Convergence of collocation methods for solving periodic boundary value problems for renewal equations defined through finite-dimensional boundary conditions

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Summary

The problem of computing periodic solutions can be expressed as a boundary value problem and solved numerically via piecewise collocation. Here, we extend to renewal equations the corresponding method for retarded funcional differential equations in [K. Engelborghs et al., SIAM J. Sci. Comput., 22 (2001), pp. 1593-1609]. The theoretical proof of the convergence of the method has been recently provided in [A. Andò and D. Breda, SIAM J. Numer. Anal., 58 (2020), pp. 3010-3039] for retarded functional differential equations and in [A. Andò and D. Breda, submitted in 2021] for renewal equations and consists in both cases in applying the abstract framework in [S. Maset, Numer. Math., 133 (2016), pp. 525-555] to a reformulation of the boundary value problem featuring an infinite-dimensional boundary condition. We show that, in the renewal case, the proof can also be carried out and even simplified when considering the standard formulation, defined by boundary conditions of finite dimension.

KEYWORDS:
renewal equations, periodic solutions, piecewise orthogonal collocation, boundary value problems

1 | INTRODUCTION

Renewal equations (REs), also known as Volterra integral equations (see, e.g., [1] for an extensive treatment), constitute an important class of delay equations which do not involve derivatives. In other words, they specify the value of the unknown function at the present time in terms of its past values. Such equations appear in a wide range of applications, from the theory of industrial replacement to that of population dynamics, where they arise from the presence of some physiological structure (e.g., age) of the relevant population [2,3,4,5].

A general RE can be written as

$$x(t) = F(x_t) \tag{1}$$

where $F : X \to \mathbb{R}^d$ is an autonomous, usually nonlinear function, the state space X is defined as $X := L^1([-\tau, 0], \mathbb{R}^d)$ [6] for some delay $\tau > 0$, and the state or history $x_t \in X$ at time t associated to (1) is defined as

$$x_t(\theta) := x(t + \theta), \quad \theta \in [-\tau, 0].$$

The infinite dimension of the state space represents a main reason why the dynamical analysis for REs is harder than that for ordinary differential equations. An important aspect of the dynamical analysis is the study of invariant sets; in particular, we are interested in

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periodic orbits. In this respect, the theory for REs has been developed only recently. In[7] an extension of the Floquet theory to REs was proposed, allowing the study of local asymptotic stabilty of periodic solutions, while[8] provides a theoretical analysis of convergence of the piecewise collocation method described in[9] to actually compute the periodic solutions. Such analysis is based on the abstract approach in[10] for general Boundary Value Problems (BVPs).

The problem of finding a periodic solution of (1) with period $\omega > 0$ can be expressed as a BVP in two different but equivalent ways (see[9] Section 2 for a comparison of the two formulations in the case of Retarded Functional Differential Equation, RFDEs).[8] considers a BVP of the form

$$\begin{cases} x(t) = F(x_t), \quad t \in [0, \omega], \\ x_0 = x_\omega \\ p(x|_{[0,\omega]}) = 0, \end{cases} \tag{2}$$

whose solution represents the restriction to two periods (say $[-\omega, \omega]$ without loss of generality) of the original solution. The boundary condition includes the periodicity condition, stating that the solution states at the extrema of the period are equal, plus an extra phase condition, defined by a scalar linear function p and necessary in order to remove translational invariance. The convergence proof in[8] is based on reformulating (2) as a fixed-point problem, similarly to what was done in[11] for RFDEs.

The aim of the present paper is to provide a simplification of the proof of convergence given in[8] by considering the alternative periodic BVP formulation, which features a finite-dimensional boundary condition, is the one which is most commonly considered in the literature on RFDEs (e.g.,[12,13,14,15]) and, most importantly, leads to the same numerical method (the reader can refer to[8] Section 4 for numerical tests to support the validity of the method). Section 2 will describe both the BVP formulation and the relevant numerical method. Section 3 will go through the steps of the convergence analysis, by stating the propositions that ensure the validity of both the theoretical assumptions (Subsection 3.1) and the numerical ones (Subsection 3.2) required in[10]. As it will be shown, some parts of the proofs of the latter can be notably simplified by resorting to the BVP with finite-dimensional boundary condition.

2 | PIECEWISE ORTHOGONAL COLLOCATION FOR PERIODIC BOUNDARY VALUE PROBLEMS

The problem of finding a periodic solution of (1) with period $\omega > 0$[1] can be expressed as a BVP whose solution represents the restriction to one period (say $[0, \omega]$ without loss of generality) of the original solution. The boundary condition includes part of the periodicity condition, stating that the solution values at the extrema of the period are equal. The rest of the periodicity condition is, in a certain sense, included within the relevant RE: indeed, whenever one needs to evaluate the solution x at points that fall off the interval $[0, \omega]$ due to the delay, one exploits the periodicity to bring back the evaluation to the domain $[0, \omega]$. More rigorously, this corresponds to defining, for $t \in [0, \omega]$, a periodic state $\bar{x}_t \in X$ as

$$\bar{x}_t(\theta) = \begin{cases} x(t + \theta), & t + \theta \in [0, \omega], \\ x(t + \theta + \omega), & t + \theta \in [-\omega, 0), \end{cases}$$

for $\theta \in [-\tau, 0] \subset [-\omega, 0]$, allowing in turn to define the periodic BVP as

$$\begin{cases} x(t) = F(\bar{x}_t), \quad t \in [0, \omega], \\ x(0) = x(\omega) \\ p(x) = 0. \end{cases} \tag{3}$$

Examples of phase conditions are the trivial one, of the form $x_k(0) = \hat{x}$ for some $k \in \{1, \dots, d\}$ for a fixed $\hat{x}$ and the integral one, of the form

$$\int_0^\omega \langle x(t), \tilde{x}'(t) \rangle \, dt = 0,$$

where $\tilde{x}$ is (an approximation of) a given reference solution[16]. Since the method described here is typically used within a numerical continuation framework, such a reference solution could be, e.g., the (piecewise polynomial approximation of) the solution obtained at the previous continuation step.

