 - P_1}{2} \qquad F_r = P_r - P_{r-2} \qquad r = 2, 3, \ldots, N \tag{4}$$

The Legendre's polynomials used for fourth order theory are:

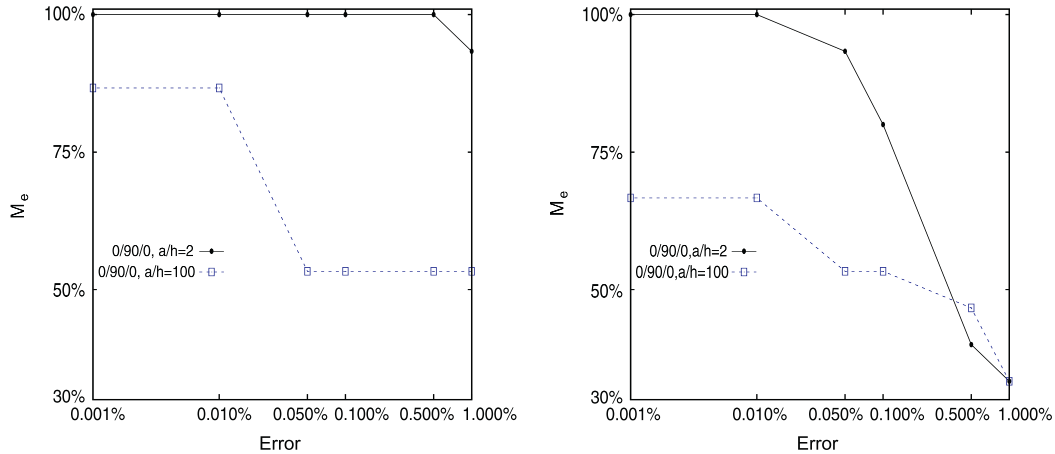
$$P_0 = 1 \quad P_1 = \zeta_k \quad P_2 = \frac{3\zeta_k^2 - 1}{2} \quad P_3 = \frac{5\zeta_k^3 - 3\zeta_k}{2}$$
$$P_4 = \frac{35\zeta_k^4}{8} - \frac{15\zeta_k^2}{4} + \frac{3}{8} \tag{5}$$

LW models ensure the compatibility of displacement at the interfaces 'zig-zag' effects by definition, that is

$$\mathbf{u}_t^k = \mathbf{u}_b^{k+1} \qquad k = 1, \ldots, N_l - 1 \tag{6}$$

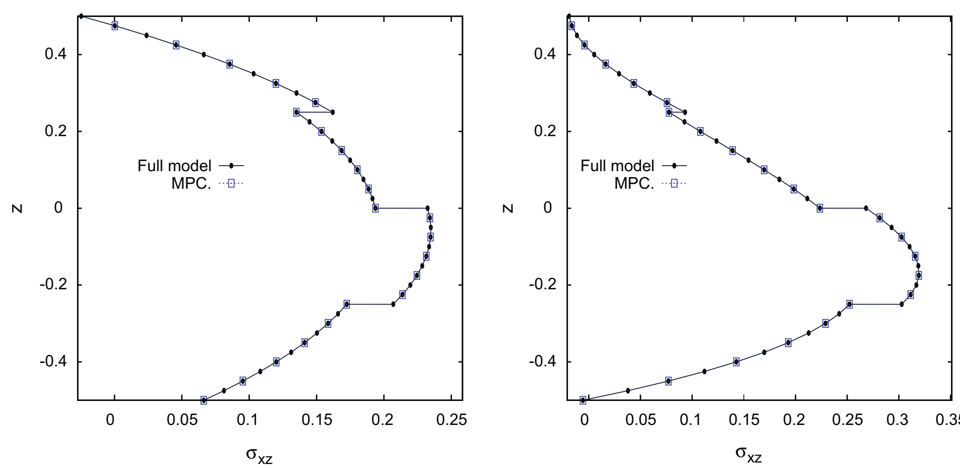
In the following LW models are denoted by the acronym as LDN, where D states that a Displacement formulation is employed and N is the expansion order. An example of LD4 layer displacement field is

$$\begin{aligned} u_x^k &= F_t\ u_{xt}^k + F_2\ u_{x2}^k + F_3\ u_{x3}^k + F_4\ u_{x4}^k + F_b\ u_{xb}^k \\ u_y^k &= F_t\ u_{yt}^k + F_2\ u_{y2}^k + F_3\ u_{y3}^k + F_4\ u_{y4}^k + F_b\ u_{yb}^k \\ u_z^k &= F_t\ u_{zt}^k + F_2\ u_{z2}^k + F_3\ u_{z3}^k + F_4\ u_{z4}^k + F_b\ u_{zb}^k \end{aligned} \tag{7}$$

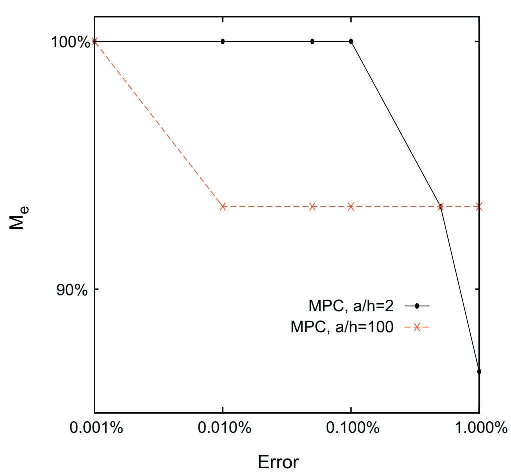


(a) MPC. (b) C1.

Fig. 6. Number of required displacement variables (M_e) vs error and a/h. ED4 approach.



