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Time-Inhomogeneous Feller-type Diffusion Process with Absorbing Boundary Condition

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Abstract
A time-inhomogeneous Feller-type diffusion process with linear infinitesimal drift $\alpha(t)x + \beta(t)$ and linear infinitesimal variance $2r(t)x$ is considered. For this process, the transition density in the presence of an absorbing boundary in the zero-state and the first-passage time density through the zero-state are obtained. Special attention is dedicated to the proportional case, in which the immigration intensity function $\beta(t)$ and the noise intensity function $r(t)$ are connected via the relation $\beta(t) = \xi\, r(t)$, with $0 \le \xi < 1$. Various numerical computations are performed to illustrate the effect of the parameters on the first-passage time density, by assuming that $\alpha(t)$, $\beta(t)$ or both of these functions exhibit some kind of periodicity.

Keywords Transient distributions · First-passage time densities · Periodic intensity functions

Mathematics Subject Classification 60J60 · 60J70 · 82C31

1 Introduction and Background

One-dimensional time-inhomogeneous diffusion processes play a relevant role in different application fields, including physics, biology, neuroscience, finance and others (cf., for instance, Giorno and Nobile [1,2], Albano and Giorno [3], Ghost and Prajneshu [4], Buonocore et al [5], Gutiérrez et al [6], Di Crescenzo et al [7], Román-Román et al. [8], Molini et al. [9], Gan and Waxman [10], Abundo [11]). In this paper, we consider a time-inhomogeneous Feller-type diffusion process, characterized by linear infinitesimal drift and linear infinitesimal variance vanishing in the zero-state (lower boundary of the process). We assume that the zero-state represents an absorbing boundary for the process.

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Let $\{X(t), t \geq t_0\}$, $t_0 \geq 0$, be a time-inhomogeneous Feller-type diffusion process with the following infinitesimal drift and infinitesimal variance

$$A_1(x,t) = \alpha(t)\,x + \beta(t), \qquad A_2(x,t) = 2\,r(t)\,x, \tag{1}$$

defined in the state-space $[0, +\infty)$, with $\alpha(t) \in \mathbb{R}$, $\beta(t) \in \mathbb{R}$, $r(t) > 0$ continuous functions for all $t \geq t_0$.

The time-homogeneous Feller diffusion process, in which $\alpha(t) = \alpha$, $\beta(t) = \beta$ and $r(t) = r$ for all $t \geq 0$, is taken in account in Feller [12], where is shown that boundary $x = 0$ changes its character depending on whether $\beta \leq 0$ (exit), $0 < \beta < r$ (regular), $\beta \geq r$ (entrance). Furthermore, as proved in Feller [13], if one knows the nature of the end points of the state-space one can decide what kind of boundary condition has to be associated with the Fokker-Planck and Kolmogorov diffusion equations to determine the transition pdf of the process. A review showing the relevance of the Feller's work on boundary classification of one-dimensional diffusion processes is provided in Peskir [14]. By following this approach, for the time-homogeneous Feller diffusion process, the transition pdf in the presence of an absorption condition or a zero-flux condition in the zero-state is explicitly obtained in Karlin and Taylor [15] and in Giorno et al. [16]. Furthermore, a class of Kolmogorov diffusion equations that can be transformed into a Kolmogorov equation for a time-homogeneous Feller process is considered in Capocelli and Ricciardi [17].

Feller diffusion process is widely used in mathematical biology to model the growth of a population (cf., Lavigne and Roques [18], Masoliver [19], Ricciardi et al. [20]), in queueing systems to describe the number of customers in a queue (cf., Di Crescenzo and Nobile [21]), in neurobiology to analyze the input-output behavior of single neurons (see, for instance, Ditlevsen and Lánský [22], Lánský et al. [23], Nobile and Pirozzi [24], Giorno et al. [25,26], Buonocore et al. [27]), in mathematical finance to model asset prices, market indices, interest rates and stochastic volatility (see, Tian and Zhang [28], Cox et al. [29], Linetsky [30], Göing-Jaeshke and Yor [31]).

Sometimes, the Feller-type diffusion process $X(t)$ is obtained as a continuous approximation of a time-inhomogeneous discrete Markov processes (see, for instance, Di Crescenzo and Nobile [21], Giorno et al. [25]). Indeed, in population dynamics the Feller-type diffusion process arises as a continuous approximation of a birth-death process with immigration (cf. Giorno and Nobile [32] and references therein). In these cases $\alpha(t)$, related to the growth intensity function, is positive (negative) when the birth intensity function is greater (less) than the death intensity function, whereas $\alpha(t) = 0$ if the birth intensity function is equal to the death intensity function. Since $\alpha(t)$ is a time dependent function, it can be positive, negative or zero at different time instants. Instead, $\beta(t)$ is related to the immigration intensity function. In particular, $\beta(t) > 0$ indicates the presence of immigrations and a zero-flux condition or an absorbing condition can be imposed in the zero-state of the diffusion process.

For a full characterization of the time-inhomogeneous Feller-type diffusion process $X(t)$, the behavior at the boundary 0 must be specified. In this paper, we assume that the zero-state is an absorbing boundary, so that the process $X(t)$ terminates when the boundary is reached. We suppose that $\alpha(t) \in \mathbb{R}$, $\beta(t) \in \mathbb{R}$, $r(t) > 0$, $\beta(t) \leq \xi\, r(t)$, with $0 \leq \xi < 1$, for all $t \geq t_0$.

We denote by

$$f_a(x,t|x_0,t_0) = \frac{\partial}{\partial x} P\{X(t) \leq x;\; X(\vartheta) > 0, \forall \vartheta < t | X(t_0) = x_0\}, \quad x > 0, y > 0 \tag{2}$$

the transition probability density function (pdf) of $X(t)$ in the presence of an absorbing boundary in the zero-state. As shown in Kolmogorov [33] and Dynkin [34], the pdf $f_a(x,t|x_0,t_0)$

satisfies the Kolmogorov equation

$$\frac{\partial f_a(x,t|x_0,t_0)}{\partial t_0} + A_1(x_0,t_0)\,\frac{\partial f_a(x,t|x_0,t_0)}{\partial x_0} + \frac{1}{2}\,A_2(x_0,t_0)\,\frac{\partial^2 f_a(x,t|x_0,t_0)}{\partial x_0^2} = 0, \quad (3)$$

with $A_1(x_0,t_0)$ and $A_2(x_0,t_0)$ given in (1), to solve imposing the initial delta condition

$$\lim_{t_0 \uparrow t} f_a(x,t|x_0,t_0) = \delta(x - x_0) \quad (4)$$

and the absorbing boundary condition in the zero-state:

$$\lim_{x_0 \downarrow 0} f_a(x,t|x_0,t_0) = 0. \quad (5)$$

Furthermore, let

$$T(x_0,t_0) = \inf_{t \ge t_0}\{t : X(t) = 0\}, \qquad X(t_0) = x_0 > 0 \quad (6)$$

be the random variable describing the first-passage time (FPT) through the zero-state starting from $X(t_0) = x_0 > 0$; we denote by

$$g(0,t|x_0,t_0) = \frac{d}{dt} P\{T(x_0,t_0) \le t\}. \quad (7)$$

We note that the FPT density $g(0,t|x_0,t_0)$ is not affected by the boundary condition on the zero-state, provided that it is attainable.

The problem of determining FPT densities for the Feller-type diffusion process arises in a variety of fields, including neurobiology, population dynamics, queueing systems and mathematical finance (cf., for instance, Linetsky [30], Masoliver and Perelló [35], Buonocore et al. [36], D'Onofrio et al. [37], Giorno et al. [38,39], Albano e Giorno [40], Di Nardo and D'Onofrio [41]). For instance, in population dynamics $g(0,t|x_0,t_0)$ describes the extinction density, whereas in queueing systems represents the busy period density. Lavigne and Roques in [18] focus on the distribution of the extinction times of a population whose size is described by a time-inhomogeneous Feller-type diffusion process with infinitesimal drift $A_1(x,t) = \alpha(t)\,x$ and infinitesimal variance $A_2(x,t) = \sigma^2\,x$, where $\alpha(t)$ is a continuous function and σ^2 is a positive constant.

The functions (2) and (7) are intimately related; indeed, one has:

$$\int_0^{+\infty} f_a(x,t|x_0,t_0)\,dx + \int_{t_0}^{t} g(0,\tau|x_0,t_0)\,d\tau = 1. \quad (8)$$

Relation (8) shows that the determination of $g(0,t|x_0,t_0)$ requires the explicit evaluation of the transition pdf $f_a(x,t|x_0,t_0)$ in the presence of an absorbing boundary at the zero-state.

Plain of the Paper

The paper is organized in five sections and seven appendices in which the proofs of the main results are reported. In Sect. 2, for the time-inhomogeneous Feller-type diffusion process $X(t)$, with infinitesimal moments (1), we give some preliminary results concerning the Laplace transform (according to x_0) of the transition pdf $f_a(x,t|x_0,t_0)$ in the presence of an absorbing boundary in the zero-state. The proportional case, in which the immigration intensity function $\beta(t)$ and the noise intensity function $r(t)$ are related as $\beta(t) = \xi\,r(t)$, with $0 \le \xi < 1$, is also analyzed. In Sect. 3, the transition pdf $f_a(x,t|x_0,t_0)$ is obtained for the process (1) in the general case, by distinguishing the case $x = 0$ (Sect. 3.1) and $x > 0$

(Sect. 3.2). In Sect. 4, we focus on the FPT of $X(t)$ through the zero-state for the general case and we determine the expression of the FPT pdf $g(0,t|x_0,t_0)$. In Sects. 3 and 4, we also show as the results of the proportional case can be derived from the general case. In Sect. 5, various numerical computations are performed making use of MATHEMATICA to illustrate the effect of periodic intensity functions on the FPT pdf $g(0,t|x_0,t_0)$. Specifically, we assume that the growth intensity function $\alpha(t)$, the immigration intensity function $\beta(t)$ or both these functions exhibit some kind of periodicity. The FPT mean $t_1(0,t|x_0,t_0)$ and the coefficient of variation $\mathrm{CV}(0|x_0,t_0)=\sqrt{\mathrm{Var}(0|x_0,t_0)}/t_1(0|x_0,t_0)$ are also analyzed.

2 Preliminary Results

In this section, we determine the Laplace transform (according to x_0) of the transition pdf $f_a(x,t|x_0,t_0)$ in the general case. Furthermore, the explicit expressions of the transition pdf and of the FPT density through the zero-state are obtained in the proportional case.

2.1 Laplace Transform

For $t\ge t_0$ and $x\ge 0$, we consider the Laplace transform:

$$Z_a(x,t|s,t_0)=\int_0^{+\infty} e^{-s x_0} f_a(x,t|x_0,t_0)\,dx_0,\quad \mathrm{Re}\,s>0. \tag{9}$$

We determine $Z_a(x,t|s,t_0)$ so that, by taking its inverse Laplace transform, we obtain $f_a(x,t|x_0,t_0)$. Multiplying both sides of (3) by e^{-sx_0}, integrating with respect to x_0 over the interval $[0,+\infty)$ and making use of the boundary condition (5), we have the following partial differential equation

$$\begin{aligned}&\frac{\partial Z_a(x,t|s,t_0)}{\partial t_0}-s\,[\alpha(t_0)+s\,r(t_0)]\,\frac{\partial Z_a(x,t|s,t_0)}{\partial s}\\&+[s\,\beta(t_0)-\alpha(t_0)-2\,s\,r(t_0)]\,Z_a(x,t|s,t_0)=0,\end{aligned} \tag{10}$$

to solve with the initial condition

$$\lim_{t_0\uparrow t} Z_a(x,t|s,t_0)=e^{-sx}, \tag{11}$$

derived from (9) by using the initial condition (4).

