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# Symmetry Groups, Quantum Mechanics and Generalized Hermite Functions

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**Abstract:** This is a review paper on the generalization of Euclidean as well as pseudo-Euclidean groups of interest in quantum mechanics. The Weyl–Heisenberg groups, $H_n$, together with the Euclidean, $E_n$, and pseudo-Euclidean $E_{p,q}$, groups are two families of groups with a particular interest due to their applications in quantum physics. In the present manuscript, we show that, together, they give rise to a more general family of groups, $K_{p,q}$, that contain $H_{p,q}$ and $E_{p,q}$ as subgroups. It is noteworthy that properties such as self-similarity and invariance with respect to the orientation of the axes are properly included in the structure of $K_{p,q}$. We construct generalized Hermite functions on multidimensional spaces, which serve as orthogonal bases of Hilbert spaces supporting unitary irreducible representations of groups of the type $K_{p,q}$. By extending these Hilbert spaces, we obtain representations of $K_{p,q}$ on rigged Hilbert spaces (Gelfand triplets). We study the transformation laws of these generalized Hermite functions under Fourier transform.

**Keywords:** Euclidean and pesudo-Euclidean symmetry groups; generalized Hermite functions; rigged Hilbert spaces

**MSC:** 22D10; 43A80



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## 1. Introduction

The theory of groups is considered the natural language for describing the symmetries or invariance properties of physical systems. In particular, Lie groups are appropriate tools used to study symmetries depending on continuous parameters and are very useful in describing the invariance properties of the physical world. Lie groups, which are the subject of interest in this manuscript, are the $n$-dimensional ($n$D) Heisenberg groups $H_n$ [1], the Euclidean group $E_n = \mathbb{R}^n \odot SO(n)$, and the pseudo-Euclidean groups $E_{p,q} = \mathbb{R}^{p,q} \odot SO_0(p,q)$ with the connected component of unity $SO_0(p,q)$. Both $E_n$ and $E_{p,q}$ are subgroups of their corresponding affine groups. Notice that $p + q = n$ throughout the paper.

The Heisenberg group has a closed connection to the indetermination principle in quantum mechanics through Fourier transform [1,2] and, hence, to the Planck constant due to the fact that the exponent in $\epsilon^{\pm ip\cdot x/\hbar}$ is dimensionless. This group also has a connection to Gabor formalism [3,4], where the uncertainty principle for time–frequency operators plays a fundamental role in wavelet expansion.

Furthermore, the invariance properties of Euclidean, $\mathbb{R}^n$, or pseudo-Euclidean spaces, $\mathbb{R}^{p,q}$, are consequences of freedom in the characterization of these affine spaces. Based on the description of the physical world, one has four options:

1. The existence of two sets of *conjugate variables* allows for equivalent descriptions of the physical systems, which permit their study on either in the position or in the momentum representations. Both representations are connected through the Weyl–Heisenberg group $H_{p,q}$.

2. *Homogeneity*, which means the freedom of choice of origin in the coordinate system, positions, or momenta. The groups $E_{p,q}$ and $H_{p,q}$ are relevant in this case.
3. *Self-similarity*, which stands for the freedom to choose the unit of length.
4. The *invariance from orientation* is the freedom to select the orientation of the unit vectors for the orthogonal bases of the physical space. In both the self-similarity and invariance groups, $E_{p,q}$ plays a role.

Options 1 and 2 are relevant in the discussion of the Heisenberg group and its relation with Fourier transform, while options 3 and 4 have to do with the choice of reference frame.

In the present contribution, we propose a unified review of some important facts concerning generalizations of Euclidean and pseudo-Euclidean groups [5–7], which are interesting to use to realize symmetries in quantum mechanics. Thus, we consider these options as an ensemble, contrary to the usual tradition of considering them separately. The natural response comes with the use of the spaces $\mathbb{R}^n$(or $\mathbb{R}^{p,q}$), the Hilbert space $L^2(\mathbb{R}^n)$ (or $L^2(\mathbb{R}^{p,q})$), as well all other spaces of functions defined on $\mathbb{R}^n$ or on $\mathbb{R}^{p,q}$. Bases either in the coordinate representation ($\{x\}$) or in the momenta conjugate representation ($\{p\}$) are equally suitable in this context. As a matter of fact, options 1–4 above are not completely independent, since Fourier transform, which gives an invertible correspondence between coordinate and momentum representations [8,9], does not allow us to independently fix self-similarity and orientation.

The assumption of these invariances may be considered a principle of relativity. Two observers located at different points of the space and using different units of length and/or momenta, and different orientations of unit vectors may give a different description of the observed events. At the same time, the mentioned invariance principle should be equivalent to the fundamental statement of relativity that establishes that both descriptions have to be completely equivalent.

In a recent paper [10], we recalled the abovementioned program for the case of the real line. We studied an extension $\widetilde{E}_{1,1}$ of the Heisenberg–Weyl group $H_1$ isomorphic to the central extension of the group of isometries of $\mathbb{R}^{1,1}$ with signature $(+,-)$. The group $\widetilde{E}_{1,1}$ may be seen as the central extension of the Poicaré group in $(1+1)$ dimensions enlarged with the $PT$ (parity-time reversal) transformation.

While in our precedent studies we used the space $\mathbb{R}$ as the point of departure, we now generalize our analysis starting with the Euclidean and pseudo-Euclidean spaces such as $\mathbb{R}^n$ and $\mathbb{R}^{p,q}$. Thus, we include in our study the Euclidean and pseudo-Euclidean groups $E_n$ and $E_{p,q}$ and the Heisenberg–Weyl groups $H_n$ or $H_{p,q}$. They provide the existence of two families of conjugate $n$D variables connected by Fourier Transform. Consequently, one may expect the existence of a related indetermination principle.

We may represent the group $H_{p,q}$ in terms of real matrices $(n+2)\times(n+2)$ as

$$H_{p,q}[\mathbf{a},\mathbf{b},c] = \begin{bmatrix} 1 & \mathbf{a}^T & c \\ \mathbf{0} & \mathbb{I}_n & \mathbf{b} \\ 0 & \mathbf{0}^T & 1 \end{bmatrix}, \tag{1}$$

where $\mathbf{a} \in \mathbb{R}^{p,q}$ describes the translations in the space of momenta ($\mathbf{a}^T$ stands for a row vector where $T$ symbolizes transposition); $\mathbf{b} \in \mathbb{R}^{p,q}$ is the translations in the space of coordinates ($\mathbf{b}$ denotes a column vector); $c$ is related to a central charge or, in other words, to the indetermination principle; and $\mathbb{I}_n$ is the identity matrix $n \times n$.

Later, in Section 4.1, we shall see that the decomposition of the pseudo-Euclidean group $E_{p,q}$ as a semidirect product $E_{p,q} = \mathbb{R}^{p,q} \odot SO_o(p,q)$ allows us to connect the groups $H_{p+q}$ and $SO(p,q)$ into a new group. Along $E_{p,q}$, which describes the transformations on the physical space, and $H_{p,q}$, a group related to the Canonical Commutation Relations, we construct the more general group $K_{p,q}$, which contains the other two as subgroups. The Lie algebra, $k_{p,q}$, of $K_{p,q}$ contains two kinds of generators: the canonical conjugate observables and the generators of the spatial symmetry. To construct $K_{p,q}$, we replace the identity matrix $\mathbb{I}_n$ that appears at the center of (1) based on a matrix that is the product of a scalar factor

$k \in \mathbb{R}^*$, which is related to the self-invariance and the general orientation of the space times a matrix $\Lambda \in SO_o(p,q)$. Thus, we obtain an $(n+2) \times (n+2)$ matrix representation of $K_{p,q}$ given by

$$K_{p,q}[\mathbf{a}, \mathbf{b}, c, k, \Lambda] = \begin{bmatrix} 1 & \mathbf{a}^T & c \\ \mathbf{0} & k\Lambda & \mathbf{b} \\ 0 & \mathbf{0}^T & 1 \end{bmatrix}. \quad (2)$$

Obviously, both representations (1) and (2) are real. The groups $K_{p,q}$ are simply connected if $p$ and $q$ are even. Otherwise, $K_{p,q}$ are doubly connected: the identity component characterized by $k > 0$, i.e, $\det K_{p,q}[\mathbf{a}, \mathbf{b}, c, k, \Lambda] > 0$, which is a subgroup of $K_{p,q}$, and a second connected component with negative determinant.

Here, we study in detail the representations of $K_{p,q}$ supported on $L^2(\mathbb{R}^{p,q})$, where we use standard bases with a closed connection to the $n$D Fourier transform and their eigenvalues. These are the bases defined by the Hermite functions. In addition, we introduce a generalization of the Hermite functions so as to describe the abovementioned invariance. The action of the group $K_{p,q}$ is defined by its transformations on $\mathbb{R}^{p,q}$. The $n$D Hermite functions, to be defined later, are complex square integrable functions giving new complete orthonormal sets in $L^2(\mathbb{R}^{p,q})$.

