Chapter 7
Conceptions of Proof – In Research and Teaching

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1 Conceptualisations of Proof

1.1 Conceptions by Mathematicians

The education of professional mathematicians very successfully transmits a practically precise conception of proof. Mathematically educated persons who specialise in and know a certain domain of mathematics will generally agree that a given piece of mathematical text is an adequate proof of a given statement. Nevertheless, no explicit general definition of a proof is shared by the entire mathematical community. Consequently, in attempting such a conceptualisation it

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is wise to resort first to mathematical logic for a definition of proof and to look afterwards at how working mathematicians comment on such a definition.

According to Rav (1999, p. 11), in a formalised theory **T** a linear derivation is a finite sequence of formulas in the language of **T**, each member of which is a logical axiom, or an axiom of **T**, or the result of applying one of the finitely many explicitly stated rules of inference to previous formulas in the sequence. A tree derivation can be similarly defined. A formula of **T** is said to be derivable if it is the end-formula of a linear or tree derivation.

Obviously, the structure characterised in this formal definition echoes the axiomatic method of Euclid's *Elements* (c. 300 B.C.). Hence, we can consider the notion of proof as some combination of the axiomatic method and formalism, the latter called 'rigour' since the time of Cauchy.

That definition may be considered as a 'projection' of the real practice of mathematical proof onto the skeleton of formal logic. A projection inherits some properties of the original, but is as a rule, poorer. Consequently, Rav distinguishes the formal idea of proof from that of a 'conceptual proof', by which he means an informal proof "of customary mathematical discourse, having an irreducible semantic content" (Rav 1999, p. 11; see also Hanna and Barbeau 2009, p. 86). As a rule, working mathematicians insist on the informal and semantic components of proof. As Rav stresses, beyond establishing the truth of a statement, proof contributes to getting new mathematical insights and to establishing new contextual links and new methods for solving problems. (Functions of proof beyond that of verification are also discussed by Bell (1976), de Villiers (1990), and many others.)

Working mathematicians also stress the *social process* of checking the validity of a proof. As Manin put it: "A proof only becomes a proof after the social act of 'accepting it as a proof'. This is true for mathematics as it is for physics, linguistics, and biology" (Manin 1977, p. 48). By studying the comments of working mathematicians Hanna came to the conclusion that the public process of accepting a proof not only involves a check of deductive validity, but is also determined by factors like 'fit to the existing knowledge', 'significance of the theorem', the 'reputation of the author' and 'existence of a convincing argument' (Hanna 1983, p. 70; see also Neubrand 1989). Bell (1976) also stressed the essentially public character of proof.

All in all, formal definitions of proof cover the meaning of the notion only incompletely, whereas mathematicians are convinced that, in practice, they know precisely what a proof is. This situation is difficult to handle in the teaching of mathematics at schools, since there exist no easy explanations of what proof and proving are that teachers could provide to their pupils. Proof is not a "stand-alone concept", as Balacheff nicely puts it (2009, p. 118), and is aligned to the concept of a "theory" (see also Jahnke 2009b, p. 30).

1.2 Conceptions by Mathematics Educators

Genuinely *didactical* conceptions of proof are determined by two clearly distinguishable sets of motives. One line of thought tries to devise genetic ideas of proof.

These ideas are pedagogically motivated in that they try to devise a learning path from a cognitive state in which an individual learner is able to construct argumentations with some deductive components to a state in which the learner manages to understand and develop mathematical proofs in their proper sense. The other line of thought builds conceptions of proof with the intention of doing empirical research. Both lines of thought define categories that allow one to classify individuals' argumentative behaviours and strategies observed in classrooms or in interview situations. In both, it is essential to distinguish between arguments which are not yet proofs from mathematical proofs proper. No wonder different researchers come to different conclusions about the demarcation between the two modes of reasoning. Balacheff (1988) and Duval (1991), for example, draw a sharp line of demarcation, whereas other authors stress the continuity between argumentation and proof – thereby embedding proof in a general theory of argumentation. In mathematics education, in recent years it has become customary to use the term 'argumentation' for reasoning which is 'not yet' proof and the term 'proof' for mathematical proof proper.

1.2.1 Genetic Ideas of Proof

Genetic conceptions distinguish between different stages in the development of proof essentially along three lines. First, the *type of warrant* for a general statement is at stake. Pupils might infer from some special cases a general rule or statement. In such cases problems of proof and generalisation are intermingled. Generalisation is an important scientific activity but, of course, different from proof. Consequently, it is an important step in these pupils' cognitive development to understand that a general statement can only be derived from other general statements. Second, if pupils argue on the level of general statements the *type of principle* they refer to differs according to their proximity to established mathematical principles and norms. For example, in comparing the lengths of different paths between two points, pupils might apply a physical argument like the stretching of an elastic band as a warrant. Whether this is accepted as a valid argument is, in principle, a matter of classroom convention. Nevertheless, some authors would not classify this as a mathematical proof, because the principle does not belong to the accepted principles of mathematics proper. Third, the *mode of representing* an argument might also be a distinguishing feature between different stages in the acquisition of proof. Though, in principle, it is extrinsic to the very heart of proof whether it is displayed in verbal or symbolic form, growth in the ability to handle the symbolic language of mathematics is an indispensable condition for growth of a learner's ability to understand and develop mathematical proofs; the complexity of most mathematical relations is such that they can hardly be expressed without recourse to symbolic representation.