Proposition 1 *We assume that $\alpha(t)\in\mathbb{R}$, $\beta(t)\in\mathbb{R}$, $r(t)>0$, $\beta(t)\le\xi\,r(t)$, with $0\le\xi<1$. For $t\ge t_0$, we have:*

$$\begin{aligned}Z_a(x,t|s,t_0)=&\frac{e^{-A(t|t_0)}}{[1+sR(t|t_0)]^2}\,\exp\Big\{-\frac{s\,x\,e^{-A(t|t_0)}}{1+sR(t|t_0)}\Big\}\\&\times\exp\left\{\int_{t_0}^t\beta(u)\,\frac{s\,e^{-A(u|t_0)}}{1+sR(u|t_0)}\,du\right\},\qquad x\ge 0,\end{aligned} \tag{12}$$

where

$$A(t|t_0)=\int_{t_0}^t\alpha(z)\,dz,\qquad R(t|t_0)=\int_{t_0}^t r(\tau)\,e^{-A(\tau|t_0)}\,d\tau. \tag{13}$$

Proof The proof is given in Appendix A. □

2.2 Proportional Case

For all $t \geq 0$, we suppose that the continuous functions $\beta(t)$ and $r(t)$ are proportional, i.e.

$$\frac{\beta(t)}{r(t)} = \xi, \qquad 0 \leq \xi < 1. \tag{14}$$

In the absence of immigration, i.e. when $\beta(t) = 0$ for all $t \geq 0$, one has $\xi = 0$.

Proposition 2 *Under the assumption (14), for $t \geq t_0$ one has:*

$$Z_a(x, t|s, t_0) = \frac{e^{-A(t|t_0)}}{[1 + s R(t|t_0)]^{2-\xi}} \exp\Big\{-\frac{s\, x\, e^{-A(t|t_0)}}{1 + s R(t|t_0)}\Big\}, \qquad x \geq 0. \tag{15}$$

Furthermore, the transition pdf of $X(t)$ in the presence of an absorbing boundary in the zero-state is:

$$f_a(x, t|x_0, t_0) = \begin{cases} \frac{e^{-A(t|t_0)}}{\Gamma(2-\xi)} \Big[\frac{1}{R(t|t_0)}\Big]^{2-\xi} x_0^{1-\xi} \exp\Big\{-\frac{x_0}{R(t|t_0)}\Big\}, & x = 0, \\ \frac{e^{-A(t|t_0)}}{R(t|t_0)} \Big(\frac{x_0}{x}\Big)^{(1-\xi)/2} \exp\Big\{-\frac{x_0 + x\, e^{-A(t|t_0)}}{R(t|t_0)}\Big\} & \\ \quad \times \exp\Big\{\frac{1-\xi}{2} A(t|t_0)\Big\} I_{1-\xi}\Big[\frac{2\sqrt{x\, x_0\, e^{-A(t|t_0)}}}{R(t|t_0)}\Big], & x > 0, \end{cases} \tag{16}$$

with $A(t|t_0)$ and $R(t|t_0)$ defined in (13) and where

$$I_\nu(z) = \sum_{k=0}^{+\infty} \frac{1}{k!\, \Gamma(\nu + k + 1)} \Big(\frac{z}{2}\Big)^{2k+\nu}, \qquad \nu \in \mathbb{R} \tag{17}$$

denotes the modified Bessel function of the first kind.

Proof The proof is given in Appendix B. □

Note that, the first of (16) follows by taking the limit as $x \downarrow 0$ in the second, recalling that for fixed ν and for $z \to 0$ one has (cf. Abramowitz and Stegun [42], p. 375, no 9.6.7):

$$I_\nu(z) \sim \frac{1}{\Gamma(\nu + 1)} \Big(\frac{z}{2}\Big)^\nu \qquad \nu \neq -1, -2, \ldots$$

If (14) holds, for $t \geq t_0$, $x > 0$ and $x_0 > 0$ from (16) it follows:

$$f_a(x, t|x_0, t_0) = \Big(\frac{x_0}{x}\Big)^{1-\xi} \exp\Big\{\frac{(x - x_0)\,[1 - e^{-A(t|t_0)}]}{R(t|t_0)}\Big\} f_a(x_0, t|x, t_0). \tag{18}$$

Proposition 3 *Under the assumption (14), for $t \geq t_0$ and $x_0 > 0$ one has:*

$$\int_0^{+\infty} f_a(x, t|x_0, t_0)\, dx = \frac{1}{\Gamma(1 - \xi)}\, \gamma\Big(1 - \xi, \frac{x_0}{R(t|t_0)}\Big), \qquad 0 \leq \xi < 1, \tag{19}$$

with $R(t|t_0)$ given in (13) and where

$$\gamma(a, z) = \int_0^z e^{-y}\, y^{a-1}\, dy, \qquad \mathrm{Re}\, a > 0 \tag{20}$$

denotes the incomplete gamma function.

Proof Recalling (16) and using the transformation $y = x\, e^{-A(t|t_0)}/R(t|t_0)$ in the integral, one obtains:

$$\int_0^{+\infty} f_a(x,t|x_0,t_0)\,dx = \exp\Big\{-\frac{x_0}{R(t|t_0)}\Big\}\Big[\frac{x_0}{R(t|t_0)}\Big]^{(1-\xi)/2} \times \int_0^{+\infty} e^{-y} y^{-(1-\xi)/2} I_{1-\xi}\Big[2\sqrt{\frac{x_0\, y}{R(t|t_0)}}\Big]\,dy, \qquad 0\le\xi<1. \tag{21}$$

Since (cf. Erdèlyi et al. [43], p. 197, no. 19)

$$\int_0^{+\infty} e^{-py}\, y^{-\nu/2}\, I_\nu(2\sqrt{ay})\,dy = a^{-\nu/2}\, p^{\nu-1}\, e^{a/p}\,\frac{\gamma(\nu,a/p)}{\Gamma(\nu)}, \qquad \mathrm{Re}\, p>0,$$

Eq. (19) follows from (21). □

Proposition 4 *Under the assumption* (14)*, for* $t\ge t_0$ *and* $x_0>0$ *the FPT pdf through the zero-state of* $X(t)$ *is:*

$$g(0,t|x_0,t_0) = \frac{1}{\Gamma(1-\xi)}\,\frac{r(t)\,e^{-A(t|t_0)}}{R(t|t_0)}\Big[\frac{x_0}{R(t|t_0)}\Big]^{1-\xi}\exp\Big\{-\frac{x_0}{R(t|t_0)}\Big\}, \tag{22}$$

with $R(t|t_0)$ *given in* (13)*. Furthermore, for* $0\le\xi<1$ *the ultimate FPT probability is:*

$$P\{T(x_0,t_0)<+\infty\} = \begin{cases} 1, & \lim\limits_{t\to+\infty} R(t|t_0) = +\infty,\\[2ex] 1-\dfrac{\gamma(1-\xi,x_0/c)}{\Gamma(1-\xi)}, & \lim\limits_{t\to+\infty} R(t|t_0) = c<+\infty. \end{cases} \tag{23}$$

Proof By virtue of (8), making use of (19), for $0\le\xi<1$ one has:

$$g(0,t|x_0,t_0) = -\frac{\partial}{\partial t}\int_0^{+\infty} f_a(x,t|x_0,t_0)\,dx = -\frac{1}{\Gamma(1-\xi)}\,\frac{\partial}{\partial t}\gamma\Big(1-\xi,\frac{x_0}{R(t|t_0)}\Big),$$

from which (22) follows. Furthermore, taking the limit as $t\to+\infty$ in (8), it results

$$P\{T(x_0,t_0)<+\infty\} = \int_0^{+\infty} g(0,t|x_0,t_0)\,dt = 1-\lim_{t\to+\infty}\int_0^{+\infty} f_a(x,t|x_0,t_0)\,dx,$$

so that, recalling (19), one is lead to (23). □

Two interesting cases occur when $\xi=0$ and $\xi=1/2$.

Indeed, by setting $\xi=0$ in (14), one considers the time-inhomogeneous Feller-type diffusion process (1) with $\beta(t)=0$. In the context of population dynamics, this case describes the absence of the immigration and it is of interest to determine for which choices of $\alpha(t)$ and $r(t)$ the population is doomed to extinction as the time increases. Recalling that $\gamma(1,z) = 1-e^{-z}$, for $t\ge t_0$, $x_0>0$ and $\xi=0$, from (19) one has

$$\int_0^{+\infty} f_a(x,t|x_0,t_0)\,dx = 1-\exp\Big\{-\frac{x_0}{R(t|t_0)}\Big\}, \tag{24}$$

and from (22) one obtains:

$$g(0,t|x_0,t_0) = \frac{r(t)\,x_0\,e^{-A(t|t_0)}}{R^2(t|t_0)}\exp\Big\{-\frac{x_0}{R(t|t_0)}\Big\}. \tag{25}$$

Instead, when $\xi = 1/2$ in (14), the FPT pdf (22) identifies with the FPT pdf through the zero-state for special time-inhomogeneous Wiener or Ornstein-Uhlenbeck diffusion processes (see, for instance, Giorno and Nobile [1]). Specifically, if $\xi = 1/2$ and $\alpha(t) = 0$, the density (22) identifies with the FPT pdf $g_W(0, t|\sqrt{x_0}, t_0)$ of a time-inhomogeneous Wiener process, with state-space in $\mathbb{R}$, having infinitesimal drift $B_1(x, t) = 0$ and infinitesimal variance $B_2(t) = r(t)/2$; instead, if $\xi = 1/2$ and $\alpha(t) \neq 0$, the density (22) identifies with the FPT pdf $g_{OU}(0, t|\sqrt{x_0}, t_0)$ of a time-inhomogeneous Ornstein-Uhlenbeck process, with state-space in $\mathbb{R}$, having infinitesimal drift $C_1(x, t) = [\alpha(t)/2]\, x$ and infinitesimal variance $C_2(t) = r(t)/2$.

Under the assumption (14), if $\lim_{t\to+\infty} R(t|t_0) = +\infty$, it is meaningful to evaluate the FPT moments through the zero-state starting from $X(t_0) = x_0 > 0$:

$$t_k(0|x_0, t_0) = \int_0^{+\infty} t^k\, g(0, t|x_0, t_0)\, dt, \qquad k = 1, 2, \ldots$$

Indeed, if $\lim_{t\to+\infty} R(t|t_0) = +\infty$, from Proposition 4 one has $P\{T(x_0, t_0) < +\infty\} = 1$ and, making use of (8) and (19), for $0 \le \xi < 1$ one has:

$$\begin{aligned} t_k(0|x_0, t_0) &= k \int_0^{+\infty} t^{k-1}\Big[\int_0^{+\infty} f_a(x, t|x_0, t_0)\, dx\Big]\, dt \\ &= \frac{k}{\Gamma(1-\xi)} \int_0^{+\infty} t^{k-1}\gamma\Big(1-\xi, \frac{x_0}{R(t|t_0)}\Big)\, dt, \qquad k = 1, 2, \ldots \end{aligned} \tag{26}$$

We finally note that, for the time-homogeneous Feller process, in which $\alpha(t) = \alpha$, $\beta(t) = \xi\, r$, $r(t) = r$, with $\alpha \in \mathbb{R}$, $r > 0$ and $0 \le \xi < 1$, the pdf $f_a(x, t|x_0, t_0)$ and the FPT pdf $g(0, t|x_0, t_0)$ can be easily obtained from (16) and (22) by setting

$$A(t|t_0) = \alpha(t - t_0), \qquad R(t|t_0) = \begin{cases} r(t - t_0), & \alpha = 0, \\ \dfrac{r}{\alpha}\big(1 - e^{-\alpha(t-t_0)}\big), & \alpha \neq 0. \end{cases} \tag{27}$$

3 General Case

We assume that $\alpha(t)$, $\beta(t)$ and $r(t)$ are continuous functions such that $\alpha(t) \in \mathbb{R}$, $\beta(t) \in \mathbb{R}$, $r(t) > 0$, $\beta(t) \le \xi\, r(t)$, with $0 \le \xi < 1$. From (12), for $t \ge t_0$ we have

$$Z_a(x, t|s, t_0) = \begin{cases} Z_a(0, t|s, t_0), & x = 0, \\ Z_a(0, t|s, t_0)\, V_a(x, t|s, t_0), & x > 0, \end{cases} \tag{28}$$

where

$$V_a(x, t|s, t_0) = \exp\Big\{-\frac{s\, x\, e^{-A(t|t_0)}}{1 + s R(t|t_0)}\Big\}, \tag{29}$$

with $A(t|t_0)$ and $R(t|t_0)$ given in (13). We note that $V_a(x, t|s, t_0)$ does not dependent upon $\beta(t)$. Therefore, to obtain the transition pdf $f_a(x, t|x_0, t_0)$ for $X(t)$ with infinitesimal moments (1), we proceed as follows:

(1) we determine the transition pdf $f_a(0, t|x_0, t_0)$ for $x_0 > 0$ and $t \ge t_0$ by taking the inverse Laplace transform of $Z_a(0, t|s, t_0)$;

(2) we find the inverse Laplace transform $v_a(x,t|x_0,t_0)$ of (29) and we calculate the transition pdf $f_a(x,t|x_0,t_0)$ as a convolution, according to x_0, between $f_a(0,t|x_0,t_0)$ and the function $v_a(x,t|x_0,t_0)$ for $x>0$, $x_0>0$ and $t\geq t_0$.