(a) $a/h = 2$ (b) $a/h = 100$



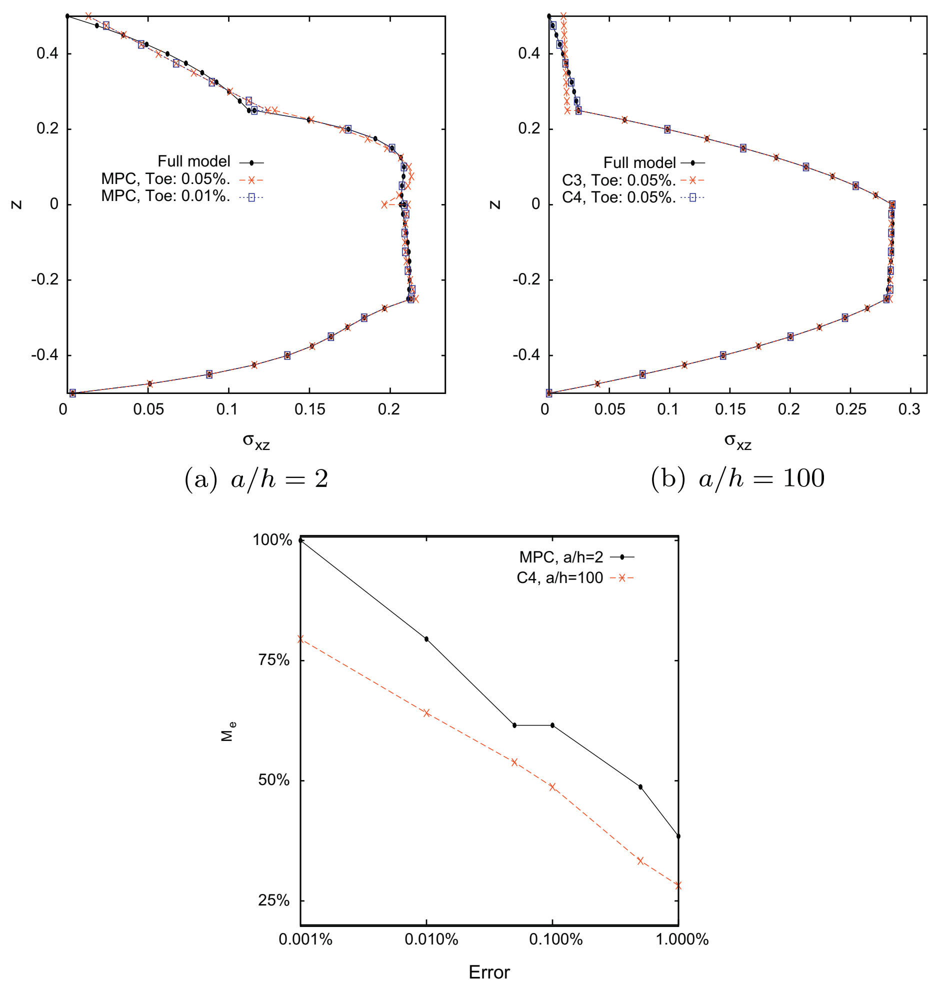
(c) Number of displacement variables vs tolerance on error

Fig. 7. ED4 results for any-symmetrical laminated plate. Tolerance = 0.05%.

Table 7
Reduced ED4 models for antisymmetrical laminated plate. Ply sequence: 0°/90°/0°/90°.

	$a/h = 2$, $M_e = 15/15$	$a/h = 100$, $M_e = 14/15$
MPC, C3	▲ ▲ ▲ ▲ ▲	▲ ▲ ▲ ▲ ▲
	▲ ▲ ▲ ▲ ▲	▲ ▲ ▲ ▲ ▲
	▲ ▲ ▲ ▲ ▲	▲ ▲ ▲ △ ▲

More details about CUF can be found in [16–18]. Governing equation are herein omitted for sake of brevity. Details can be found in the already mentioned CUF works and books. It is however underlined that the attention has been here restricted to the case of closed form solution related to simply supported, cross-ply rectangular plates loaded by a transverse distribution of harmonic loadings. The displacement are therefore expressed in the following harmonic form:



(a) $a/h = 2$ (b) $a/h = 100$

(c) Number of displacement variables vs tolerance on error

Fig. 8. LD4 results for antisymmetrical laminated plate.

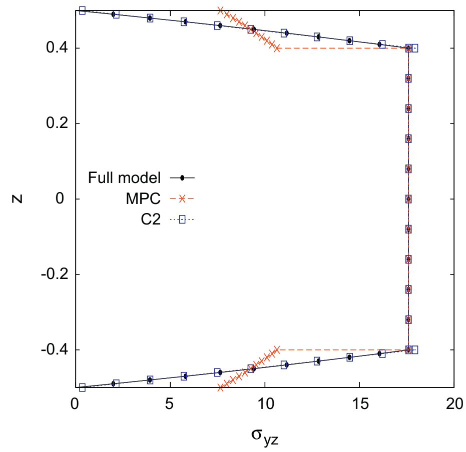
Table 8
Reduced LD4 models for antisymmetrical laminated plate. Ply sequence: 0°/90°/0°/90°.

Tolerance= 0.05%

	$a/h = 2$	$a/h = 100$
MPC	$M_e : 36/51$	$M_e : 21/51$
	■ ▲ △ △ ■ ▲ ▲ △ ■ ▲ △ △ ■ ▲ ▲ ▲ ■	■ △ △ △ ■ △ △ △ ■ ▲ △ △ ■ △ △ △ ■
	■ ▲ ▲ ▲ ■ ▲ △ △ ■ ▲ ▲ ▲ ■ ▲ △ △ ■	■ △ △ △ ■ ▲ △ △ ■ △ △ △ ■ △ △ △ ■
	■ ▲ ▲ ▲ ■ ▲ △ △ ■ ▲ △ △ ■ △ ▲ △ ■	■ ▲ ▲ ▲ ■ △ △ △ ■ △ △ △ ■ ▲ △ △ ■
C4	$M_e : 43/51$	$M_e : 33/51$
	■ ▲ ▲ △ ■ ▲ ▲ ▲ ■ ▲ △ △ ■ ▲ ▲ ▲ ■	■ ▲ △ △ ■ ▲ ▲ △ ■ △ △ △ ■ ▲ ▲ △ ■
	■ ▲ ▲ ▲ ■ ▲ △ △ ■ ▲ ▲ ▲ ■ ▲ ▲ △ ■	■ ▲ ▲ △ ■ △ △ △ ■ ▲ ▲ △ ■ ▲ △ △ ■
	■ ▲ △ △ ■ ▲ ▲ ▲ ■ ▲ ▲ ▲ ■ ▲ ▲ ▲ ■	■ ▲ △ △ ■ ▲ ▲ △ ■ ▲ ▲ △ ■ ▲ ▲ ▲ ■

Tolerance on error: 0.01%

	$a/h = 2$	$a/h = 100$
MPC	$M_e : 43/51$	$M_e : 29/51$
	■ ▲ △ △ ■ ▲ ▲ ▲ ■ ▲ △ △ ■ ▲ ▲ ▲ ■	■ △ △ △ ■ ▲ △ △ ■ ▲ ▲ △ ■ ▲ △ △ ■
	■ ▲ ▲ ▲ ■ ▲ △ △ ■ ▲ ▲ ▲ ■ ▲ ▲ △ ■	■ ▲ △ △ ■ ▲ ▲ △ ■ ▲ △ △ ■ △ △ △ ■
	■ ▲ ▲ ▲ ■ ▲ ▲ △ ■ ▲ ▲ ▲ ■ ▲ ▲ ▲ ■	■ ▲ ▲ ▲ ■ ▲ △ △ ■ ▲ △ △ ■ ▲ △ △ ■