[1] One can always consider $\omega \geq \tau$ without loss of generality, since a solution with period ω is also a solution with period $k\omega$ for any positive integer k.

The following description of the numerical method to solve (3) assumes, for the time being, that the function F defining the right-hand side can be exactly computed. The first step (see, e.g., [9]) is to reformulate (3) so that the interval of definition of the RE remains fixed and no longer depends on the unknown ω. This is achieved by defining the map $s_\omega : \mathbb{R} \to \mathbb{R}$ as

$$s_\omega(t) := \frac{t}{\omega},$$

which allows to write (3) equivalently as

$$\begin{cases} x(t) = F(\bar{x}_t \circ s_\omega), \quad t \in [0,1] \\ x(0) = x(1) \\ p(x) = 0. \end{cases} \tag{4}$$

As explained in [9] and then reported in [8], (4) can be solved through piecewise orthogonal collocation, which is now a standard approach (originally developed for ODEs, see `MatCont` [17]). For L and m positive integers, the numerical solution u in $[0,1]$ is a piecewise continuous polynomial obtained by solving the following system having dimension $(1+Lm)\times d+1$:

$$\begin{cases} u(t_{i,j}) = F(\bar{u}_{t_{i,j}} \circ s_w), \quad j \in \{1,\dots,m\}, \quad i \in \{1,\dots,L\} \\ u(0) = u(1) \\ p(u) = 0 \end{cases}$$

for a given mesh $0 = t_0 < \cdots < t_L = 1$ and collocation points

$$t_{i-1} < t_{i,1} < \cdots < t_{i,m} < t_i$$

for all $i \in \{1,\dots,L\}$. The unknowns are, other than w approximating ω, those of the form $u_{i,j} := u(t_{i,j})$ for $(i,j) = (1,0)$ and $i \in \{1,\dots,L\}$, $j \in \{1,\dots,m\}$. In the context of piecewise collocation, one typically refers to the corresponding Finite Element Method (FEM), which consists in considering a fixed value for m while the mesh size L increases, in view of convergence. This is indeed the case for the rest of the present paper.

In realistic applications, right-hand sides F usually feature an integral, therefore cannot be exactly computed in general. For instance, F can be of the form

$$F(\varphi) := \int_{-\tau}^{0} K(\sigma, \varphi(\sigma))\, d\sigma \tag{5}$$

for some integration kernel $K : [-\tau, 0] \times \mathbb{R}^d \to \mathbb{R}^d$, or

$$F(\varphi) := f\left(\int_{0}^{\tau} k(\sigma)\varphi(-\sigma)\, d\sigma \right), \tag{6}$$

for some integration kernel $k : [0,\tau] \to \mathbb{R}^d$ and some function $f : \mathbb{R}^d \to \mathbb{R}^d$. The analysis that follows focuses on (5), but one can apply the same ideas to extend it to (6). By rescaling time via the function s_ω one obtains

$$F(x_t \circ s_\omega) = \int_{-\frac{\tau}{\omega}}^{0} \omega K(\omega\theta, x_t(s_\omega(\omega\theta)))\, d\theta = \int_{-\frac{\tau}{\omega}}^{0} \omega K(\omega\theta, x(t+\theta))\, d\theta. \tag{7}$$

If the integrand can be exactly computed, as it is usually the case in applications, then (7) is approximated as

$$F_M(x_t \circ s_\omega) := \omega \sum_{i=0}^{M} w_i K(\omega\alpha_i, x(t+\alpha_i)),$$

where M is a given approximation level and the quadrature nodes $-\frac{\tau}{\omega} = \alpha_0 < \cdots < \alpha_M = 0$ are allowed to be completely independent of the collocation nodes mentioned earlier. Indeed, such nodes and the corresponding weights $w_0, \dots, w_M$ are meant to define a quadrature formula which exploits the possible irregularities in K.

3 | CONVERGENCE ANALYSIS

The convergence analysis of the numerical method described in Section 2 follows the abstract approach [10], intended for neutral functional differential equations. Note that REs can be treated as equations of this kind by interpreting the relevant solutions as derivatives of other functions.This may recall the approach used in [18], which involves the integration of the relevant REs.

The general fixed-point problem described in [10] consists in finding $(\mathsf{v}^*, \beta^*) \in \mathbb{V} \times \mathbb{B}$ with $\mathsf{v}^* := \mathcal{G}(\mathsf{u}^*, \alpha^*)$ and $(\mathsf{u}^*, \alpha^*, \beta^*) \in \mathbb{U} \times \mathbb{A} \times \mathbb{B}$ such that

$$(\mathsf{u}^*, \alpha^*, \beta^*) = \Phi(\mathsf{u}^*, \alpha^*, \beta^*)$$

for $\Phi : \mathbb{U} \times \mathbb{A} \times \mathbb{B} \to \mathbb{U} \times \mathbb{A} \times \mathbb{B}$ given by

$$\Phi(\mathsf{u}, \alpha, \beta) := \begin{pmatrix} \mathcal{F}(\mathcal{G}(\mathsf{u}, \alpha), \mathsf{u}, \beta) \\ (\alpha, \beta) - \mathcal{B}(\mathcal{G}(\mathsf{u}, \alpha), \mathsf{u}, \beta) \end{pmatrix}, \tag{8}$$

where $\mathcal{F} : \mathbb{V} \times \mathbb{U} \times \mathbb{B} \to \mathbb{U}$ in the first line defines the right-hand side of the functional equation of the relevant BVP and $\mathcal{B} : \mathbb{V} \times \mathbb{U} \times \mathbb{B} \to \mathbb{A} \times \mathbb{B}$ in the second one represents the boundary conditions. The solution $\mathsf{v} = \mathcal{G}(\mathsf{u}, \alpha)$ lies in a normed space of functions $\mathbb{V}$ and is obtained through the operator $\mathcal{G} : \mathbb{U} \times \mathbb{A} \to \mathbb{V}$ which reconstructs it given a function u from a Banach space $\mathbb{U}$ and an initial value/state α from a Banach space $\mathbb{A}$. β is a vector of possible parameters, living in a Banach space $\mathbb{B}$ of finite dimension.