3.1 General Case: $x = 0$

In this section, we obtain the transition pdf in the presence of an absorbing boundary in the zero-state when the process $X(t)$ reaches $x=0$ at time $t\geq t_0$. By setting $x=0$ in (12), for $t\geq t_0$ we obtain:

$$Z_a(0,t|s,t_0)=\frac{e^{-A(t|t_0)}}{[1+sR(t|t_0)]^2}\,\exp\left\{\int_{t_0}^{t}\beta(u)\frac{s\,e^{-A(u|t_0)}}{1+sR(u|t_0)}\,du\right\},\tag{30}$$

with $A(t|t_0)$ and $R(t|t_0)$ defined in (13).

In the sequel, we denote by $B_n(d_1,d_2,\ldots,d_n)$ the complete Bell polynomials, recursively defined as follows:

$$B_0=1,\quad B_{n+1}(d_1,d_2,\ldots,d_{n+1})=\sum_{i=0}^{n}\binom{n}{i}B_{n-i}(d_1,d_2,\ldots,d_{n-i})\,d_{i+1},\quad n\in\mathbb{N}_0,\tag{31}$$

with

$$d_k=\frac{k!}{[R(t|t_0)]^k}\int_{t_0}^{t}\beta(u)e^{-A(u|t_0)}[R(t|t_0)-R(u|t_0)]^{k-1}\,du,\qquad k=1,2,\ldots\tag{32}$$

Proposition 5 *Under the assumption of Proposition 1, for $t\geq t_0$ and $x_0>0$ the transition pdf of the time-inhomogeneous Feller-type diffusion process $X(t)$ with an absorbing boundary in the zero-state is*

$$f_a(0,t|x_0,t_0)=\frac{x_0\,e^{-A(t|t_0)}}{R^2(t|t_0)}\,\exp\left\{-\frac{x_0}{R(t|t_0)}\right\}\Psi(t|x_0,t_0),\tag{33}$$

where

$$\Psi(t|z,t_0)=\sum_{n=0}^{+\infty}\frac{B_n(d_1,d_2,\ldots,d_n)}{(n+1)!}\,L_n^{(1)}\left[\frac{z}{R(t|t_0)}\right],\qquad z>0,\tag{34}$$

with $A(t|t_0)$ and $R(t|t_0)$ defined in (13), $B_n(d_1,d_2,\ldots,d_n)$ given in (31) and in (32), and

$$L_n^{(a)}(y)=\sum_{k=0}^{n}(-1)^k\binom{n+a}{n-k}\frac{y^k}{k!},\qquad a\geq 0,\ n=0,1,\ldots\tag{35}$$

denoting the Laguerre polynomials.

Proof The proof is given in Appendix C. □

Remark 1 *(Proportional case)* We assume that (14) holds. We prove that the first of (16) follows from (33).

Indeed, from (31) and (32) one has

$$d_n=\xi\,(n-1)!,\qquad B_0=1,\quad B_n(d_1,d_2,\ldots,d_n)=(\xi)_n,\qquad n=1,2,\ldots,\tag{36}$$

where $(\xi)_n$ denotes the Pochhammer symbol defined as $(\xi)_0 = 1$ and $(\xi)_n = \xi\,(\xi+1)\cdots(\xi+n-1)$ for $n = 1, 2, \ldots$ Recalling that the series of Laguerre polynomials satisfies the following identity (cf. Erdèlyi et al [44], p. 213, no. 16):

$$\sum_{n=0}^{+\infty} \frac{(\xi)_n}{\Gamma(a+n+1)}\, L_n^{(a)}(y) = \frac{y^{-\xi}}{\Gamma(a-\xi+1)}, \quad a > 0,\, y > 0,\, 0 \le \xi < a+1, \tag{37}$$

under the assumption (14), from (34) one has:

$$\Psi(t|z, t_0) = \sum_{n=0}^{+\infty} \frac{(\xi)_n}{(n+1)!}\, L_n^{(1)}\Big[\frac{z}{R(t|t_0)}\Big] = \frac{1}{\Gamma(2-\xi)}\Big[\frac{z}{R(t|t_0)}\Big]^{-\xi}, \quad z > 0. \tag{38}$$

Hence, if (14) holds, Eq. (33) identifies with the first of (16). ◊

3.2 General Case: $x > 0$

In this section, we obtain the transition pdf $f_a(x, t|x_0, t_0)$ for $x > 0$ and $t \ge t_0$. From (29), for $t \ge t_0$, $x > 0$ and $\mathrm{Re}\, s > 0$ we have $V_a(x, t|s, t_0) \ge 0$ and

$$\lim_{s \downarrow 0} V_a(x, t|s, t_0) = 1.$$

We show that $V_a(x, t|s, t_0)$ is the Laplace transform of a function $v_a(x, t|x_0, t_0)$, i.e.

$$V_a(x, t|s, t_0) = \int_0^{+\infty} e^{-s\,x_0}\, v_a(x, t|x_0, t_0)\, dx_0, \qquad \mathrm{Re}\, s > 0. \tag{39}$$

Proposition 6 *Under the assumption of Proposition 1, for $x_0 > 0$ and $t \ge t_0$, one has:*

$$\begin{aligned} v_a(x, t|x_0, t_0) &= \exp\Big\{-\frac{x\, e^{-A(t|t_0)}}{R(t|t_0)}\Big\}\, \delta(x_0) + \frac{1}{R(t|t_0)}\sqrt{\frac{x}{x_0}\, e^{-A(t|t_0)}} \\ &\times \exp\Big\{-\frac{x_0 + x\, e^{-A(t|t_0)}}{R(t|t_0)}\Big\}\, I_1\Big[\frac{2\sqrt{x\, x_0\, e^{-A(t|t_0)}}}{R(t|t_0)}\Big], \quad x > 0, \end{aligned} \tag{40}$$

with $A(t|t_0)$ and $R(t|t_0)$ defined in (13), whereas $\delta(x)$ denotes the delta Dirac function and $I_\nu(z)$ represents the Bessel function modified of first kind.

Proof The proof is given in Appendix D. □

The function $v_a(x, t|x_0, t_0)$ in (40) is the sum of two terms. The second term in (40) identifies with $x\, f_a(x, t|x_0, t_0)/x_0$, where $f_a(x, t|x_0, t_0)$ is given in (16) for $x > 0$ and $\xi = 0$ (absence of immigration). Since,

$$\int_0^{+\infty} \frac{x}{x_0}\, f_a(x, t|x_0, t_0)\, dx_0 = 1 - \exp\Big\{-\frac{x\, e^{-A(t|t_0)}}{R(t|t_0)}\Big\}, \qquad x > 0,$$

from (40) it follows that

$$\int_0^{+\infty} v_a(x, t|x_0, t_0)\, dx_0 = 1.$$

For $x > 0$, the transition pdf $f_a(x, t|x_0, t_0)$ can be obtained via a convolution, according to x_0, between the pdf $f_a(0, t|x_0, t_0)$ and the function $v_a(x, t|x_0, t_0)$, determined in Propositions 5 and 6, respectively:

$$f_a(x,t|x_0,t_0)=\int_0^{x_0} f_a(0,t|z,t_0)\, v_a(x,t|x_0-z,t_0)\, dz, \quad x>0, x_0>0. \tag{41}$$

Proposition 7 *Under the assumption of Proposition 1, for $t\ge t_0$, $x>0$ and $x_0>0$ one has:*

$$\begin{aligned}f_a(x,t|x_0,t_0)&=\frac{e^{-A(t|t_0)}}{R^2(t|t_0)}\exp\Big\{-\frac{x_0+x\,e^{-A(t|t_0)}}{R(t|t_0)}\Big\}\Big\{x_0\,\Psi(t|x_0,t_0)\\&+\frac{\sqrt{x\,e^{-A(t|t_0)}}}{R(t|t_0)}\int_0^{x_0}\frac{z}{\sqrt{x_0-z}}\,I_1\Big[\frac{2\sqrt{x\,(x_0-z)\,e^{-A(t|t_0)}}}{R(t|t_0)}\Big]\Psi(t|z,t_0)\,dz\Big\},\end{aligned} \tag{42}$$

with $A(t|t_0)$ and $R(t|t_0)$ given in (13) and $\Psi(t|z,t_0)$ defined in (34).

Proof It follows from (41), by virtue of (33) and (40). □

Note that, by taking the limit as $x\downarrow 0$ in (42), we obtain (33).

Remark 2 *(Proportional case)* We assume that (14) holds. We prove that the second of (16) follows from (42).

Indeed, recalling (36) and (38), from (42) for $t\ge t_0$, $x>0$ and $x_0>0$ one has:

$$\begin{aligned}f_a(x,t|x_0,t_0)&=\frac{e^{-A(t|t_0)}}{\Gamma(2-\xi)}\Big[\frac{1}{R(t|t_0)}\Big]^{2-\xi}\exp\Big\{-\frac{x_0+x\,e^{-A(t|t_0)}}{R(t|t_0)}\Big\}\\&\times\Big\{x_0^{1-\xi}+\frac{\sqrt{xe^{-A(t|t_0)}}}{R(t|t_0)}\int_0^{x_0}\frac{(x_0-y)^{1-\xi}}{\sqrt{y}}\,I_1\Big[\frac{2\sqrt{x\,y\,e^{-A(t|t_0)}}}{R(t|t_0)}\Big]dy\Big\}.\end{aligned} \tag{43}$$

By virtue of (17), one obtains

$$\int_0^{x_0}\frac{(x_0-y)^{1-\xi}}{\sqrt{y}}\,I_1\big(2a\sqrt{y}\big)\,dy=\frac{x_0^{-\xi}}{a^2}\Big\{-ax_0+a^{\xi}x_0^{(1+\xi)/2}I_{1-\xi}(2a\sqrt{x_0})\Gamma(2-\xi)\Big\},$$

for $a>0$, $\xi<2$ and $x_0>0$. Hence, under the assumption (14), Eq. (43) leads to the second of (16). ♢

4 The First-Passage Time Through the Zero-State

We now focus on the distribution function of the FPT through the zero-state for the time-inhomogeneous Feller-type diffusion process $X(t)$, with infinitesimal moments (1), when $\alpha(t)$, $\beta(t)$ and $r(t)$ are continuous functions such that $\alpha(t)\in\mathbb{R}$, $\beta(t)\in\mathbb{R}$, $r(t)>0$, $\beta(t)\le\xi\, r(t)$, with $0\le\xi<1$. The FPT problem of $X(t)$ through the zero-state can be studied starting from Eq. (8) and making use of (42).

Proposition 8 *Under the assumption of Proposition 1, for $t\ge t_0$ and $x_0>0$ one has:*

$$\begin{aligned}\int_0^{+\infty}f_a(x,t|x_0,t_0)\,dx&=1-\exp\Big\{-\frac{x_0}{R(t|t_0)}\Big\}+\frac{x_0}{R(t|t_0)}\exp\Big\{-\frac{x_0}{R(t|t_0)}\Big\}\\&\times\sum_{n=1}^{+\infty}\frac{B_n(d_1,d_2,\ldots,d_n)}{n!}\,\Phi\Big(1-n,2;\frac{x_0}{R(t|t_0)}\Big),\end{aligned} \tag{44}$$

with $R(t|t_0)$ defined in (13), $B_n(d_1, d_2, \ldots, d_n)$ given in (31) and (32) and where

$$\Phi(a, b; x) = 1 + \sum_{n=1}^{+\infty} \frac{(a)_n}{(b)_n} \frac{x^n}{n!} \tag{45}$$

denotes the confluent hypergeometric function (Kummer's function).