A famous paper by van Dormolen (1977) exemplifies genetic conceptions of proof. In regard to different proof situations, van Dormolen gives possible solutions which reflect different stages in the development of argumentative skills. One task asks for a proof of the statement that the diagonals in an isosceles trapezium are equal (Fig. 7.1). A beginning pupil might react by measuring the diagonals and finding them equal. On a second level, the pupil might mentally cut out the trapezium,

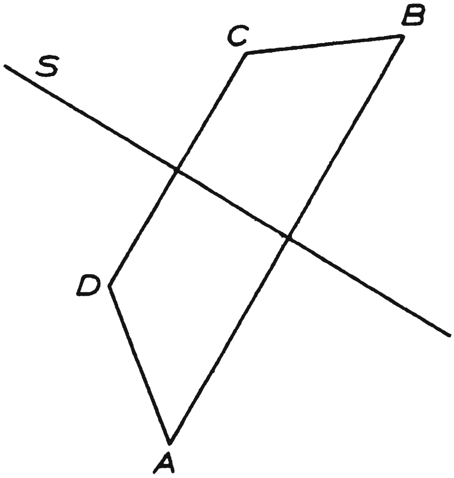

Fig. 7.1 Van Dormolen's trapezium problem

turn it about and put it back into its hole, with the result that each diagonal is now in the position of the other. On a third level, a pupil might formally argue from the symmetry of the figure and apply a reflection, so that one diagonal is mapped on the other.

Van Dormolen considers these three stages in the framework of van Hiele's theory of levels of thinking. On a first level the student is bound to special objects. On a second level the student can think about properties of classes of objects, and on the third level the student is able to logically organise an argument.

As a further example of such a pedagogically motivated genetic conception, Kirsch (1979) distinguished between 'pre-mathematical' and 'mathematical' proofs, referring to the earlier paper (Semadeni n.d.). Later, Kirsch changed his terminology and spoke about 'pre-formal' versus 'formal' proofs (Blum and Kirsch 1991).

1.2.2 Conceptions from Empirical Research

In discussing conceptions of proof motivated by empirical research, Balacheff (1988) states explicitly that his view rests on an experimental approach and that he is interested in studying pupils' practices. For this purpose, he distinguishes between 'pragmatic' and 'conceptual' proofs. Pragmatic proofs have recourse to real actions, whereas conceptual proofs deal with properties and the relations between them and, consequently, do not involve actions. However, there can and should be bridges between the two types, insofar as pragmatic proofs might have a generic quality and be a step towards conceptual proofs. Though Balacheff's primary interest lies in the classification of pupils' practices, these distinctions interfere with genetic ideas, since pragmatic proofs operate on a lower, earlier level than conceptual proofs. Balacheff explicitly says that his categories "form a hierarchy … Where a particular type of proof falls in this hierarchy depends on how much the demands of generality and the conceptualisation of knowledge are involved," (Balacheff 1988, p. 218). It is also plausible to see in the distinction between pragmatic and conceptual proofs an influence of Piaget's cognitive psychology which is based on the idea that thoughts are preceded by actions.

Balacheff has further refined the distinction between pragmatic and conceptual proofs into sub-categories. Pragmatic proofs split into 'naïve empiricism' and 'crucial experiment': 'Naïve empiricism' refers to asserting the truth of a result by verifying special cases. 'Crucial experiment' means that a pupil considers a special example and argues that the proposition in question must be true if it is true for this (extreme) example. Argumentative behaviour of this kind shows some attention to the problem of a statement's generality, but is still bound to the special case.

The 'generic example', Balacheff's first sub-category of a conceptual proof, is different. The generic example makes the reasons for the truth of an assertion explicit by operations or transformations on an object that is a characteristic representative of its class. The difference from the 'crucial experiment' is subtle but decisive. The final category, 'thought experiment', invokes internalised actions and is detached from particular representations. If we understand Balacheff, this level still falls short of a professional approach to proof but represents the best students can attain in their school lives.

Harel and Sowder (1998) have proposed another influential system of categories. They give priority to the function of proof as a *convincing argument* and derive from this idea categories for classifying individuals' argumentative behaviours. Their starting point is a pedagogical motive. "The goal is to help students refine their own conception of what constitutes justification in mathematics: from a conception that is largely dominated by surface perceptions, symbol manipulation, and proof rituals, to a conception that is based on intuition, internal conviction, and necessity" (1998, p. 237). Consequently, their categories reflect students' ideas about what a convincing argument might be. These categories they call 'proof schemes': Individuals' ways of thinking associated with the proving act and consisting of "what constitutes ascertaining and persuading for that person" (1998, p. 244).

Harel and Sowder's (1998) whole system of categories rests on a number of empirical studies and splits into three large domains: 'External conviction proof schemes', 'Empirical proof schemes' and 'Analytical proof schemes'. In the domain of 'External conviction proof schemes' the authors distinguish between 'Authoritarian', 'Ritual' and 'Symbolic' proof schemes. Authoritarian means that students refer to an authority – be it their teacher, a book or whatever – to convince themselves of the truth of a statement. Ritual and Symbolic proof schemes reflect the fact that many students come to the conclusion that ritual and form constitute mathematical justification. The Symbolic schemes describe a behaviour of approaching a solution without first comprehending the meaning of the symbols involved.

In the domain of Empirical proof schemes, the authors distinguish between 'inductive' and 'perceptual' proof schemes (Harel and Sowder 1998). 'Inductive' means obtaining a general statement from some special examples or measurements, whereas 'perceptual' designates uncritically taking for granted some visual property of an object or a configuration.