In the case of REs, the right-hand side prescribes the values of the solution, rather than those of its derivative. This means that no further step is required to retrieve the solution v^*, once one has the solution $(\mathsf{u}^*, \alpha^*, \beta^*)$ of (8). Thus, in principle, one could choose to work with $\mathbb{A} = \emptyset$ - instead of with an infinite-dimensional $\mathbb{A}$, as it is needed in [8] - and $\mathcal{G}(\mathsf{u}) = \mathsf{u}$. This is indeed the case, as it will be shown in the rest of this section.

In order to prove that the BVP (4) is amenable of the analysis in [10], the first step is to observe that it can be rewritten as a fixed point problem of the form (8). The only parameter involved is the period ω, therefore $\mathbb{B} = \mathbb{R}$. In order to work with the space $\mathbb{A} = \emptyset$, the boundary conditions need to be scalar and limited to the phase condition, unlike in the formulation (2) considered in [8], where infinite-dimensional boundary conditions need to be imposed. As a consequence, the periodicity condition must be moved from the BVP to the definition of the space, i.e., $\mathbb{U}$ can only contain functions satisfying the periodicity condition. By choosing $\mathbb{U} \subseteq \{\mathsf{u} : [0, 1] \to \mathbb{R}^{\mathsf{d}} \,|\, \mathsf{u}(0) = \mathsf{u}(1)\}$, the solutions of (4) are exactly the pairs $(\mathsf{u}^*, \omega^*) \in \mathbb{U} \times \mathbb{R}$ with $(\mathsf{u}^*, \omega^*) \in \mathbb{U} \times (0, +\infty)$ the fixed point of the map $\Phi : \mathbb{U} \times \mathbb{R} \to \mathbb{U} \times \mathbb{R}$ defined by

$$\Phi(\mathsf{u}, \omega) := \begin{pmatrix} \mathsf{F}(\overline{\mathsf{u}}_{(\cdot)} \circ \mathsf{s}_\omega) \\ \omega - \mathsf{p}(\mathsf{u}) \end{pmatrix}. \tag{9}$$

Consequently, (4) leads to an instance of (8) with $\mathbb{A} = \emptyset$, $\mathcal{F} : \mathbb{U} \times \mathbb{R} \to \mathbb{U}$ and $\mathcal{B} : \mathbb{U} \times \mathbb{R} \to \mathbb{R}$ given respectively by

$$\mathcal{F}(\mathsf{u}, \omega) := \mathsf{F}(\overline{\mathsf{u}}_{(\cdot)} \circ \mathsf{s}_\omega) \tag{10}$$

and

$$\mathcal{B}(\mathsf{u}, \omega) := \mathsf{p}(\mathsf{u}).$$

The operator $\mathcal{B}$ is linear and independent of ω. Provided that $\mathbb{U}$ only contains function satisfying the periodicity condition, and does actually contain the sought solution of (4), its precise definition does not play a role up to this point. However, the fixed point problem (9) needs to satisfy several assumptions in order to apply the general framework in [10], and the validity of such assumptions depend on both $\mathbb{U}$ and the regularity of the right-hand side F, given in turn by the regularity of the integrand K in (5). Unlike the theoretical assumptions, the numerical ones also depend on the discretization of the problem. Subsections 3.1 and 3.2 contain the propositions stating the validity of the theoretical and numerical assumptions respectively. As it is eventually remarked in Subsection 3.3, the validity of both the theoretical and the numerical assumptions implies that the relevant numerical method is convergent.

3.1 | Theoretical assumptions

The hypotheses on the regularity of the problem (9) that are needed to satisfy the theoretical assumptions are listed below. B^∞ denotes the set of measurable and bounded functions and it has been chosen - instead of the classical L^1 space - in order to deal with collocation, which requires to evaluate the relevant functions pointwise. X denotes the state space relevant to the rescaled BVP (4). Note that, with respect to the similar hypotheses needed in [8], the only difference is in 3.1, concerning the specific choices of the spaces $\mathbb{U}$ and $\mathbb{A}$.

(T1) $\mathsf{x} = \mathsf{B}^\infty([-\tau, 0], \mathbb{R}^{\mathsf{d}})$, $\mathsf{X} = \mathsf{B}^\infty([-1, 0], \mathbb{R}^{\mathsf{d}})$.

(T2) $\mathbb{U} = \{\mathsf{u} \in \mathsf{B}^\infty([0, 1], \mathbb{R}^{\mathsf{d}}) \,|\, \mathsf{u}(0) = \mathsf{u}(1)\}$, $\mathbb{A} = \emptyset$.

(T3) $\mathsf{K} : [-\tau, 0] \times \mathbb{R}^{\mathsf{d}} \to \mathbb{R}^{\mathsf{d}}$ is continuous and has partial derivatives $\mathsf{D}_1\mathsf{K}$, $\mathsf{D}_2\mathsf{K}$.

(T4) $D_2K : \mathbb{R}^d \to \mathbb{R}^d$ is continuous.

(T5) There exist $r > 0$ and $\kappa \geq 0$ such that

$$\|DK(\omega\cdot, v_t) - DK(\omega^*\cdot, v_t^*)\|_{\mathbb{R}^d \leftarrow \mathcal{L}(\mathbb{R}\times\mathbb{R}^d, \mathbb{R}^d)} \leq \kappa \|(v_t, \omega) - (v_t^*, \omega^*)\|_{X\times\mathbb{R}}$$

for every $(v_t, \omega) \in \overline{B}((v_t^*, \omega^*), r)$, uniformly with respect to $t \in [0, 1]$.

The first theoretical assumption ($A\mathfrak{FB}$,[10] page 534) states that the the operators $\mathcal{F}$ and $\mathcal{B}$ appearing in (8) are Fréchet-differentiable. In our case, $\mathcal{B}$ is given by a linear function, thus the assumption is a consequence of the following proposition, which can be proved by simply following the steps in the proof of[8] Proposition 3.2.