Proof The proof is given in Appendix E. □

Remark 3 *(Proportional case)* We assume that (14) holds. We prove that from (44) one obtains (19).

Indeed, recalling (36) and making use of the relation $\Phi(1, 2; z) = (e^z - 1)/z$, from (44) for $t \geq t_0$ and $x_0 > 0$ one has:

$$\int_0^{+\infty} f_a(x, t|x_0, t_0)\, dx = \frac{x_0}{R(t|t_0)} \exp\Big\{-\frac{x_0}{R(t|t_0)}\Big\} \sum_{n=0}^{+\infty} \frac{(\xi)_n}{n!} \Phi\Big(1 - n, 2; \frac{x_0}{R(t|t_0)}\Big). \tag{46}$$

Since (Tricomi [45], p. 31, no. 10)

$$\sum_{n=0}^{+\infty} \frac{(b - c)_n}{n!} \Phi(a - n, b; z) = \frac{\Gamma(b)}{\Gamma(c)} z^{c-b}\, \Phi(a, c; z), \quad b > 0, c > 0, b - c > 0, \tag{47}$$

from (46) one has:

$$\int_0^{+\infty} f_a(x, t|x_0, t_0)\, dx = \frac{e^{-x_0/R(t|t_0)}}{\Gamma(2 - \xi)} \Big[\frac{x_0}{R(t|t_0)}\Big]^{1-\xi} \Phi\Big(1, 2 - \xi; \frac{x_0}{R(t|t_0)}\Big). \tag{48}$$

The incomplete gamma function (20) can be expressed in terms of the Kummer's function (cf. Tricomi [45]p. 160, no. 7):

$$\gamma(a, z) = \frac{1}{a} e^{-z} z^a\, \Phi(1, a + 1, z), \qquad \mathrm{Re}\, a > 0,$$

so that Eq. (19) follows from (48). ◊

Relation (44) plays an important role in the determination of the FPT distribution function and of the FPT density through the zero-state. Indeed, by virtue of (8), for $t \geq t_0$ and $x_0 > 0$ the FPT distribution function is

$$P\{T(x_0, t_0) < t\} = \int_{t_0}^{t} g(0, \tau|x_0, t_0)\, d\tau = 1 - \int_0^{+\infty} f_a(x, t|x_0, t_0)\, dx, \tag{49}$$

so that the FPT density through the zero-state can be obtained as

$$g(0, t|x_0, t_0) = -\frac{\partial}{\partial t} \int_0^{+\infty} f_a(x, t|x_0, t_0)\, dx. \tag{50}$$

Proposition 9 *Under the assumption of Proposition 1, for $t \geq t_0$ and $x_0 > 0$ one has:*

$$\begin{aligned} g(0, t|x_0, t_0) = \frac{x_0}{R(t|t_0)} \exp\Big\{-\frac{x_0}{R(t|t_0)}\Big\} \Bigg\{ \frac{r(t)\, e^{-A(t|t_0)}}{R(t|t_0)} \\ \times \sum_{n=0}^{+\infty} \frac{B_n(d_1, d_2, \ldots, d_n)}{n!} \Phi\Big(-n, 1; \frac{x_0}{R(t|t_0)}\Big) \\ - \sum_{n=1}^{+\infty} \frac{1}{n!} \Phi\Big(1 - n, 2; \frac{x_0}{R(t|t_0)}\Big) \frac{d}{dt} B_n(d_1, d_2, \ldots, d_n) \Bigg\}, \end{aligned} \tag{51}$$

with $A(t|t_0)$ *and* $R(t|t_0)$ *defined in* (13) *and* $B_n(d_1, d_2, \ldots, d_n)$ *given in* (31) *and* (32).

Proof The proof is given in Appendix F. $\square$

Remark 4 *(Proportional case)* We assume that (14) holds. We prove that from (51) one obtains (22).

Indeed, from (36) one has $B_0 = 1$ and $B_n(d_1, d_2, \ldots, d_n) = (\xi)_n$ for $n = 1, 2, \ldots$, so that, under the assumption (14), for $t \ge t_0$ and $x_0 > 0$ from (51) one has:

$$g(0, t|x_0, t_0) = \frac{x_0\, r(t)\, e^{-A(t|t_0)}}{R^2(t|t_0)} \exp\Big\{-\frac{x_0}{R(t|t_0)}\Big\} \sum_{n=0}^{+\infty} \frac{(\xi)_n}{n!}\, \Phi\Big(-n, 1; \frac{x_0}{R(t|t_0)}\Big). \tag{52}$$

Making use of (47), it results

$$\sum_{n=0}^{+\infty} \frac{(\xi)_n}{n!}\, \Phi\Big(-n, 1; \frac{x_0}{R(t|t_0)}\Big) = \frac{1}{\Gamma(1-\xi)} \Big[\frac{x_0}{R(t|t_0)}\Big]^{-\xi},$$

so that (22) follows from (52). $\diamondsuit$

5 Special Cases

Under the assumption (14), we analyze the cases in which the growth intensity function $\alpha(t)$, or the immigration intensity function $\beta(t)$ or both of them have some kind of periodicity. These cases are of interest in various applied fields, such as in population growth and in queueing systems. Indeed, periodic immigration intensity functions play an important role in the description of the evolution of dynamic for systems influenced by seasonal immigration or other regular environmental cycles. Furthermore, periodic growth intensity functions express the existence of fluctuation in the population dynamics and the presence of rush hours occurring on a daily basis in queueing systems.

5.1 Periodic Immigration Intensity Function

We consider the time-inhomogeneous Feller-type process $X(t)$ such that

$$A_1(x, t) = \alpha\, x + \xi\, r(t), \qquad A_2(x, t) = 2\, r(t)\, x, \tag{53}$$

with $\alpha \in \mathbb{R}$, $0 \le \xi < 1$ and

$$r(t) = \nu\Big[1 + c\, \sin\Big(\frac{2\pi t}{Q}\Big)\Big], \qquad t \ge 0, \tag{54}$$

where $\nu > 0$ is the average of the periodic function $r(t)$ of period Q, c is the amplitude of the oscillations, with $0 \le c < 1$. From (13), for $t \ge t_0$ one has $A(t|t_0) = \alpha\,(t - t_0)$ and

$$R(t|t_0) = \begin{cases} \nu(t - t_0) + \frac{c\,\nu\,Q}{2\pi}\Big[\cos\Big(\frac{2\pi t_0}{Q}\Big) - \cos\Big(\frac{2\pi t}{Q}\Big)\Big], & \alpha = 0, \\[2ex] \frac{\nu}{\alpha}\Big(1 - e^{-\alpha(t-t_0)}\Big) + \frac{c\,\nu\,Q}{4\pi^2 + Q^2\alpha^2}\Big\{2\pi\,\cos\Big(\frac{2\pi t_0}{Q}\Big) & \\ +\alpha\,Q\,\sin\Big(\frac{2\pi t_0}{Q}\Big) - e^{-\alpha(t-t_0)}\Big[2\pi\,\cos\Big(\frac{2\pi t}{Q}\Big) + \alpha\,Q\,\sin\Big(\frac{2\pi t}{Q}\Big)\Big]\Big\}, & \alpha \ne 0. \end{cases} \tag{55}$$

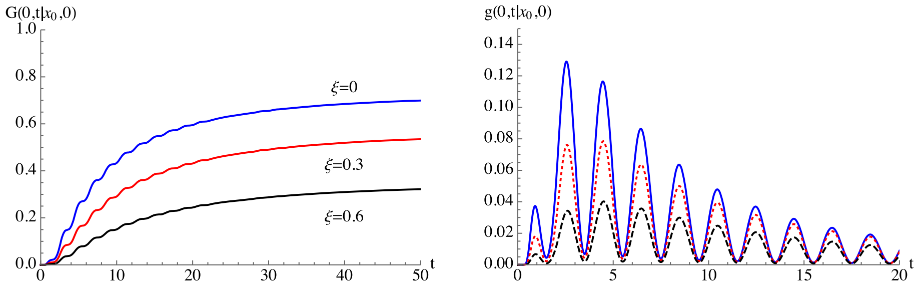


Fig. 1 The FPT distribution (on the left) and the FPT density (on the right) through the zero-state starting from $X(0) = 5$ are plotted as function of t for the process (53), with $\alpha = 0.05$, $r(t)$ given in (54) with $\nu = 0.75$, $c = 0.9$, $Q = 2$ with $\xi = 0$ (blue solid curve), $\xi = 0.3$ (red dotted curve) and $\xi = 0.6$ (black dashed curve)

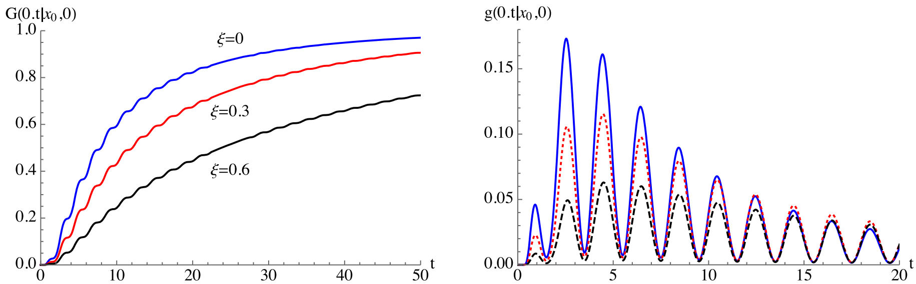


Fig. 2 As in Fig. 1 for $\alpha = -0.05$

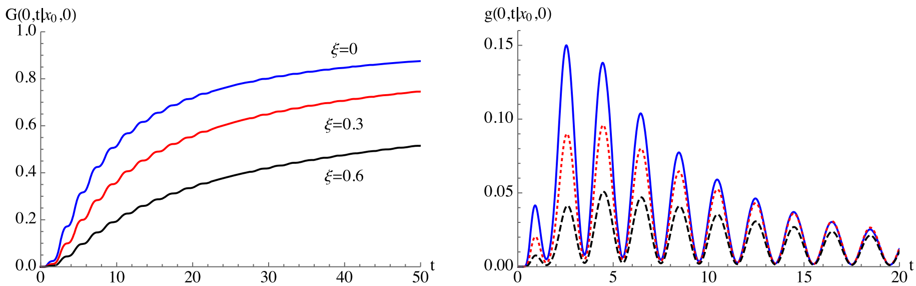


Fig. 3 As in Fig. 1 for $\alpha = 0$

Then, from (55) one obtains:

$$\lim_{t\to+\infty} R(t|t_0) = \begin{cases} +\infty, & \alpha \le 0, \\ \frac{\nu}{\alpha} + \frac{c\,\nu\,Q}{4\pi^2+Q^2\alpha^2}\left[2\pi\cos\left(\frac{2\pi t_0}{Q}\right) + \alpha\,Q\sin\left(\frac{2\pi t_0}{Q}\right)\right], & \alpha > 0, \end{cases}$$

so that, by virtue of (23), the FPT through the zero-state is a certain event for $\alpha \le 0$. Moreover, for $\alpha = 0$ the FPT moments (26) are divergent.