In general, Harel and Sowder define an Analytical proof scheme as one “that validates conjectures by means of logical deductions” (1998, p. 258). At first glance, it seems surprising that this scheme splits into a ‘Transformational’ and an ‘Axiomatic’ proof scheme. The ‘Axiomatic’ can be considered as what an educated mathematician might mean when speaking about proof. The ‘Transformational’ refers to arguments which identify an invariant by systematically changing a configuration. For example, consider angles inscribed in a circle over a fixed chord. According to a well-known theorem these angles are equal. If the vertex of an angle is moved on the circumference to one of the end-points of the chord, then in the limiting position one of the legs coincides with the tangent and the other with the chord. Consequently, the inscribed angle becomes the angle between chord and tangent and equal to the latter. Of course, this is also a well-known theorem in elementary geometry. Obviously, such a transformational argument is not a rigorous proof; yet it operates on a general level and can rightly be considered as belonging to an analytical domain.

Both Balacheff’s (1988) and Harel and Sowder’s (1998) systems of categories have been applied with some modifications in a number of empirical studies. Both systems share the problem that it is not always straightforward to relate an argumentative behaviour to a certain category; rather, it may require a considerable amount of interpretation. What distinguishes the two systems is the fact that Balacheff’s system involves a genetic sequence whereas Harel and Sowder do not seem to be interested in such a sequence; their point of view is more the co-existence or even competition between conceptions of proof and argumentation in various fields of human life (everyday life and other sciences). This difference might be due to the fact that their subjects are college students, whereas Balacheff is mostly interested in grade-school teaching.

2 Proving and Beliefs of Teachers and Students

2.1 The System of Beliefs

This sub-chapter examines the epistemological and pedagogical beliefs about the practice of proof in the classroom:

- beliefs about the nature and role of proof in mathematics.
- beliefs about the role of proof in school mathematics.
- beliefs about difficulties in proving.
- beliefs about how proof should be taught in school.
- beliefs about oneself as mathematical thinker in the context of proof.

Moreover, due to the central role of proof in mathematical activity, beliefs involved in the practice of proof are not confined to the subject of proof but also include beliefs about mathematics, about mathematics teaching and learning, about oneself in relation to mathematics, and so on.

Furinghetti and Pehkonen (2002) have shown to what extent beliefs and related concepts are controversial issues. They also have pointed out the different uses of the term ‘conception’ and the mutual relationship of beliefs, conceptions and knowledge. Following their recommendations, below we clarify our assumptions about the meanings of the terms we use and their mutual relationships.

As regards conceptions and beliefs, we follow Philipp (2007, p. 259) who describes conception as “a general notion or mental structure encompassing beliefs, meanings, concepts, propositions, rules, mental images, and preferences”. As regards beliefs and knowledge, our position is expressed by the following passage by Leatham:

> Of all the things we believe, there are some things that we “just believe” and other things we “more than believe – we know”. Those things we “more than believe” we refer to as knowledge and those things we “just believe” we refer to as beliefs. Thus beliefs and knowledge can profitably be viewed as complementary subsets of the things we believe. (2006, p. 92)

For us, “things we know” are those that rely on a social agreement inside a given community (for mathematics, the community of mathematicians).

In the same vein, Philipp describes knowledge as “beliefs held with certainty or justified true belief. What is knowledge for one person may be belief for another, depending upon whether one holds the conception as beyond question.” (2007, p. 259). Philipp goes on to describe beliefs as:

> Psychologically held understandings, premises, or propositions about the world that are thought to be true. Beliefs are more cognitive, are felt less intensely, and are harder to change than attitudes. Beliefs might be thought of as lenses that affect one’s view of some aspect of the world or as dispositions toward action. Beliefs, unlike knowledge, may be held with varying degrees of conviction and are not consensual. Beliefs are more cognitive than emotions and attitudes. (I do not indent this definition under affect because, although beliefs are considered a component of *affect* by those studying affect, they are not seen in this way by most who study teachers’ beliefs.) (ibid., p. 259)

We mainly use the term ‘belief’; we use ‘conception’ (in the sense explained above) when referring to a set of beliefs.

2.2 Teachers’ Epistemological and Pedagogical Beliefs

Researchers have investigated the beliefs about proof of pre-service and in-service elementary and secondary school teachers. We have organised their findings around four major themes: teachers’ knowledge of proof, teachers’ beliefs about the nature and role of proof in mathematics, teachers’ beliefs about the role of proof in school mathematics and teachers’ beliefs about themselves as mathematical thinkers in the context of proof.

2.2.1 Teachers’ Knowledge of Proof

The majority of researchers who have investigated teachers’ knowledge of proof have focused on teachers’ acceptance of empirical versus deductive arguments as valid proofs. Knuth (2002a), who investigated what constitutes proof for 16 in-service secondary school mathematics teachers, and Martin and Harel (1989), who assessed the notions of proof held by 101 pre-service elementary school teachers, gave their participants statements accompanied by predetermined arguments and asked them to rate these in terms of their validity. Whereas Martin and Harel asked for written responses only, Knuth conducted in-depth interviews with his participants.

Both Knuth (2002a) and Martin and Harel (1989) concluded that although most teachers correctly identify a valid argument, they also wrongly accept invalid arguments as proofs. Several pre-service elementary teachers accepted empirically based arguments as proofs (Martin and Harel 1989; see also Morselli 2006; Simon and Blume 1996). What secondary school teachers find convincing in an argument – inclusion of a concrete feature, specific examples and visual reference – (Knuth 2002a) might also explain why elementary teachers accepted empirical arguments as proofs. (On visualisation, see Biza et al. 2009)

The criteria teachers used to evaluate an argument differed widely but there were some commonalities. Several teachers adopted Symbolic or Ritual proof schemes (Harel and Sowder 1998; see Sect. 1.2.2 above). For example, some teachers focused on the correctness of the algebraic manipulations or on the form of an argument as opposed to its nature, (Knuth 2002a), whilst others accepted false proofs based on their ritualistic aspects (Martin and Harel 1989). Although they rated correct deductive arguments as valid proofs, teachers still did not find them convincing (Knuth 2002a). Treating the proof of a particular case as the proof for the general case was also common amongst most teachers (Knuth 2002a; Martin and Harel 1989).