Proposition 1. Under 3.1, 3.1 and 3.1, $\mathcal{F}$ in (10) is Fréchet-differentiable at every $(\hat{u}, \hat{\omega}) \in \mathbb{U} \times (0, +\infty)$, and

$$D\mathcal{F}(\hat{u}, \hat{\omega})(u, \omega) = \mathfrak{L}(\cdot; \hat{u}, \hat{\omega})[\overline{u}_{(\cdot)} \circ s_{\hat{\omega}}] + \omega \mathfrak{M}(\cdot; \hat{u}, \hat{\omega}) \tag{11}$$

for $(u, \omega) \in \mathbb{U} \times (0, +\infty)$, where, for $t \in [0, 1]$,

$$\mathfrak{L}(t; \hat{u}, \hat{\omega})[u_{(\cdot)} \circ s_{\hat{\omega}}] := \hat{\omega} \int_{-\frac{\tau}{\hat{\omega}}}^{0} D_2K(\hat{\omega}\theta, \overline{\hat{u}}(t+\theta)) u(t+\theta)\, d\theta \tag{12}$$

and

$$\mathfrak{M}(t; u, \omega) := \int_{-\frac{\tau}{\omega}}^{0} K(\omega\theta, \overline{u}(t+\theta))\, d\theta - \frac{\tau}{\omega} K\left(-\tau, \overline{u}\left(t - \frac{\tau}{\omega}\right)\right) + \omega \int_{-\frac{\tau}{\omega}}^{0} D_1K(\omega\theta, \overline{u}(t+\theta))\theta\, d\theta.$$

The second theoretical assumption ($A\mathfrak{G}$,[10] page 534) concerns the boundedness of the Green operator $\mathcal{G}$ appearing in (8). However, in the case of (9), such operator is not involved, as explained at the beginning of Section 3. Thus, formulation (9) allows us to skip the corresponding technical step in the convergence proof.

The third theoretical assumption (Ax*1,[10] page 536) states that the Fréchet derivative of Φ in (8) is locally Lipschitz continuous at its fixed points. From this point on, (u^*, ω^*) denotes a fixed point of Φ in (9). The validity of the assumption is a consequence of the following proposition, whose proof goes as the one of[8] Proposition 3.4.

Proposition 2. Under 3.1, 3.1, 3.1 and 3.1, there exist $r \in (0, \omega^*)$ and $\kappa \geq 0$ such that

$$\|D\Phi(u, \omega) - D\Phi(u^*, \omega^*)\|_{\mathbb{U}\times\mathbb{R} \leftarrow \mathbb{U}\times(0,+\infty)} \leq \kappa \|(u, \omega) - (u^*, \omega^*)\|_{\mathbb{U}\times\mathbb{R}}$$

for all $(u, \omega) \in \mathcal{B}((u^*, \omega^*), r)$.

In order to prove the validity of the last theoretical assumption (Ax*2,[10] page 536), one needs an additional requirement, other than the hypotheses stated at the beginning of this subsection. Such requirement is anyway a consequence of the hyperbolicity of the periodic solution of interest, which is in turn a standard assumption in the context of application of the principle of linearized stability (see[7]). In particular, if the periodic solution is hyperbolic and $\mathfrak{L}$ in (12) is continuous, then the initial value problem associated to the linear homogeneous RE

$$x(t) = \mathfrak{L}(t; u^*, \omega^*)[x_t \circ s_{\omega^*}] \tag{13}$$

is well-posed.

Proposition 3. Let $T^*(t, s) : X \to X$ be the evolution operator for the linear homogeneous RE (13). Under 3.1, 3.1, 3.1 and 3.1, if 1 is a simple eigenvalue of $T^*(1, 0)$, then the linear bounded operator $I_{\mathbb{U}\times\mathbb{R}} - D\Phi(u^*, \omega^*)$ is invertible, i.e., for all $(u_0, \omega_0) \in \mathbb{U} \times \mathbb{R}$ there exists a unique $(u, \omega) \in \mathbb{U} \times \mathbb{R}$ such that

$$\begin{cases} u = \mathfrak{L}^*[\overline{u}_\cdot \circ s_{\omega^*}] + \omega \mathfrak{M}^* + u_0 \\ p(u) = \omega_0. \end{cases} \tag{14}$$

Proof. The proof is based on treating (14) as a boundary value problem for v the periodic extension of u to $[-1,1]$, i.e.,

$$\begin{cases} v(t) = \mathfrak{L}^*(t)[v_t \circ s_{\omega^*}] + \omega \mathfrak{M}^*(t) + u_0(t) \\ v_1 = v_0 \\ p(v|_{[0,1]}) = \omega_0. \end{cases} \tag{15}$$

for $t \in [0,1]$. (15) is, in turn, an instance of the boundary value problem[8] (19) in Proposition 4 with $\varphi_0 = 0$. By[8] Proposition 4, there is a unique solution (v,ω) to (15), which means that $(v|_{[0,1]},\omega) \in \mathbb{U} \times \mathbb{R}$ is the unique solution to (14). □

3.2 | Numerical assumptions

The present subsection deals with the numerical assumptions required by[10], that is, those concerning the discretization scheme used to reduce (9) to a finite-dimensional problem. Before stating the hypotheses needed to prove such numerical assumptions, we describe below the discretization scheme, given by both the primary and the secondary discretization. The former concerns the approximation of the Banach spaces involved, and is defined as in[8] for the space $\mathbb{U}$, except for the fact that in the present case it needs to take into account the boundary condition defining $\mathbb{U}$ in 3.1. The latter, on the other hand, concerns the approximation of the right-hand side (10) whenever such an approximation is needed, such as in the case (5).