In Figs. 1, 2 and 3, the FPT distribution $G(0, t|x_0, t_0) = 1 - \int_0^{+\infty} f_a(x, t|x_0, t_0)\, dx$, obtained making use of (19), and the FPT pdf $g(0, t|x_0, t_0)$, given in (22), are plotted as function of t for the diffusion process (53) for some choices of parameters. In Fig. 4, the mean $t_1(0|x_0, t_0)$ and the coefficient of variation $\mathrm{CV}(0|x_0, t_0)$, obtained making use of (26),

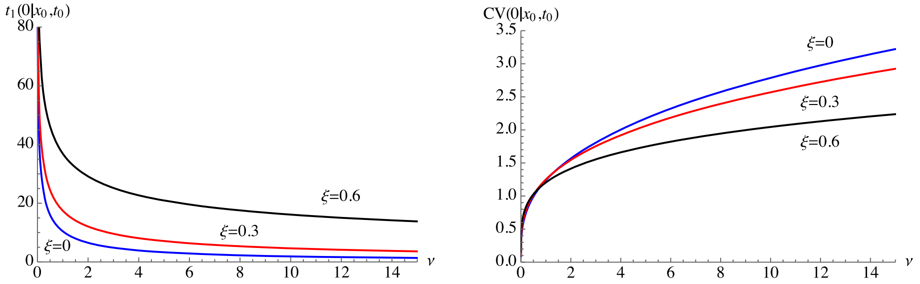


Fig. 4 The mean (on the left) and the coefficient of variation (on the right) of FPT from $X(0) = 5$ through the zero-state are plotted as function of ν for the diffusion process (53) with $\alpha = -0.05$, $c = 0.9$, $Q = 2$

are plotted as function of ν for $\xi = 0, 0.3, 0.6$. We note that as ν increases, the FPT mean $t_1(0|x_0, t_0)$ decreases whereas the coefficient of variation increases. Instead, as ξ increases in $[0, 1)$, the FPT mean increases and the coefficient of variation decreases, due to a raise of the immigration intensity function.

5.2 Periodic Growth Intensity Function

We consider the time-inhomogeneous Feller-type process $X(t)$ such that

$$A_1(x,t) = \alpha(t)\,x + \xi\,r, \qquad A_2(x) = 2\,r\,x, \tag{56}$$

with $r > 0$, $0 \le \xi < 1$ and

$$\alpha(t) = \eta - \frac{2\pi\,b}{Q_1}\,\frac{\cos\Big(\frac{2\pi t}{Q_1}\Big)}{1 + b\,\sin\Big(\frac{2\pi t}{Q_1}\Big)}, \qquad t \ge 0, \tag{57}$$

where $\eta \in \mathbb{R}$ is the average of the periodic function $\alpha(t)$ of period Q_1, b determines the amplitude of the oscillations, with $0 \le b < 1$. In Fig. 5, the intensity function (57) is plotted as function of t for some choices of parameters η, b and Q_1. The dotted lines refer to the average cases, in which $\alpha(t) = \eta$ with $\eta = -5$ (bottom) and $\eta = 5$ (top). From (13), for $t \ge t_0$ one has

$$A(t|t_0) = \eta\,(t - t_0) - \ln\Big[1 + b\,\sin\Big(\frac{2\pi t}{Q_1}\Big)\Big] + \ln\Big[1 + b\,\sin\Big(\frac{2\pi t_0}{Q_1}\Big)\Big], \tag{58}$$

and

$$R(t|t_0) = \begin{cases} \dfrac{r}{1+b\,\sin\Big(\frac{2\pi t_0}{Q_1}\Big)}\,\Big\{t - t_0 - \frac{b\,Q_1}{2\pi}\Big[\cos\Big(\frac{2\pi t}{Q_1}\Big) - \cos\Big(\frac{2\pi t_0}{Q_1}\Big)\Big]\Big\}, & \eta = 0,\\[2ex] \dfrac{r}{1+b\,\sin\Big(\frac{2\pi t_0}{Q_1}\Big)}\,\Big\{\frac{1-e^{-\eta(t-t_0)}}{\eta} - \frac{2\,\pi\,b\,Q_1}{4\pi^2+Q_1^2\eta^2}\Big[e^{-\eta(t-t_0)}\cos\Big(\frac{2\pi t}{Q_1}\Big) & \\ +\frac{Q_1\eta}{2\pi}e^{-\eta(t-t_0)}\sin\Big(\frac{2\pi t}{Q_1}\Big) - \cos\Big(\frac{2\pi t_0}{Q_1}\Big) - \frac{Q_1\eta}{2\pi}\,\sin\Big(\frac{2\pi t_0}{Q_1}\Big)\Big\}, & \eta \ne 0. \end{cases} \tag{59}$$

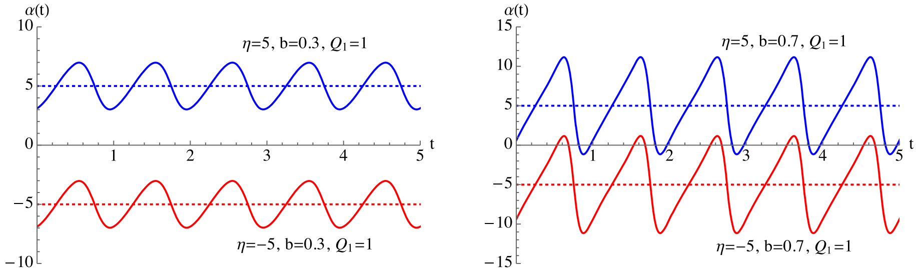


Fig. 5 The intensity function $\alpha(t)$, given in (57), is plotted as function of t for some choices of parameters. The dotted lines refer to the average cases

Then, from (59) one obtains:

$$\lim_{t\to+\infty} R(t|t_0) = \begin{cases} +\infty, & \eta \le 0, \\ \dfrac{r\left\{\frac{1}{\eta}+\frac{2\pi\, b\, Q_1}{4\pi^2+Q_1^2\,\eta^2}\left[\cos\left(\frac{2\pi t_0}{Q_1}\right)+\frac{Q_1\eta}{2\pi}\,\sin\left(\frac{2\pi t_0}{Q_1}\right)\right]\right\}}{1+b\,\sin\left(\frac{2\pi t_0}{Q_1}\right)}, & \eta > 0, \end{cases}$$

so that, by virtue of (23), the FPT through zero-state is a certain event for $\eta \le 0$. Moreover, for $\eta = 0$ the FPT moments (26) are divergent.

In Fig. 6, the FPT pdf $g(0,t|x_0,t_0)$, given in (22), is plotted as function of t for the process (56) for some choices of parameters. Instead, in Fig. 7, the mean $t_1(0|x_0,t_0)$ and the coefficient of variation $\mathrm{CV}(0|x_0,t_0)$, obtained making use of (26), are plotted as function of r for $\xi = 0, 0.3, 0.6$. We note that as r increases, the FPT mean $t_1(0|x_0,t_0)$ decreases, whereas the coefficient of variation increases. Moreover, the FPT mean and the coefficient of variation increase with ξ in $[0,1)$.

5.3 Periodic Immigration and Growth Intensity Functions

We consider the time-inhomogeneous Feller-type process $X(t)$ such that

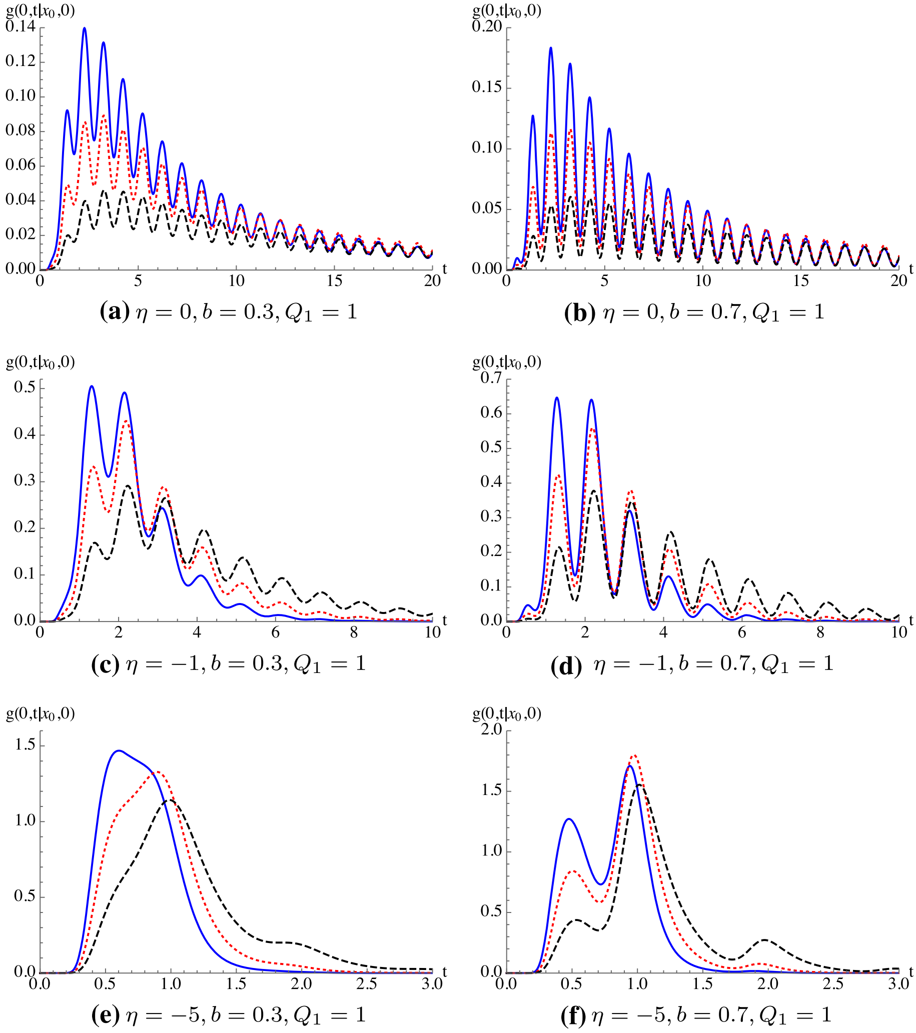
$$A_1(x,t) = \alpha(t)\,x + \xi\,r(t), \qquad A_2(x,t) = 2\,r(t)\,x, \tag{60}$$

with $0 \le \xi < 1$, $r(t)$ defined in (54) and $\alpha(t)$ given in (57). Recalling (13), for $t \ge t_0$ one obtains $A(t|t_0)$ given in (58) and

$$R(t|t_0) = \frac{\nu}{1+b\,\sin\left(\frac{2\pi t_0}{Q_1}\right)} \int_{t_0}^{t} e^{-\eta(\tau-t_0)}\left[1+c\,\sin\left(\frac{2\pi\,\tau}{Q}\right)\right]\left[1+b\,\sin\left(\frac{2\pi\,\tau}{Q_1}\right)\right] d\tau. \tag{61}$$

The explicit expression of $R(t|t_0)$ in (61) is obtained in Appendix G. We note that $\lim_{t\to+\infty} R(t|t_0)$ diverges as $\eta \le 0$, so that, due to (23), the FPT through the zero-state is a certain event for $X(t)$.

