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Linear evasion differential game of one evader and several pursuers with integral constraints

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Abstract
An evasion differential game of one evader and many pursuers is studied. The dynamics of state variables $x_1, \ldots, x_m$ are described by linear differential equations. The control functions of players are subjected to integral constraints. If $x_i(t) \neq 0$ for all $i \in \{1, \ldots, m\}$ and $t \geq 0$, then we say that evasion is possible. It is assumed that the total energy of pursuers doesn't exceed the energy of evader. We construct an evasion strategy and prove that for any positive integer m evasion is possible.

Keywords Evasion differential game · Evader · Many Pursuers · Integral constraint · Evasion · Strategy

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1 Introduction

The field of differential games was pioneered by Isaacs (1965) in the 1960s and since then enormous amount of work has been devoted to its study (for example, see Friedman 1971; Hajek 1975; Pontryagin 1988; Krasovskii and Subbotin 1988; Petrosyan 1993; Başar and Olsder 1999; Buckdahn et al. 2011 and references therein). A substantial part of these works concern simple motion pursuit and evasion differential games of many players. Often, either geometric or integral constraints are imposed on the control parameters of players. In Croft (1964) Croft showed that in the n-dimensional Euclidean ball n lions can catch the man, while the man can escape from $n-1$ lions when the controls of the players are subject to geometric constraints. A similar game problem was studied by Ivanov (1980) on any convex compact set and an estimate from above was obtained for guaranteed pursuit time.

Interesting results were obtained in Alexander et al. (2009), Azamov (2008), Bakolas and Tsiotras (2011), Berkovitz (1986), Bhattacharya et al. (2016), Konstantinidis and Kehagias (2016) and Sun and Tsiotras (2014) for various games in unbounded regions as well as on graphs. Evasion problem on time interval $[t_0, \infty)$ was introduced and studied by Pontryagin and Mischenko (1971). In Mishchenko et al. (1977) Mishchenko et al. proposed a new manoeuvre for evasion in the game of many pursuers.

Chernous'ko (1976) studied an evasion game of one evader and several pursuers with a state constraint, i.e. the evader was supposed to remain in a neighborhood of a given ray for the duration of the game. He proved that if the evader is faster than the pursuers then evasion is possible. This result was extended by Chernous'ko and Zak (1985) and Zak (1978, 1981, 1982) to more general differential game problems. Related problems of evasion from a group of pursuers were studied in Borowko et al. (1988) and Chodun (1989).

In Pshenichnii (1976) Pshenichnii considered a simple motion differential game of many pursuers and one evader in $\mathbb{R}^n$, when all players have the same dynamic possibilities. He proved that if the initial state of the evader belongs to the interior of convex hull of pursuers' initial states, then pursuit can be completed, otherwise evasion is possible. Based on this work, Pshenichnii et al. (1981) developed the method of resolving functions for solving linear pursuit problems with many pursuers. Later on, the results of paper (Pshenichnii 1976) were extended by many researchers to cover various cases. For example, when control sets of players are convex compact sets, Grigorenko (1990) obtained the necessary and sufficient conditions of evasion of one evader from several pursuers.

The papers Chikrii and Prokopovich (1992) and Kuang (1986) are also extensions of Pshenichnii (1976). In Kuchkarov et al. (2012) the game problem of many pursuers and one evader was studied on a cylinder. In the recent work of Kuchkarov et al. (2016), the results of Pshenichnii (1976) were extended to differential games on manifolds with Euclidean metric. In Blagodatskikh and Petrov (2009) Blagodatskikh and Petrov obtained necessary and sufficient condition of evasion in a simple motion differential game of a group of pursuers and a group of evaders in $\mathbb{R}^n$ where all evaders use the same control. By definition, pursuit is considered completed if the state of a pursuer coincides with the state of at least one evader. Also, the works (Bannikov and Petrov

2010; Vagin and Petrov 2001) related to such games. Recently in Scott and Leonard (2018) the authors consider a pursuit-evasion game involving one pursuer and multiple evaders motivated by the seminal "selfish herd" model of Hamilton (1971). The pursuer can freely move in any direction with bounded speed and evaders move with bounded speed and bounded turning speed. Using Isaacs' heuristic argument they constructed an optimal strategy for the pursuer and concluded that the optimal strategy for the pursuer is to focus on a single evader that can be captured in minimum time. Moreover, "non-targeted" evaders are always able to escape. We refer to Kumkov et al. (2017) for a survey of results on differential games of many players with geometric constraints. In the case of integral constraints, simple motion evasion games of many players were solved in Alias et al. (2016), Ibragimov et al. (2012) and Ibragimov et al. (2018).

In the present paper, we study a linear evasion differential game of many pursuers and one evader. The controls of players are subjected to integral constraints. To the best of our knowledge no previous study has investigated the linear evasion game problem stated in the present paper. The main difficulties in solving the problem are the construction of evasion strategy and to prove the fact that the objects go around the origin on some specified time interval $[0, T]$ maintaining some distances from the origin. Note that we employ non-anticipative strategies in the present game model (see, for example, Cardaliaguet et al. 2000).

Note that there is a similarity between the constructed evasion strategies and proofs of the main results of current and existing works (Ibragimov et al. 2012, 2018). They are (i) the definition of time intervals $[\tau_i, \tau_i')$, (ii) the construction of a strategy for the evader which allows the evader to use a manoeuvre on $[\tau_i, \tau_i')$ against the i-th pursuer, (iii) estimating the distances between the evader and pursuers, and establishing that evasion is possible. We believe that these steps will be common for the most of open evasion differential game problems of many players with integral constraints described by the linear system of equations as well. However, the main difficulties in solving the open problems will remain to overcome the steps (i)–(iii) listed above.

In the case of linear equations studied in the present paper, the strategies of existing papers do not work since, for the linear equation describing the game, we have to find its own τ_i. Also, by contrast, we need bounded τ_i, τ_i' and new techniques to estimate $x_p(t)$. Note that according to the strategy of the present paper, in contrast to previous works, each object, for any control functions of pursuers, moves with a positive speed in the direction of y-axis on the time interval $[0, T]$. Moreover, all the objects will become on the upper half plane by the time T, and then evasion is established. The fact that each time interval where the evader uses a manoeuver is contained in the interval $[0, T]$ plays a key role in establishing a number of estimates in the proof of the main result.

This work is a milestone study to undertake a detailed analysis of linear evasion differential game problem of many pursuers and one evader with integral constraints. We are confident that the construction in the present paper will be a stepping stone to open problems and will open prospects for general multi person linear evasion differential games with integral constraints and this study makes a major contribution to research on general linear evasion differential games.

2 Statement of problem

Let $x_1, \dots, x_m$, $m \geq 1$, be the points moving in $\mathbb{R}^n$ whose dynamics are described by the equations

$$\dot{x}_i = -\lambda_i x_i + v - u_i, \quad x_i(0) = x_i^0, \quad i = 1, 2, \dots, m, \tag{1}$$

where $u_1, \dots, u_m$ are the control parameters of pursuers and v is that of evader, $\lambda_i > 0$, $x_i, x_i^0, u_i, v \in \mathbb{R}^n$, $n \geq 2$, $x_i^0 \neq 0$, $i = 1, \dots, m$.

Definition 2.1 Measurable functions $u_i(t)$ and $v(t)$, $t \geq 0$, that satisfy the following integral constraints

$$\int_0^\infty |u_i(t)|^2 dt \leq \rho_i^2, \; i = 1, \dots m; \quad \int_0^\infty |v(t)|^2 dt \leq \sigma^2. \tag{2}$$

are called controls of the ith pursuer and evader, respectively.