Employing a different method, Jones (2000) asked recent mathematics graduates enrolled in a 1-year course to become secondary school teachers to construct concept maps reflecting their conceptions of mathematical proof. Analysis of the concept maps revealed that participants who had barely received pass degrees in mathematics courses needed "considerable support in developing a secure knowledge base of mathematics" (2000, p. 57). On the other hand, Jones reported, technical fluency in writing proofs did not necessarily imply richly connected knowledge of proof.

2.2.2 Teachers' Beliefs About the Nature and Role of Proof in Mathematics

Chazan stated that "many teachers do not seem to understand why mathematicians place such a premium on proof" (1993, p. 359). However, all of the in-service secondary school teachers in Knuth's (2002a) study indicated that the role of proof in mathematics was to establish the truth of a statement. They also suggested various other roles: explaining why something is true with a procedural focus rather than promoting understanding; the communicative role of proof (social interaction, communicating and convincing others); and the creation of knowledge and systematisation of results.

In a survey study with 30 pre-service elementary school teachers and 21 students majoring in mathematics with an emphasis in secondary education, Mingus and

Grassl (1999) asked the participants what constitutes a proof and asked about the role of proof in mathematics. In their definitions of proof, the secondary-education majors emphasised explanatory power, whereas the elementary-education majors focused on verification. The majority of the participants also pointed out the importance of proofs in helping “students understand the mathematics they are doing” (1999, p. 441). Furthermore, the secondary-education majors “also considered the role of *proof* for maintaining and advancing the structure of mathematics” (ibid., p. 441).

Although the teachers in Knuth’s (2002a) study could identify the roles of proof in mathematics and the explanatory power of proofs was mentioned by the secondary-education majors in the Mingus and Grassl (1999) study, Harel and Sowder later concluded from a review of the literature that teachers “do not seem to understand other important roles of proof, most noticeably its explanatory role” (2007, p. 48). Also, one important question is how teachers’ beliefs about proof relate to other aspects of their classroom practice. In a small qualitative study, Conner (2007) found that three student teachers’ conceptions of proof (particularly their beliefs about the purpose and role of proof in mathematics) aligned closely with how they supported argumentation (not proof in particular, but asking for and providing data and warrants for claims) in secondary classrooms.

2.2.3 Teachers’ Beliefs About the Role of Proof in School Mathematics

The roles of proof in school mathematics that the secondary teachers in Knuth’s (2002b) study talked about included all the roles they mentioned for proof in mathematics in general (Knuth 2002a) except for systematising statements into an axiomatic system. In the subsequent report (Knuth 2002b), the secondary teacher participants added some new roles for proof when they discussed it in the context of school mathematics: developing logical-thinking skills and displaying student thinking.

Although the roles that the teachers attached to proof in secondary-school mathematics seemed promising, their beliefs about the centrality of proof were limited (Knuth 2002a, b). Several teachers did not think that proof should be a central idea throughout secondary school mathematics, but only for advanced mathematics classes and students studying mathematics-related fields. On the other hand, all the teachers considered that *informal* proof should be a central idea *throughout* secondary-school mathematics. This is consistent with Healy and Hoyles’ reporting: “For many teachers it was more important that the argument was clear and uncomplicated than that it included any algebra” (2000, p. 413).

The majority of the teachers in Knuth’s research (2002b) viewed Euclidean geometry or upper-level mathematics classes as appropriate places to introduce proof to students. All of them said that they would accept an empirically based argument as a valid argument from students in a lower-level math class. Two of them, however, explained that they would discuss its limitations. Probably these beliefs were shaped by the teachers’ own experiences with proofs, since high-school

Euclidean geometry is the "usual locus" (Sowder and Harel 2003, p. 15) for introducing proof in U.S. curricula. Furthermore, "the only substantial treatment of proof in the secondary mathematics curriculum occurs" (Moore 1994, p. 249) in this (usually 1-year) geometry course. Knuth's findings also point out that teachers view proof as a subject to be taught separately rather than as a learning tool that can be integrated throughout mathematics.

On the other hand, the majority (69%) of the pre-service teachers in Mingus and Grassl's (1999) study advocated the introduction of proof before 10th-grade geometry classes. Furthermore, the participants who had taken college-level mathematics courses believed that proofs needed to be introduced earlier, in the elementary grades, in contrast to the participants who had only experienced proofs at the high-school level. Mingus and Grassl argued that the former group "may have recognized that a lack of exposure to formal reasoning in their middle and high school backgrounds affected their ability to learn how to read and construct *proofs*" (1999, p. 440).

Other studies reveal different beliefs about the role of proof. Furinghetti and Morselli (2009a) investigated how secondary teachers treat proofs, and which factors (especially beliefs) affect that treatment, in a qualitative study of ten cases via individual, semi-structured interviews. Nine of the ten teachers declared that they teach proof in the classroom. The other said that she does not because Euclidean geometry is not in her school curriculum. The other teachers also referred mainly to Euclidean geometry as the most suitable domain for teaching proof. Sowder and Harel (2003) already pointed out beliefs about geometry being the ideal domain for the teaching of proof or, even more, the teaching of proof being confined to geometry. Concerning the way proof is treated in the classroom, Furinghetti and Morselli (2009a) identified two tendencies: teaching theorems versus teaching via the proof. The first sees proof as a means for convincing and systematising mathematical facts, whilst the second uses proof mainly to promote mathematical understanding. The first focuses on proof as a product, the second as a process.