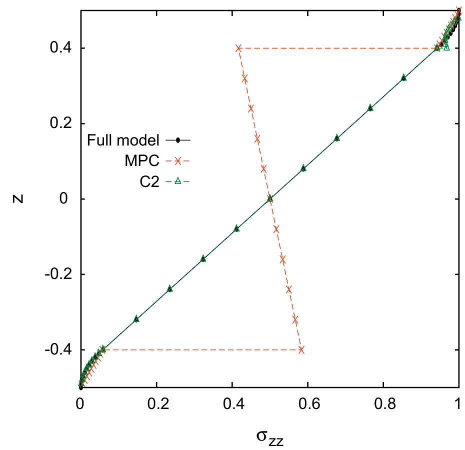


Fig. 9. Sandwich plate, benchmark 1. Tolerance = 0.05%, a/h = 100.

Table 9
Sandwich plate. $\bar{u}_z = \frac{100}{p_z}\frac{u_z}{a^4}\frac{E_T}{}\frac{h^3}{}$, $(\bar{\sigma}_{xx}, \bar{\sigma}_{xz}) = (\sigma_{xx}, \sigma_{xz})/p_z$.

Benchmark	$\bar{u}_z(z=0)$ 1	2	3	$\bar{\sigma}_{xx}(z=h/2)$ 1	2	3	$\bar{\sigma}_{xz}(z=0)$ 1	2	3
a/h = 100									
3D	7.1881	149.51	3.1589	4288.0	14535	–	17.594	15.743	–
CLT	5.5798	5.5814	2.9382	4172.0	4173.2	7708.6	0.0655	0.001	0.3701
FSDT	5.5866	5.5882	2.9432	4172.0	4173.2	7708.0	0.0655	0.001	0.3701
ED4	5.7498	5.7693	3.0244	4192.6	4177.9	7771.6	2.4239	0.0272	9.0942
LD4	7.1880	149.51	3.1589	4287.9	14535.0	7751.8	17.594	15.743	18.093
a/h = 4									
3D	590.54	1370.6	124.69	67.958	408.06	–	0.4053	0.0094	–
CLT	5.5799	5.5814	2.9382	6.6752	6.6771	12.334	0.0026	0.00003	0.0148
FSDT	9.8085	9.8239	6.0535	6.6752	6.6771	11.989	0.0026	0.00003	0.0145
ED4	101.72	112.5	52.165	6.2903	10.614	30.045	0.0896	0.0010	0.3467
LD4	590.53	1370.6	124.6938	67.9580	408.05	14535.0	0.4053	0.0095	15.7430

$$
\begin{aligned}
u_{x_\tau}^k &= \sum_{m,n} U_{x_\tau}^k \cdot \cos\left(\frac{m\pi_k}{a_k}\right)\sin\left(\frac{n\pi y_k}{b_k}\right) \qquad k = 1, N_l \\
u_{y_\tau}^k &= \sum_{m,n} U_{y_\tau}^k \cdot \sin\left(\frac{m\pi x_k}{a_k}\right)\cos\left(\frac{n\pi y_k}{b_k}\right) \qquad \tau = 1, N \\
u_{z_\tau}^k &= \sum_{m,n} U_{z_\tau}^k \cdot \sin\left(\frac{m\pi x_k}{a_k}\right)\sin\left(\frac{n\pi y_k}{b_k}\right)
\end{aligned} \tag{8}
$$

where $U_{x_\tau}^k, U_{y_\tau}^k$ and $U_{z_\tau}^k$ are the amplitudes, m and n are the number of waves (they range from 0 to ∞) and a_k and b_k are the dimensions of the plate in the x and y directions, respectively. The same solution can be applied to ESL approach, in this case displacement variables appears without superscript k. With these assumptions the static response could be written as a system of algebraic equations:

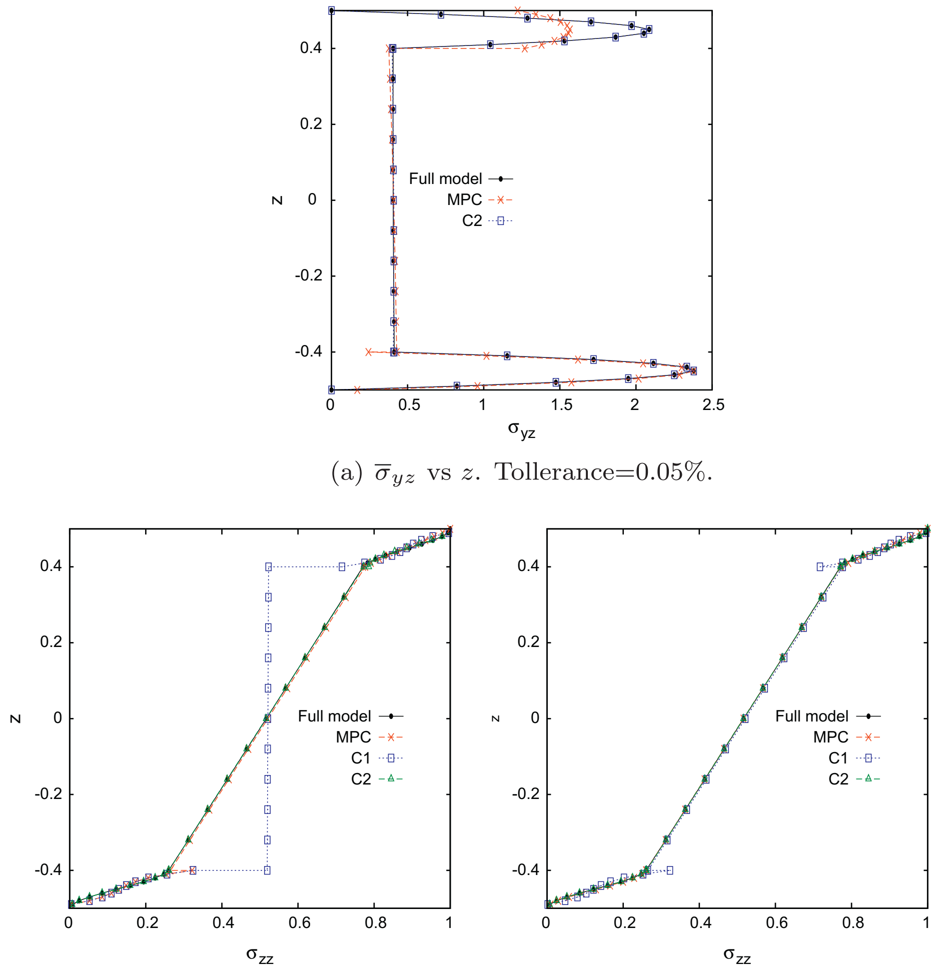
$$
\delta \mathbf{u}_s^{k^\tau}: \ \mathbf{K}_{uu}^{k\tau s}\mathbf{u}_\tau^k = \mathbf{P}_{u\tau}^k \tag{9}
$$

which is used to provide the numerical analysis provided in the present work. Explicit forms of the above arrays can be found in the already mentioned work as well as in the recent book [17].