Definition 2.2 A function $(t, t_1, \dots, t_k, x_1, \dots, x_m, u_1, \dots, u_m) \mapsto V(t, t_1, \dots, t_k, x_1, \dots, x_m, u_1, \dots, u_m)$, $V : [0, \infty)^{k+1} \times \mathbb{R}^{2nm} \to \mathbb{R}^n$, where $t_1, \dots, t_k$, $0 < t_1 < \dots < t_k < \infty$, are some positive numbers (unspecified) and k is a positive integer, is called strategy of evader if the following system of equations

$$\begin{aligned} \dot{x}_i &= -\lambda_i x_i + V(t, t_1, \dots, t_k, x_1, \dots, x_m, u_1, \dots, u_m) - u_i, \\ x_i(0) &= x_i^0, \; i = 1, \dots, m, \end{aligned} \tag{3}$$

has a unique solution $(x_1(t), \dots, x_m(t))$, $t \geq 0$, for any controls $(u_1(t), \dots, u_m(t))$, of pursuers and along this solution

$$\int_0^\infty |V(t, t_1, \dots, t_k, x_1(t), \dots x_m(t), u_1(t), \dots, u_m(t))|^2 dt \leq \sigma^2.$$

The strategy $V(t, t_1, \dots, t_k, x_1, \dots, x_m, u_1, \dots, u_m)$ is nonanticipatively defined with respect to the strictly increasing finite sequence of numbers $t_1, \dots, t_k$ as follows. Let the time t_i ($t_0 = 0$), $i = 0, 1, \dots, k$, be occurred. The strategy of the evader is defined on the time interval $[t_i, t_{i+1})$, $i = 0, 1, \dots, k$, where $t_{k+1} = +\infty$, as a function $V = V^i(t, t_1, \dots, t_i, x_1, \dots, x_m, u_1, \dots, u_m)$. The trajectories of objects $x_1(t), \dots, x_m(t)$ i.e. the solution of (3) generated by this strategy and arbitrary controls of pursuers $u_1(t), \dots, u_m(t)$ are then defined as the solution of the initial value problem

$$\begin{aligned} \dot{x}_j &= -\lambda_j x_j + V^i(t, t_1, \dots, t_i, x_1, \dots, x_m, u_1, \dots, u_m) - u_j, \quad x_j(t)|_{t=t_i} \\ &= x_j(t_i), \; j = 1, \dots, m, \end{aligned}$$

until the time t_{i+1}, $i = 0, 1, \dots, k$, occurs. The number t_{i+1} is defined as the first time when the points $x_1(t), \dots, x_m(t)$ satisfy a certain condition. In this way, we define the solution of (3) on the intervals $[t_i, t_{i+1})$, $j = 0, 1, \dots, k$. It should be noted that the evader can predict neither the values of $t_1, \dots, t_k$ nor the length of the interval $[t_i, t_{i+1})$, $i = 0, 1, \dots, (k-1)$.

Definition 2.3 If there exists a strategy V of evader such that for any controls of pursuers $x_i(t) \neq 0$, $i = 1, \dots, m$, $t \geq 0$, then we say that evasion is possible.

The problem is to find a condition for evasion to be possible.

Thus, the evader knows the values $x_1(t), \dots, x_m(t)$, $u_1(t), \dots, u_m(t)$ of parameters $x_1, \dots, x_m$, $u_1, \dots, u_m$ at the current time t. Pursuers apply arbitrary controls $u_1(t), \dots, u_m(t)$, $t \geq 0$, and try to realize the equation $x_i(t) = 0$ at least for one $i \in \{1, 2, \dots, m\}$, whereas the evader tries to maintain the inequalities $x_i(t) \neq 0$ for all $i = 1, \dots, m$ and $t \geq 0$.

3 The main result

In this section we prove a theorem about evasion. To this end, we specify the conditions to define the numbers t_i and construct an explicit nonanticipative strategy for the evader.The following is the main result of the current paper.

Theorem 3.1 *If*

$$\rho_1^2 + \cdots + \rho_m^2 \leq \sigma^2, \tag{4}$$

then evasion is possible in game (1)–(2).

We prove the theorem in several subsections. The proof strategy is as follows. The solution of the initial value problem (1) is given by

$$x_i(t) = e^{-\lambda_i t} y_i(t), \quad y_i(t) = x_i^0 + \int_0^t e^{\lambda_i s}(v(s) - u_i(s))ds. \tag{5}$$

Since $x_i(t) = 0$ if and only if $y_i(t) = 0$, below we study the evolution of $y_i(t)$. We construct an evasion strategy such that the second coordinate of each point $y_i(t)$ strictly increases all the time. It remains to look at the situation with initial state $y_i(0)$ with $y_{i2}(0) < 0$ for some i.

Define an a_i-approach time $t = \tau_i$ as the first time for which $|y_i(t)| = a_i$ and $y_{i2}(t) < 0$. (a_i) is a strictly decreasing sequence so τ_i (if ever defined) is increasing. Time $\tau_i' > \tau_i$ is specified to ensure that $y_{i2}(\tau_i') > 0$ if a "manoeuvre" is deployed on the time interval $[\tau_i, \tau_i')$.

More essential for the proof of the theorem is that the sequence $\{\tau_i\}$ is bounded such that the set $I_1 = \bigcup_{i=1}^{m_0}[\tau_i, \tau_i')$ is contained in $[0, T]$ for some T. This is essentially implied by the fact that $y_{i2}(t)$ is strictly increasing for all i and at any time $t > 0$, and the definition for the a_i-approach time τ_i requires $y_{i2}(t) < 0$.

The "manoeuvre" is defined (see (16)) such that once some object $y_i(\tau_p)$ is close to the origin, i.e. $|y_i(\tau_p)| = a_p$ while $y_{i2}(\tau_p) < 0$ for some τ_p, some energy is allocated on the first coordinate in order to increase $|y_{i1}(t)|$ (such that $|y_i(t)| \geq a_{p+1}$ on $[\tau_p, \tau'_p]$, (see (55)), avoiding the origin). Also, $y_{i2}(\tau_p)$ increases on $[\tau_p, \tau'_p]$ (such that $y_{i2}(t) \geq a_p$ on $[\tau'_p, \infty)$ (see (56))). The objective is that after the distance of point $y_i(t)$ from the origin is a_p at $t = \tau_p$, it is not possible for it to be again at an a_q distance at some τ_q with $q \geq p + 1$ (see (53)-(55))

The auxiliary trajectory $z_p(t)$ is the trajectory of $y_p(t)$ if the evader applies the "manoeuvre" against the p-pursuer on the whole interval $[\tau_p, \tau'_p)$. In the end, estimations of $|y_p(t) - z_p(t)|$ and $z_p(t)$ are needed to estimate $|y_p(t)|$.

3.1 Notations

It is sufficient to consider the case when $n = 2$ and

$$\rho^2 := \rho_1^2 + \cdots + \rho_m^2 < \sigma^2 \tag{6}$$

(see, for example, (Ibragimov et al. 2012, 2018).