The presence of both discrete and continuous bases for infinite dimensional representations of $K_{p,q}$ justifies the introduction of a structure such as rigged Hilbert spaces, also called Gelfand triplets [11,12]. Gelfand triplets also serve to give good topological properties, such as continuity, to the elements of the Lie algebras when represented by linear operators, which in general are not bounded on a Hilbert space.

Additionally, structures such that $K_{p,q}$, $E_{p,q}$ and $H_{p,q}$ are well defined are of obvious interest. They are also interesting because of their physical applications. For example, the group $K_3$ has interesting applications to 3D quantum physics. This group has only spatial dimensions. What if a group such as $K_{p,q}$, with $q > 1$, has more than one time-like dimension? A similar question arises when one deals with a anti-de Sitter group, such as $SO(3,2)$. Then, the situation is far from being clear.

On the other hand, if $q = 1$, we are in a different scenario where the situation on Minkowski spaces of the type $\mathbb{R}^{3,1}$ or $\mathbb{R}^{1,1}$ is well known. These Minkowski spaces have a time-like variable, which could be associated to a time. This induces a coordinate and its conjugate momentum, which on spaces such as $K_{p,1}$, are represented by a coordinate operator $X_0$ and its conjugate $P_0$. They satisfy a commutation relation of the form $[X_0, P_0] = \mathcal{I}$. The interpretation of $X_0$ and $P_0$ as a time operator and an energy operator, respectively, seems to come naturally. Such an interpretation may only be possible outside the world of ordinary non-relativistic quantum mechanics, where the energy operator is a semibounded Hamiltonian, Semiboundedness of the Hamiltonian prohibits commutation relations of the type $[T, H] = \mathcal{I}$. Additionally, non-relativistic quantum mechanics moves on spaces of the form $\mathbb{R}^n$ and not of the form $\mathbb{R}^{p,q}$. Thus, a possible interpretation of the commutation relation $[X_0, P_0] = \mathcal{I}$ as a relation between the time and energy operators may only be possible on relativistic quantum theories. Note that this commutation relation yields an exact uncertainty time–energy relation. Observe that any representation for both $X_0$ and $P_0$ as operators on an infinite dimensional Hilbert space must have an absolutely continuous spectrum covering the whole real line $\mathbb{R}$.

Is this applicable to the physical world? Does this relation of uncertainty time–energy have a link to physical reality? Any answer to these questions is highly conjectural. One may think that symmetries such as $K_{p,1}$ are just local symmetries acting on a neighbourhood of each point on the space–time continuum. Another point of view may state that these groups express symmetries on the physical word excluding energy translation, in which case, we should just keep $K_p$. In the present paper, we use finite dimensional representations only. Using infinite dimensional representations gives additional complications due to the absence of unitary equivalence of the CCR representations without the additional conditions leading to the Stone–von Neumann theorem.

The paper is organized as follows: In Section 2, we summarize and generalize the results starting with the 1D space introduced in [10]. In Section 3 and to prepare for the study of the most general case, we introduce 2D spaces both in the plane and the hyperbolic plane. As in the general case, the generators are a sum of the generators related to the CCR and those describing invariance on the physical space. We left the study of the most general symmetric spaces $\mathbb{R}^{p,q}$ for Section 4, where we construct a unique algebra, and a unique representation of this algebra, putting together canonical conjugate observables and generators of invariance, as noted before. In Section 5, we introduce the rigged Hilbert space structures associated with these generalized Hermite functions and the structures discussed along the present manuscript. We close the paper with a short discussion and a few concluding remarks.

## 2. Heisenberg–Weyl Group in the Real Line $\mathbb{R}$

As mentioned before, in a recent paper [10], we considered the 1D Heisenberg–Weyl group, $H_1$ and its connection with the group of transformations of the real line, $E_1$. Thereafter, we obtained a new group establishing a relation between the indetermination principle, the Fourier Transform, and the Hermite functions.

As is well known, the Heisenberg–Weyl group $H_1$ realizes in one dimension on the coordinate space the basic commutation relation of Quantum Physics $[x, p] \equiv [x, -i\hbar\frac{\partial}{\partial x}] = i\hbar$. Conditions on unitary equivalence of this representation of the Canonical Commutation Relations (CCR) and the Stone–von Neumann theorem are analyzed in [13]. One of the matrix representations of the Heisenberg–Weyl group, $H_1$, is given in terms of the group $M_3(\mathbb{R})$ real $3 \times 3$ upper triangular matrices [14,15]. Typically,

$$H_1[a,b,c] \;=\; \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}, \qquad a,b,c \in \mathbb{R}\,. \tag{3}$$

Here, we easily obtain the multiplication law based on matrix multiplication. If we wish to include self-similarity and orientation, we have to upgrade $H_1$ and extend it to the group $K_1$, which is also a subgroup of $M_3(\mathbb{R})$. A representation of $K_1$ may be realized by the matrices of the form

$$K_1[a,b,c,k] \;=\; \begin{bmatrix} 1 & a & c \\ 0 & k & b \\ 0 & 0 & 1 \end{bmatrix}, \qquad a,b,c \in \mathbb{R}, \;\; k \in \mathbb{R}^*\,. \tag{4}$$

From (4), we obtain the group law through matrix multiplication, so that

$$K_1[a',b',c',k'] \cdot K_1[a,b,c,k] = K_1[ka'+a, k'b+b', c'+c+a'b, k']\,. \tag{5}$$

It is noteworthy that $K_1$ has two connected components. These are (i) the connected component of the identity, which is characterized by $k > 0$ and is a subgroup of $K_1$, which we henceforth denote by $K_1^o$, and (ii) a second component for which its elements are labelled by $k < 0$. The elements belonging to this component can be obtained by multiplication of the elements of $K_1^o$ by the matrix $\mathcal{P}_1 := \text{Diagonal}[1,-1,1]$, which represents the "parity" or space-inversion operator.

The real parameters $a, b$, and $c$ of $H_1$ are in correspondence with the three generators $X$, $P$, and $I$ of the Lie algebra of $H_1$ and $\mathfrak{H}_1$. In addition, the Lie algebra $\mathcal{K}_1$ of $K_1$ also contains a generator $D$ associated with the parameter $k$. The explicit form of these generators in the representation (4) is

$$X=\frac{\partial K_1[\dots]}{\partial a}\bigg|_{\mathbb{I}_3}=\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix},\qquad P=\frac{\partial K_1[\dots]}{\partial b}\bigg|_{\mathbb{I}_3}=\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix},$$
$$I=\frac{\partial K_1[\dots]}{\partial c}\bigg|_{\mathbb{I}_3}=\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix},\qquad D=\frac{\partial K_1[\dots]}{\partial k}\bigg|_{\mathbb{I}_3}=\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}. \tag{6}$$

The commutation relations for $\mathcal{K}_1$ are

$$[X,P]\,=\,I\,,\quad [D,X]\,=\,-X\,,\quad [D,P]\,=\,P,\quad [I,\bullet\,]\,=\,0\,. \tag{7}$$

The real line $\mathbb{R}$ is a metric space that supports two important continuous conjugates (in the sense of position–momentum conjugation) bases $\{|x\rangle\}_{x\in\mathbb{R}}$ and $\{|p\rangle\}_{p\in\mathbb{R}}$ for $L^2(\mathbb{R})$, constructed with the generalized eigenvectors of the operators $X$ and $P$

$$X\,|x\rangle = x\,|x\rangle\,,\qquad P\,|p\rangle = p\,|p\rangle\,. \tag{8}$$

At this point, it is necessary to underline that this notion of *continuous basis* does not have anything to do with the notion of a Hamel basis for a linear space, a orthonormal basis (complete orthonormal set) for a Hilbert space, or a Schauder basis in a Banach space. It is instead a system functional that spans the vectors on a locally convex space, dense as a subspace of the Hilbert space, through some integral formula involving these functionals, very much in the fashion of spectral decompositions of self adjoint operators. We delay the precise meaning of this notion to Section 5. See also the references quoted therein.

The first of the bases, $\{|x\rangle\}_{x\in\mathbb{R}}$, satisfies the following relations:

$$\langle\, x|x'\,\rangle\;=\;\sqrt{2\pi}\;\delta(x-x')\,,\qquad \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} dx\,|x\rangle\langle x| = \mathbb{I}\,. \tag{9}$$

These generalized bases are well defined on certain extensions of the Hilbert space, the Gelfand triplets, as discussed later.