3. Axiomatic/asympotic technique

Plate analysis take advantage of CUF by the possibility to introduce any higher order terms in a plate model. The introduction of high order terms offer significant advantages in terms of improved structural response analysis but at the same time the computational cost increases. In order to preserve the accuracy of a high order model and to minimize the computational cost it is possible to employ the axiomatic/asymptotic approach which aims to evaluate the effectiveness of each term. According to what reported in [35,37] the following steps must be performed:

1. Plate parameters such as the geometry, BC, loadings, materials and layer layouts, are fixed.
2. A set of output parameters is chosen, such as displacement and stress components.
3. A theory is fixed, that is the displacement variables to be analyzed are defined; in this paper all displacement variables of ED4 and LD4 models are analyzed.
4. A reference solution is defined; in the present work ED4 and LD4 approaches are adopted, since the fourth order models offer an excellent agreement with the three-dimensional solutions as highlighted in [35], in [21] and in the next analyzes.
5. CUF is used to generate the governing equations for the theories considered.
6. Each displacement variable effectiveness is numerically established measuring the loss of accuracy on the chosen output parameters compared with the reference solution.
7. Any displacement variable which does not alter the mechanical response is considered not effective for the kinematics model.
8. The most suitable kinematics model for a specific parameter is then obtained discarding the non-effective displacement variables. The reduced combined models are created discarding all displacement variables which are not crucial for all parameters at the same time.



(a) $\overline{\sigma}_{yz}$ vs z. Tollerance=0.05%.

(b) $\overline{\sigma}_{zz}$ vs z. Tollerance=0.05%.

(c) $\overline{\sigma}_{zz}$ vs z. Tollerance=0.01%.

Fig. 10. $\bar{\sigma}_{yz}$ and $\bar{\sigma}_{zz}$ vs z for LD4 model. Sandwich plate, benchmark 1. $a/h = 4$.

An example is given in order to explain how the evaluation of the effectiveness of the displacement variables is carried out. Let's consider a LD4 model for a single layer plate:

$$\begin{aligned} u_x &= F_t\ u_{xy} + F_2\ u_{x2} + F_3\ u_{x3} + F_4\ u_{x4} + F_b\ u_{xb} \\ u_y &= F_t\ u_{yy} + F_2\ u_{y2} + F_3\ u_{y3} + F_4\ u_{y4} + F_b\ u_{yb} \\ u_z &= F_t\ u_{zy} + F_2\ u_{z2} + F_3\ u_{z3} + F_4\ u_{z4} + F_b\ u_{zb} \end{aligned} \tag{10}$$

If the effectiveness of u_{z2} term has to be analyzed the relative reduced model is:

$$\begin{aligned} u_x &= F_t\ u_{xy} + F_2\ u_{x2} + F_3\ u_{x3} + F_4\ u_{x4} + F_b\ u_{xb} \\ u_y &= F_t\ u_{yy} + F_2\ u_{y2} + F_3\ u_{y3} + F_4\ u_{y4} + F_b\ u_{yb} \\ u_z &= F_t\ u_{zy} + \qquad + F_3\ u_{z3} + F_4\ u_{z4} + F_b\ u_{zb} \end{aligned} \tag{11}$$

The static response offered by (10) is considered as a reference solution. The static response given by the model reported in (11) is compared with the reference values according to the equation:

$$\delta_Q = \max \left| 1 - \frac{Q_i}{Q^i_{\text{ref}}} \right| \times\ 100 > \text{tolerance} \tag{12}$$

where Q^i_{ref} and Q_i denote respectively the reference value and the actual value of any variable under exam, as displacement u_z or stress σ_{xz}, evaluated at the generic i point denoted in the criteria described below. A generic contribution (term) to the displacement field is considered to be effective if the quantity δ_Q is greater than a specific level. The suppression of a term is obtained by means of a penalty technique. In the following analysis the tolerance on error is considered as a parameter. The symbols reported in Table 1 are used. To be noticed that interface values in the LW cases cannot be removed as definition. An example of reduced model related to Eq. (11) is given in Table 2.

4. Description of single-point and multi-points error criteria

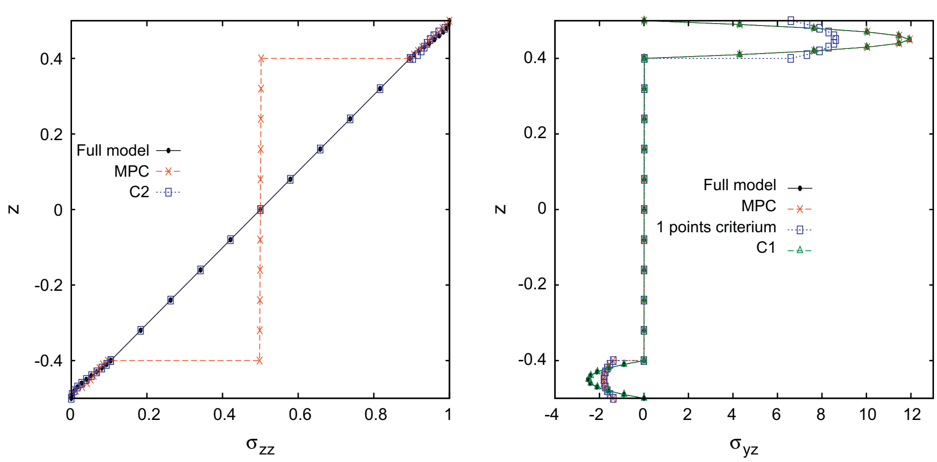
The asymptotic/axiomatic approach proposed in [35] evaluates stress and displacement components in fixed points $[x, y, z]$. In particular $[a/2, b/2, h/2]$ was used for u_z, σ_{xx} and σ_{zz} while the position $[0, b/2, 0]$ was used for the shear stress σ_{xz} and $[a/2, 0, 0]$ for the shear σ_{yz} (values in brackets are x, y, z coordinates, respectively). The reduced models were derived according to this procedure. However it was observed that in some cases the distribution along the thickness of the reduced model may show a significant deviation from the reference solution. In order to overcome this limitation in this paper an improved multi-points evaluation along the thickness is proposed. That is various positions along z coordinate are used to build the asymtotic/aximoatic reduced models.