2.2.4 Teachers' Beliefs About Proof and Themselves as Mathematical Thinkers

Teacher attitudes towards using mathematical reasoning, their abilities in constructing proofs, and their abilities to deal with novel ideas are especially important, because "ideas that surprise and challenge teachers are likely to emerge during instruction" (Fernandez 2005, p. 267). In such situations teachers should be able to "reason, not just reach into their repertoire of strategies and answers" (Ball 1999, p. 27). However, the U.S. teachers in Ma's (1999) study were not mathematically confident to deal with a novel idea and investigate it. Like students, these teachers relied on some authority – a book or another teacher – to be confident about the truth of a statement.

Although it was not their main purpose, Simon and Blume (1996) also found evidence that prospective elementary teachers appealed to authority. Their study differed from others in the sense that they investigated pre-service elementary-school teachers' conceptions of proof in the context of a mathematics course "which was run as a whole class constructivist teaching experiment" (1996, p. 3). The participants had previously experienced mathematics only in traditional classrooms where the authority was the teacher; the goal of the instructor in the study was to shift the "authority for verification and validation of mathematical ideas from teacher and textbook to the mathematical community (the class as a whole)" (ibid., p. 4). The authors argued that this shift was significant because it "can result in the students' sense that they are capable of creating mathematics and determining its validity" (ibid., p. 4).

Simon and Blume's (1996) findings illustrate how pre-service teachers' prior experiences with proofs (or the lack thereof) and views about mathematics influence how they initially respond to situations where proof is necessary. At the beginning of the semester when the study's instructor asked them to justify mathematical ideas, the participants referred to their previous mathematics courses or provided empirical reasons. They also did not necessarily make sense of the others' general explanations if they were not operating at the same level of reasoning. However, Simon and Blume claimed that "norms were established over the course of the semester, that ideas expressed by community members were expected to be justified and that those listening to the justification presented would be involved in evaluating them" (1996, p. 29).

More recently, Smith (2006) compared the perceptions of, and approaches to, mathematical proof by undergraduates enrolled in lecture and problem-based "transition to proof" courses, the latter using the "modified Moore method," (2006, p. 74). Their key finding: "while the students in the lecture-based course demonstrated conceptions of proof that reflect those reported in the research literature as insufficient and typical of undergraduates, the students in the problem-based course were found to hold conceptions of, and approach the construction of, proofs in ways that demonstrated efforts to make sense of mathematical ideas" (ibid., p. 73). These promising results "suggest that such a problem-based course may provide opportunities for students to develop conceptions of proof that are more meaningful and robust than does a traditional lecture-based course" (ibid., p. 73).

2.3 Students' Beliefs About Proof

Many studies deal with students' approaches to proof. They mainly focus on students' difficulties; only rarely do they address the issue of beliefs directly and explicitly. Their most prevalent findings on students' beliefs about proof are that students find giving proofs difficult and that their views of the purpose and role of proof are very limited (Chazan 1993; Harel and Sowder 1998; Healy and Hoyles 2000). Some students are ignorant of the need to give a mathematical proof to verify a statement; others appeal to an authority – a teacher, a book or a theorem – to establish a truth (Carpenter et al. 2003).

According to Ball and Bass (2000), third-grade students "did not have the mathematical disposition to ask themselves about the completeness of their results when working on a problem with finitely many solutions" (2000, p. 910) early in the year. Similarly, Bell (1976) found that 70% of the 11- to 13-year-olds in his study could recognise and describe patterns or relationships but showed no attempt to justify or deduce them. Even some students at the university level may believe that "proof is only a formal exercise for the teacher; there is no deep necessity for it" (Alibert 1988, p. 31).

A common research finding is that students accept empirical arguments as proofs. They believe that checking a few cases is sufficient (Bell 1976). Healy and Hoyles (2000) found that 24% of 14- and 15-year-old algebra students, assigned a familiar mathematical problem accompanied by different arguments, indicated that the empirical argument would be the most similar to their own approach (39% for an unfamiliar problem). Chazan (1993), in his study of high-school geometry students' preferences between empirical and deductive reasoning, documented similar results.

Some students are aware that checking a few cases is not tantamount to proof, but believe that checking more varied and/or randomly selected examples *is* proof. They try to minimise the limitations of checking a few examples in a number of ways, including use of a pattern, extreme cases or special cases. In Chazan's (1993) study, some students believed that if they tried different kinds of triangles – acute, obtuse, right, equilateral, and isosceles – they would verify a given statement about triangles. One of Ball's third graders gave as a reason for accepting the truth of the statement that the sum of two odd numbers is even that she tried "almost 18 of them and even some special cases" (Ball and Bass 2003, p. 35).

On the other hand, some students are aware of the fact that checking a few examples is not enough and are also not satisfied with trying different cases. Interviews in Healy and Hoyles's (2000) study revealed that some students who chose empirical arguments as closest to their own were not really satisfied with these arguments but believed that they could not make better ones. There are also students who realise that some problems contain infinitely many numbers, so one cannot try them all; consequently, rather than considering a general proof, these students believe that no proof is possible (Ball and Bass 2003).