A similar result can be obtained for the second basis, $\{|p\rangle\}_{p\in\mathbb{R}}$, in (8). We know that the Fourier transform (FT) and its inverse (IFT) connect both bases [9]:

$$FT[|x\rangle,x,p]=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} dx\,e^{ipx}|x\rangle\;=|p\rangle,\quad IFT[|p\rangle,p,x]=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} dp\,e^{-ipx}|p\rangle\;=|x\rangle\,. \tag{10}$$

Obviously, from (10), we have that $\langle\, x|p\,\rangle = e^{i\,p\,x}$.

The proper meaning of these bases is discussed later in Section 5. In any case, recall that there exists a representation of the Heisenberg–Weyl group $H_1$ by unbounded operators on $L^2(\mathbb{R})$. Let us call $P$ and $X$ the operators satisfying the commutation relation $[X,P]\,=\,I$. On $L^2(\mathbb{R})$, these operators may be represented by $[Pf](x) = -i\,df(x)/dx$ and $[Xf](x) = xf(x)$. We also may choose an abstract representation of these operators on an abstract infinite dimensional separable Hilbert space $\mathcal{H}$; see our previous comment on the unitary equivalence of CCR representations. Since there is always a unitary mapping between $\mathcal{H}$ and $L^2(\mathbb{R})$, it gives the relation between the representations of $P$ and $X$ on $L^2(\mathbb{R})$ and the operators satisfying the same commutation relation on $\mathcal{H}$. In order to avoid notational complexity, we also denote the latter operators by $P$ and $X$.

Then, following [9], being given an arbitrary vector $|f\rangle$ in the abstract Hilbert space $\mathcal{H}$, we may write that

$$|f\rangle = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} dx\,f(x)\,|x\rangle = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} dp\,\hat{f}(-p)\,|p\rangle\,, \tag{11}$$

with $f(x) = \langle x|f\rangle$ and $\hat{f}(-p) = \langle p|f\rangle$, and we have to take into account that $\hat{f}(p) = FT[f(x);x,p]$. Note that, if $f(x)$ is real, then $\hat{f}(-p) = \hat{f}(p)^*$.

Relation (11) gives the vectors in the abstract Hilbert space $\mathcal{H}$ in terms of both conjugate continuous bases. The components of both spans in (11) gives respective square integrable functions that are Fourier transforms of each other. We have to underline that not all vectors $|f\rangle \in \mathcal{H}$ may be written as in (11), only those belonging to a dense in $\mathcal{H}$ space of test vectors, as shall be explained in Section 5.

The action of the group elements $e^{-iPb}$ and $e^{-iXa}$ on the continuous bases is given by

$$e^{-iPb}\,|x\rangle = |x+b\rangle\,, \qquad e^{-iXa}\,|p\rangle = |p-a\rangle\,, \qquad a,b\in\mathbb{R}\,, \tag{12}$$

From the well-known relations (12), we conclude that the continuous basis $\{|x\rangle\}$ is equivalent to the continuous basis $\{|x+b\rangle\}$ and, similarly, in the momentum representation for $\{|p\rangle\}$ and $\{|p-a\rangle\}$.

The basis $\{|x\rangle\}_{x\in\mathbb{R}}$ as well as the basis $\{|p\rangle\}_{p\in\mathbb{R}}$ support each infinite dimensional unitary irreducible representation (UIR) of the Heisenberg–Weyl group $H_1$. We denote this representation as $U_h(g)$ no matter which of the two conjugate continuous bases we are using. The representations are labelled by $h\in\mathbb{R}^*\equiv\mathbb{R}/\{0\}$. Explicitly [10,16,17],

$$U_h(g)\equiv U_h(c,a,b) := e^{ihcI}\,e^{ih(aX-bP)} = e^{ih(c-ab/2)I}\,e^{ihaX}\,e^{-ihbP}\,. \tag{13}$$

The action of (13) on the continuous basis is clear. For instance,

$$U_h(g)\,|x\rangle = e^{ihc}\,e^{iha(x+b/2)}\,|x+b\rangle\,. \tag{14}$$

The corresponding expression for $|p\rangle$ is left to the reader. Additionally, the action of this UIR $U_h(g)$ translated to the functions of $L^2(\mathbb{R})$ can be straightforwardly computed by taking into account (11). Its explicit expression is given by

$$(\mathcal{U}_h(g)f)(x) = e^{ihc}\,e^{iha(x-b/2)}\,f(x-b)\,. \tag{15}$$

Nevertheless, the group $H_1$ does not exhaust the invariances of the real line if we add the hypothesis of self-similarity and orientation. Hence, the continuous basis $\{|x\rangle\}$ is equivalent to the continuous basis $\{|k\,x\rangle\}$ with $k\in\mathbb{R}^*$; $\{|p\rangle\}$ and $\{|k'p\rangle\}$ show the same equivalency with $k'\in\mathbb{R}^*$. The relation between $k$ and $k'$ is obvious: $k'=k^{-1}$. Thus, the real line $\mathbb{R}$ supports a UIR, $U_{h,\mathcal{C}}$, of $K_1$. Let us start with the connected component $K_1^o$ of $K_1$. In this case, we have $k=e^d>0,\ d\in\mathbb{R}$. We have that (see Formula (53) of [10])

$$e^{idD}\,|x\rangle = e^{d/2}\,|e^d\,x\rangle\,. \tag{16}$$

Therefore,

$$U_{h,\mathcal{C}}(\tilde{g})\,|x\rangle = e^{d/2}\,e^{ih(c+\mathcal{C})}\,e^{iha(e^d\,x+b/2)}\,|e^d\,x+b\rangle\,, \qquad \tilde{g}=(a,b,c,d)\in K_1^o\,. \tag{17}$$

Here, $\mathcal{C}$ is a real number giving the eigenvalues of the quadratic Casimir of $K_1^o$, which is $\mathcal{C}=X\,P-I\,D$. To study the dilations given by negative $k$, we have to introduced the parity operator $\mathcal{P}:x\to -x$. Thus, when the operator $\mathcal{P}$ enters into the game and is realized as a unitary operator, a UIR of $K_1$ acts on the continuous spatial bases as

$$\mathcal{U}_{h,\mathcal{C}}(\tilde{g},\alpha)\,|x\rangle = \mathcal{U}_{h,\mathcal{C}}(\tilde{g})\,|x^\alpha\rangle = e^{d/2}\,e^{ih(c+\mathcal{C})}\,e^{iha(e^d\,x^a+b/2)}\,|e^d\,x^\alpha+b\rangle\,. \tag{18}$$

Here, $\alpha$ stands either for an identity ($x^\alpha\equiv x^{\mathcal{I}}=x$) or for the parity ($x^\alpha\equiv x^{\mathcal{P}}=-x$). While the elements of the form $(\tilde{g},\mathcal{I})$ belong to $K_1^o$, those of the form $(\tilde{g},\mathcal{P})$ belong to the second connected component of $K_1$. We can rewrite (18) in terms of $k\in R^*$ with $|k|=e^d$ and $d\in\mathbb{R}$

$$\mathcal{U}_{h,\mathcal{C}}(c,a,b,k)\,|x\rangle = \sqrt{|k|}\,e^{ih(c+\mathcal{C})}\,e^{iha(k\,x+b/2)}\,|k\,x+b\rangle\,. \tag{19}$$

The corresponding action on the functions of $L^2(\mathbb{R})$ is given by

$$(\mathcal{U}_{h,C}(\tilde{g},\alpha)f)(x) = \frac{1}{\sqrt{|k|}}\, e^{ih(c+C)}\, e^{i\,h\,a(x-b/2)}\; f\Big(k^{-1}\,(x-b)\Big)\,. \tag{20}$$

As is well known, the Hermite functions $\{\psi_m(x)\}_{m\in\mathbb{N}}$ and their Fourier transforms $\{\psi_m(p)\}_{m\in\mathbb{N}}$, which are also Hermite functions satisfying the properties,

$$FT[\psi_m(x), x, p] = i^m\, \psi_m(p)\;, \quad IFT[\psi_m(p), p, x] = (-i)^m\, \psi_m(x)\,. \tag{21}$$

are complete orthonormal sets (bases) in $L^2(\mathbb{R})$, [18,19]. We recall that, regardless of the complex character of the Hilbert space $L^2(\mathbb{R})$, all Hermite functions are real.

The invariance properties of $K_1$ are shown by a generalization of the Hermite functions obtained using the UIR's of $K_1$. The explicit form of these generalized Hermite functions in the coordinate representation is

$$\chi_m(x,a,b,k) := |k|^{1/2}\; e^{-ia(k\,x+b/2)}\, \psi_m(k\,x+b)\,, \qquad a,b\in\mathbb{R}\,,\; k\in\mathbb{R}^*\,, \tag{22}$$

similar to the momentum representation. We have obtained two sequences of functions depending on three parameters, which we denote here as $\{\chi_m(x,a,b,k)\}$ and $\{\chi_m(p,a,b,k)\}$ for all fixed values of $k\neq 0, a, b\in\mathbb{R}$. The well-known orthonormal and completeness relations of the Hermite functions produce similar relations for these generalized Hermite functions, so these families of functions are orthonormal bases in $L^2(\mathbb{R})$.