The first criteria consists to compare the absolute maximum values of the involved stress and displacement components along the thickness. This criterium is defined as maximum point criterium (MPC). In this case the number and the position of the points along the thickness is not known a priori, see Fig. 2.

Table 10
Reduced LD4 models for sandwich plate, benchmark 1.

Tolerance on error: 0.05%													
$a/h = 100$													
MPC (M_e : 16/39)	■	△	△	△	■	△	△	△	■	△	△	△	■
	■	△	△	△	■	△	△	△	■	△	△	△	■
	■	▲	▲	▲	■	△	△	△	■	▲	△	△	■
C1 (M_e : 15/39)	■	△	△	△	■	△	△	△	■	▲	△	△	■
	■	△	△	△	■	△	△	△	■	▲	△	△	■
	■	△	△	△	■	△	△	△	■	△	▲	△	■
C2 (M_e : 20/39)	■	▲	△	△	■	△	△	△	■	▲	△	△	■
	■	▲	△	△	■	△	△	△	■	▲	△	△	■
	■	▲	△	△	■	▲	△	△	■	▲	▲	△	■
$a/h = 4$													
MPC (M_e : 19/39)	■	△	△	△	■	△	△	△	■	△	▲	△	■
	■	△	△	△	■	△	△	△	■	△	▲	△	■
	■	▲	▲	▲	■	▲	△	△	■	▲	△	△	■
C1 (M_e : 19/39)	■	△	▲	△	■	△	△	△	■	△	▲	△	■
	■	△	▲	△	■	△	△	△	■	△	▲	△	■
	■	▲	△	△	■	△	△	△	■	▲	▲	△	■
C2 (M_e : 28/39)	■	▲	▲	△	■	▲	△	△	■	▲	▲	△	■
	■	▲	▲	△	■	▲	△	△	■	▲	▲	△	■
	■	▲	△	▲	■	▲	△	△	■	▲	▲	▲	■

Tolerance on error: 0.01%													
$a/h = 100$													
MPC (M_e : 18/39)	■	△	△	△	■	△	▲	△	■	△	△	△	■
	■	△	△	△	■	△	▲	△	■	△	△	△	■
	■	▲	▲	▲	■	△	△	△	■	▲	△	△	■
C1 (M_e : 22/39)	■	△	▲	△	■	△	△	△	■	▲	▲	△	■
	■	△	▲	△	■	△	△	△	■	▲	▲	△	■
	■	▲	▲	△	■	△	△	△	■	▲	▲	△	■
C2 (M_e : 26/39)	■	▲	▲	△	■	△	△	△	■	▲	▲	△	■
	■	▲	▲	△	■	△	△	△	■	▲	▲	△	■
	■	▲	▲	△	■	▲	△	△	■	▲	▲	▲	■
$a/h = 4$													
MPC (M_e : 23/39)	■	△	△	△	■	▲	△	△	■	△	▲	△	■
	■	△	△	△	■	▲	△	△	■	△	▲	△	■
	■	▲	▲	▲	■	▲	△	△	■	▲	▲	▲	■
C1 (M_e : 22/39)	■	△	▲	△	■	△	△	△	■	▲	▲	△	■
	■	△	▲	△	■	△	△	△	■	▲	▲	△	■
	■	▲	△	△	■	▲	△	△	■	▲	▲	△	■
C2 (M_e : 31/39)	■	▲	▲	△	■	▲	▲	△	■	▲	▲	△	■
	■	▲	▲	△	■	▲	▲	△	■	▲	▲	△	■
	■	▲	▲	▲	■	▲	△	△	■	▲	▲	▲	■



(a) $\overline{\sigma}_{zz}$ vs z, $a/h = 100$ (b) $\overline{\sigma}_{yz}$ vs z, $a/h = 4$

Fig. 11. Sandwich plate, benchmark 2. Tollerance = 0.05%.

A few other additional strategies are also implemented. These consider the values of the involved stress and displacement components at some specific points distributed along the thickness direction, that is:

- The points at the middle of each layer, this criterium will be defined as criterium 1 (C1);
- The points at the interfaces of each layer, this criterium will be defined as criterium 2 (C2); Top and bottom values of the plates are excluded.
- The points at the middle and at the two-interfaces of layers, this criterium will be defined as criterium (C3). Also in the case top–bottom surface values are not considered.
- The points located at every $z_k = -h_k/4$, this criterium is denotes as criterium (C4).

C1 and C3 cases are depicted in Fig. 2 where only the two layers k and $k+1$ are shown.

5. Results

Results are restricted to composite cross-ply and sandwich simply supported plates, loaded by a transverse pressure applied at the top-surface with the following harmonic form:

$$p = \bar{p}_z \cdot sin\left(\frac{mx}{a}\right) \cdot \sin\left(\frac{ny}{b}\right) \qquad (13)$$

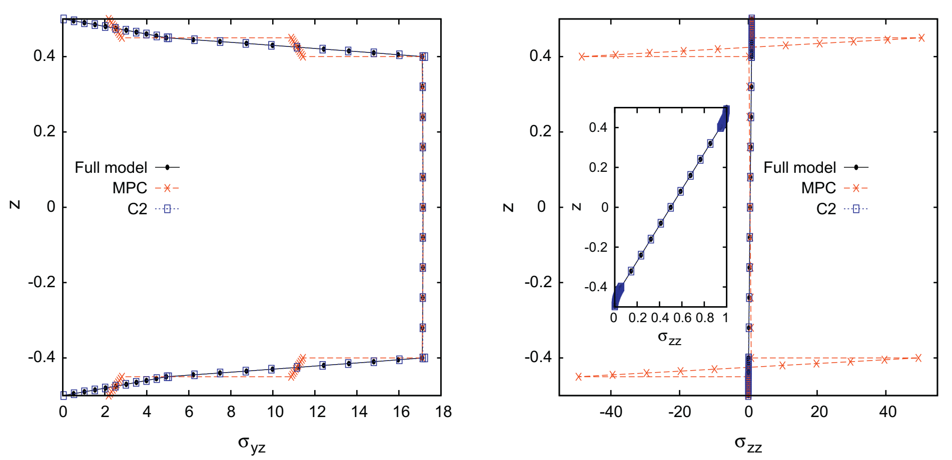
Table 11
Reduced LD4 models for sandwich plate, benchmark 2.