Let α be any number satisfying the condition

$$0 < \alpha < \frac{(\sigma - \rho)^2}{2(\max_{1 \leq i \leq m} |x_i^0| + 1)}. \tag{7}$$

We choose a number a_1 such that

$$0 < a_1 < \min \left\{ \frac{1}{2}, \frac{(\sigma - \rho)^2}{4\alpha}, \frac{\sigma^2}{32\alpha}, \frac{\alpha}{2\Lambda}, \min_i |x_i^0| \right\}, \tag{8}$$

where $\Lambda = \max_{1 \leq i \leq m} \lambda_i$. Let

$$T_0 = \frac{1}{\alpha} \max_{1 \leq i \leq m} |x_i^0|, \; T = T_0 + \frac{2a_1}{\alpha}, \quad \kappa = \min \left\{ \frac{1}{2}, \frac{\alpha}{16\sigma^2 e^{\Lambda T}}, \frac{\alpha^3}{8 \cdot 6^4 \sigma^6 e^{4\Lambda T}} \right\}. \tag{9}$$

Let a sequence $\{a_i\}_{i=1}^{\infty}$ be defined by the formula $a_{i+1} = \kappa \cdot a_i^4$. It is not difficult to see that this sequence has the following

Property 3.2 $\sum_{i=k}^{\infty} a_i \leq 2a_k$ *for any* $k \geq 1$.

Let $y_i = (y_{i1}, y_{i2})$, $v = (v_1, v_2)$, and $u_i = (u_{i1}, u_{i2})$. Define a_i-approach time τ_i to be the first time such that

$$|y_j(\tau_i)| = a_i, \quad y_{j2}(\tau_i) < 0, \quad i = 1, \ldots, m_0, \tag{10}$$

for some $j \in \{1, \ldots, m\}$, where m_0 is a positive integer. In Sect. 3.2 we'll show that τ_i are defined for some points y_i, $i = 1, \ldots, m_0$, $m_0 \leq m$. Note that a_i-approach times τ_i may not be defined as well ($m_0 = 0$).

First we define τ_1 if relations (10) are satisfied at $i = 1$ for some j. Then, we define τ_2 and so on. Therefore, $\tau_1 < \tau_2 < \cdots < \tau_{m_0}$. Note that times τ_i are unspecified and depend on the evader's strategy and the controls of the pursuers. It is important to note the fact that all the numbers τ_i will be in the interval $[0, T]$, which will be established in Sect. 3.2.

Without loss of generality we relabel y_j for which $|y_j(\tau_i)| = a_i$, $y_{j2}(\tau_i) < 0$ by y_i. Note that the condition (10) can occur at the time τ_i for several j. If so, we label any one of them by y_i. Let

$$\tau_i' = \frac{1}{\lambda_i} \ln \left(e^{\lambda_i \tau_i} + \frac{2\lambda_i a_i}{\alpha} \right), \quad i = 1, \dots, m_0.$$

Property 3.3 *For any* $i, k \in \{1, \dots, m_0\}$,

(1) $\tau_i' - \tau_i \le \frac{2a_i}{\alpha}$.
(2) $\sum_{i=k}^{m_0} (\tau_i' - \tau_i) \le \frac{4a_k}{\alpha}$.

Proof To prove item (1), we have

$$\tau_i' - \tau_i = \frac{1}{\lambda_i} \ln \left(e^{\lambda_i \tau_i} + \frac{2\lambda_i a_i}{\alpha} \right) - \tau_i. \tag{11}$$

Since for any $a \ge 1$ and $b \ge 0$ we have $\ln(a + b) \le \ln a + b$, therefore (11) implies that

$$\tau_i' - \tau_i \le \frac{1}{\lambda_i} \left(\lambda_i \tau_i + \frac{2\lambda_i a_i}{\alpha} \right) - \tau_i = \frac{2a_i}{\alpha}. \tag{12}$$

The proof of item (2) follows from (12) as follows

$$\sum_{i=k}^{m_0} (\tau_i' - \tau_i) \le \sum_{i=k}^{\infty} \frac{2a_i}{\alpha} < \frac{4a_k}{\alpha} \tag{13}$$

using Property 3.2. □

Further, we define a function $r : [0, \infty) \to \{0, 1, \dots, m_0\}$ as follows: set $r(t) = i$, if $t \in [\tau_i, \tau_i') \backslash I_{i+1}$, $i = 0, \dots, m_0$, where $\tau_0' = \infty$, $I_i = \cup_{j=i}^{m_0} [\tau_j, \tau_j')$, $I_{m_0+1} = \emptyset$. The function r has the following useful property:

Property 3.4 *For* $i = 1, 2, \dots, (m_0 - 1)$,

(1) $r(t) = i$ *for* $\tau_i \le t < \tau_i'$ *if* $\tau_i' \le \tau_{i+1}$.
(2) $r(t) = i$ *for* $\tau_i \le t < \tau_{i+1}$ *if* $\tau_{i+1} \le \tau_i'$.

Proof Suppose that $\tau_i' \le \tau_{i+1}$. Then $[\tau_i, \tau_i') \backslash I_{i+1} = [\tau_i, \tau_i')$. Therefore, $r(t) = i$ for $t \in [\tau_i, \tau_i')$. This proves item (1).

To prove item (2), suppose that $\tau_{i+1} \le \tau_i'$. Since $\tau_i < \tau_{i+1} < \dots < \tau_{m_0}$, we have $[\tau_i, \tau_{i+1}) \subset [\tau_i, \tau_i') \backslash I_{i+1}$. Therefore, $r(t) = i$ for $t \in [\tau_i, \tau_{i+1})$ by definition. □

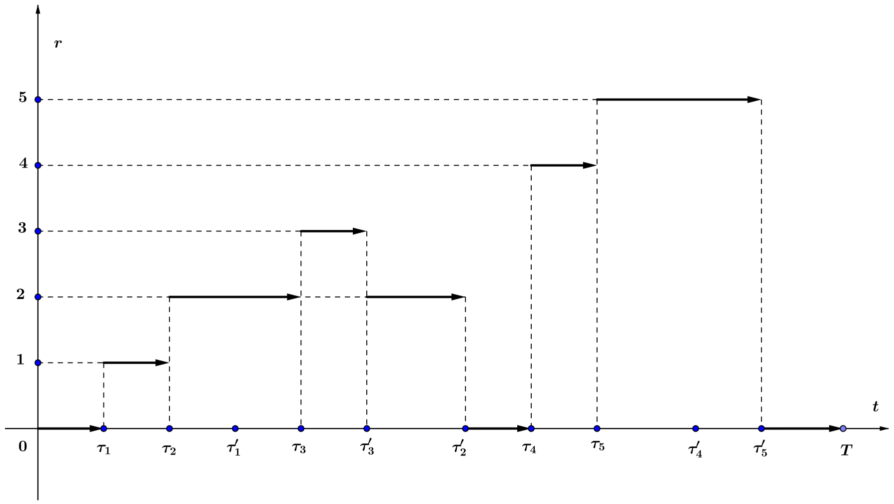

Fig. 1 The graph of function $r(x)$

Example 3.5 If

$$0 = \tau_0 < \tau_1 < \tau_2 < \tau_1' < \tau_3 < \tau_3' < \tau_2' < \tau_4 < \tau_5 < \tau_4' < \tau_5',$$

then $r(t)$ has the graph shown in Fig. 1.

3.2 Strategy for the evader

Now we are ready to construct a strategy for the evader. Let $u_j(t)$, $j = 1, \dots, m$, be arbitrary controls of pursuers. Set

$$v(t) = V_0(t) = \left(0, \alpha + \left(\sum_{j=1}^{m} |u_j(t)|^2\right)^{1/2}\right), \quad t \in [0, T] \setminus I_1, \tag{14}$$

$$v(t) = V_r(t) = (V_{r1}(t), U(t)), \ t \in [0, T] \cap I_1, \tag{15}$$

where $r = r(t)$, $V_i(t) = (V_{i1}(t), U(t))$, $\tau_i \le t < \tau_i'$, $i = 1, \dots, m_0$, is defined as follows

$$V_{i1}(t) = \begin{cases} \alpha + |u_{i1}(t)|, & y_{i1}(\tau_i) \ge 0, \\ -(\alpha + |u_{i1}(t)|), & y_{i1}(\tau_i) < 0, \end{cases} \tag{16}$$

$$U(t) = \alpha + \left(\sum_{j=1}^{m} u_{j2}^2(t)\right)^{1/2}.$$

Note that $U(t)$ doesn't depend on i. Finally, let

$$v(t) = \left(0, \left(\sum_{j=1}^{m} |u_j(t)|^2\right)^{1/2}\right), \quad t > T. \tag{17}$$

Equation (15) shows that the function $r = r(t)$ assigns the control $V_r(t)$ for $v(t)$.