However, these generalized Hermite functions are not eigenfunctions of the Fourier transform and its inverse, contrarily to the ordinary Hermite functions (21), since they transform under the Fourier transform as its inverse:

$$\begin{aligned} FT[\chi_m(x,,a,b,k), x, p] &= i^m\, \chi_m(p,b,-a,k^{-1})\,, \\ IFT[\chi_m(p,a,b,k), p, x] &= (-i)^m\, \chi_m(x,-b,a,k^{-1})\,. \end{aligned} \tag{23}$$

## 3. Euclidean and Pseudo-Euclidean Plane Cases

In this section, we consider a generalization of our results relative to the analysis on the real line. In this case, we enter 2D configuration spaces, which are either the Euclidean plane $\mathbb{R}^2$ and the pseudo-Euclidean plane $\mathbb{R}^{1,1}$. We study both cases separately.

### *3.1. The Groups $H_2$ and $K_2$ on the Plane*

The Heisenberg–Weyl group on 2D, $H_2$, admits a representation by real $4\times 4$ upper triangular matrices as follows:

$$H_2[\mathbf{a},\mathbf{b},c] = \begin{pmatrix} 1 & \mathbf{a}^T & c \\ \mathbf{0} & \mathbb{I}_2 & \mathbf{b} \\ 0 & \mathbf{0}^T & 1 \end{pmatrix} \equiv \begin{pmatrix} 1 & a_1 & a_2 & c \\ 0 & 1 & 0 & b_1 \\ 0 & 0 & 1 & b_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad a_1,a_2,b_1,b_2,c\in\mathbb{R}\,. \tag{24}$$

As with the 1D case studied in the previous section, the 2D Heisenberg–Weyl group, $H_2$, can be extended by adding the group of proper rotations $SO(2)$ and the dilations on the plane, $\mathbb{R}^*$, to obtain the group $K_2$, which admits a representation by real $4\times 4$ matrices given by

$$K_2[\mathbf{a},\mathbf{b},c,k,R(\theta)] = \begin{pmatrix} 1 & \mathbf{a}^T & c \\ \mathbf{0} & k\,R(\theta) & \mathbf{b} \\ 0 & \mathbf{0}^T & 1 \end{pmatrix}, \quad R(\theta)\in SO(2)\,, k\in\mathbb{R}^*\,, \tag{25}$$

with

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \qquad \theta \in [0, 2\pi). \tag{26}$$

The group law, obtained by matrix multiplication, is given by

$$K_2(\mathbf{a}, \mathbf{b}, c, k, R) \cdot K_2(\mathbf{a}', \mathbf{b}', c', k', R') = K_2(\mathbf{a}' + k' R'^T \mathbf{a},\ \mathbf{b} + kR\,\mathbf{b}',\ c + c' + \mathbf{a}\cdot\mathbf{b}',\ kk',\ R\,R'), \tag{27}$$

where $R\,R' \equiv R(\theta)\,R(\theta') = R(\theta + \theta')$. The identity element is $\mathrm{Id} = K_2(\mathbf{0}, \mathbf{0}, 0, 1, \mathbb{I}_2)$, which is the identity matrix on $GL(4)$. The inverse of $K_2(\mathbf{a}, \mathbf{b}, c, k, R)$ is

$$K_2(\mathbf{a}, \mathbf{b}, c, k, R)^{-1} = K_2(-k^{-1}\,R\,\mathbf{a}, -k^{-1}\,R^{-1}\,\mathbf{b}, -c + k^{-1}\,\mathbf{a}\cdot R^{-1}\,\mathbf{b}, -k^{-1}, R^{-1}). \tag{28}$$

### 3.2. The Groups $H_{1,1}$ and $K_{1,1}$ on the Pseudo-Plane

Another interesting generalization of the 2D Heisenberg–Weyl group, $H_2$, can be obtain by replacing the Eucliden plane $\mathbb{R}^2$ with the pseudo-Euclidean plane $\mathbb{R}^{(1,1)}$ using the metric of signature $(+,-)$, obtaining $H_{1,1}$, which formally is like $H_2$ (24). Thus, we may obtain the group $K_{1,1}$ from $H_{1,1}$ by adding the connected component of the identity of $O(1,1)$ and $SO_0(1,1)$, and the dilations $\mathbb{R}^*$. Hence, we have that

$$K_{1,1}[\mathbf{a}, \mathbf{b}, c, k, \Lambda(\eta)] = \begin{pmatrix} 1 & \mathbf{a}^T & c \\ \mathbf{0} & k\,\Lambda(\eta) & \mathbf{b} \\ 0 & \mathbf{0}^T & 1 \end{pmatrix}, \quad \Lambda(\eta) \in SO_0(1,1)\,, k \in \mathbb{R}^*\,, \tag{29}$$

with

$$\Lambda(\eta) = \begin{pmatrix} \cosh\eta & \sinh\eta \\ \sinh\eta & \cosh\eta \end{pmatrix}, \qquad \eta \in \mathbb{R}\,. \tag{30}$$

The group law is given by an expression similar to (27) provided that rotations $R$ are replaced by pseudo-rotations $\Lambda$. The inverse of $K_{1,1}[\mathbf{a}, \mathbf{b}, c, k, \Lambda(\eta)]$ may be computed similarly as in (28).

### 3.3. The Lie Algebras of $K(2)$ and $K(1,1)$

At this point, we move the discussion from the Lie group $K(2)$ and $K(1,1)$ to their respective Lie algebras $\mathcal{K}_2$ and $\mathcal{K}_{1,1}$. Both algebras are 7D with infinitesimal generators given by $X_1$, $X_2$, $P_1$, $P_2$, $I$, $D$ and either $J$ or $K$ if the algebra is either $\mathcal{K}_2$ or $\mathcal{K}_{1,1}$, respectively. Note that the generator $J$ come from the Lie algebra $so(2)$ and the generator $K$ from $so(1,1)$. A $4 \times 4$ realization of these generators is the following:

$$\begin{aligned} X_\alpha &= \left.\frac{\partial K_-}{\partial a^\alpha}\right|_{I_d} = \begin{pmatrix} 0 & \boldsymbol{\alpha}^T & 0 \\ \mathbf{0} & \mathbb{O}_2 & 0 \\ 0 & \mathbf{0}^T & 0 \end{pmatrix}, \quad P_\alpha = \left.\frac{\partial K_-}{\partial b^\alpha}\right|_{I_d} = \begin{pmatrix} 0 & \mathbf{0}^T & 0 \\ \mathbf{0} & \mathbb{O}_2 & \boldsymbol{\alpha} \\ 0 & \mathbf{0}^T & 0 \end{pmatrix}, \\ I &= \left.\frac{\partial K_-}{\partial c}\right|_{I_d} = \begin{pmatrix} 0 & \mathbf{0}^T & c \\ \mathbf{0} & \mathbb{O}_2 & 0 \\ 0 & \mathbf{0}^T & 0 \end{pmatrix}, \quad D = \left.\frac{\partial K_-}{\partial k}\right|_{I_d} = \begin{pmatrix} 0 & \mathbf{0}^T & 0 \\ \mathbf{0} & \mathbb{I}_2 & 0 \\ 0 & \mathbf{0}^T & 0 \end{pmatrix}, \end{aligned} \tag{31}$$

where $\boldsymbol{\alpha}$ is either the column vector $(1,0)^T$ for $\alpha = 1$ or $(0,1)^T$ for $\alpha = 2$, and $\mathbb{O}_2$ is the $2 \times 2$ zero matrix. In comparison with the 2D case studied in the previous subsection, we replace the symbol $K_2[\mathbf{a}, \mathbf{b}, c, k, R(\theta)]$ in (25) with $K_-$ in (31) if the considered Lie algebra is $\mathcal{K}_2$. If the algebra were instead $\mathcal{K}_{1,1}$, then $K_-$ replaces to $K_{1,1}[\mathbf{a}, \mathbf{b}, c, k, \Lambda(\eta)]$ in (29).