	$a/h = 100$												
	$M_e : 20/39$												
MPC	■	△	△	△	■	△	△	△	■	▲	▲	△	■
	■	△	△	△	■	△	△	△	■	▲	▲	△	■
	■	▲	▲	▲	■	△	△	△	■	▲	△	△	■
	$M_e : 25/39$												
C2	■	▲	▲	△	■	△	△	△	■	▲	▲	△	■
	■	▲	▲	△	■	△	△	△	■	▲	▲	△	■
	■	▲	△	△	■	▲	△	△	■	▲	▲	▲	■
	$a/h = 4$												
	$M_e : 20/39$												
MPC	■	△	▲	△	■	▲	△	△	■	△	△	△	■
	■	△	▲	△	■	▲	△	△	■	△	△	△	■
	■	▲	▲	▲	■	△	△	△	■	▲	△	△	■
	$M_e : 18/39$												
C1	■	△	▲	△	■	△	△	△	■	△	▲	△	■
	■	△	▲	△	■	△	△	△	■	△	▲	△	■
	■	▲	△	△	■	△	△	△	■	▲	△	△	■

Here the amplitude $\bar{p}_z$ is equal to 1000 (Pa). m and n are the number of waves along x and y directions. In all analyzes these parameters are equal to $m = n = 1$. Geometrical notations and reference system ar given in Fig. 1.

5.1. Symmetrically and unsymmetrically laminated plates

Here its is considered a composite with equal thickness layers. For sake of simplicity the attention has been restricted to fourth order case for both LW and ESL theories, these will be denoted as LD4 and ED4, respectively. A preliminary assessment of ED4 and LD4 results with three-dimensional 3D solutions in [38] is given in Table 3. Mechanical/geometrical data are those in [38]. It is clearly shown that accuracy of Layer-Wise analysis is independent of the stacking sequence except for the ED4 cases. LD4 results are very closed to 3D results, such results permit us to consider LD4 as reference-results whenever 3D analysis is not available.



(a) $\bar{\sigma}_{yz}$ vs z. Tollerance on error: 0.05%. (b) $\bar{\sigma}_{zz}$ vs z. Tollerance on error: 0.05%.

Fig. 12. $\bar{\sigma}_{yz}$ and $\bar{\sigma}_{zz}$ vs z for LD4 model. Sandwich plate, benchmark 3. $a/h = 100$.

Table 12
Reduced LD4 models for sandwich plate, benchmark 3.

	$a/h = 100$																					
	$M_e : 23/63$																					
MPC	■	△	△	△	■	△	△	△	■	△	▲	△	■	△	△	△	■	△	△	△	■	Tolerance= 0.05%
	■	△	△	△	■	△	△	△	■	△	▲	△	■	△	△	△	■	△	△	△	■	
	■	▲	▲	△	■	△	△	△	■	△	△	△	■	△	△	△	■	▲	△	△	■	
	$M_e : 33/63$																					
C2	■	▲	△	△	■	▲	△	△	■	△	△	△	■	▲	△	△	■	▲	△	△	■	
	■	▲	△	△	■	▲	△	△	■	△	△	△	■	▲	△	△	■	▲	△	△	■	
	■	▲	△	△	■	▲	△	△	■	▲	△	△	■	▲	▲	△	■	▲	▲	△	■	
	$a/h = 4$																					
	$M_e : 27/63$																					
MPC	■	△	△	△	■	△	△	△	■	△	△	△	■	△	△	△	■	▲	▲	△	■	Tolerance =0.05%
	■	△	△	△	■	△	△	△	■	△	△	△	■	△	▲	△	■	△	△	△	■	
	■	▲	▲	▲	■	△	△	△	■	▲	△	△	■	▲	△	△	■	▲	△	△	■	
	$M_e : 33/63$																					
	■	△	△	△	■	△	△	△	■	▲	△	△	■	▲	△	△	■	▲	▲	△	■	Tolerance = 0.01%
	■	△	△	△	■	△	△	△	■	▲	△	△	■	△	▲	△	■	▲	△	△	■	
	■	▲	▲	▲	■	▲	△	△	■	▲	△	△	■	▲	△	△	■	▲	▲	△	■	
	$M_e : 42/63$																					
C2	■	▲	▲	△	■	▲	▲	△	■	▲	△	△	■	▲	▲	△	■	▲	▲	△	■	Tolerance = 0.05%
	■	▲	▲	△	■	△	▲	△	■	▲	△	△	■	▲	▲	△	■	▲	▲	△	■	
	■	▲	△	△	■	▲	△	△	■	▲	△	△	■	▲	▲	△	■	▲	▲	△	■	
	$M_e : 49/63$																					
	■	▲	▲	△	■	▲	▲	△	■	▲	▲	△	■	▲	▲	△	■	▲	▲	△	■	Tolerance= 0.01%
	■	▲	▲	△	■	▲	▲	△	■	▲	▲	△	■	▲	▲	△	■	▲	▲	△	■	
	■	▲	△	△	■	▲	▲	▲	■	▲	△	△	■	▲	▲	▲	■	▲	▲	▲	■	

The analyzes given in the subsequent part of the paper refer to plates whose gemetrical/mechanical properties are taken from [35]. Layers properties are: $E_L = 40 \times 10^9$ (Pa), $E_T = E_z = 1 \times 10^9$ [Pa], $G_{LT} = 0.5 \times 10^9$ (Pa), $G_z = 0.6 \times 10^9$ (Pa), $\nu_{LT} = \nu_{Lz} = 0.25$. Sides a and b are equal. The orientation of the plies are: 0°/90°/0° and 0°/90°/0°/90°. Results are given in Table 4. Classical Lamination Theory CLT and First order Shear Deformation Theory FSDT results have been quoted for comparison purposes.

Reduced LD4 models for thin and thick symmetrically laminated plates are considered in Table 5, C1 and C2 criteria are addressed with two tolerance values. Corresponding reduced models are depicted and the related number of terms is quoted. M_e denotes (in percentage), the number of effective terms with respect to the total number of terms in the LD4 theories. To be noticed that interface displacement variables of LD4 analysis cannot be discarded by the reduced model, these are always used and depicted with a black-square. The trend of stress σ_{xx} and σ_{yz} trends along thickness direction evaluated through these reduced models are depicted in Fig. 3. The following comments can be made:

- The number of terms is very much affected by geometry, tolerance as well error criteria.
- For a given criterium a decrease in the error-tolerance it leads to an increase of the number of displacement variables and it improves the stress evaluation.
- MPC and C1 criteria leads to different reduced models.