Bell (1976) found that students, although unable to give complete proofs, showed different levels of deductive reasoning, ranging from weak to strong. Some students can follow a deductive argument but believe that "deductive proof is simply evidence" (Chazan 1993, p. 362). Fischbein (1982) found that although 81.5% of them believed that a given proof was fully correct, just 68.5% of a student population accepted the theorem. Some students, given a statement claiming the same result for a subset of elements from an already generally proven category, think that another specific proof is necessary (Healy and Hoyles 2000). Some students also think that either further examples are necessary or that a deductive argument is subject to counterexamples. In Fischbein's (1982) study, only 24.5% of the students accepted the correctness of a given proof and "*at the same time*" (1982, p. 16) thought that they did not need additional checks.

Student-constructed proofs may take various forms. Even students who can produce valid mathematical proofs do not tend to give formal arguments using symbols. Healy and Hoyles found that students who went beyond a pragmatic approach were "more likely to give arguments expressed informally in a narrative style than to use algebra formally" (2000, p. 408). In Bell's (1976) study, none of the students used algebra in their proofs. Porteous (1990) also reported that it was "disappointing to find an almost total absence of algebra" (1990, p. 595). According to Healy and Hoyles, students did not use algebraic arguments because "it offered them little in the way of explanation … and [they] found them hard to follow" (2000, p. 415).

All the aforementioned studies mainly refer to pre-secondary school. Studies carried out at the more sophisticated high-school and college levels have had to consider further elements and difficulties.

As Moore claims, "the ability to read abstract mathematics and do proofs depends on a complex constellation of beliefs, knowledge, and cognitive skills" (1994, p. 250). Furinghetti and Morselli's study (2007, 2009b) instantiates how beliefs may intervene in this constellation and, at the same time, hints at the interpretative difficulties linked to this kind of investigation. Their analyses show the weight and role of beliefs as driving forces throughout the proving process as well as their mutual relationship with cognitive factors. For instance, in one case study (Furinghetti and Morselli 2009b) the student's choice of the algebraic representation, and the revision of such a choice after a difficulty is met, are hindered by the student's beliefs about self (low self-confidence), about mathematical activity as an automatic activity, and about the role of algebra as a proving tool. In another case (Furinghetti and Morselli 2007) the proving process is supported by the student's self-confidence and his belief about proof as a process aimed not only at proving but also at explaining. In this latter case, the authors underline the positive role of beliefs in supporting the construction of a proving process as well the final systematisation of the product.

3 Metaknowledge About Proof

3.1 Metaknowledge

Literally, metaknowledge is knowledge about knowledge. We use the term 'metaknowledge about proof' to designate the knowledge needed to reflect about, teach and learn proof. We distinguish metaknowledge from beliefs by placing beliefs closer to individuals' opinions, emotions and attitudes, whereas "metaknowledge" refers to consensually held ideas. Metaknowledge about proof includes concepts which refer to:

- the *structure of mathematical theories*, like axiom/hypothesis, definition, theorem;
- formal *logic*, like truth, conditional, connectives, quantifiers;

- *modes of representation*, like symbolic, pictorial and verbal reasoning; and
- relationships between proof in mathematics and related processes of argumentation in *other fields*, especially the empirical sciences.

Above, in Sects. 2.2 and 2.3, we discussed research findings about problematic dimensions of teachers' and students' beliefs about proof – above all, the preponderance of empirical ways of justification. These findings are internationally valid; consequently, we have to consider the problematic beliefs as outcomes of the usual way of teaching proof at school and university. Here, we expound our *thesis* that these shortcomings of teaching can be successfully overcome only when metaknowledge about proof is made a theme of mathematics teaching, beginning with the explicit introduction of the very notion of proof. The opinions of many educators, for example Healy and Hoyles (2000) and Hemmi (2008) support this view.

3.2 Place of Metaknowledge in the Curriculum

The role of proof in the curriculum varies across different countries. Of course, there is a broad international consensus that learning mathematical argumentation should start with the very beginning of mathematics in the primary grades (e.g. Ball and Bass 2000; Bartolini 2009; Wittmann 2009). However, the situation is different with regard to the explicit introduction of the notion of proof *per se*. In some countries, such as France, Germany and Japan, proof is seen as something to be explicitly taught. Cabassut (2005) notes that the introduction of proof in France and Germany takes place mostly in grade 8, which is also the situation in Japan (Miyazaki and Yumoto 2009). In these countries, the official syllabus makes explicit what should be taught about proof and/or textbooks contain chapters about proof (Cabassut 2009; Fujita, Jones, and Kunimune 2009). In other countries, such as Italy (Furinghetti and Morselli 2009a) and the United States (National Council of Teachers of Mathematics [NCTM] 2000), proof remains a more informal concept but is nevertheless made a theme by individual teachers. Furinghetti and Morselli report that most of the Italian teachers they have interviewed respond that they treat proof in their classes. In the United States, 'reasoning and proof' is identified as a process standard (NCTM 2000) to be integrated across content and grade levels rather than taught explicitly as one object of study. Regardless of whether proof is explicitly treated, it is important that mathematics teachers have well-founded metaknowledge about proof in order to communicate an adequate image of mathematics to their students.

3.3 Basic Components of Metaknowledge About Proof

In the practice of teaching, the attitude seems frequently to prevail that metaknowledge about proof emerges spontaneously from examples. Only a few ideas are available about how to provide metaknowledge about proof explicitly to pupils or teachers. For example, Arsac et al. (1992) give explicit 'rules' for discussions with pupils of the lower secondary level (11–15 years old): a mathematical assertion is either true or false; a counter-example is sufficient for rejection of an assertion; in mathematics people agree on clearly formulated definitions and properties as warrant of the debate; in mathematics one cannot decide that an assertion is true merely because a majority of persons agree with it; in mathematics numerous examples confirming an assertion are not sufficient to prove it; in mathematics an observation on a drawing is not sufficient to prove a geometrical assertion. However, listing such rules is not sufficient to develop metaknowledge, because the latter is broader and includes relationships to other fields.