Since $\mathcal{K}_2$ is an Euclidean algebra with metric signature $(+,+)$, covariant and contravariant coordinates coincide on it. This is not the case for the algebra $\mathcal{K}_{1,1}$, which has a signature $(+,-)$. Here, the contravariant, $x^\mu$, and the covariant, $x_\mu$, coordinates are related via the metric tensor $g_{\mu\nu}$ in the sense that $x_\mu = g_{\mu\nu}x^\nu$, so that $X_1 = X^1$, $X_2 = -X^2$ and analogously for $P_1$ and $P_2$. Finally, the $4 \times 4$ matrix representation of the other two generators $J$ of $\mathcal{K}_2$ and $K$ of $\mathcal{K}_{1,1}$ is

$$J = \frac{\partial K_-}{\partial \theta}\Big|_{Id} = \begin{pmatrix} 0 & \mathbf{0}^T & 0 \\ \mathbf{0} & -i\,\sigma_2 & \mathbf{0} \\ 0 & \mathbf{0}^T & 0 \end{pmatrix}, \qquad K = \frac{\partial K_-}{\partial \eta}\Big|_{Id} = \begin{pmatrix} 0 & \mathbf{0}^T & 0 \\ \mathbf{0} & \sigma_1 & \mathbf{0} \\ 0 & \mathbf{0}^T & 0 \end{pmatrix}, \tag{32}$$

where $\sigma_i$ are Pauli matrices.

From (31) and (32), we may obtain the commutation relations for the algebra elements. For those belonging to both algebras $\mathcal{K}_2$ and $\mathcal{K}_{1,1}$, the commutators are

$$[X_\alpha, P_\beta] = \delta_{\alpha\beta} I, \quad [D, X_\alpha] = -X_\alpha, \quad [D, P_\alpha] = +P_\alpha\,. \tag{33}$$

In addition to these common generators, we have to add $J$ to $\mathcal{K}_2$ with commutation relations

$$[J, X_\alpha] = \epsilon_{\alpha\beta} X_\beta, \quad [J, P_\alpha] = \epsilon_{\alpha\beta} P_\beta\,, \tag{34}$$

and $K$ to $\mathcal{K}_{1,1}$, with new commutation relations given by

$$[K, X_\alpha] = (-1)^\alpha\,\epsilon_{\alpha\beta} X_\beta, \quad [K, P_\alpha] = (-1)^{\alpha+1}\,\epsilon_{\alpha\beta} P_\beta\,, \tag{35}$$

where $\epsilon_{\alpha\beta}$ is the skew-symmetric tensor.

We conclude this note with an interesting remark. The matrices $\mathcal{P}_1 = \text{Diagonal}[1,-1,1,1]$ and $\mathcal{P}_2 = \text{Diagonal}[1,1,-1,1]$ act on any generator $Y$ of either $\mathcal{K}_2$ and $\mathcal{K}(1,1)$ as $\mathcal{P}_\alpha\, Y\, \mathcal{P}_\alpha^{-1}$. This is $\{X_\alpha, P_\alpha\} \to \{-X_\alpha, -P_\alpha\}$ on each of these two algebras. Observe that $\mathcal{P}_1\mathcal{P}_2 = \text{Diagonal}[1,-1,-1,1]$, and this corresponds to the effect of the negative sign of the parameter $k$ for negative values. Moreover, we have that $\text{Diagonal}[-1,-1] = R(\pi) \in SO(2)$, which is not in $SO_0(1,1)$ but instead in $SO(1,1)$. As a consequence, $\mathcal{K}_2$ has only a connected component while $\mathcal{K}(1,1)$ has two.

The Adjoint Action

The adjoint action of the group $K_2$ on its Lie algebra $\mathcal{K}_2$ is given by $g_2 Y g_2^{-1}$, where $g_2 \equiv K_2[\mathbf{a}, \mathbf{b}, c, d, R(\theta)] \in K_2^o$, $e^d = k$ with $d \in \mathbb{R}$, $R \in SO(2)$ and $Y \in \mathcal{K}_2$. This action produces the following transformations:

$$g_2\,\mathbf{X}\,g_2^{-1} = \; e^{-d}\,R^{-1}\mathbf{X} - e^{-d}\,R^{-1}\mathbf{b}\,I\,, \qquad g_2\,\mathbf{P}\,g_2^{-1} = \; e^d\,R^{-1}\mathbf{P} + \mathbf{a}\,I\,. \tag{36}$$

where $\mathbf{Y} = (Y_1, Y_2)^T$.

Analogously, the adjoint action of $K_{1,1}^0$ on $\mathbf{X}\,, \mathbf{P} \in \mathcal{K}_{1,1}$ is explicitely given by

$$g_{1,1}\,\mathbf{X}\,g_{1,1}^{-1} = \; e^{-d}\,\Lambda^{-1}\mathbf{X} - e^{-d}\,\Lambda^{-1}\mathbf{b}\,I\,, \qquad g_{1,1}\,\mathbf{P}\,g_{1,1}^{-1} = \; e^d\,\Lambda\mathbf{P} + \mathbf{a}\,I\,, \tag{37}$$

where, now, $g_{1,1} \equiv K_{1,1}^0[\mathbf{a}, \mathbf{b}, c, d, \Lambda(\eta)] \in K_{1,1}^0$ and $\Lambda \in SO_0(1,1)$.

Note that $g_2 = e^{cI}\, e^{\mathbf{b}\cdot\mathbf{P}}\, e^{\theta J} e^{dD}\, e^{\mathbf{a}\cdot\mathbf{X}}$ and similarly for $g_{1,1}$, after the replacement $e^{\theta J} \to e^{\eta K}$.

### *3.4. Bases on the Plane and the Hyperplane*

To begin with, let us consider two 2D real vector spaces $\mathbb{R}^2$ and $\mathbb{R}^{1,1}$. On these spaces, we defined respective metrics with respective signatures $(+,+)$ and $(+,-)$. Let us consider the Hilbert spaces $L^2(\mathbb{R}^2)$ and $L^2(\mathbb{R}^{1,1})$, on which we define the coordinate or position operators $\mathbf{X} \equiv (X_1, X_2)$ and their conjugate momentum operators $\mathbf{P} \equiv (P_1, P_2)$. These operators act on the generalized eigenvectors $|\mathbf{x}\rangle \equiv |x_1, x_2\rangle = |x_1\rangle \otimes |x_2\rangle$ and $|\mathbf{p}\rangle \equiv |p_1, p_2\rangle = |p_1\rangle \otimes |p_2\rangle$, respectively, as (see Section 5)

$$X_\alpha\,|\mathbf{x}\rangle = x_\alpha |\mathbf{x}\rangle\,, \qquad P_\alpha\,|\mathbf{p}\rangle = p_\alpha |\mathbf{p}\rangle\,, \qquad \alpha = 1,2\,. \tag{38}$$

These generalized eigenvectors are transformed into each other by means of Fourier type transformations (10) such as

$$|\mathbf{p}\rangle = \frac{1}{2\pi}\int_{\mathbb{R}^2|\mathbb{R}^{1,1}} d\mathbf{x}\, e^{i\mathbf{p}\cdot\mathbf{x}}\, |\mathbf{x}\rangle\,, \qquad |\mathbf{x}\rangle = \frac{1}{2\pi}\int_{\mathbb{R}^2|\mathbb{R}^{1,1}} d\mathbf{p}\, e^{-i\mathbf{p}\cdot\mathbf{x}}\, |\mathbf{p}\rangle\,. \tag{39}$$

The scalar product $\mathbf{p}\cdot\mathbf{x}$ depends on the basic space considered, either $\mathbb{R}^2$ or $\mathbb{R}^{1,1}$ since they have different metrics.

As for the 1D case (12), we have similar relations

$$e^{-i\mathbf{b}\cdot\mathbf{P}}\, |\mathbf{x}\rangle = |\mathbf{x}+\mathbf{b}\rangle\,, \qquad e^{-i\mathbf{a}\cdot\mathbf{X}}\, |\mathbf{p}\rangle = |\mathbf{p}-\mathbf{a}\rangle\,. \tag{40}$$

Hence, the continuous basis $\{|\mathbf{x}\rangle\}$ is equivalent to the continuous basis $\{|\mathbf{x}+\mathbf{b}\rangle\}$ with $\mathbf{b}\in\mathbb{R}^2$ (or $\mathbb{R}^{1,1}$). The same result is valid with respect to the bases in the momentum representation, $\{|\mathbf{p}\rangle\}$ and $\{|\mathbf{p}-\mathbf{a}\rangle\}$.