In Fig. 8, the FPT pdf $g(0,t|x_0,t_0)$, given in (22), is plotted as function of t for the process (60) for some choices of parameters. Comparing Figs. 6 and 8 , we note the effect of the different periodicities of the growth intensity function $\alpha(t)$, with $Q_1 = 1$, and of the immigration intensity function $\beta(t) = \xi\,r(t)$, with $Q = 2$. In Fig. 9, the mean $t_1(0|x_0,t_0)$ and the coefficient of variation $\mathrm{CV}(0|x_0,t_0)$, obtained making use of (26), are plotted as function



(a) $\eta = 0, b = 0.3, Q_1 = 1$ **(b)** $\eta = 0, b = 0.7, Q_1 = 1$

(c) $\eta = -1, b = 0.3, Q_1 = 1$ **(d)** $\eta = -1, b = 0.7, Q_1 = 1$

(e) $\eta = -5, b = 0.3, Q_1 = 1$ **(f)** $\eta = -5, b = 0.7, Q_1 = 1$

Fig. 6 FPT densities through the zero-state starting from $X(0) = 5$ are plotted as function of t for the process (56), with $r = 1, \alpha(t)$ given in (57) and with $\xi = 0$ (blue solid curve), $\xi = 0.3$ (red dotted curve) and $\xi = 0.6$ (black dashed curve)

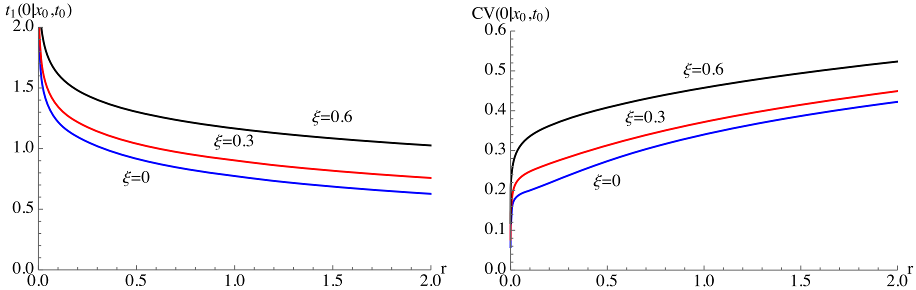


Fig. 7 The mean (on the left) and the coefficient of variation (on the right) of FPT from $X(0) = 5$ to the zero-state are plotted as function of r for the process (56) with $\eta = -5$, $b = 0.3$, $Q_1 = 1$

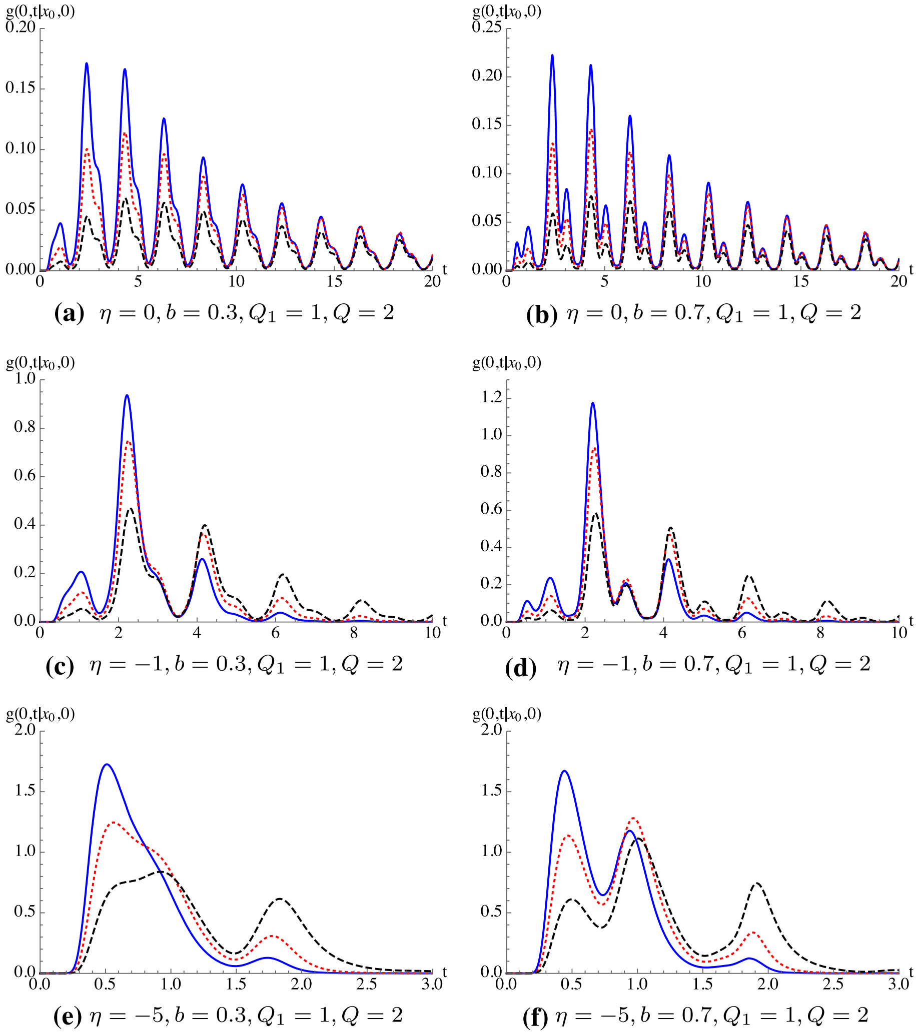


Fig. 8 FPT densities through the zero-state starting from $X(0) = 5$ are plotted as function of t for the process (60), being $r(t)$ defined in (54), with $\nu = 0.75$ and $c = 0.9$, and $\alpha(t)$ given in (57) with $\xi = 0$ (blue solid curve), $\xi = 0.3$ (red dotted curve) and $\xi = 0.6$ (black dashed curve)

of ν for $\xi = 0, 0.3, 0.6$. As ν increases, the FPT mean $t_1(0|x_0, t_0)$ decreases whereas the coefficient of variation increases. Instead, as ξ increases in $[0, 1)$, both the FPT mean and the coefficient of variation increase.

6 Concluding Remarks

In this paper, we have considered a time-inhomogeneous Feller-type diffusion process $\{X(t), t \geq t_0\}$, $t_0 \geq 0$, with infinitesimal drift $A_1(x, t) = \alpha(t)\, x + \beta(t)$ and infinitesimal variance $A_2(x, t) = 2\, r(t)\, x$, defined in the state-space $[0, +\infty)$, with $\alpha(t) \in \mathbb{R}$, $\beta(t) \in \mathbb{R}$,

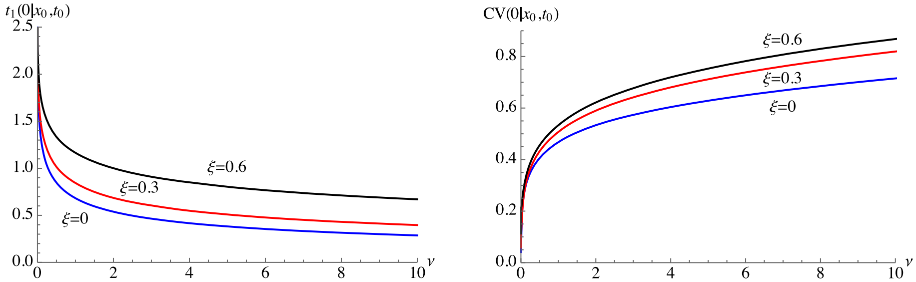


Fig. 9 The mean (on the left) and the coefficient of variation (on the right) of FPT from $X(0) = 5$ to the zero-state are plotted as function of ν for the process (60), being $r(t)$ defined in (54), with $c = 0.9$ and $Q = 2$, and $\alpha(t)$ given in (57), with $\eta = -5$, $b = 0.3$ and $Q_1 = 1$

$r(t) > 0$, $\beta(t) \leq \xi\, r(t)$, with $0 \leq \xi < 1$, for all $t \geq t_0$. We have assumed that the zero-state represents an absorbing boundary for $X(t)$. This process plays a relevant role in different fields, including physics, biology, neuroscience, finance and others. For instance, in population biology $\alpha(t)$ represents the growth intensity function and can be positive, negative or zero at different time instants, $\beta(t)$ describes the immigration intensity function; instead, the noise intensity function $r(t)$ takes into account the environmental fluctuations. For this process, the transition density $f_a(x, t|x_0, t_0)$ in the presence of an absorbing boundary in zero-state and the FPT density $g(0, t|x_0, t_0)$ from $X(t_0) = x_0$ to the zero-state are obtained. Special attention is dedicated to the proportional case, in which the immigration intensity function and the noise intensity function are related as $\beta(t) = \xi\, r(t)$, with $0 \leq \xi < 1$. Various numerical computation are performed to illustrate the effect of periodic intensity functions on the FPT pdf $g(0, t|x_0, t_0)$, by assuming that $\alpha(t)$, $\beta(t)$ or both these functions exhibit some kind of periodicity.

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Appendix

A Proof of Proposition 1

To solve (10) with initial condition (11), we use the method of characteristics (cf., for instance, Williams [46]) and we consider the following differential equations:

$$\frac{dt_0}{d\xi} = 1, \qquad \frac{ds}{d\xi} = -s\,[\alpha(t_0) + s\,r(t_0)], \qquad \frac{dZ_a}{d\xi} = [\alpha(t_0) + 2\,s\,r(t_0) - s\,\beta(t_0)]\,Z_a, \tag{A1}$$

with the initial conditions:

$$t_0(w, \xi = t) = t, \qquad s(w, \xi = t) = w, \qquad Z_a(w, \xi = t) = e^{-wx}. \tag{A2}$$

The first equation of (A1), with the related initial condition in (A2), leads to $t_0 = \xi$. Then, solving the second equation in (A1) with $t_0 = \xi$ and making use of the second of (A2), one has:

$$s = \frac{w\,e^{-A(\xi|t)}}{1 + w\,R(\xi|t)}. \tag{3}$$

Moreover, solving the third equation in (A1) with $t_0 = \xi$ and s given in (3), we have

$$Z_a(w, \xi) = e^{-wx} \exp\left\{A(\xi|t) + \int_t^{\xi} [2\,r(u) - \beta(u)]\,\frac{w\,e^{-A(u|t)}}{1 + w\,R(u|t)}\,du\right\}, \tag{4}$$

where the third of (A2) has been used. From (3) with $\xi = t_0$, we also obtain

$$w = \frac{s\,e^{-A(t|t_0)}}{1 + s\,R(t|t_0)}. \tag{5}$$

Hence, recalling that $\xi = t_0$ and making use of (5), from (4) it follows:

$$\begin{aligned} Z_a(x, t|s, t_0) &= e^{-A(t|t_0)} \exp\left\{-\frac{s\,x\,e^{-A(t|t_0)}}{1 + s\,R(t|t_0)}\right\} \\ &\times \exp\left\{-\int_{t_0}^{t} [2\,r(u) - \beta(u)]\,\frac{s\,e^{-A(u|t_0)}}{1 + s\left[R(t|t_0) + e^{-A(t|t_0)}\,R(u|t)\right]}\,du\right\}. \end{aligned} \tag{6}$$

Since

$$R(t|t_0) + e^{-A(t|t_0)}\,R(u|t) = R(u|t_0), \qquad t_0 \le u \le t,$$

one has:

$$\begin{aligned} &\exp\left\{-\int_{t_0}^{t} [2\,r(u) - \beta(u)]\,\frac{s\,e^{-A(u|t_0)}}{1 + s\left[R(t|t_0) + e^{-A(t|t_0)}\,R(u|t)\right]}\,du\right\} \\ &= \exp\left\{-2\int_{t_0}^{t} r(u)\,\frac{s\,e^{-A(u|t_0)}}{1 + s\,R(u|t_0)}\,du\right\} \exp\left\{\int_{t_0}^{t} \beta(u)\,\frac{s\,e^{-A(u|t_0)}}{1 + s\,R(u|t_0)}\,du\right\}. \end{aligned} \tag{7}$$

We note that

$$\exp\left\{-2\int_{t_0}^{t} r(u)\,\frac{s\,e^{-A(u|t_0)}}{1 + s\,R(u|t_0)}\,du\right\} = \frac{1}{[1 + s\,R(t|t_0)]^2}, \tag{8}$$

being

$$\frac{d}{du}\ln[1 + s\,R(u|t_0)] = \frac{s\,r(u)\,e^{-A(u|t_0)}}{1 + s\,R(u|t_0)}, \qquad t_0 \le u \le t. \tag{9}$$

Making use of (7) and (8) in (6), one obtains (12). Finally, we note that the assumptions on the functions $\alpha(t)$, $\beta(t)$ and $r(t)$ in Proposition 1 imply that

$$0 \le Z_a(x, t|s, t_0) \le \frac{e^{-A(t|t_0)}}{[1 + s R(t|t_0)]^{2-\xi}} \exp\left\{-\frac{s\,x\,e^{-A(t|t_0)}}{1 + s R(t|t_0)}\right\}, \quad 0 \le \xi < 1,$$

so that

$$\lim_{x_0 \downarrow 0} f_a(x, t|x_0, t_0) = \lim_{s \uparrow +\infty} s\, Z_a(x, t|s, t_0) = 0,$$

i.e. the condition (5) is satisfied. □

B Proof of Proposition 2

We note that

$$\exp\left\{\int_{t_0}^{t} \beta(u)\, \frac{s\, e^{-A(u|t_0)}}{1 + s\, R(u|t_0)}\, du\right\} = \exp\left\{\xi \int_{t_0}^{t} \frac{s\, r(u)\, e^{-A(u|t_0)}}{1 + s\, R(u|t_0)}\, du\right\} = [1 + s\, R(t|t_0)]^{\xi}, \tag{B1}$$

where the last identity follows by virtue of (9). Hence, making use of (B1) in (12), one obtains (15). To derive (16), we consider the inverse Laplace transform of (15) distinguishing two cases: *(i)* $x = 0$ and *(ii)* $x > 0$.
Case (i) If $x = 0$, Eq. (15) becomes:

$$Z_a(0, t|s, t_0) = \frac{e^{-A(t|t_0)}}{[R(t|t_0)]^{2-\xi}} \left[s + \frac{1}{R(t|t_0)}\right]^{\xi - 2}, \qquad 0 \le \xi < 1. \tag{B2}$$