The primary discretization consists in reducing the space $\mathbb{U}$ to finite-dimensional space $\mathbb{U}_L$ given a level of discretization L. This happens by means of a restriction operator $\rho_L^+ : \mathbb{U} \to \mathbb{U}_L$ and a prolongation operator $\pi_L^+ : \mathbb{U}_L \to \mathbb{U}$, which extend respectively to

$$R_L : \mathbb{U} \times \mathbb{R} \to \mathbb{U}_L \times \mathbb{R}, \quad R_L(u,\omega) := (\rho_L^+ u, \omega)$$

and

$$P_L : \mathbb{U}_L \times \mathbb{R} \to \mathbb{U} \times \mathbb{R}, \quad P_L(u_L,\omega) := (\pi_L^+ u_L, \omega).$$

All of them are linear and bounded. In the following we describe the specific choices we make in this context, based on piecewise polynomial interpolation.

Recall that $\mathbb{U}$ concerns the interval $[0,1]$. We choose the uniform outer mesh

$$\Omega_L^+ := \{t_i^+ = ih \ : \ i = 0,1,\dots,L,\ h = 1/L\} \subset [0,1], \tag{16}$$

and inner meshes

$$\Omega_{L,i}^+ := \{t_{i,j}^+ := t_{i-1}^+ + c_j h \ : \ j = 0,\dots,m\} \subset [t_{i-1}^+, t_i^+], \quad i = 1,\dots,L, \tag{17}$$

where $0 = c_0 < \cdots < c_m = 1$ are given abscissae for m a positive integer and can be chosen arbitrarily for the purposes of the present proof. Correspondingly, we define

$$\mathbb{U}_L := \mathbb{R}^{Lm \times d}, \tag{18}$$

whose elements u_L are indexed as

$$u_L := (u_{1,1},\dots,u_{1,m},\dots,u_{L,1}\dots,u_{L,m})^T \tag{19}$$

with components in $\mathbb{R}^d$. Note that $\mathbb{U}_L$ (as well as its elements u_L and the relevant restriction and prolongation operators) depend indeed on both L and m. However, only L is being used for indexing, in order to recall that L is in fact what determines the discretization level, since m remains constant when using FEM. Finally, we define, for $u \in \mathbb{U}$,

$$\rho_L^+ u := (u(t_{1,1}^+),\dots,u(t_{1,m}^+),\dots,u(t_{L,1}^+)\dots,u(t_{L,m}^+))^T \in \mathbb{U}_L \tag{20}$$

and, for $u_L \in \mathbb{U}_L$, $\pi_L^+ u_L \in \mathbb{U}$ as the unique element of the space

$$\Pi_{L,m}^+ := \{p \in C([0,1],\mathbb{R}^d) \ : \ p|_{[t_{i-1}^+, t_i^+]} \in \Pi_m,\ i = 1,\dots,L\} \tag{21}$$

such that

$$\pi_L^+ u_L(0) = u_{L,m}, \quad \pi_L^+ u_L(t_{i,j}^+) = u_{i,j}, \quad j = 1,\dots,m,\ i = 1,\dots,L. \tag{22}$$

Above Π_m is the space of $\mathbb{R}^d$-valued polynomials having degree m and, when needed, we represent $p \in \Pi^+_{L,m}$ through its pieces as

$$p|_{[t^+_{i-1},t^+_i]}(t) = \sum_{j=0}^{m} \ell_{m,i,j}(t)p(t^+_{i,j}), \quad t \in [0,1], \tag{23}$$

where, for ease of notation, we implicitly set

$$t^+_{i,0} := t^+_{i-1}, \quad i = 1,\dots,L,$$

and $\{\ell_{m,i,0}, \ell_{m,i,1}, \dots, \ell_{m,i,m}\}$ is the Lagrange basis relevant to the nodes $\{t^+_{i,0}\} \cup \Omega^+_{L,i}$. Observe that the latter is invariant with respect to i as long as we fix the abscissae c_j, $j = 1,\dots,m$, defining the inner meshes (17). Indeed, for every $i = 1,\dots,L$,

$$\ell_{m,i,j}(t) = \ell_{m,j}\left(\frac{t - t^+_{i-1}}{h}\right), \quad t \in [t^+_{i-1}, t^+_i],$$

where $\{\ell_{m,0}, \ell_{m,1}, \dots, \ell_{m,m}\}$ is the Lagrange basis in $[0,1]$ relevant to the abscissae $c_0, c_1, \dots, c_m$ with $c_0 := 0$.

The secondary discretization is defined as in[8] and reported below for convenience. Such discretization is only needed whenever $\mathcal{F}$ in the first of (8) cannot be computed exactly. In this case, it is approximated with an operator $\mathcal{F}_M$ that can be computed, for a given level of discretization M. In our case, given by (5), the secondary discretization basically consists in performing a quadrature. Thus, we define

$$\mathcal{F}_M(u,\omega) = F_M(\overline{u}_{(\cdot)} \circ s_\omega) := \omega \sum_{i=0}^{M} w_i K(\omega\alpha_i, \overline{u}_{\alpha_i}), \tag{24}$$

where $-\frac{\tau}{\omega} = \alpha_0 < \cdots < \alpha_M = 0$, as done in[8]. The operator $\mathcal{B}$ in the second of (8) is not subject to a secondary discretization, since in our case it is entirely defined by p in the second of (9), which can typically be evaluated exactly in $\pi^+_L \mathbb{U}_L$. Thus, by setting

$$\Phi_M(u,\omega) := \begin{pmatrix} F_M(\overline{u}_{(\cdot)} \circ s_\omega) \\ \omega - p(u) \end{pmatrix}, \tag{25}$$

the discrete version $\Phi_{L,M} := R_L \Phi_M P_L : \mathbb{U}_L \times \mathbb{R} \to \mathbb{U}_L \times \mathbb{R}$ of the fixed point operator Φ in (9) is defined as

$$\Phi_{L,M}(u_L,\omega) := \begin{pmatrix} \rho^+_L F_M(\overline{\pi^+_L u_L}_{(\cdot)} \circ s_\omega) \\ \omega - p(\pi^+_L u_L) \end{pmatrix}.$$

We are now able to state the further hypotheses on the regularity of the right-hand side and on the chosen discretization scheme which are needed to prove the validity of the numerical assumptions.