The use of the 2D Fourier transform serves us to realize that the five operators given by $\mathbf{X}$, $\mathbf{P}$, and $I$ determine a UIR representation of $H_2$ or $H_{1,1}$ by exponentiation. As in the 1D case, where the basic linear space is the line $\mathbb{R}$, we may add some other invariances to those of $H_2$ or $H_{1,1}$. Thus, let us consider a transformation on $\mathbb{R}^2$ or in $\mathbb{R}^{1,1}$ of the type $K_-^0[\mathbf{a},\mathbf{b},\Lambda,d]$, where $\Lambda$ is either a rotation $R(\theta)$ on the Euclidean plane $\mathbb{R}^2$ or pseudo-rotation $\Lambda(\eta)$ on the pseudo-Euclidean plane $\mathbb{R}^{1,1}$. Then, if $|\mathbf{x}'\rangle$ and $|\mathbf{p}'\rangle$ are the transformed generalized vectors resulting from these transformations on coordinates and momenta, see (20), we have that

$$|\mathbf{x}\rangle \xrightarrow{K_-[\mathbf{a},\mathbf{b},c,\Lambda,d]} |\mathbf{x}'\rangle := |e^d\,\Lambda\mathbf{x}+\mathbf{b}\rangle\,, \qquad |\mathbf{p}\rangle \xrightarrow{K_-[\mathbf{a},\mathbf{b},c,\Lambda,d]} |\mathbf{p}'\rangle := |e^{-d}\,\Lambda^{-1T}\mathbf{p}-\mathbf{a}\rangle\,. \tag{41}$$

Let $\mathcal{H}$ be an abstract infinite dimensional separable Hilbert space and $S:\mathcal{H}\longmapsto L^2(\mathbb{R}^2)$, or alternatively, $S:\mathcal{H}\longmapsto L^2(\mathbb{R}^{1,1})$, a unitary map. Let $|f\rangle\in\mathcal{H}$ and $U|f\rangle = f(x)$. Following the ideas developed in [9] or Section 5, we have the following decomposition for vectors $|f\rangle$ in a suitable dense subspace of $\mathcal{H}$:

$$|f\rangle = \int_{\mathbb{R}^2|\mathbb{R}^{1,1}} d\mathbf{x}\, f(\mathbf{x})\, |\mathbf{x}\rangle\,, \qquad f(\mathbf{x}) = \langle\mathbf{x}|f\rangle\,. \tag{42}$$

The action of the extended groups $K_2$ and $K_{1,1}$ on the generalized kets $|\mathbf{x}'\rangle$ and $|\mathbf{p}'\rangle$, given by (41) permits us to calculate the action of a UIR of these groups, $U(g)$, on functions of $L^2(\mathbb{R}^2)$ or $L^2(\mathbb{R}^{1,1})$, which is

$$(U(g)f)(\mathbf{x}) = f(k^{-1}\Lambda^{-1}(\mathbf{x}-\mathbf{b})) \qquad (U(g)f)(\mathbf{p}) = f(k\Lambda^T(\mathbf{p}+\mathbf{a}))\,. \tag{43}$$

Let us recall that, for $K(2)$, we have $\Lambda\equiv R$ and $(\Lambda^{-1})^T=\Lambda$, while for $K_{1,1}$, we have instead $(\Lambda^{-1})^T = G\Lambda G^{-1}$, where $G$ is the matrix associated to the metric tensor.

### *3.5. Based on Functions Defined in the Euclidean and Pseudo-Euclidean Planes*

Let $\{\psi_\alpha(x_\alpha)\}$ be the set of all Hermite functions on $\mathbb{R}$, which form an orthonormal basis on $L^2(\mathbb{R})$. Consequently, the set of functions

$$\Psi_{\mathbf{m}}(\mathbf{x}) := \psi_{m_1}(x^1)\,\psi_{m_2}(x^2)\,, \qquad \mathbf{m}=(m_1,m_2)\in\mathbb{N}^2\,, \tag{44}$$

is a orthonormal complete set (orthonormal basis) on $L^2(\mathbb{R}^2)$ or $L^2(\mathbb{R}^{1,1})$. Consequently, for any $f(x)\in L^2(\mathbb{R}^2)$ ($L^2(\mathbb{R}^{1,1})$), we have that

$$f(\mathbf{x}) = \sum_{\mathbf{m}\in\mathbb{N}^2}^{\infty} c^{\mathbf{m}}\,\Psi_{\mathbf{m}}(\mathbf{x}) = \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty} c^{m_1,m_2}\;\psi_{m_1}(x^1)\,\psi_{m_2}(x^2)\,, \quad c^{m_1,m_2}\in\mathbb{C}\,. \tag{45}$$

In addition, functions $\Psi_{\mathbf{n}}(\mathbf{x})$ verify the following normalization and completeness relations, which can be easily obtained from the corresponding relations of the 1D case,

$$\int_{\mathbb{R}^2} d\mathbf{x}\, [\Psi_{\mathbf{m}'}(\mathbf{x})]^* \,\Psi_{\mathbf{m}}(\mathbf{x}) = \delta_{\mathbf{m},\mathbf{m}'} \equiv \delta_{m_1,m_1'}\,\delta_{m_2,m_2'}\,,$$
$$\sum_{\mathbf{m}\in\mathbb{N}^2} [\Psi_{\mathbf{m}}(\mathbf{x})]^* \,\Psi_{\mathbf{m}}(\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y}) \equiv \delta(x^1-y^1)\,\delta(x^2-y^2)\,. \tag{46}$$

We may call the functions $\Psi_{\mathbf{n}}(\mathbf{x})$, the *double Hermite functions or the 2D Hermite functions*. A symmetrized version of these $n$D Hermite functions can be found in Ref. [20] (see also [21]). They are real functions and eigenfunctions of the Fourier transform and of its inverse, i.e.,

$$\begin{aligned} FT[\Psi_{\mathbf{m}}(\mathbf{x});\mathbf{x};\mathbf{p}] &= \prod_{\alpha=1}^{2} FT[\psi_{m_\alpha}(x^\alpha),x^\alpha,p^\alpha] = i^{\widetilde{\mathbf{m}}}\,\Psi_{\mathbf{m}}(\mathbf{p})\,,\\ IFT[\Psi_{\mathbf{m}}(\mathbf{p});\mathbf{p};\mathbf{x}] &= \prod_{\alpha=1}^{2} IFT[\psi_{m_\alpha}(p^\alpha),p^\alpha,x^\alpha] = (-i)^{\widetilde{\mathbf{m}}}\,\Psi_{\mathbf{m}}(\mathbf{x})\,. \end{aligned} \tag{47}$$

where $\widetilde{\mathbf{m}} := \sum_\alpha m_\alpha$.

As we have proceeded with the 1D Hermite functions, we use the invariance properties to 2D Hermite functions to construct a representation of the groups $K_2$ and $K_{1,1}$ supported on a kind of *generalized* Hermite functions, to be defined next. To begin with this construction, let us define the following functions:

$$\mathfrak{X}_{\mathbf{m}}(\mathbf{x},\mathbf{a},\mathbf{b},k,\Lambda) := |k|\, e^{-i\,\mathbf{a}(k\Lambda+\mathbf{b}/2)}\Psi_{\mathbf{m}}(k\Lambda\mathbf{x}+\mathbf{b})\,. \tag{48}$$

Using (22), (44) and (48), we obtain an explicit form of the 2D generalized Hermite functions in terms of the 1D generalized Hermite functions, $\chi_m(x,a,b,k)$, as

$$\mathfrak{X}_{\mathbf{m}}(\mathbf{x},\mathbf{a},\mathbf{b},k,\Lambda) = \chi_{m_1}((\Lambda\mathbf{x})^1,a^1,b^1,k)\,\chi_{m_2}((\Lambda\mathbf{x})^2,a^2,b^2,k)\,, \tag{49}$$

where $(\Lambda\mathbf{x})^\alpha$ denotes the $\alpha$-th contravariant component of the vector $\Lambda\mathbf{x}$. The 2D generalized Hermite functions (48) determine a complete orthonormal set (orthonormal basis) on both $L^2(\mathbb{R}^2)$ (or $L^2(\mathbb{R}^{1,1})$). In fact, it is very simple to show that

$$\int_{\mathbb{R}^2} d\mathbf{x}\, \mathfrak{X}_{\mathbf{m}}(\mathbf{x},\mathbf{a},\mathbf{b},k,\Lambda)\,[\mathfrak{X}_{\mathbf{m}'}(\mathbf{x},\mathbf{a},\mathbf{b},k,\Lambda)]^* = \delta_{\mathbf{m},\mathbf{m}'}\,,$$
$$\sum_{\mathbf{m}\in\mathbb{N}^2} \mathfrak{X}_{\mathbf{m}}(\mathbf{x},\mathbf{a},\mathbf{b},k,\Lambda)\,[\mathfrak{X}_{\mathbf{m}}(\mathbf{y},\mathbf{a},\mathbf{b},k,\Lambda)]^* = \delta(\mathbf{x}-\mathbf{y})\,. \tag{50}$$

In addition, for the Fourier transform in 2D and its inverse, respectively, we have the following relations:

$$\begin{aligned} FT[\mathfrak{X}_{\mathbf{m}}(\mathbf{x},\mathbf{a},\mathbf{b},k,\Lambda)\,;\mathbf{x},\mathbf{p}] &= \prod_{\alpha=1}^{2} FT[\chi_{m_\alpha}(x^\alpha,a^\alpha,b^\alpha),x^\alpha,p^\alpha]\\ &= i^{\widetilde{\mathbf{m}}}\,[\mathfrak{X}_{\mathbf{m}}(\mathbf{p},\mathbf{b},-\mathbf{a},k^{-1},\Lambda^{-1T})\,,\\ IFT[\mathfrak{X}_{\mathbf{m}}(\mathbf{p},\mathbf{a},\mathbf{b},k,\Lambda)\,;\mathbf{p};\mathbf{x}] &= \prod_{i=\alpha}^{2} IFT[\chi_{m_\alpha}(p^\alpha,a^\alpha,b^\alpha);p^\alpha,x^\alpha]\\ &= (-i)^{\widetilde{\mathbf{m}}}\,\mathfrak{X}_{\mathbf{m}}(\mathbf{x},\mathbf{b},-\mathbf{a},k^{-1},\Lambda^{-1T})\,, \end{aligned} \tag{51}$$

which closes the discussion in 2D.