Since (cf. Erdèlyi et al. [43], p. 144, no. 3)

$$\int_0^{+\infty} e^{-s x_0}\, x_0^{\nu-1}\, e^{-a x_0}\, dx_0 = \Gamma(\nu)\, (s + a)^{-\nu}, \qquad \mathrm{Re}\, \nu > 0,$$

taking the inverse Laplace transform in (B2), for $t \ge t_0$ the first of (16) immediately follows.
Case (ii) Let $x_0 > 0$. By setting

$$1 + s\, R(t|t_0) = z, \qquad \frac{x_0}{R(t|t_0)} = y, \tag{B3}$$

in (15), making use of (9), one has:

$$\begin{aligned}\int_0^{+\infty} e^{-zy} \left\{e^{y}\, f_a\big[x, t|R(t|t_0)y, t_0\big]\right\} dy &= \frac{e^{-A(t|t_0)}}{R(t|t_0)} \exp\left\{-\frac{x\, e^{-A(t|t_0)}}{R(t|t_0)}\right\} \\ &\times z^{\xi-2} \exp\left\{\frac{x\, e^{-A(t|t_0)}}{z\, R(t|t_0)}\right\}, \qquad 0 \le \xi < 1.\end{aligned} \tag{B4}$$

Since (cf. Erdèlyi et al. [43], p. 197, no. 18)

$$\int_0^{+\infty} e^{-zy}\, a^{-\nu/2}\, y^{\nu/2}\, I_\nu(2\sqrt{a\, y})\, dy = z^{-\nu-1}\, e^{a/z}, \qquad \mathrm{Re}\, \nu > -1,$$

taking the inverse Laplace transform in (B4), for $t \ge t_0$ one obtains:

$$\begin{aligned}f_a\big[x, t|R(t|t_0)y, t_0\big] &= e^{-y}\, \frac{e^{-A(t|t_0)}}{R(t|t_0)} \exp\left\{-\frac{x\, e^{-A(t|t_0)}}{R(t|t_0)}\right\} \left[\frac{x\, e^{-A(t|t_0)}}{R(t|t_0)}\right]^{-(1-\xi)/2} \\ &\times y^{(1-\xi)/2}\, I_{1-\xi}\left[2\sqrt{\frac{x\, y\, e^{-A(t|t_0)}}{R(t|t_0)}}\right], \qquad 0 \le \xi < 1,\end{aligned} \tag{B5}$$

from which, applying again the transformation $x_0 = R(t|t_0)\, y$, the second of (16) follows. □

C Proof of Proposition 5

Let $x_0 > 0$ and $t \geq t_0$. Making use of (B3) in (30) and recalling (9), one has:

$$\int_0^{+\infty} e^{-zy} \Big\{ e^y \, f_a\big[0, t | R(t|t_0) y, t_0\big] \Big\} \, dy = \frac{e^{-A(t|t_0)}}{R(t|t_0)\, z^2} \times \exp\left\{ (z-1) \int_{t_0}^{t} \frac{\beta(u)\, e^{-A(u|t_0)}}{R(t|t_0) + (z-1)\, R(u|t_0)} \, du \right\}. \tag{C1}$$

We note that

$$\begin{aligned} &\exp\left\{ (z-1) \int_{t_0}^{t} \frac{\beta(u)\, e^{-A(u|t_0)}}{R(t|t_0) + (z-1)\, R(u|t_0)} \, du \right\} \\ &= \exp\left\{ \frac{z-1}{z} \int_{t_0}^{t} \frac{\beta(u)\, e^{-A(u|t_0)}}{R(t|t_0) \Big[1 - \Big(\frac{z-1}{z}\Big) \Big(1 - \frac{R(u|t_0)}{R(t|t_0)}\Big) \Big]} \, du \right\} \\ &= \exp\left\{ \sum_{k=1}^{+\infty} \Big(1 - \frac{1}{z}\Big)^k \frac{1}{[R(t|t_0)]^k} \int_{t_0}^{t} \beta(u)\, e^{-A(u|t_0)}\, [R(t|t_0) - R(u|t_0)]^{k-1} \, du \right\}, \end{aligned} \tag{C2}$$

where the last equality follows being

$$0 < \frac{z-1}{z} \Big(1 - \frac{R(u|t_0)}{R(t|t_0)} \Big) < 1, \qquad t_0 \leq u \leq t.$$

Since (cf., for instance, Comtet [47]):

$$\exp\Bigg\{ \sum_{r=1}^{+\infty} \frac{d_r}{r!} \, \vartheta^r \Bigg\} = \sum_{n=0}^{+\infty} \frac{B_n(d_1, d_2, \ldots, d_n)}{n!} \, \vartheta^n,$$

where $B_n(d_1, d_2, \ldots, d_n)$ are the complete Bell polynomials defined in (31), with d_k given in (32), from (C2) one obtains:

$$\exp\left\{ (z-1) \int_{t_0}^{t} \frac{\beta(u)\, e^{-A(u|t_0)}}{R(t|t_0) + (z-1)\, R(u|t_0)} \, du \right\} = \sum_{n=0}^{+\infty} \frac{B_n(d_1, d_2, \ldots, d_n)}{n!} \Big(1 - \frac{1}{z}\Big)^n. \tag{C3}$$

Then, making use of (C3) in (C1) one has:

$$\int_0^{+\infty} e^{-zy} \Big\{ e^y \, f_a\big[0, t | R(t|t_0) y, t_0\big] \Big\} \, dy = \frac{e^{-A(t|t_0)}}{R(t|t_0)\, z^2} \sum_{n=0}^{+\infty} \frac{B_n(d_1, d_2, \ldots, d_n)}{n!} \Big(\frac{z-1}{z}\Big)^n. \tag{C4}$$

Finally, since (cf. Gradshteyn and Ryzhik [48], p. 809, no. 8)

$$\int_0^{+\infty} e^{-zy} y^{\alpha} \, L_n^{(\alpha)}(y) \, dy = \frac{\Gamma(\alpha + n + 1)\, (z-1)^n}{n!\, z^{\alpha+n+1}} \qquad \mathrm{Re}\, \alpha > -1, \; \mathrm{Re}\, z > 0,$$

by setting $\alpha = 1$, Eq. (C4) leads to:

$$f_a\big[0, t|R(t|t_0)y, t_0\big] = e^{-y}\,\frac{e^{-A(t|t_0)}}{R(t|t_0)}\sum_{n=0}^{+\infty}\frac{B_n(d_1, d_2, \ldots, d_n)}{n!}\,\frac{y}{n+1}\,L_n^{(1)}(y), \qquad y > 0.$$

Applying again the transformation $x_0 = R(t|t_0)\,y$, one obtains (33). □

D Proof of Proposition 6

We use (B3) in (29), so that, by virtue of (39), for $t \geq t_0$ we obtain:

$$\int_0^{+\infty} e^{-zy}\Big\{e^y\,v_a\big[x, t|R(t|t_0)y, t_0\big]\Big\}\,dy = \frac{1}{R(t|t_0)}\,\exp\Big\{-\frac{x\,e^{-A(t|t_0)}}{R(t|t_0)}\Big\}\exp\Big\{\frac{x\,e^{-A(t|t_0)}}{R(t|t_0)\,z}\Big\}$$
$$y > 0,\, x > 0. \tag{D1}$$

Since (cf. Erdèlyi et al. [43], p. 197, no. 16)

$$\int_0^{+\infty} e^{-z\,y}\left[\delta(y) + \frac{\sqrt{a}\,I_1(2\sqrt{a\,y})}{\sqrt{y}}\right]dy = e^{a/z}, \qquad \mathrm{Re}\,z > 0, \mathrm{Re}\,a > 0,$$

from (D1) for $t \geq t_0$ and $x > 0$ one has:

$$v_a\big[x, t|R(t|t_0)y, t_0\big] = \frac{e^{-y}}{R(t|t_0)}\,\exp\Big\{-\frac{x\,e^{-A(t|t_0)}}{R(t|t_0)}\Big\}$$
$$\times\Bigg\{\delta(y) + \sqrt{\frac{x\,e^{-A(t|t_0)}}{y\,R(t|t_0)}}\;I_1\Bigg[2\sqrt{\frac{x\,y\,e^{-A(t|t_0)}}{R(t|t_0)}}\Bigg]\Bigg\}, \quad y > 0. \tag{D2}$$

Then, applying the transformation $x_0 = R(t|t_0)\,y$, Eq. (40) follows from (D2), recalling that $\delta(a\,x) = \delta(x)/|a|$ and $g(x)\,\delta(x-a) = g(a)\,\delta(x-a)$. □

E Proof of Proposition 8

From (42), we obtain:

$$\int_0^{+\infty} f_a(x, t|x_0, t_0)\,dx = \left[\frac{1}{R(t|t_0)}\right]^{3/2}\exp\Big\{-\frac{x_0}{R(t|t_0)}\Big\}\Bigg\{x_0\sqrt{R(t|t_0)}\,\Psi(t|x_0, t_0)$$
$$+\int_0^{x_0} dz\,\frac{z\,\Psi(t|z, t_0)}{\sqrt{x_0 - z}}\int_0^{+\infty} e^{-y}\sqrt{y}\,I_1\Big[2\,\frac{\sqrt{x_0 - z}}{R(t|t_0)}\,\sqrt{y}\Big]\,dy\Bigg\}. \tag{E1}$$

We note that (cf. Erdèlyi et al [43], p. 197, no. 18)

$$\int_0^{+\infty} e^{-py}\,y^{\nu/2}\,I_\nu(2\sqrt{ay})\,dy = a^{\nu/2}p^{-\nu-1}e^{a/p}, \qquad \mathrm{Re}\,p > 0, \mathrm{Re}\,\nu > -1,$$

so that from (E1), by virtue of (34), it follows:

$$\int_0^{+\infty} f_a(x, t|x_0, t_0)\,dx = \frac{1}{R(t|t_0)}\Bigg[x_0\,\exp\Big\{-\frac{x_0}{R(t|t_0)}\Big\}\,\Psi(t|x_0, t_0)$$
$$+R(t|t_0)\sum_{n=0}^{+\infty}\frac{B_n(d_1, d_2, \ldots, d_n)}{(n+1)!}\int_0^{x_0/R(t|t_0)} y\,e^{-y}\,L_n^{(1)}(y)\,dy\Bigg]. \tag{E2}$$

Recalling the expression of the Laguerre polynomials (35), one has:

$$\int_0^z y\, e^{-y}\, L_n^{(1)}(y)\, dy = \begin{cases} 1-(1+z)\, e^{-z}, & n=0, \\ z^2\, e^{-z}, & n=1, \\ \dfrac{z^2 e^{-z}}{n}\, L_{n-1}^{(2)}(z), & n=2,3,\dots \end{cases} \tag{E3}$$

Then, making use of (E3) in (E2), for $t \geq t_0$ and $x_0 > 0$ one obtains:

$$\begin{aligned} \int_0^{+\infty} f_a(x,t|x_0,t_0)\, dx &= 1-\exp\Big\{-\frac{x_0}{R(t|t_0)}\Big\} + \frac{x_0}{R(t|t_0)}\, \exp\Big\{-\frac{x_0}{R(t|t_0)}\Big\} \\ &\times \sum_{n=1}^{+\infty} \frac{B_n(d_1,d_2,\dots,d_n)}{(n+1)!} \Big\{L_n^{(1)}\Big[\frac{x_0}{R(t|t_0)}\Big] + \frac{1}{n}\, \frac{x_0}{R(t|t_0)} L_{n-1}^{(2)}\Big[\frac{x_0}{R(t|t_0)}\Big]\Big\}. \end{aligned} \tag{E4}$$