For example, if there are 5 pursuers and the numbers $\tau_i, \tau_i', i = 1, \ldots, 5$, are arranged as in Example 3.5, then using the values of $r(t)$ in Fig. 1 we obtain from (15) that

$$v(t) = \begin{cases} V_1(t) & \text{if} \quad t \in [\tau_1, \tau_2), \\ V_2(t) & \text{if} \quad t \in [\tau_2, \tau_3) \cup [\tau_3', \tau_2') \\ V_3(t) & \text{if} \quad t \in [\tau_3, \tau_3'), \\ V_4(t) & \text{if} \quad t \in [\tau_4, \tau_5), \\ V_5(t) & \text{if} \quad t \in [\tau_5, \tau_5'). \end{cases} \tag{18}$$

On the intervals $[0, \tau_1)$, $[\tau_2', \tau_4)$, and $[\tau_5', T]$, where $r(t) = 0$ on $[0, T]$, the evader's strategy is defined by (14).

If $r(t) = i > 0$ on some time interval, then we say that evader is applying a manoeuvre $V_i(t)$ against the ith pursuer, or the evader is under the attack of the i-th pursuer on that interval. In Example 3.5, $r(t) = 0$ on intervals $[0, \tau_1)$ and $[\tau_2', \tau_4)$ and so the evader is not under the attack of any pursuer on these intervals. Since $r(t) = 1$ on $[\tau_1, \tau_2)$, therefore the evader is under the attack of the first pursuer on this interval, and so evader applies the manoeuvre $V_1(t)$ against the first pursuer on this interval. Also, we can see other manoeuvres of evader in formula (18).

The evader chooses his maneuvers nonanticipatively stage by stage as the game progresses. For the Example 3.5, the numbers $t_1, t_2, \ldots, t_k \in [0, T]$ in Definition 2.2 are defined as follows: $t_0 = 0$, $t_1 = \tau_1$, $t_2 = \tau_2$, $t_3 = \tau_3$, $t_4 = \tau_3'$, $t_5 = \tau_2'$, $t_6 = \tau_4$, $t_7 = \tau_5$, $t_8 = \tau_5'$, $t_9 = T$. As the time t_i, $i = 1, 2, \ldots, 8$ ($t_0 = 0$) occurs, the function $r(t)$ assigns the strategy $V_{r_i}(t)$ for the evader defined by (14), (15), where $r_i = r(t_i)$. The evader uses this strategy until t_{i+1} occurs, that is, on the interval $t_i \leq t < t_{i+1}$. Note that t_{i+1} is unspecified and defined as the first time when $r(t_i) \neq r(t_{i+1})$.

In general, the numbers $t_1, t_2, \ldots, t_k \in [0, T]$ are defined as follows. By (14) the evader uses strategy $v(t) = V_0(t)$, $\tau_0 \leq t < \tau_1$. This means the evader applies $v(t) = V_0(t)$ until τ_1 occurs. Set $t_1 = \tau_1$, $t_k = T$. The numbers $t_2, \ldots, t_{k-1}$ are defined inductively. Let the time $t_i \in \{\tau_1, \tau_1', \ldots, \tau_p, \tau_p'\}$, $i, p \geq 1$, occur and let $r_i = r(t_i)$. Then by (14) and (15) the evader applies the strategy $v(t) = V_{r_i}(t)$ starting from t_i until the time t_{i+1} occurs, for which $r(t_i) \neq r(t_{i+1})$ for the first time, where $t_{i+1} \in \{\tau_1, \tau_1', \ldots, \tau_p, \tau_p', \tau_{p+1}, \tau_{p+1}'\}$. As the time t_{i+1} occurs the evader uses the strategy $v(t) = V_{r_{i+1}}(t)$ starting from the time t_{i+1} where $r_{i+1} = r(t_{i+1})$ and so on. To determine the times τ_i, $i = 1, 2, \ldots, m_0$, the evader uses the current values of states $y_i(t)$, $i = 1, 2, \ldots, m$. To this end, it suffices for the evader to know the current time t and $x_i(t)$, $i = 1, 2, \ldots, m$. Also, we can see from (14) and (15) that the strategy of evader has the form $v(t) = V_{r_i}(t)$ on the intervals $t_i \leq t < t_{i+1}$, $i = 0, 1, \ldots, k$.

We now show that the strategy defined by the equations (14)–(17) is admissible. Indeed, let

$$f(t) = \begin{cases} (0,\alpha), & t \in [0,T)\backslash I_1 \\ (\alpha,\alpha), & t \in I_1 \\ (0,0), & t > T \end{cases},$$

$$g(t) = \begin{cases} \left(0, \left(\sum_{j=1}^m |u_j(t)|^2\right)^{1/2}\right), & t \in [0,T]\backslash I_1, \\ \left(|u_{r1}(t)|, \left(\sum_{j=1}^m u_{j2}^2(t)\right)^{1/2}\right), & t \in I_1, \\ \left(0, \left(\sum_{j=1}^m |u_j(t)|^2\right)^{1/2}\right), & t > T. \end{cases}$$

Note that

$$\int_0^\infty |f(s)|^2 ds \le 2\alpha^2 T, \quad |g(t)|^2 \le \sum_{j=1}^m |u_j(t)|^2. \tag{19}$$

Clearly, for $v(t)$ defined by (14)–(17) we have $v_1^2(t) + v_2^2(t) = |f(t) + g(t)|^2$. Therefore, using the Minkowskii inequality and (19) we obtain

$$\begin{aligned} \left(\int_0^\infty |v(s)|^2 ds\right)^{1/2} &= \left(\int_0^\infty |f(s) + g(s)|^2 ds\right)^{1/2} \\ &\le \left(\int_0^\infty |f(s)|^2 ds\right)^{1/2} + \left(\int_0^\infty |g(s)|^2 ds\right)^{1/2} \\ &\le (2\alpha^2 T)^{1/2} + \left(\int_0^\infty \sum_{j=1}^m |u_j(s)|^2 ds\right)^{1/2} \\ &\le \alpha\sqrt{2T} + \left(\sum_{j=1}^m \rho_i^2\right)^{1/2} = \alpha\sqrt{2T} + \rho \le \sigma \end{aligned}$$

since by definition of T, T_0 and α

$$\begin{aligned} \alpha\sqrt{2T} &= \alpha\sqrt{2\left(T_0 + \frac{2a_1}{\alpha}\right)} = \sqrt{2\alpha\left(\max_{i=1,\ldots,m} |x_{i0}| + 2a_1\right)} \\ &\le \sqrt{2\alpha\left(\max_{i=1,\ldots,m} |x_{i0}| + 1\right)} \le \sigma - \rho. \end{aligned}$$

Here, in the last inequality we used (7). Thus, the evasion strategy (14)–(17) is admissible.

Next, we prove the following statement.