### *3.6. Free Relativistic Particle in One Dimension*

In this subsection, we intend to lift to the relativistic context our comments on Section 3.3. For simplicity, we refer here just to the Lie algebra, $k_{1,1}$, of the group $K_{1,1}$ with the same infinitesimal generators fulfilling the same commutation relations. Nevertheless, in order to underline the relativistic character of the discussion, we introduce the following notation for these generators: $\{X, T, P, E, I, D, K\}$, so that we have called $T$ and $E$ to $X_2$ and $P_2$, respectively, in order to underline their respective character of "time" and energy. We keep $X_1$ as $X$ and $P_1$ as $P$. We use the notation in Section 3.3 for the remainder generators. The discussion may be lifted to $p$ spatial dimensions without further conceptual complications.

In the Introduction, we listed four invariance options for the description of the physical world. In the realm of special relativity, we drop the homogeneity on the space of momenta and the self-similarity. These are not invariant in special relativity due to the zero point energy and the mass, which fixes the origin and the scale of the energy.

The group of invariance of the special relativity is the subgroup of $K_{1,1}$, which is obtained by eliminating dilations of impulses and self-similarity. We call $\widetilde{K}_{1,1}$ the resulting subgroup. It admits a $4 \times 4$ matrix representation as in (29), with $\mathbf{a} = 0$ and $k = 1$. The adjoint action of $g \in \widetilde{K}_{1,1}$ on $\{X, P\}$ and $\{T, E\}$ has essentially been given in (37), choosing $\mathbf{a} = 0$ and $k = 1$. Explicitly,

$$gXg^{-1} = \gamma X - \beta\gamma T - b_1' I\,, \qquad gTg^{-1} = \gamma T - \beta\gamma X - b_2' I\,, \tag{52}$$

$$gPg^{-1} = \gamma P + \beta\gamma X\,, \qquad gEg^{-1} = \gamma E + \beta\gamma P\,. \tag{53}$$

Here, $\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$ as usual in special relativity.

It is noteworthy that the representations of $\widetilde{K}_{1,1}$ contains the representations of the one dimensional Lorentz group on the space of impulses (or momenta) $\{E, P\}$, as a matter of fact $(gEg^{-1})^2 - (gPg^{-1})^2 = E^2 - P^2$, and the representation of the one dimensional Poincarè group on the "configuration space" $\{X, T\}$. As is well known, the unitary representations of a group are also unitary representations (possibly reducible) of their subgroups.

The generators $\{X, P\}$ and $\{T, E\}$ may be represented by self-adjoint operators on $L^2(\mathbb{R}^{1,1})$, with orthonormal basis given in (49), which generate a unitary irreducible representation of $K_{1,1}$ satisfying the exponential commutation relations,

$$e^{ia_1X}\, e^{ib_1P} = e^{-ia_1b_1}\, e^{ib_1P}\, e^{ia_1X}\,, \quad e^{ia_2T}\, e^{ib_2E} = e^{-ia_2b_2}\, e^{ib_2E}\, e^{ia_2P}\,. \tag{54}$$

Taking into account the Stone–von Neumann Theorem, relations (54) are equivalent to the commutation relations $[X, P] = I$ and $[T, E] = I$, which are properly defined on respective dense subspaces in $L^2(\mathbb{R}^{1,1})$, save for a unitary equivalence. Note that all operators $\{X, TP, T, E\}$ are self adjoint on this space. With the equivalence between these operators and the generators on Section 3.3, we can say that they admit matrix representations such as those given in (31).

## 4. The $n$-Dimensional Case

The purpose of the present section is the generalization of the contents of previous sections to the $n$-dimensional ($n$D) case.

### *4.1. The Pseudo-Orthogonal Groups $O(p,q)$ and Some of Their Extensions*

We begin with the $p+q$ dimensional, $(p+q)$D, real vector space $\mathbb{R}^{p,q}$, which is the real, $n$D, vector space $\mathbb{R}^n$, $n = p+q$, endowed with the quadratic form (pseudometric) defined for any pair $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ as

$$\langle \mathbf{x}, \mathbf{y} \rangle = g_{\alpha\beta}\, x^\alpha\, x^\beta = x_\alpha\, y^\beta\,, \qquad \alpha, \beta = 1, 2, \ldots, n\,, \tag{55}$$

where $g_{\alpha\beta}$ are the components of the metric tensor $G$ with signature $(\overbrace{+,+,\ldots,+}^{p},\overbrace{-,-,\ldots,-}^{q})$.

The Lie group of linear transformations leaving invariant this quadratic form is $O(p,q)$. Needless to say that, for $q = 0$, we recover the orthogonal group $O(n)$. The groups $O(p,q)$ and $SO(p,q)$ are not connected as they have four and two connected components, respectively. Recall that $O(n)$ $(q = 0)$ has just two connected components. For each $\Lambda \in O(p,q)$, the invariance of the pseudometrics $G$ with respect to the group $O(p,q)$ means that

$$\Lambda^T G \Lambda = G\,. \tag{56}$$

The connected component of the identity is the subgroup $SO_0(p,q)$ with Lie algebra $so_o(p,q)$, so that each $\Lambda \in SO_o(p,q)$ has the form $\Lambda = e^{tA}$ with $A \in so_0(p,q)$ and

$$A^T G + G A = 0\,, \tag{57}$$

so that $GA$ is antisymmetric. The elements $A$ of the orthogonal Lie algebras $so(n)$ are antisymmetric, since here, $G = \mathbb{I}_n$. In the general case, $so_0(p,q)$, we have

$$G \cdot A = \begin{pmatrix} \mathbb{I}_p & O \\ \mathbb{O} & -\mathbb{I}_q \end{pmatrix} \cdot \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ -A_{21} & -A_{22} \end{pmatrix} = \begin{pmatrix} -A_{11}^T & -A_{21}^T \\ -A_{12}^T & -A_{22}^T \end{pmatrix}, \tag{58}$$

where $A_{11}$ and $A_{22}$ are submatrices of dimensions $p^2$ and $q^2$, respectively. The submatrices $A_{12}$ and $A_{21}$ have dimensions $q \times p$ and $p \times q$, respectively. The remaining submatrices $A_{11}$ and $A_{22}$ are antisymmetric and correspond to rotations. We also have $A_{21} = A_{12}^T$ a characteristic of pseudo-rotations.

The dimension of $so(p,q)$ is $n(n-1)/2$ with $n = p+q$. A representation of each element of a basis of the Lie algebra $so_0(p,q)$ is given by all matrices $M_{\alpha\beta}$ with the property

$$\left(M_{\alpha\beta}\right)^{\gamma}_{\delta} = -g^{\gamma}_{\alpha}\, g_{\beta\delta} + g^{\gamma}_{\beta}\, g_{\alpha\delta}\;, \qquad \alpha < \beta\,,\ \alpha,\beta = 1,2,\ldots,n\,, \tag{59}$$

with $g^{\beta}_{\alpha} = \delta_{\alpha\beta}$. Each of the $\left(M_{\alpha\beta}\right)^{\gamma}_{\delta}$ is the infinitesimal generator of a pseudo-rotation on the plane $\alpha\beta$ fulfilling $M_{\alpha\beta} = -M_{\beta\alpha}$. Their commutations relations are given by

$$[M_{\alpha\beta}, M_{\gamma\delta}] = -\left(g_{\beta\gamma}\, M_{\alpha\delta} - g_{\alpha\gamma}\, M_{\beta\delta} - g_{\beta\delta}\, M_{\alpha\gamma} + g_{\alpha\delta}\, M_{\beta\gamma}\right), \quad \alpha < \delta\,,\ \beta < \gamma\,. \tag{60}$$

Formulas (59) and (60) are also valid for $so(n)$ with $G = \mathbb{I}_n$. Since $so(n)$ is a particular case of $so_0(p,q)$, we always use the notation $so_0(p,q)$ in both cases in the sequel.