(N1) The primary discretization of the space $\mathbb{U}$ is based on the choices (16)–(22).

(N2) The nodes $\alpha_0,\dots,\alpha_M$, together with the weights $w_0,\dots,w_M$ chosen for the secondary discretization as in (24) define an interpolatory quadrature formula which is convergent in $B^\infty([0,1],\mathbb{R}^d)$.

The validity of the first numerical assumption (A$\mathfrak{F}_K\mathfrak{B}_K$,[10] page 535) can be proven as done in[8], thanks to the following proposition.

Proposition 4. Under 3.1,3.1 and 3.1 $\mathcal{F}_M$ is Fréchet-differentiable, from the right with respect to ω, at every point $(\hat{u},\hat{\omega}) \in \mathbb{U} \times (0,+\infty)$ and

$$D\mathcal{F}_M(\hat{u},\hat{\omega})(u,\omega) = \mathfrak{L}_M(\cdot;\hat{u},\hat{\omega})[\overline{u}_{(\cdot)} \circ s_{\hat{\omega}}] + \omega\mathfrak{M}_M(\cdot;\hat{u},\hat{\omega})$$

for $(u,\omega) \in \mathbb{U} \times (0,+\infty)$, where, for $t \in [0,1]$,

$$\mathfrak{L}_M(t;\hat{u},\hat{\omega})[u_{(\cdot)} \circ s_{\hat{\omega}}] := \hat{\omega} \sum_{i=0}^{M} w_i D_2 K(\hat{\omega}\alpha_i, \overline{\hat{u}}(t+\alpha_i))u(t+\alpha_i)$$

and

$$\mathfrak{M}_M(t;u,\omega) := \sum_{i=0}^{M} w_i K(\omega\alpha_i, \overline{u}(t+\alpha_i)) - \frac{\tau}{\omega} K\left(-\tau, \overline{u}\left(t - \frac{\tau}{\omega}\right)\right) + \omega \sum_{i=0}^{M} w_i D_1 K(\omega\alpha_i, \overline{u}(t+\alpha_i))\alpha_i.$$

For the sequel it is useful to define $\Psi, \Psi_{L,M} : \mathbb{U} \times \mathbb{R} \to \mathbb{U} \times \mathbb{R}$ as

$$\Psi := I_{\mathbb{U}\times\mathbb{R}} - \Phi, \qquad \Psi_{L,M} := I_{\mathbb{U}\times\mathbb{R}} - P_L R_L \Phi_M. \tag{26}$$

Indeed, the second numerical assumption (CS1,[10] page 536) concerns the Lipschitz continuity of $\Psi_{L,M}$ in (26), and is a consequence of the following proposition, which in turn can be proved as[8] Proposition 3.8.

Proposition 5. Under 3.1, 3.1, 3.1, 3.1 and 3.2, there exist $r_1 \in (0, \omega^*)$ and $\kappa \geq 0$ such that

$$\|D\Psi_{L,M}(u,\omega) - D\Psi_{L,M}(u^*,\omega^*)\|_{\mathbb{U}\times\mathbb{R}\leftarrow\mathbb{U}\times(0,+\infty)} \leq \kappa\|(u,\omega) - (u^*,\omega^*)\|_{\mathbb{U}\times\mathbb{R}}$$

for all $(u,\omega) \in \overline{B}((u^*,\omega^*), r_1)$ and for all positive integers L and M.

Finally, the last numerical assumption (CS2,[10] page 537) ensures in particular that the chosen discretization scheme is both stable and consistent. The stability part is given by the next proposition, which can be proved much more easily than in[8] due to the possibility to apply Banach's perturbation lemma more directly. This represents, in particular, the greatest simplification in the proof of the convergence analysis with respect to that presented in[8]. The proof of such proposition makes use of some standard results on the primary discretization, stated below as a lemma for convenience.

Lemma 1. Under 3.2, let ρ_L^+, π_L^+ be defined respectively in (20), (22) and let Λ_m be the Lebesgue constant corresponding to the nodes used for discretization in a single mesh interval. Then, under 3.1,

$$\|\pi_L^+\rho_L^+\|_{\mathbb{U}\leftarrow\mathbb{U}} \leq \Lambda_m \tag{27}$$

holds for all positive integers L and

$$\lim_{L\to\infty} \|\pi_L^+\rho_L^+ u - u\|_{\mathbb{U}} = 0 \tag{28}$$

holds for all $u \in C([0,1],\mathbb{R}^d)$.

Proof. By (23),

$$\pi_L^+\rho_L^+ u(t) = \sum_{j=0}^{m} \ell_{m,i,j}(t)u(t_{i,j}^+)$$

holds for $u \in \mathbb{U}$ and $t \in [t_{i-1}^+, t_i^+]$, $i = 1,\dots,L$. Then (27) follows from

$$\|\pi_L^+\rho_L^+ u\|_{\mathbb{U}} \leq \max_{i=1,\dots,L} \max_{t\in[t_{i-1}^+,t_i^+]} \sum_{j=0}^{m} |\ell_{m,i,j}(t)|\|u\|_{\mathbb{U}} = \Lambda_m\|u\|_{\mathbb{U}}.$$

As for (28),

$$\begin{aligned}
\pi_L^+\rho_L^+ u(t) - u(t) &= \sum_{j=0}^{m} \ell_{m,i,j}(t)u(t_{i,j}^+) - \sum_{j=0}^{m} \ell_{m,i,j}(t)u(t) + \sum_{j=0}^{m} \ell_{m,i,j}(t)u(t) - u(t) \\
&= \sum_{j=0}^{m} \ell_{m,i,j}(t)[u(t_{i,j}^+) - u(t)] + \left(\sum_{j=0}^{m} \ell_{m,i,j}(t) - 1\right) u(t) \\
&= \sum_{j=0}^{m} \ell_{m,i,j}(t)[u(t_{i,j}^+) - u(t)]
\end{aligned}$$

holds always for $t \in [t_{i-1}^+, t_i^+]$, $i = 1,\dots,L$. Therefore $\|\pi_L^+\rho_L^+ u - u\|_{\mathbb{U}} \leq \Lambda_m\omega(u;h)$, where ω denotes the modulus of continuity. The latter vanishes as $h \to 0$ only if u is at least continuous. □

Proposition 6. Under 3.1, 3.1, 3.1, 3.2 and 3.2, $D\Psi_{L,M}(u^*,\omega^*)$ is invertible and its inverse is uniformly bounded with respect to both L and M.