Moreover, since the Laguerre polynomials satisfy the following functional relations (cf. Gradshteyn and Ryzhik [48], p. 1001, no. 8.971.4 and no. 8.971.5)

$$\begin{aligned} &y\, L_n^{(a+1)}(z) = (n+a)\, L_{n-1}^{(a)}(z) - (n-y)\, L_n^{(a)}(z), \\ &L_n^{(a-1)}(z) = L_n^{(a)}(z) - L_{n-1}^{(a)}(z), \end{aligned}$$

one also has:

$$n\, L_n^{(1)}(z) + y\, L_{n-1}^{(2)}(z) = (n+1)\, L_{n-1}^{(1)}(z).$$

Hence, (E4) can be rewritten as

$$\begin{aligned} \int_0^{+\infty} f_a(x,t|x_0,t_0)\, dx &= 1-\exp\Big\{-\frac{x_0}{R(t|t_0)}\Big\} + \frac{x_0}{R(t|t_0)}\, \exp\Big\{-\frac{x_0}{R(t|t_0)}\Big\} \\ &\times \sum_{n=1}^{+\infty} \frac{B_n(d_1,d_2,\dots,d_n)}{n\, n!}\, L_{n-1}^{(1)}\Big[\frac{x_0}{R(t|t_0)}\Big]. \end{aligned} \tag{E5}$$

Finally, since (cf. Gradshteyn and Ryzhik [48], p. 1001, no. 8.972.1)

$$L_n^{(a)}(z) = \binom{n+a}{n}\, \Phi(-n, a+1; z), \qquad a \geq 0, n = 0,1,\dots, \tag{E6}$$

Eq. (44) follows immediately from (E5). □

F Proof of Proposition 9

Making use of (44) in (50), one has

$$
\begin{aligned}
g(0,t|x_0,t_0) &= \frac{x_0}{R(t|t_0)}\,\exp\Big\{-\frac{x_0}{R(t|t_0)}\Big\}\Big\{\frac{r(t)\,e^{-A(t|t_0)}}{R(t|t_0)} \\
&\times\Big[1+\Big(1-\frac{x_0}{R(t|t_0)}\Big)\sum_{n=1}^{+\infty}\frac{B_n(d_1,d_2,\ldots,d_n)}{n!}\,\Phi\Big(1-n,2;\frac{x_0}{R(t|t_0)}\Big)\Big] \\
&-\sum_{n=1}^{+\infty}\frac{B_n(d_1,d_2,\ldots,d_n)}{n!}\,\frac{d}{dt}\Phi\Big(1-n,2;\frac{x_0}{R(t|t_0)}\Big) \\
&-\sum_{n=1}^{+\infty}\frac{1}{n!}\,\Phi\Big(1-n,2;\frac{x_0}{R(t|t_0)}\Big)\frac{d}{dt}B_n(d_1,d_2,\ldots,d_n).
\end{aligned}
\tag{F1}
$$

Since (cf. Gradshteyn and Ryzhik [48], p. 1023, no. 9213)

$$
\frac{d}{dz}\Phi(a,b;z)=\frac{a}{b}\,\Phi(a+1,b+1;z),
$$

one obtains:

$$
\frac{d}{dt}\Phi\Big(1-n,2;\frac{x_0}{R(t|t_0)}\Big)=\frac{n-1}{2}\,\frac{x_0\,r(t)\,e^{-A(t|t_0)}}{R^2(t|t_0)}\,\Phi\Big(2-n,3;\frac{x_0}{R(t|t_0)}\Big).
$$

Therefore, Eq. (F1) can be rewritten as:

$$
\begin{aligned}
g(0,t|x_0,t_0) &= \frac{x_0}{R(t|t_0)}\,\exp\Big\{-\frac{x_0}{R(t|t_0)}\Big\}\Big\{\frac{r(t)\,e^{-A(t|t_0)}}{R(t|t_0)}\sum_{n=0}^{+\infty}\frac{B_n(d_1,d_2,\ldots,d_n)}{n!} \\
&\times\Big[\Big(1-\frac{x_0}{R(t|t_0)}\Big)\Phi\Big(1-n,2;\frac{x_0}{R(t|t_0)}\Big)-\frac{n-1}{2}\,\frac{x_0}{R(t|t_0)}\,\Phi\Big(2-n,3;\frac{x_0}{R(t|t_0)}\Big)\Big] \\
&-\sum_{n=1}^{+\infty}\frac{1}{n!}\,\Phi\Big(1-n,2;\frac{x_0}{R(t|t_0)}\Big)\frac{d}{dt}B_n(d_1,d_2,\ldots,d_n)\Big\},
\end{aligned}
\tag{F2}
$$

where the use of the following relations

$$
\begin{aligned}
&\Phi(a,a;z)=e^z, \\
&\frac{z}{b}\Phi(a+1,b+1;z)=\Phi(a+1,b;z)-\Phi(a,b;z)
\end{aligned}
\tag{F3}
$$

has been made. Finally, recalling that

$$
a\,\Phi(a+1,b+1;z)=(a-b)\,\Phi(a,b+1;z)+b\,\Phi(a,b;z),
$$

the expression in square bracket in Eq. (F2) becomes:

$$
\begin{aligned}
&\Big(1-\frac{x_0}{R(t|t_0)}\Big)\Phi\Big(1-n,2;\frac{x_0}{R(t|t_0)}\Big)-\frac{n-1}{2}\,\frac{x_0}{R(t|t_0)}\,\Phi\Big(2-n,3;\frac{x_0}{R(t|t_0)}\Big) \\
&=\Phi\Big(1-n,1;\frac{x_0}{R(t|t_0)}\Big)-\frac{x_0}{R(t|t_0)}\,\Phi\Big(1-n,2;\frac{x_0}{R(t|t_0)}\Big) \\
&=\Phi\Big(-n,1;\frac{x_0}{R(t|t_0)}\Big),
\end{aligned}
\tag{F4}
$$

where the last identity follows from (F3). Then, substituting (F4) in (F2), we obtain Eq. (51). □

G Evaluation of $R(t|t_0)$ in (61)

From (61) one has:

$$R(t|t_0) = \frac{\nu}{1+b\,\sin\left(\frac{2\pi t_0}{Q_1}\right)}\Big[R_1(t|t_0)+c\,R_2(t|t_0)+b\,c\,R_3(t|t_0)\Big],$$

with $0 \le b < 1$ and $0 \le c < 1$, where

$$\begin{aligned}
R_1(t|t_0) &= \int_{t_0}^{t} e^{-\eta(\tau-t_0)}\left[1+b\,\sin\left(\frac{2\pi\tau}{Q_1}\right)\right]d\tau \\
&= \begin{cases}
t-t_0-\frac{b\,Q_1}{2\pi}\left[\cos\left(\frac{2\pi t}{Q_1}\right)-\cos\left(\frac{2\pi t_0}{Q_1}\right)\right], & \eta=0,\\[2mm]
\frac{1-e^{-\eta(t-t_0)}}{\eta}-\frac{2\pi\,b\,Q_1}{4\pi^2+Q_1^2\eta^2}\Big[e^{-\eta(t-t_0)}\cos\left(\frac{2\pi t}{Q_1}\right) & \\
\quad+\frac{Q_1\eta}{2\pi}e^{-\eta(t-t_0)}\sin\left(\frac{2\pi t}{Q_1}\right)-\cos\left(\frac{2\pi t_0}{Q_1}\right)-\frac{Q_1\eta}{2\pi}\,\sin\left(\frac{2\pi t_0}{Q_1}\right)\Big], & \eta\neq 0,
\end{cases}\\
R_2(t|t_0) &= \int_{t_0}^{t} e^{-\eta(\tau-t_0)}\,\sin\left(\frac{2\pi\tau}{Q}\right)d\tau \\
&= \begin{cases}
\frac{Q}{2\pi}\left[\cos\left(\frac{2\pi t_0}{Q}\right)-\cos\left(\frac{2\pi t}{Q}\right)\right], & \eta=0,\\[2mm]
\frac{2\pi\,Q}{4\pi^2+Q^2\eta^2}\Big\{\cos\left(\frac{2\pi t_0}{Q}\right)+\frac{Q\eta}{2\pi}\,\sin\left(\frac{2\pi t_0}{Q}\right) & \\
\quad-e^{-\eta(t-t_0)}\Big[\cos\left(\frac{2\pi t}{Q}\right)+\frac{Q\eta}{2\pi}\,\sin\left(\frac{2\pi t}{Q}\right)\Big]\Big\}, & \eta\neq 0,
\end{cases}
\end{aligned}$$

and

$$\begin{aligned}
R_3(t|t_0) &= \int_{t_0}^{t} e^{-\eta(\tau-t_0)}\,\sin\left(\frac{2\pi\tau}{Q}\right)\,\sin\left(\frac{2\pi\tau}{Q_1}\right)d\tau \\
&= \begin{cases}
\frac{QQ_1}{4\pi}\Big\{\frac{1}{Q-Q_1}\Big[\sin\left(2\pi t\,\frac{Q-Q_1}{QQ_1}\right)-\sin\left(2\pi t_0\,\frac{Q-Q_1}{QQ_1}\right)\Big] & \\
\quad-\frac{1}{Q+Q_1}\Big[\sin\left(2\pi t\,\frac{Q+Q_1}{QQ_1}\right)-\sin\left(2\pi t_0\,\frac{Q+Q_1}{QQ_1}\right)\Big]\Big\}, & \eta=0,\\[3mm]
\frac{QQ_1}{2}\Big\{e^{-\eta(t-t_0)}\Big[\frac{QQ_1\eta\cos\left(2\pi t\,\frac{Q+Q_1}{QQ_1}\right)-2\pi(Q+Q_1)\sin\left(2\pi t\,\frac{Q+Q_1}{QQ_1}\right)}{4\pi^2(Q+Q_1)^2+Q^2Q_1^2\eta^2} & \\
\quad-\frac{QQ_1\eta\cos\left(2\pi t\,\frac{Q-Q_1}{QQ_1}\right)-2\pi(Q-Q_1)\sin\left(2\pi t\,\frac{Q-Q_1}{QQ_1}\right)}{4\pi^2(Q-Q_1)^2+Q^2Q_1^2\eta^2}\Big] & \\
\quad-\frac{QQ_1\eta\cos\left(2\pi t_0\,\frac{Q+Q_1}{QQ_1}\right)-2\pi(Q+Q_1)\sin\left(2\pi t_0\,\frac{Q+Q_1}{QQ_1}\right)}{4\pi^2(Q+Q_1)^2+Q^2Q_1^2\eta^2} & \\
\quad+\frac{QQ_1\eta\cos\left(2\pi t_0\,\frac{Q-Q_1}{QQ_1}\right)-2\pi(Q-Q_1)\sin\left(2\pi t_0\,\frac{Q-Q_1}{QQ_1}\right)}{4\pi^2(Q-Q_1)^2+Q^2Q_1^2\eta^2}\Big\}, & \eta\neq 0,
\end{cases}
\end{aligned}$$

for $Q \neq Q_1$, whereas

$$
\begin{aligned}
R_3(t|t_0) &= \int_{t_0}^{t} e^{-\eta(\tau-t_0)} \sin^2\Big(\frac{2\pi\tau}{Q}\Big)\, d\tau \\
&= \begin{cases}
\frac{t-t_0}{2} - \frac{Q}{8\pi}\sin\Big(\frac{4\pi t}{Q}\Big) + \frac{Q}{8\pi}\sin\Big(\frac{4\pi t_0}{Q}\Big), & \eta = 0,\\[2ex]
\frac{1-e^{-\eta(t-t_0)}}{2\eta} + \frac{Q}{2[16\pi^2+Q^2\eta^2]}\Big\{4\pi\sin\Big(\frac{4\pi t_0}{Q}\Big) - Q\pi\cos\Big(\frac{4\pi t_0}{Q}\Big) & \\
\quad -e^{-\eta(t-t_0)}\Big[4\pi\sin\Big(\frac{4\pi t}{Q}\Big) - Q\pi\cos\Big(\frac{4\pi t}{Q}\Big)\Big]\Big\}, & \eta \neq 0,
\end{cases}
\end{aligned}
$$

for $Q = Q_1$. □

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