Here, in identifying basic components of metaknowledge about proof, we confine ourselves to metaknowledge which should be made a theme already in the lower secondary grades and which (*a fortiori*) should be provided to future teachers of mathematics. We leave aside metaknowledge related to formal logic, since this topic is treated in other chapters of this volume and is appropriate only for a more advanced level. (We also exclude 'modes of representation'; see Cabassut 2005, 2009).

When introducing proof in the mathematics classroom, teachers usually say two things to their students: first, that proofs produce *certain knowledge*, "We think that this statement might be true, but to be sure we have to prove it"; second, that proof establishes *generally valid statements* – statements true not only for special cases but for all members of a class (e.g., all natural numbers or all triangles). Teachers all over the world thus try to explain proof to their students; we take those two messages as basic components of the necessary metaknowledge about proof. However, many teachers and educators are unaware that the two messages are incomprehensible by themselves and need further qualifications. One reason is the difference between these statements and statements made in science courses. Conner and Kittleson (2009) point out that students encounter similar problem situations in mathematics and science, but the ways in which results are established differ between these disciplines. In mathematics, a proof is required to establish a result; in science, results depend on a preponderance of evidence (not accepted as valid in mathematics).

3.3.1 The Certainty of Mathematics

It is important to convey to students the idea that proofs do not establish facts but 'if-then-statements'. We do not prove a 'fact' *B* but an implication 'If *A* then *B*'. Boero, Garuti and Lemut (2007, p. 249 *ff*) rightly speak about the conditionality of mathematical theorems. For example, we do not prove the 'fact' that all triangles have an angle sum of 180°; rather we prove that in a certain theory this consequence can be derived. The angle-sum theorem is an 'if-then-statement' whose 'if' part consists ultimately of the axioms/hypotheses of Euclidean geometry. Thus the *absolute certainty* of mathematics resides not in the facts but in the logical inferences, which are often implicit.

Whether mathematicians believe in the 'facts' of a theory is dependent on their confidence in the truth of the hypotheses/axioms. This confidence is the result of a more or less conscious *process of assessment*. Mathematicians find the axioms of arithmetic highly reliable and therefore can believe that there are infinitely many prime numbers. The situation is different in geometry; with 'medium-sized' objects Euclidean geometry is the best available theory, but in cosmological dimensions Riemannian geometry is taken as the appropriate model. The situation is even more complicated with applied theories in physics and other sciences.

The issues of the potential certainty of mathematical proof and of the conditionality of the theorems have to be made frequent themes in mathematics education, beginning at the secondary level. Teachers should discuss them with students in various situations if they expect the students to get an adequate understanding of mathematical proof. In particular, they should make students aware of the necessary process of assessing the reliability of a theory.

3.3.2 Universally Valid Statements

To an educated mathematician, it seems nearly unimaginable that the phrase "for all objects x with a certain property the statement A is true" should present any difficulty of understanding to a learner. Many practical experiences and some recent empirical studies show, however, that it does exactly that. Lee and Smith, in a recent study (2008, 2009) of college students, found that some of their participants held the notion that "true rules could always allow exceptions" or that "true means mostly true" or that there might be an "unknown exception to the rule." (Lee and Smith 2009, pp. 2–24). This is consistent with the experiences of students frequently not understanding that *one* counterexample suffices for rejection of a theorem. Galbraith (1981) found, for example, that one third of his 13- to 15-year-old students did not understand the role of counterexamples in refuting general statements (see also Harel and Sowder 1998).

Frequently, students do not think that the set to which a general statement refers has a definite extension but assume tacitly that under special circumstances an exception might occur. From the point of view of classical mathematics this is a 'misconception'; however considering general statements outside of classical mathematics one finds that concepts are generally seen as having indefinite extensions. Both everyday knowledge and the empirical sciences consider general statements which under certain conditions might include exceptions. To cover this phenomenon, one can distinguish between *open* and *closed* general statements, having respectively indefinite and definite domains of validity (Durand-Guerrier 2008; Jahnke 2007, 2008). In principle, closed general statements can occur only in mathematics, whereas disciplines outside of mathematics operate with open general statements with the tacit assumption that under certain conditions exceptions might occur. At the turn from the eighteenth to the nineteenth centuries even mathematicians spoke of "theorems which might admit exceptions" (on this issue see Sørensen 2005). Also, intuitionistic mathematics does not consider the concept of the set of all subsets (e.g., of the natural numbers) as a totality of definite extension. We well know that in cases where the domain for which a statement is valid does not have a definite extension the usual logical rules, especially the rule of the excluded middle, are no longer valid.

All in all, the seemingly simple phrase "for all" used in the formulation of mathematical theorems is not an obvious concept for the beginner. Rather, it is a sophisticated theoretical construct whose elaboration has taken time in history and needs time in individuals' cognitive development. Durand-Guerrier (2008, pp. 379–80) provides a beautiful didactic example about how to work on this concept with younger pupils.

3.3.3 Definitions

The theories which mathematicians construct by way of proof are hypothetical and consist of 'if-then-statements'. This fact implies that mathematical argumentation requires and presupposes *rigour*. Hypotheses/axioms and definitions have to be understood and applied in their exact meanings. This requirement sharply contrasts with everyday discourse, which does not commonly use definitions, at least in the mathematical sense. Consequently, the development of a conscious use of definitions is an important component of proof competence.