The same analysis has been conducted for the ED4 plate theories. Stress for thin and thick symmetric plates are reported in Fig. 4. The relative reduced ED4 models are reported in Table 6. In some cases the reduced model is not affected by error criteria, that is MPC and C2 cases lead to the same reduced model. In some cases (mostly related to thick geometry) C1 and C2 criteria can offer a remarkable reduction of the number of the effective displacement variables comparing to MPC, as reported in Table 6; the reduced displacement field enriches by tolerance decreasing. A summary of the two case-theories under consideration is proposed in Figs. 5 and 6. To notice that symmetrically laminated thick plate leads to sensible reduction of displacement variables when C1 is adopted instead of maximum point criterium.

Cross-ply asymmetrically laminated composite plate are analyzed in Fig. 7. and in Table 7. Reduced models related to ED4 analysis are quoted. No significant displacement variable reduction is obtained in this lay-out configuration. LD4 analysis results are given in Fig. 8 and in Table 8. Quite different results are obtained by the various criteria. Comparison of the number of displacement variables for ED4 and LD4 cases is reported in Figs. 7 and 8. As in the previous cases LW models show a more significant reduction of the number of the displacement variables than ESL analysis.

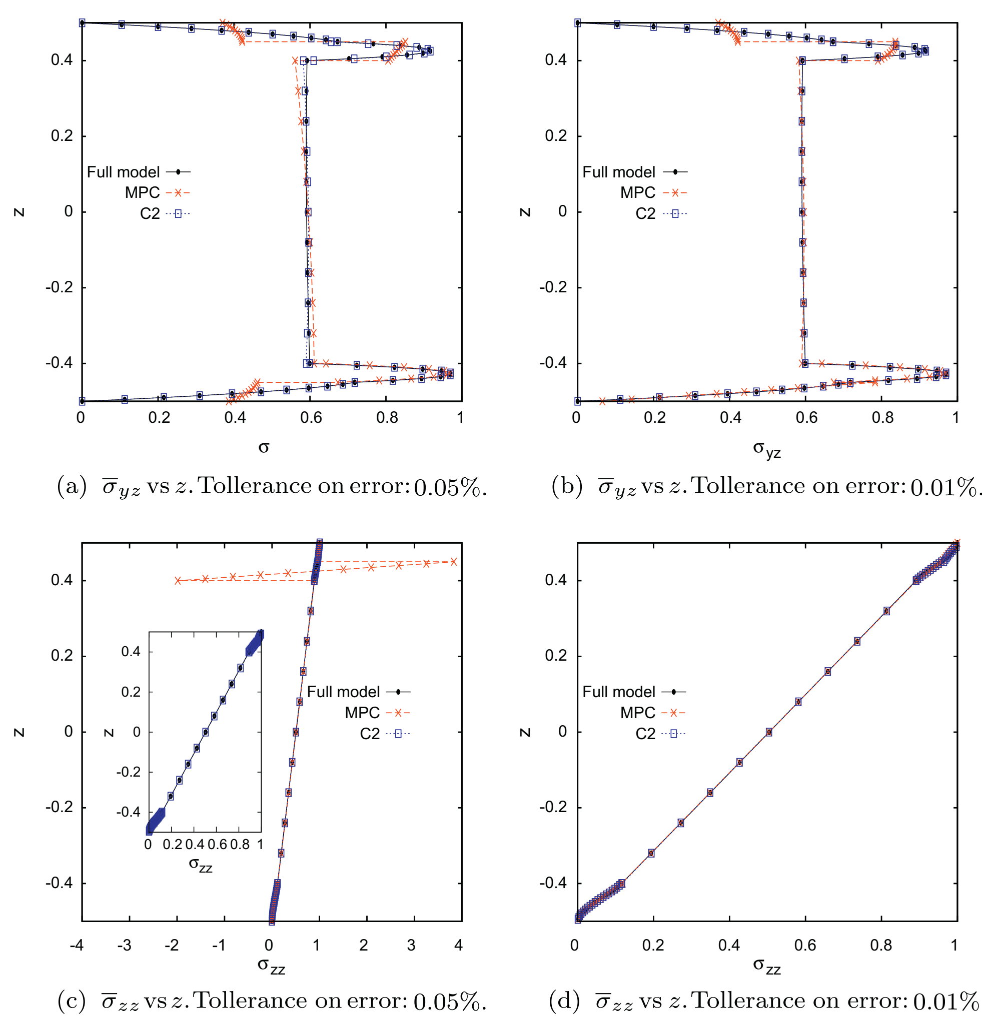


Fig. 13. $\bar{\sigma}_{yz}$ and $\bar{\sigma}_{zz}$ vs z for LD4 results. Sandwich plate, benchmark 3. $a/h = 4$.

5.2. Sandwich plates

Three different sandwich square plates are analyzed: Benchmarks 1, 2 and 3. Only LD4 results are considered. $h = 0.01$ (m), a and b are considered as parameter.

Benchmark 1 plate has a Nomex core ($E_L = E_T = 0.01 \times 10^6$ (Pa), $E_z = 75.85 \times 10^6$ (Pa), $G = 22.5 \times 10^6$ (Pa), $\mu = 0.25$, thickness $h_2 = 0.008$ (m)) and two isotropic skins (aluminum, $E = 73$ (GPa), $\mu = 0.33$, thickness $h_1 = h_3 = 0.001$ (m)). The ratio E_f/E_c, which is the ratio of skin E_L value over core E_L value, is equal to 7.3×10^6.

Benchmark 2 consists of two isotropic skins (aluminum, $h_1 = h_2 = 0.001$ (m)) and a core which is 100 times more flexible than the core of sandwich benchmark 1,thai is $E_f/E_c = 7.3 \times 10^8$.

Benchmark 3 is four layer composite skins plate ($E_L = 50 \times 10^9$ (Pa), $E_T = E_z = 10 \times 10^6$ (Pa), $G = 5 \times 10^9$ (Pa), $\mu = 0.25$); thickness are $h_1 = h_2 = h_4 = h_5 = 0.005$ (m); ply sequence is $0°/90° - 90°/0°$, and $E_f/E_c = 50 \times 10^6$. Mechanics and geometry of the core is the same as benchmark 1.