Lemma 3.6 *The following are true*

1) *For all* $i = 1, \dots, m$, *we have (i)* $y_{i2}(t) > 0$ *for* $t \ge T_0$ *and (ii)* $\tau_i' \le T$.
2) (i) *if* $x_{j2}^0 < 0$ *for some* $j \in \{1, \dots, m\}$, *then* $y_{j2}(\theta) = 0$ *at some unique* θ, $0 < \theta < T_0$, *and* $y_{j2}(t) > 0$ *for all* $t > \theta$.
   (ii) *if* $x_{j2}^0 \ge 0$ *for some* $j \in \{1, \dots, m\}$, *then* $y_{j2}(t) > 0$ *for all* $t > 0$.

Proof We first show that $y_{i2}(T_0) > 0$ for all $i = 1, \dots, m$. Indeed, by (14)–(15) we have

$$v_2(t) \ge \alpha + |u_{i2}(t)|,\ 0 \le t \le T_0, \tag{20}$$

and therefore,

$$\dot{y}_{i2}(t) = e^{\lambda_i t}(v_2(t) - u_{i2}(t)) \ge \alpha e^{\lambda_i t} > 0. \tag{21}$$

Hence, $y_{i2}(t)$, $0 \le t \le T_0$, increases strictly. By (21) we have

$$\begin{aligned} y_{i2}(T_0) &= x_{i2}^0 + \int_0^{T_0} e^{\lambda_i s}(v_2(s) - u_{i2}(s))ds \\ &\ge x_{i2}^0 + \alpha \int_0^{T_0} e^{\lambda_i s} ds \ge x_{i2}^0 + \frac{\alpha}{\lambda_i}\left(e^{\lambda_i T_0} - 1\right) \\ &> x_{i2}^0 + \frac{\alpha}{\lambda_i}\lambda_i T_0 \ge -|x_i^0| + \max_{1 \le j \le m} |x_j^0| \ge 0. \end{aligned}$$

Thus, $y_{i2}(T_0) > 0$ for all $i = 1, \dots, m$. Since $v_2(t) \ge |u_i(t)| \ge |u_{i2}(t)|$ for $t > T_0$, therefore for $t > T_0$ we have

$$\begin{aligned} y_{i2}(t) &= y_{i2}(T_0) + \int_{T_0}^{t} e^{\lambda_i s}(v_2(s) - u_{i2}(s))ds \\ &\ge y_{i2}(T_0) + \int_{T_0}^{t} e^{\lambda_i s}(|u_{i2}(s)| - u_{i2}(s))ds \ge y_{i2}(T_0) > 0. \end{aligned}$$

Thus, $y_{i2}(t) > 0$, for all $t \ge T_0$ and $i = 1, \dots, m$. In particular, we obtain that there is no a_k-approach time τ_k in the time interval $[T_0, \infty)$, since by definition (10) of an a_k-approach time τ_k one has to have $y_{k2}(\tau_k) < 0$. This is impossible for $\tau_k \ge T_0$ since by item 1) (i) $y_{k2}(t) > 0$ for all $t \ge T_0$. Thus, $\tau_i \le T_0$ for all $i = 1, \dots, m_0$.

Next, by item 1) (ii) of Property 3.3 we have

$$\tau_i' \le \tau_i + \frac{2a_i}{\alpha} \le T_0 + \frac{2a_1}{\alpha} = T, \tag{22}$$

and the proof of item 1) of Lemma 3.6 follows. In particular, (22) implies that $I_1 \subset [0, T]$.

Remark 3.7 Due to the inclusion $I_1 \subset [0, T]$ the set $[0, T] \cap I_1$ in (15) is equal to I_1.

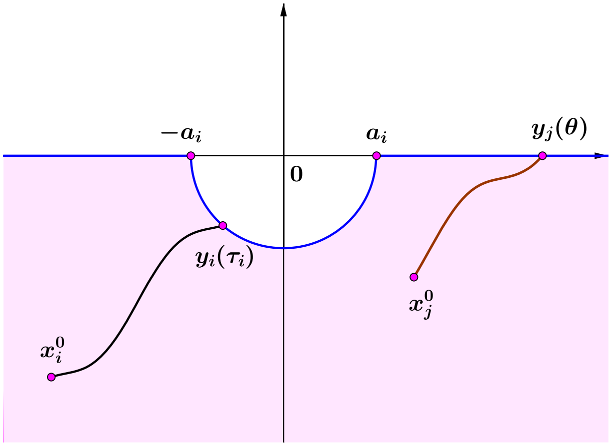

Fig. 2 Initial states have negative y-coordinates

Next, we prove item 2) (i). Since by (21) $y_{j2}(t)$, $0 \le t \le T_0$, increases strictly and as shown above $y_{j2}(T_0) > 0$, then we necessarily have that $y_{j2}(\theta) = 0$ at some unique θ, $0 < \theta < T_0$. In view of (20) we then obtain for $t > \theta$ that

$$y_{j2}(t) = y_{j2}(\theta) + \int_\theta^t e^{\lambda_j s}(v_2(s) - u_{j2}(s))ds$$
$$\ge \int_{[\theta,t]\cap[\theta,T_0]} e^{\lambda_j s}(\alpha + |u_{j2}(s)| - u_{j2}(s))ds > 0,$$

which is the desired result. To show item 2) (ii), using $x_{j2}^0 \ge 0$ we observe that for $t \ge 0$,

$$y_{j2}(t) = x_{j2}^0 + \int_0^t e^{\lambda_j s}(v_2(s) - u_{j2}(s))ds$$
$$\ge \int_{[0,t]\cap[0,T_0]} e^{\lambda_j s}(\alpha + |u_{j2}(s)| - u_{j2}(s))ds > 0.$$

This completes the proof of Lemma 3.6. □

Item 1) (i) of Lemma 3.6 implies, in particular, that for the point y_j with initial state x_j^0 for which $x_{j2}^0 < 0$, the inequality $y_{j2}(T_0) > 0$ is satisfied. Thus, we necessarily have either $|y_j(\tau_j)| = a_j$ and $y_{j2}(\tau_j) < 0$ at some $0 < \tau_j < T_0$ (see the point y_i in Fig. 2) or $y_{j2}(\theta) = 0$ and $|y_{j1}(\theta)| \ge a_j$ at some $0 < \theta < T_0$ (see the point y_j in Fig. 2). The former case will be studied in the following subsections in detail. In the latter case, we ignore the point $y_j(t)$ starting from the time θ since by Lemma 3.6 2) (i) we have $y_{j2}(t) > 0$ and so $y_j(t) \ne 0$ for all $t > \theta$. That is why in definition of

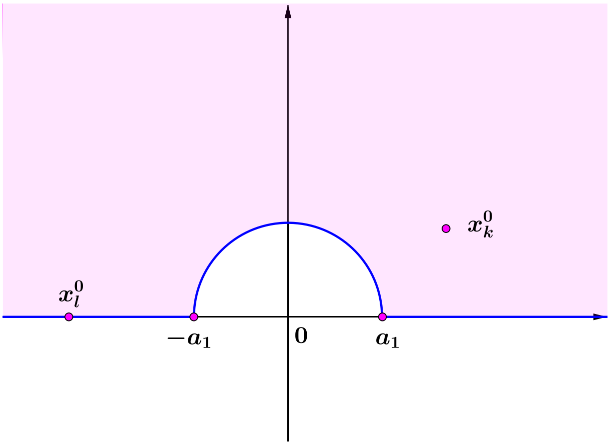

Fig. 3 Initial states have non negative y-coordinates

a_i-approach time τ_i (10) we required the inequality $y_{j2}(\tau_i) < 0$. The initial states x_k^0 with $x_{k2}^0 > 0$, and x_l^0 with $x_{l2}^0 = 0$, are shown in Fig. 3. We ignore the corresponding points $y_k(t)$ and $y_l(t)$ as well for $t \geq 0$ since by Lemma 3.6 2) (ii) $y_{k2}(t) > 0$ and $y_{l2}(t) > 0$ for all $t \geq 0$.