Most students at the end of their school careers do not understand the importance and meaning of mathematical definitions, even many university students (Lay 2009). Explicit efforts in teaching are required to develop a habit of using definitions correctly in argumentations. Since such a habit does not emerge spontaneously, students need metaknowledge about definitions. They should know that definitions are conventions but are not arbitrary; in general, a definition is constructed the way it is for good reasons.

Beginning university students of mathematics encounter an impressive example of the importance and meaning of definitions when they first operate with infinite sets: namely, how can one determine the 'size' of an infinite set? If one compares sets by way of the relation '$\subset$', then the set N of natural numbers is a proper subset of the set Q of rational numbers: $N \subset Q$, and N is 'smaller' than Q. If, however one compares sets by means of bijective mappings, a fundamental theorem of Cantorian set theory says that N and Q have the same cardinality. Hence, the outcome of a comparison of two sets depends on the definition of 'size'. Numerous further examples occur in analysis: for instance, whether an infinite series is convergent depends on the definition of convergence.

Not many examples of this type arise in secondary teaching. One instance of the importance and relevance of alternative definitions is the definition of a trapezoid (trapezium) as having at least two parallel sides versus having exactly two parallel sides. Asking whether a rectangle is a trapezium requires a student to look past the standard figures depicting the two types of quadrangle. If they apply a particular set of definitions, they conclude that answer is affirmative. Proof and deduction enter the game when the student realises that consequently the formulae for the perimeter and area of a trapezium must also give the perimeter and area of a rectangle. (For further ideas about teaching the construction of definitions, see Ouvrier-Buffet 2004 and 2006.)

3.3.4 Mini-theories as a Means to Elaborate Metaknowledge About Proof

We have suggested three basic components of metaknowledge about proof which naturally emerge in the teaching of proof and which should be more deeply elaborated in teaching: the *certainty* (*conditionality*) of mathematical theorems, the *generality* of the theorems and the conscious use of *definitions*. One possible method to further this learning is to develop *mini-theories* accessible to learners and sufficiently substantial to discuss meta-issues. The idea of such mini-theories, not completely new, resembles Freudenthal's (1973) concept of 'local ordering' or the use of a finite geometry as a surveyable example of an axiomatic theory. However, the study of finite geometries is not feasible in secondary teaching. Besides, our idea of a mini-theory differs in two aspects from Freudenthal's concept of local ordering. First, we would include in the teaching of a mini-theory phases of explicit reflection about the structure of axiomatic theories, the conditionality of mathematical theorems and the set of objects to which a theorem applies. Second, we would also take into account 'small theories' from physics, like Galileo's law of free fall and its consequences, and other examples of mathematised empirical science (Jahnke 2007).

Treating a mathematised empirical theory would provide new opportunities to make students aware of the process of assessing the truth of a theory (see above on certainty; Conner and Kittleson 2009; Jahnke 2009a, b). Usually, teachers only tell students that the axioms are intuitively true and that therefore all the theorems which can be derived from them are true; however, this is a one-sided image. In many other cases, one believes in the truth of a theory because its consequences agree with empirical evidence or because it explains what one wants to explain. For example, teachers generally treat Euclidean geometry in the latter way, at least at pre-tertiary levels. In the philosophy of science, this way of constructing and justifying a theory is called the 'hypothetico-deductive method'.

Barrier et al. (2009) developed a related idea for teaching the metaknowledge of proof. They discuss a dialectic between an 'indoor game' and an 'outdoor game'. The indoor game refers to the proper process of deduction, whereas the outdoor game deals with "the truth of a statement inside an interpretation domain" (2009, p. 78).

4 Conclusion

Our discussion in Sect. 1 has shown how strongly conceptualisations of proof are dependent of the professional background and aims of the respective researcher. Practising mathematicians, whilst agreeing on the acceptance of certain arguments as proof, stray from a formal definition of proof when explaining what one is.

Consequently, it is difficult to explain precisely what a proof is, especially to one who is a novice at proving (such as a child in school). Mathematics educators also differ in their distinctions between argumentation and proof (or inclusion of one in the other). Regardless of the classification scheme of the researchers, research reveals that students and teachers often classify arguments as proofs differently from the classifications accepted in the field of mathematics.

Existing research on beliefs about proof has focused on investigating proof conceptions of prospective and practising elementary and secondary school teachers (Sect. 2). Their beliefs about proving are wrapped around two main issues: what counts as proof in the classroom and whether the focus of teaching proof is on the product or on the process. Research has clearly hinted at the fact that quite a number of teachers tend to accept empirical arguments as proofs and have limited views about the role of proof in school mathematics. Given the influence of beliefs on the teaching and learning process at all levels of schooling continued research on beliefs about proof that focuses not only on detecting beliefs but also on understanding their origins seems highly necessary.

Research strongly suggests that beliefs about proof should be addressed more intensely in undergraduate mathematics and mathematics education courses and during professional development programmes in order to overcome the shortcomings which have been identified in the beliefs about proof. Consequently, we discuss in the last section of the chapter (Sect. 3) which type of metaknowledge about proof should be provided to students and how this can be done. We identify three components of metaknowledge about proof which should be made a theme in teacher training as well as in school teaching. These are the certainty of mathematics, universally valid statements and the role of definitions in mathematical theories. The elaboration of teaching units which allow an honest discussion of metaknowledge about proof seems an urgent desideratum of future work. Mini-theories could be one possible way of achieving this, and further research is necessary both to examine the feasibility of the use of mini-theories and to develop other ways of developing metaknowledge about proof.

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