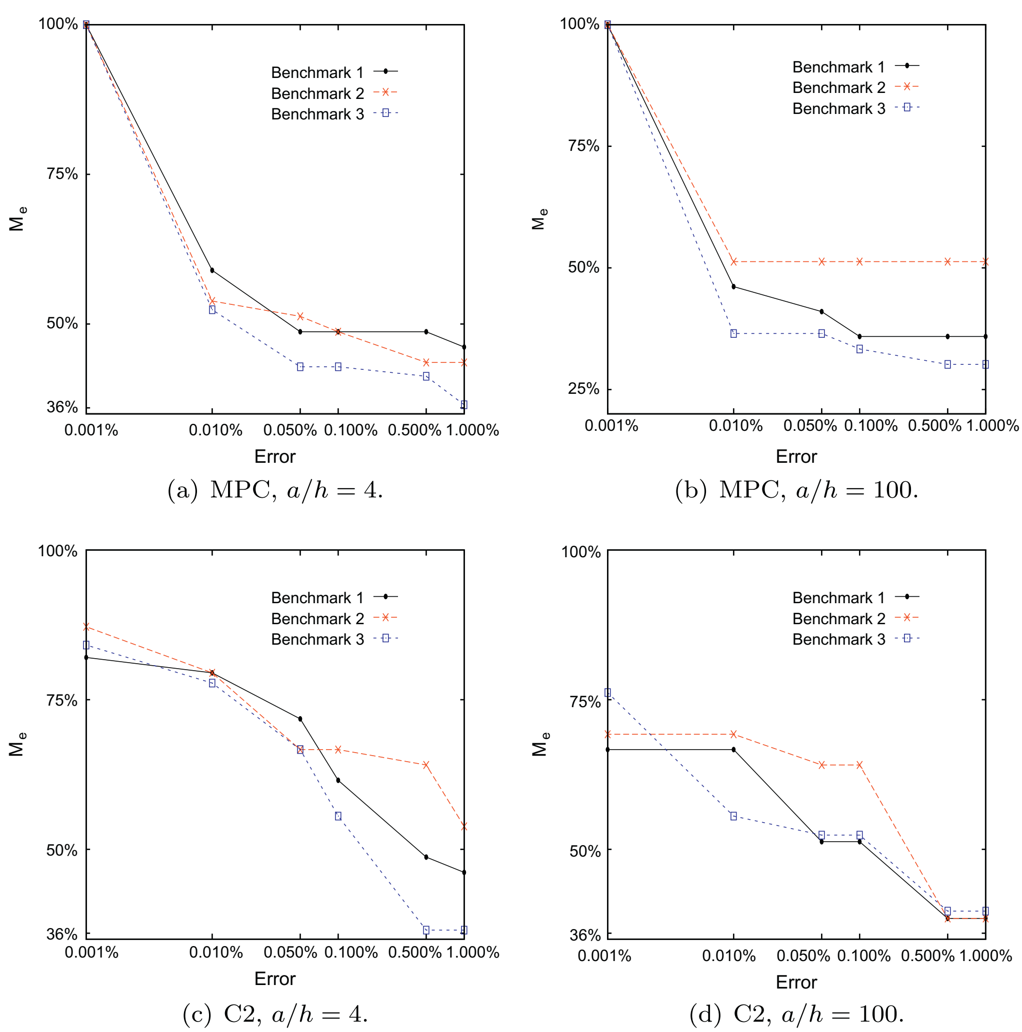
The analysis considers four values of a/h equal to 4, 10, 100 and 1000. Preliminary assessment with available results are given in Table 9. 3D reference results are reported in [21].

Results related to benchmark 1 are given in Figs. 9 and 10. Reduced LD4 models are depicted in Table 10. Benchmark 2 results are reported in Fig. 11 and in Table 11. It is possible to observe that:

- The number of effective displacement variables is very much influenced by the number of points employed by the used error criteria; that conclusion makes the analyzes in this paper of interest.
- By increasing E_f/E_c ratio the number of required terms for the reduced model may increase significantly, such an increase is very much dependent on adopted criterium.
- As expected, in some cases related to thin sandwich plates, it happens that only the linear terms u_b and u_t are needed for the core displacement field.

Figs. 12 and 13 show stresses $\bar{\sigma}_{yz}$ and $\bar{\sigma}_{zz}$ distribution along z in both cases of thin and thick plates related to benchmark 3. Reduced LD4 models are quoted in Table 12. The best stress evaluation is given by C2 and the number of required displacement variables could reach 1/2 of the original LD4 terms in both cases of thick and thin sandwich plates.

A summary of sandwich plate analysis is given in Fig. 14. The percentage of the effective terms with respect to full LD4 model is given for the three benchmarks in case of thick and thin geometries as well as for the C2 and MPC criterium to detect the error in the thickness distribution of the stress/displacement variables. It appears clear that very accurate results (corresponding to very low value of tolerance/error in the horizontal axis of the four diagrams) can be obtained with models that are significantly reduced.



(a) MPC, $a/h = 4$. (b) MPC, $a/h = 100$.

(c) C2, $a/h = 4$. (d) C2, $a/h = 100$.

Fig. 14. Reduced models vs tolerance for LD4 model. Sandwich plates.

In any case if the exact value is requested in the whole thickness domain, that is a multi-point criteria is used, the use of full model could become mandatory.

6. Conclusion

Asymptotic/axiomatic technique has been employed for the analysis of refined plate models by means of the Carrera Unified Formulation. Navier-type solutions has been obtained in the case of simply supported orthotropic plates loaded by a bisinusoidal transverse pressure. Composite and sandwich lay-outs have been investigated by both ESL and LW approaches. The effectiveness of each displacement variable has been verified by using single-point and multi-points criteria related to different choices of points (along the thickness direction) in which the error of stress and displacement variables is computed. The following main conclusions can be remarked.

1. Reduced models depends on geometrical and material properties (for example, a/h and E_f/E_c).
2. Accurate stress and displacement trends can be derived applying asymptotic/axiomatic in conjunction with appropriate choice of the various considered error criteria MPC, C1, C2, C3, C4. It has been found that the best error criteria (which leads to the reduced model with minimum number of displacement variables) is problem-dependent.
3. For symmetrically laminated composite plates C1 and C2 are the most adequate. In the case of an asymmetrically laminated plate MPC leads to proper reduced model, although in some cases the use of multi-points criterium could become mandatory.
4. Sandwich plates reduced model can be properly evaluated by employing criterium C2.

Future work could consider shell geometries, dynamics, multi-field problems as well finite element applications.

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