Property 3.8 *For any $i \in \{1, \ldots, m_0\}$ and $p \in \{1, \ldots, m\}$,*

$$\int_{\tau_i}^{\tau_i'} e^{\lambda_p s} ds \leq e^{\Lambda T} \cdot \frac{4a_i}{\alpha}. \tag{23}$$

Proof We use (12) and $\tau_i \leq T_0 < T$ to obtain

$$\begin{aligned} \int_{\tau_i}^{\tau_i'} e^{\lambda_p s} ds &= \frac{1}{\lambda_p}\left(e^{\lambda_p \tau_i'} - e^{\lambda_p \tau_i}\right) \\ &= \frac{e^{\lambda_p \tau_i}}{\lambda_p}\left(e^{\lambda_p(\tau_i' - \tau_i)} - 1\right) \leq \frac{1}{\lambda_p} e^{\Lambda T}\left(e^{\frac{2\lambda_p a_i}{\alpha}} - 1\right). \end{aligned} \tag{24}$$

Since by (8) $\frac{2\lambda_p a_i}{\alpha} \leq \frac{2\Lambda a_1}{\alpha} \leq 1$, then using the inequality $e^x - 1 \leq 2x$, $0 \leq x \leq 1$, in (24) we obtain (23). This completes the proof of the property. □

3.3 Auxiliary point z_p

Take any $p \in \{1, \ldots, m_0\}$ and estimate $|y_p(t)|$ on $[\tau_p, \tau_p']$ assuming that a_p-approach was occurred at time τ_p with the point y_p. To this end we introduce an auxiliary point

z_p whose dynamics is described by the following equation

$$z_p(t) = y_p(\tau_p) + \int_{\tau_p}^{t} e^{\lambda_p s}(V_p(s) - u_p(s))ds,\ \ \tau_p \le t \le \tau_p'. \tag{25}$$

Note that the point $z_p(t)$ is defined only on the interval $[\tau_p, \tau_p']$. Since by (15) $v_2(t) = U(t)$, therefore

$$\begin{aligned} z_{p2}(t) &= z_{p2}(\tau_p) + \int_{\tau_p}^{t} e^{\lambda_p s}(U(s) - u_{p2}(s))ds \\ &= y_{p2}(\tau_p) + \int_{\tau_p}^{t} e^{\lambda_p s}(v_2(s) - u_{p2}(s))ds = y_{p2}(t),\ \ \tau_p \le t \le \tau_p'. \end{aligned} \tag{26}$$

Next, we show that

$$\int_{\tau_p}^{\tau_p'} |V_p(s)|^2 ds \le \sigma^2. \tag{27}$$

Indeed, denoting

$$f_1(t) = (\alpha, \alpha),\ \ g_1(t) = \left(|u_{p1}(t)|, \left(\sum_{j=1}^{m} u_{j2}(t)^2 \right)^{1/2} \right)$$

we obtain

$$\begin{aligned} |V_p(t)|^2 = V_{p1}^2(t) + U^2(t) &= (\alpha + |u_{p1}(t)|)^2 + \left(\alpha + \left(\sum_{j=1}^{m} u_{j2}^2(t) \right)^{1/2} \right)^2 \\ &= |f_1(t) + g_1(t)|^2. \end{aligned}$$

Therefore, using the Minkowskii inequality and then item (1) of Property 3.3 we obtain

$$\begin{aligned} \left(\int_{\tau_p}^{\tau_p'} |V_p(s)|^2 ds \right)^{1/2} &= \left(\int_{\tau_p}^{\tau_p'} |f_1(s) + g_1(s)|^2 ds \right)^{1/2} \\ &\le \left(\int_{\tau_p}^{\tau_p'} |f_1(s)|^2 ds \right)^{1/2} + \left(\int_{\tau_p}^{\tau_p'} |g_1(s)|^2 ds \right)^{1/2} \\ &\le (2\alpha^2(\tau_p' - \tau_p))^{1/2} + \left(\int_{\tau_p}^{\tau_p'} \sum_{j=1}^{m} |u_j(s)|^2 ds \right)^{1/2} \\ &\le 2\sqrt{\alpha a_p} + \rho < \sigma, \end{aligned}$$

since by (8) $a_p \le a_1 < \frac{(\sigma-\rho)^2}{4\alpha}$, and hence (27) is true.

3.4 Estimation of $|z_p(t)|$

Let $\tau_p \le t < \tau_p'$ and for definiteness assume that $y_{p1}(\tau_p) \ge 0$. Then by (16) we have $V_{p1}(t) = \alpha + |u_{p1}(t)|$. Therefore,

$$\begin{aligned}|z_p(t)| \ge z_{p1}(t) &= y_{p1}(\tau_p) + \int_{\tau_p}^{t} e^{\lambda_p s}(V_{p1}(t) - u_{p1}(t))ds \\ &\ge \int_{\tau_p}^{t} e^{\lambda_p s}(\alpha + |u_{p1}(t)| - u_{p1}(t))ds \\ &\ge \alpha \int_{\tau_p}^{t} e^{\lambda_p s} ds = \frac{\alpha}{\lambda_p}(e^{\lambda_p t} - e^{\lambda_p \tau_p}). \end{aligned} \tag{28}$$

On the other hand,

$$|z_p(t)| \ge |y_p(\tau_p)| - \int_{\tau_p}^{t} e^{\lambda_p s}|V_p(s) - u_p(s)|ds. \tag{29}$$

The integral in (29) can be estimated by using the Cauchy-Schwartz inequality as follows

$$\begin{aligned}\int_{\tau_p}^{t} e^{\lambda_p s}|V_p(s) - u_p(s)|ds &\le \left(\int_{\tau_p}^{t} e^{2\lambda_p s}ds\right)^{1/2} \left(\int_{\tau_p}^{t} |V_p(s) - u_p(s)|^2 ds\right)^{1/2} \\ &\le \left(\int_{\tau_p}^{t} e^{2\lambda_p s}ds\right)^{1/2} \left(\int_{\tau_p}^{t} 2(|V_p(s)|^2 + |u_p(s)|^2)ds\right)^{1/2}. \end{aligned} \tag{30}$$

Since by (27) and the admissibility of control $u_p(s)$

$$\int_{\tau_p}^{t} |V_p(s)|^2 ds \le \sigma^2, \quad \int_{\tau_p}^{t} |u_p(s)|^2 ds \le \rho_p^2 \le \sigma^2,$$

then it follows from (30) that

$$\int_{\tau_p}^{t} e^{\lambda_p s}|V_p(s) - u_p(s)|ds \le 2\sigma \left(\int_{\tau_p}^{t} e^{2\lambda_p s}ds\right)^{1/2}. \tag{31}$$

Since by (22) $\tau_p \le t \le \tau_p' \le T$, therefore

$$\int_{\tau_p}^{t} e^{2\lambda_p s}ds = \frac{1}{2\lambda_p}\left(e^{\lambda_p t} + e^{\lambda_p \tau_p}\right)\left(e^{\lambda_p t} - e^{\lambda_p \tau_p}\right) \le \frac{e^{\Lambda T}}{\lambda_p}\left(e^{\lambda_p t} - e^{\lambda_p \tau_p}\right).$$