Proof. The result follows from the Banach's perturbation lemma, once we show that

$$\lim_{L,M\to\infty} \|D\Psi_{L,M}(u^*,\omega^*) - D\Psi(u^*,\omega^*)\|_{\mathbb{U}\times\mathbb{R}\leftarrow\mathbb{U}\times(0,+\infty)} = 0.$$

Since the phase condition is not affected by the discretization scheme, we are left to prove that

$$\lim_{L,M\to\infty} \|\pi_L^+\rho_L^+\mathfrak{L}_M^*[\overline{u}_{(\cdot)} \circ s_{\omega^*}] - \mathfrak{L}^*[\overline{u}_{(\cdot)} \circ s_{\omega^*}]\|_{\mathbb{U}} = 0 \tag{29}$$

for all $u \in \mathbb{U}$, and that

$$\lim_{L,M\to\infty} \|\pi_L^+\rho_L^+\mathfrak{M}_M^* - \mathfrak{M}^*\|_{\mathbb{U}} = 0. \tag{30}$$

As for (29), one gets

$$\|(\pi_L^+\rho_L^+\mathfrak{L}_M^* - \mathfrak{L}^*)[\overline{u}_{(\cdot)} \circ s_{\omega^*}]\|_{\mathbb{U}} \leq \|\pi_L^+\rho_L^+(\mathfrak{L}_M^* - \mathfrak{L}^{*\prime})[\overline{u}_{(\cdot)} \circ s_{\omega^*}]\|_{\mathbb{U}} + \|(\pi_L^+\rho_L^+ - I_{\mathbb{U}})\mathfrak{L}^*[\overline{u}_{(\cdot)} \circ s_{\omega^*}]\|_{\mathbb{U}}.$$

The first addend in the right-hand side above vanishes thanks to 3.2, (27) in Lemma 1 as well as standard results such as[19] Corollary of Theorem Ia. The second addend vanishes as well thanks to (28) of Lemma 1, given that $\mathfrak{L}^*[\overline{u}_{(\cdot)} \circ s_{\omega^*}] \in C([0,1],\mathbb{R}^d)$ for all $u \in \mathbb{U}$, as it follows through the definition of $\mathfrak{L}^*$ in (12) under 3.1. Similar arguments hold for (30). □

Eventually, to prove the validity of CS2, one can prove the following proposition, similarly to[8] Proposition 3.9.

Proposition 7. Under 3.1, 3.1, 3.1, 3.2 and 3.2,

$$\lim_{L,M\to\infty} \frac{1}{r_2(L,M)} \|[D\Psi_{L,M}(u^*,\omega^*)]^{-1}\|_{\mathbb{U}\times\mathbb{R}\leftarrow\mathbb{U}\times\mathbb{R}} \cdot \|\Psi_{L,M}(u^*,\omega^*)\|_{\mathbb{U}\times\mathbb{R}} = 0,$$

where

$$r_2(L,M) := \min\left\{r_1, \frac{1}{2\kappa\|[D\Psi_{L,M}(u^*,\omega^*)]^{-1}\|_{\mathbb{U}\times\mathbb{R}\leftarrow\mathbb{U}\times\mathbb{R}}}\right\}$$

with r_1 and κ as in Proposition 5.

3.3 | Convergence results

As anticipated, the results of the present section on the theoretical and numerical assumptions imply that the FEM for the fixed point problem (9) converges, according to[10]. Note that such method corresponds exactly to that analyzed in[8], despite the formulation of the fixed point problem being formally different. Theorem 3.1 therein can be reformulated for (9) as follows.

Theorem 1 ([10] Theorem 2, page 539). Under 3.1, 3.1, 3.1, 3.2 and 3.2, there exists a positive integer $\hat{N}$ such that, for all $L,M \geq \hat{N}$, the operator $R_L\Phi_M P_L$ has a fixed point $(u^*_{L,M},\omega^*_{L,M})$ and

$$\|(u^*_{L,M},\omega^*_{L,M}) - (u^*,\omega^*)\|_{\mathbb{U}\times\mathbb{R}} \leq 2\|[D\Psi_{L,M}(u^*,\omega^*)]^{-1}\|_{\mathbb{U}\times\mathbb{R}\leftarrow\mathbb{U}\times\mathbb{R}} \cdot \|\Psi_{L,M}(u^*\omega^*)\|_{\mathbb{U}\times\mathbb{R}}.$$

The consequent results on the order of convergence in[8] Subsection 3.3 remain unchanged. Concerning the primary discretization error

$$\varepsilon_L := \|(I_{\mathbb{U}\times\mathbb{R}} - P_L R_L)(u^*,\omega^*)\|_{\mathbb{U}\times\mathbb{R}}$$

the following holds.

Theorem 2 ([8] Theorem 3.11). Let $K \in \mathcal{C}^p([-\tau,0]\times\mathbb{R}^d,\mathbb{R}^d)$ for some integer $p \geq 0$. Then, Under 3.1, 3.1, 3.1, 3.2 and 3.2, it holds that $u^* \in C^{p+1}([0,1],\mathbb{R}^d)$, $\psi^* \in C^{p+1}([-1,0],\mathbb{R}^d)$, $v^* \in C^{p+1}([-1,1],\mathbb{R}^d)$ and

$$\varepsilon_L = O\left(h^{\min\{m,p\}}\right).$$

Observe that ϵ_L determines entirely the final order of convergence provided that one chooses a suitable quadrature formula, as explained in[8] Subsection 3.3. Thus, the final order m of convergence is guaranteed if K is sufficiently smooth.