Then by (31) we can see that

$$\int_{\tau_p}^{t} e^{\lambda_p s}|V_p(s)-u_p(s)|ds \le 2\sigma\sqrt{\frac{e^{\Lambda T}}{\lambda_p}(e^{\lambda_p t}-e^{\lambda_p \tau_p})}. \tag{32}$$

Combining (29) and (32), and using the equation $|y_p(\tau_p)| = a_p$ yields that

$$|z_p(t)| \ge a_p - 2\sigma\sqrt{\frac{e^{\Lambda T}}{\lambda_p}(e^{\lambda_p t}-e^{\lambda_p \tau_p})}. \tag{33}$$

It is easily seen from (28) and (33) that

$$|z_p(t)| \ge f(t) = \max\{f_1(t), f_2(t)\}, \quad t \ge \tau_p, \tag{34}$$

where

$$f_1(t) = \frac{\alpha}{\lambda_p}(e^{\lambda_p t}-e^{\lambda_p \tau_p}), \quad f_2(t) = a_p - 2\sigma\sqrt{\frac{e^{\Lambda T}}{\lambda_p}(e^{\lambda_p t}-e^{\lambda_p \tau_p})}.$$

Note that $f_1(t)$, $t \ge \tau_p$, increases from 0 to ∞ and $f_2(t)$, $t \ge \tau_p$, decreases from a_p to $-\infty$. It is not difficult to see that $f(t)$, $t \ge \tau_p$, attains its minimum at the point $t = t_*$, where

$$f_1(t) = f_2(t), \quad t \ge \tau_p. \tag{35}$$

Let $\left(e^{\lambda_p t}-e^{\lambda_p \tau_p}\right)^{1/2} = z$. Then Eq. (35) takes the form

$$\frac{\alpha}{\lambda_p}z^2 = a_p - 2\sigma\sqrt{\frac{e^{\Lambda T}}{\lambda_p}}z,$$

or $\alpha z^2 + 2\sigma\sqrt{e^{\Lambda T}\lambda_p}z - a_p\lambda_p = 0$. This equation has the following positive root

$$\begin{aligned} z_* &= \frac{\sqrt{\lambda_p}}{\alpha}\left(-\sigma\sqrt{e^{\Lambda T}} + \sqrt{\sigma^2 e^{\Lambda T} + a_p\alpha}\right) \\ &= \frac{a_p\sqrt{\lambda_p}}{\sigma\sqrt{e^{\Lambda T}} + \sqrt{\sigma^2 e^{\Lambda T} + a_p\alpha}}. \end{aligned}$$

Then

$$\min_{t\ge\tau_p} f(t) = f(t_*) = f_1(t_*) = \frac{\alpha}{\lambda_p}z_*^2 = \frac{\alpha}{\lambda_p}\cdot\frac{a_p^2\lambda_p}{(\sigma\sqrt{e^{\Lambda T}} + \sqrt{\sigma^2 e^{\Lambda T} + a_p\alpha})^2}. \tag{36}$$

By (8) we have $a_1 < \frac{\sigma^2}{32\alpha} < \frac{1}{\alpha}\sigma^2 e^{\Lambda T}$, and so $a_p \alpha \le a_1 \alpha < \sigma^2 e^{\Lambda T}$, therefore (36) implies that

$$|z_p(t)| \ge \min_{t \ge \tau_p} f(t) > \frac{\alpha a_p^2}{6\sigma^2 e^{\Lambda T}}. \tag{37}$$

Next, since by definition

$$\tau_p' = \frac{1}{\lambda_p} \ln\left(e^{\lambda_p \tau_p} + \frac{2\lambda_p a_p}{\alpha}\right),$$

using the fact that $y_{p2}(\tau_p) \ge -|y_p(\tau_p)| = -a_p$ we obtain

$$\begin{aligned} z_{p2}(\tau_p') &= y_{p2}(\tau_p) + \int_{\tau_p}^{\tau_p'} e^{\lambda_p s}(U(s) - u_{p2}(s))ds \\ &\ge -a_p + \int_{\tau_p}^{\tau_p'} e^{\lambda_p s}\left(\alpha + \left(\sum_{i=1}^m u_{i2}(s)^2\right)^{1/2} - u_{p2}(s)\right) ds \\ &\ge -a_p + \alpha \int_{\tau_p}^{\tau_p'} e^{\lambda_p s} ds = -a_p + \frac{\alpha}{\lambda_p}\left(e^{\lambda_p \tau_p'} - e^{\lambda_p \tau_p}\right) \\ &= -a_p + \frac{\alpha}{\lambda_p}\left(e^{\lambda_p \tau_p} + \frac{2\lambda_p a_p}{\alpha} - e^{\lambda_p \tau_p}\right) = a_p. \end{aligned} \tag{38}$$

Finally, let $t \ge \tau_p'$. By (26) $y_{p2}(\tau_p') = z_{p2}(\tau_p')$, and by (14), (15) and (17), $v_2(t) \ge |u_{p2}(t)|$. Then using (17), (38) we get

$$y_{p2}(t) = z_{p2}(\tau_p') + \int_{\tau_p'}^{t} e^{\lambda_p s}(v_2(s) - u_{p2}(s))ds \ge z_{p2}(\tau_p') \ge a_p.$$

Thus, we have the following inequalities

$$|z_p(t)| > \frac{\alpha a_p^2}{6\sigma^2 e^{\Lambda T}}, \quad \tau_p \le t \le \tau_p', \tag{39}$$

$$y_{p2}(t) \ge a_p, \qquad t \ge \tau_p'. \tag{40}$$

3.5 Estimation of $|y_p(t) - z_p(t)|$

We have

$$|y_p(t) - z_p(t)| = \left| \int_{\tau_p}^{t} (v(s) - V_p(s))ds \right|, \quad \tau_p \le t \le \tau_p'. \tag{41}$$

By (15) and (26)

$$v(t) = (V_{r1}(t), U(t)), \quad V_p(t) = (V_{p1}(t), U(t)), \quad \tau_p \le t < \tau_p'. \tag{42}$$

Consider two cases: (i) $\tau_p' \le \tau_{p+1}$ and (ii) $\tau_{p+1} \le \tau_p'$.

Case (i). Let $\tau_p' \le \tau_{p+1}$. Then by item (1) of Property 3.4 $r = r(t) = p$ for $\tau_p \le t < \tau_p'$. Therefore by (42) we have $v(t) = V_p(t)$, $\tau_p \le t < \tau_p'$. Hence, by (41)

$$|y_p(t) - z_p(t)| = 0. \tag{43}$$

Case (ii). Assume now $\tau_{p+1} \le \tau_p'$. Then by item (2) of Property 3.4 we have $v(t) = (V_{p1}(t), U(t))$, $\tau_p \le t < \tau_{p+1}$. Therefore, (41) leads to

$$\begin{aligned} &|y_p(t) - z_p(t)| \\ &= \left| \int_{\tau_{p+1}}^t e^{\lambda_p s}(v(s) - V_p(s))ds \right| \le \int_{\tau_{p+1}}^t e^{\lambda_p s}|v(s) - V_p(s)|ds \\ &\le \int\limits_{[\tau_{p+1},t)\setminus I_{p+1}} e^{\lambda_p s}|v(s) - V_p(s)|ds + \int\limits_{[\tau_{p+1},t)\cap I_{p+1}} e^{\lambda_p s}|v(s) - V_p(s)|ds. \end{aligned} \tag{44}$$

Since by definition $r(t) = p$, $t \in [\tau_p, \tau_p')\setminus I_{p+1}$ and $[\tau_{p+1}, t)\setminus I_{p+1} \subset [\tau_p, \tau_p')\setminus I_{p+1}$, therefore we have $r = r(t) = p$, and hence, $v(t) = V_p(t)$ for $t \in [\tau_{p+1}, t)\setminus I_{p+1}$. Consequently, the first integral in (44) is 0, and so (44) takes the form

$$|y_p(t) - z_p(t)| \le \int_{[\tau_{p+1},t]\cap I_{p+1}} e^{\lambda_p s}|v(s) - V_p(s)|ds. \tag{45}$$

By (16) and (42)

$$|v(s) - V_p(s)| = |V_{r1}(s) - V_{p1}(s)| \le 2\alpha + |u_{r1}(s)| + |u_{p1}(s)|,$$

and therefore (45) implies that

$$|y_p(t) - z_p(t)| \le \int_{I_{p+1}} e^{\lambda_p s}(2\alpha + |u_{r1}(s)| + |u_{p1}(s)|)ds. \tag{46}$$

To estimate the integral in (46), we need to estimate the integrals

$$\int_{I_{p+1}} e^{\lambda_p s}ds, \quad \int_{I_{p+1}} e^{\lambda_p s}|u_{r1}(s)|ds, \text{ and } \int_{I_{p+1}} e^{\lambda_p s}|u_{p1}(s)|ds. \tag{47}$$

The first integral can be estimated using (23) and Property 3.2 as follows

$$\int_{I_{p+1}} e^{\lambda_p s}ds \le \sum_{i=p+1}^{m} \int_{\tau_i}^{\tau_i'} e^{\lambda_p s}ds \le \sum_{i=p+1}^{m} e^{\Lambda T}\frac{4a_i}{\alpha} \le e^{\Lambda T}\frac{8a_{p+1}}{\alpha}. \tag{48}$$

Next, we estimate the second integral in (47). Using the Cauchy-Schwartz inequality we have

$$\int_{I_{p+1}} e^{\lambda_p s}|u_{p1}(s)|ds \le \left(\int_{I_{p+1}} e^{2\lambda_p s}ds\right)^{1/2} \cdot \left(\int_{I_{p+1}} |u_{p1}(s)|^2 ds\right)^{1/2}. \quad (49)$$

Since

$$\int_{I_{p+1}} |u_{p1}(s)|^2 ds \le \sum_{i=1}^{m} \int_0^\infty |u_i(s)|^2 ds \le \sigma^2$$

and similar to (48) we get

$$\int_{I_{p+1}} e^{2\lambda_p s}ds \le \sum_{i=p+1}^{m} \int_{\tau_i}^{\tau_i'} e^{2\lambda_p s}ds \le e^{2\Lambda T}\frac{8a_{p+1}}{\alpha}.$$

Then it follows from (49) that

$$\int_{I_{p+1}} e^{\lambda_p s}|u_{p1}(s)|ds \le \sigma e^{\Lambda T}\sqrt{\frac{8a_{p+1}}{\alpha}}. \quad (50)$$

Similarly, for the third integral in (47), we have

$$\int_{I_{p+1}} e^{\lambda_p s}|u_{p1}(s)|ds \le \sigma e^{\Lambda T}\sqrt{\frac{8a_{p+1}}{\alpha}}. \quad (51)$$

Combining (48), (50), and (51) we obtain from (46) that

$$|y_p(t) - z_p(t)| \le 2\alpha \cdot \frac{8a_{p+1}}{\alpha} \cdot e^{\Lambda T} + 2\sigma e^{\Lambda T}\sqrt{\frac{8a_{p+1}}{\alpha}} \le 3\sigma e^{\Lambda T}\sqrt{\frac{8a_{p+1}}{\alpha}}$$

using the inequality

$$16a_{p+1} < \sigma\sqrt{\frac{8a_{p+1}}{\alpha}}$$

which follows from the inequalities $a_{p+1} \le a_1 < \frac{\sigma^2}{32\alpha}$ (see (8)).

Thus,

$$|y_p(t) - z_p(t)| \le 3\sigma e^{\Lambda T}\sqrt{\frac{8a_{p+1}}{\alpha}},\ \tau_p \le t \le \tau_p'. \quad (52)$$

3.6 Estimation of $|y_p(t)|$

Using (39) and (52) we obtain

$$|y_p(t)| \geq |z_p(t)| - |y_p(t) - z_p(t)| \geq \frac{\alpha a_p^2}{6\sigma^2 e^{\Lambda T}} - 3\sigma e^{\Lambda T}\sqrt{\frac{8a_{p+1}}{\alpha}} \geq \frac{\alpha a_p^2}{12\sigma^2 e^{\Lambda T}}, \tag{53}$$

for $t \in [\tau_p, \tau_p']$ since by (9)

$$a_{p+1} \leq \frac{\alpha^3}{8 \cdot 6^4 \sigma^6 e^{4\Lambda T}} a_p^4.$$

Also, it follows from the definition of κ and the inequality $a_p < 1$ that

$$a_{p+1} \leq \frac{\alpha}{16\sigma^2 e^{\Lambda T}} a_p^4 \leq \frac{\alpha}{12\sigma^2 e^{\Lambda T}} a_p^2.$$

Therefore, (53) implies that $|y_p(t)| > a_{p+1}$. Also, by (40)

$$y_{p2}(t) \geq a_p, \quad t \geq \tau_p'.$$

Thus,

$$|y_p(t)| \geq a_p, \text{ for } 0 \leq t \leq \tau_p, \text{ (by definition of } \tau_p) \tag{54}$$

$$|y_p(t)| \geq \frac{\alpha a_p^2}{12\sigma^2 e^{\Lambda T}} > a_{p+1}, \text{ for } \tau_p \leq t \leq \tau_p', \text{ (by (53))} \tag{55}$$

$$y_{p2}(t) \geq a_p, \text{ for } t \geq \tau_p', \text{ (by (40))} \tag{56}$$

Thus we conclude:

If an a_p-approach time τ_p occurs with the point y_p (see the point y_i in Fig. 2), then $y_p(t) \neq 0$, for all $t \geq 0$ (see (54)–(56)). Moreover, for any $i \geq p+1$, there is no a_i-approach time τ_i for the point y_p

(1) on the time interval $[\tau_p, \tau_p']$ since $|y_p(t)| > a_{p+1} \geq a_i$ for any $i \geq p+1$ (see (55)).
(2) on the time interval $[\tau_p', \infty)$, since $|y_p(t)| \geq y_{p2}(t) \geq a_p > a_i$ for any $i \geq p+1$ (see (56)).

The proof of Theorem 3.1 is completed.

4 Conclusion

We have studied a linear evasion differential game of many pursuers and one evader. We have constructed a strategy for the evader and proved the possibility of evasion. The evader uses a manoeuvre on the set I_1 and on the set $[0, T]\backslash I_1$ evader uses the control $v(t) = \left(0, \alpha + \left(\sum_{j=1}^{m} |u_j(t)|^2\right)^{1/2}\right)$. The measure of the set I_1 can be made by choosing parameters a_1 and α as small as we wish. We have also shown that all the approach times τ_i can occur only before a specified time T_0, moreover $\tau_i' \leq T$. The total number of approach times τ_i of all pursuers doesn't exceed the number of pursuers m. For $t \geq T$, the evader uses the control $v(t) = \left(0, \left(\sum_{j=1}^{m} |u_j(t)|^2\right)^{1/2}\right)$ and there is no longer an approach time occurs. The main contributions of the paper are (i) the construction of evasion strategy, (ii) estimating the distances of objects from the origin, (iii) the possibility of evasion from many pursuers.

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