Remark 1. As stated in[8], the entire convergence analysis can as well be carried out for right-hand sides of the form (6). In this case, the different theoretical and numerical assumptions read

(T3) $k \in \mathcal{C}(\mathbb{R},\mathbb{R}^d)$.

(T4) $f \in \mathcal{C}^1(\mathbb{R}^d,\mathbb{R}^d)$.

(T5) There exist $r > 0$ and $\kappa \geq 0$ such that

$$\left\| f'\left(\omega \int_0^{\frac{\tau}{\omega}} k(\theta)v(t-\theta)\,d\theta\right) - f'\left(\omega^* \int_0^{\frac{\tau}{\omega^*}} k(\theta)v^*(t-\theta)\,d\theta\right)\right\|_{\mathbb{R}^d} \leq \kappa\|(v_t,\omega) - (v^*_t,\omega^*)\|_{X\times\mathbb{R}}$$

for every $(v_t,\omega) \in \overline{B}((v^*_t,\omega^*),r)$, uniformly with respect to $t \in [0,1]$.

Moreover, the above can be easily further generalized to the case

$$F(\psi) = f\left(\int_0^{\tau_1} k_1(\sigma)\psi(-\sigma)\,d\sigma,\ldots,\int_0^{\tau_n} k_n(\sigma)\psi(-\sigma)\,d\sigma\right).$$

4 | CONCLUSIONS

The recent work[8] provided the first complete theoretical proof of the convergence of the piecewise collocation method to compute periodic solutions of REs. Such proof is based on formulating the problem as a BVP of the form (2), in view of applying the abstract approach in[10] for general BVPs. The aim of the present paper is to describe how the proof can be shortened and simplified when one applies the approach in[10] to the BVP (3), which is equivalent to (2) while being defined by a finite-dimensional boundary condition. In particular, as it turns out, the proof of the stability of the method can be notably simplified.

As observed in[11], the periodic BVP corresponding to (3) for RFDEs is not amenable of the analysis in[10]. However, one could in principle try to further reformulate the problem as a different - albeit equivalent - BVP featuring a finite-dimensional boundary condition (e.g., as the one considered in[20]). The considerations above suggest that doing so might be worth the effort, especially in view of extending the convergence analysis to classes of more complex equations, e.g., neutral equations or differential equations defined by non-constant delays.

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References

1. Gripenberg G, Londen S-O, Staffans O. Volterra integral and functional equations. Encyclopedia of Mathematics and its ApplicationsCambridge: Cambridge University Press; first ed.2009.
2. Breda D, Diekmann O, Graaf W, Pugliese A, Vermiglio R. On the formulation of epidemic models (an appraisal of Kermack and McKendrick). J. Biol. Dyn.. 2012;6(2):103-117.
3. Diekmann O, Gyllenberg M, Huang H, Kirkilionis M, Metz J A J, Thieme H R. On the formulation and analysis of general deterministic structured population models. II. Nonlinear theory. J. Math. Biol.. 2001;43:157-189.
4. Diekmann O, Gyllenberg M, Metz J A J, Thieme H R. On the formulation and analysis of general deterministic structured population models. I. Linear theory. J. Math. Biol.. 1998;36:349-388.
5. Feller Willy. On the Integral Equation of Renewal Theory. Ann. Math. Statist.. 1941;12(3):243–267. .
6. Diekmann O, Getto P, Gyllenberg M. Stability and bifurcation analysis of Volterra functional equations in the light of suns and stars. SIAM J. Math. Anal.. 2008;39(4):1023-1069.
7. Breda D, Liessi D. Floquet theory and stability of periodic solutions of renewal equations. J Dyn Diff Equat. 2020;. DOI: 10.1007/s10884-020-09826-7.
8. Andò A, Breda D. Numerical computation of periodic solutions of renewal equations from population dynamics submitted, arxiv.org/abs/2105.09199; .
9. Engelborghs K, Luzyanina T, Hout K J, Roose D. Collocation methods for the computation of periodic solutions of delay differential equations. SIAM J. Sci. Comput.. 2001;22(5):1593-1609.

10. Maset S. An abstract framework in the numerical solution of boundary value problems for neutral functional differential equations. Numer. Math.. 2016;133(3):525-555.

11. Andò A, Breda D. Convergence analysis of collocation methods for computing periodic solutions of retarded functional differential equations. SIAM Journal of Numerical Analysis. 2020;58(5):3010-3039.

12. Bader G. Solving boundary value problems for functional-differential equations by collocation. In: Numerical boundary value ODEs, (Vancouver, B.C., 1984), Birkhäuser; 1985:227-243.

13. Barton David A W, Krauskopf Bernd, Wilson R Eddie. Collocation schemes for periodic solutions of neutral delay differential equations. J. Differ. Equ. Appl.. 2006;12(11):1087-1101.

14. Engelborghs K, Doedel E. Stability of piecewise polynomial collocation for computing periodic solutions of delay differential equations. Numer. Math.. 2002;91(4):627-648.

15. Reddien G W, Travis C C. Approximation methods for boundary value problems of differential equations with functional arguments. J. Math. Anal. Appl.. 1974;46:62-74.

16. Doedel E.. Lecture notes on numerical analysis of nonlinear equations. In: Osinga H M, Krauskopf B, Galán-Vioque J, eds. Numerical continuation methods for dynamical systems, Understanding Complex Systems. Springer; 2007:1-49.

17. MatCont https://sourceforge.net/projects/matcont/.

18. Scarabel F, Diekmann O, Vermiglio R. Numerical bifurcation analysis of renewal equations via pseudospectral approximation. J. Comput. Appl. Math.. 2020;397:113611.

19. Erdös P, Turán P. On interpolation, (I) quadrature and mean convergence in the Lagrange interpolation. Annals of Math.. 1937;38:142-155.

20. Sieber J. Finding periodic orbits in state-dependent delay differential equations as roots of algebraic equations. Discrete Contin. Dyn. S. Ser. S. 2012;32(8):2607-2561.

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