*4.2. On the Groups Related to $E(p,q)$*

Let $T_{p,q}$ be the group of translations isomorphic to $\mathbb{R}^{p,q}$. Then, we may consider the semidirect product $E(p,q) = T_{p,q} \odot SO_0(p,q)$. The action of each element $(\mathbf{a},\Lambda)$ of $E(p,q)$, with $\Lambda \in SO_0(p,q)$ and $\mathbf{a} \in T_{p,q}$, on $\mathbf{x} \in \mathbb{R}^{p,q}$, is given by $(\mathbf{a},\Lambda)\,\mathbf{x} := \Lambda\,\mathbf{x} + \mathbf{a}$. The multiplication law on $E(p,q)$ is

$$(\mathbf{a}',\Lambda') \cdot (\mathbf{a},\Lambda) = (\mathbf{a}' + \Lambda'\,\mathbf{a}, \Lambda'\,\Lambda)\,. \tag{61}$$

The Lie algebra $e(p,q)$ contains the $n(n-1)/2$ generators $M_{\alpha\beta}$ of $so_0(p,q)$ shown in (59) plus $n$ generators $P_\alpha$ associated to the translations. The commutators of $e(p,q)$ are, in addition to those in (60), those involving translations, i.e.,

$$[P_\alpha, P_\beta] = 0\,, \quad [M_{\alpha\beta}, P_\gamma] = \left(g_{\alpha\gamma}\, g_{\beta\delta} - g_{\alpha\delta}\, g_{\beta\gamma}\right) P_\delta = g_{\alpha\gamma}\, P_\beta - g_{\beta\gamma}\, P_\alpha\,. \tag{62}$$

There is an interesting property according to which we may imbed $E(p,q)$ as subgroup of $GL(n+1)$ so that the left affine action $\mathbf{x}' = (\mathbf{a},\Lambda)\,\mathbf{x} := \Lambda\,\mathbf{x} + \mathbf{a}$ becomes linear:

$$\mathbf{x}' = (\mathbf{a},\Lambda)\,\mathbf{x} \equiv \begin{pmatrix} \Lambda & \mathbf{a} \\ \mathbf{0}^T & 1 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} = \begin{pmatrix} \Lambda\,\mathbf{x} + \mathbf{a} \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{x}' \\ 1 \end{pmatrix}. \tag{63}$$

The right action and the right affine action of $E(p,q)$ are given by, respectively,

$$\left(\mathbf{x}^T\right)' = \mathbf{x}^T\,\Lambda, \qquad \left(\mathbf{x}^T\right)' = \mathbf{x}^T\,\Lambda + \mathbf{a}^T\,. \tag{64}$$

From this point of view, we may write the multiplication law as

$$(\mathbf{a},\Lambda)\cdot(\mathbf{a}',\Lambda') = (\mathbf{a}' + \mathbf{a}\,\Lambda', \Lambda\,\Lambda')\,, \tag{65}$$

and the embedding of $E(p,q)$ into $GL(n+1)$ gives now the following action:

$$\mathbf{x}^T\,(\mathbf{a},\Lambda) \equiv \begin{pmatrix} 1 & \mathbf{x}^T \end{pmatrix} \begin{pmatrix} 1 & \mathbf{a}^T \\ \mathbf{0} & \Lambda \end{pmatrix}. \tag{66}$$

By transposing (66), we obtain

$$\mathbf{x}' = \begin{pmatrix} 1 & \mathbf{0}^T \\ \mathbf{a} & \Lambda^T \end{pmatrix} \begin{pmatrix} 1 \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} 1 \\ \mathbf{a} + \Lambda^T \mathbf{x} \end{pmatrix} = \begin{pmatrix} 1 \\ \mathbf{x}' \end{pmatrix}. \tag{67}$$

The Lie commutators of $e(p,q)$ from the perspective of the right action are not exactly those from the perspective of the left action. Now, in addition to (60), we have

$$[X_\alpha, X_\beta] = 0\,, \quad [M_{\alpha\beta}, X_\gamma] = \varepsilon_{\alpha\beta}\left(g_{\alpha\gamma}\,g_{\beta\delta} - g_{\alpha\delta}\,g_{\beta\gamma}\right) X_\delta = \varepsilon_{\alpha\beta}\,g_{\alpha\gamma}\,X_\beta - \varepsilon_{\alpha\beta}\,g_{\beta\gamma}\,X_\alpha\,, \tag{68}$$

where $X_\alpha$ are the generators of translations and $\varepsilon_{\alpha\beta} = \pm 1$. The plus sign for $\varepsilon_{\alpha\beta}$ correspond to infinitesimal generators of rotations, while the minus sign corresponds to hyperbolic rotations or pseudo-rotations on the plane $\alpha\beta$. For standard rotations, the commutator (68) coincides with the commutator for left actions (62). For hyperbolic rotations, $(\Lambda^{-1})^T \neq \Lambda$ and a minus sign appears in (68).

Combining left and right affine actions, we arrive at a new subgroup of $GL(n+2)$ with dimension $n(n+3)/2+1$. Each element of this subgroup is given by one $\Lambda \in SO_0(p,q)$, two $\mathbf{a},\mathbf{b} \in \mathcal{T}_{p,q}$, and a new real parameter $c \in \mathbb{R}$ associated with a central charge. Each group element is denoted here either as $(\mathbf{a},\mathbf{b},c,\Lambda)$ or as $H_{p,q}[\mathbf{a},\mathbf{b},c,\Lambda]$ with matrix representation given by

$$(\mathbf{a},\mathbf{b},c,\Lambda) \equiv H_{p,q}[\mathbf{a},\mathbf{b},c,\Lambda] \equiv \begin{pmatrix} 1 & \mathbf{a}^T & c \\ \mathbf{0} & \Lambda & \mathbf{b} \\ 0 & \mathbf{0}^T & 1 \end{pmatrix}, \tag{69}$$

This is the so-called double group of $E(p,q)$ and denoted by ${}^dE(p,q)$ or $K_{p,q}$. If, in addition, we include dilations $k \in \mathbb{R}^*$, we obtain an extended group that we call either $\overline{{}^dE}(p,q)$ or $\overline{K}_{p,q}$. Any group element $(\mathbf{a},\mathbf{b},c,k,\Lambda) \in \overline{{}^dE}(p,q)$ admits the following matrix representation

$$(\mathbf{a},\mathbf{b},c,k,\Lambda) \equiv K_{p,q}[\mathbf{a},\mathbf{b},c,k,\Lambda] \equiv \begin{pmatrix} 1 & \mathbf{a}^T & c \\ \mathbf{0} & k\,\Lambda & \mathbf{b} \\ 0 & \mathbf{0}^T & 1 \end{pmatrix}. \tag{70}$$

Obviously, when $k=1$, $\overline{K}_{p,q}$ becomes $K_{p,q}$. Moreover, $\overline{K}_{p,q}$ has one or two connected components depending on if $p$ and $q$ are both even or not.

It is easy to obtain the group law for $\overline{K}_{p,q}$ just based on matrix multiplication

$$(\mathbf{a},\mathbf{b},c,k,\Lambda)\cdot(\mathbf{a}',\mathbf{b}',c',k',\Lambda') = (\mathbf{a}' + k'\Lambda'^{\,T}\mathbf{a},\ \mathbf{b} + k\Lambda\,\mathbf{b}',\ c + c' + \mathbf{a}\cdot\mathbf{b}',\ ,kk',\ \Lambda\,\Lambda')\,. \tag{71}$$

The identity is $I_d = (\mathbf{0},\mathbf{0},0,1,\mathbb{I}_n)$, which coincides with the identity in $GL(n+2)$, as should be. The inverse of an arbitrary element $(\mathbf{a},\mathbf{b},c,k,\Lambda) \in \overline{K}_{p,q}$ is

$$(\mathbf{a},\mathbf{b},c,k,\Lambda)^{-1} = (-k^{-1}\,\Lambda^{-1\,T}\,\mathbf{a}, -k^{-1}\,\Lambda^{-1}\,\mathbf{b}, -c + k^{-1}\,\mathbf{a}\cdot\Lambda^{-1}\,\mathbf{b}, -k^{-1}, \Lambda^{-1})\,. \tag{72}$$

### 4.3. The Lie Algebra $\overline{\mathcal{K}}_{p,q}$

As previously mentioned, the dimension of the Lie algebra $\overline{\mathcal{K}}_{p,q}$ of the Lie group $\overline{K}_{p,q}$ is $n(n+3)/2+2$. A basis of $\overline{\mathcal{K}}_{p,q}$ includes $n$ right-translation operators $X_\alpha$, $n = p+q$ left-translation operators $P_\alpha$, a central operator $I$, a dilation operator $D$, and $n(n-1)/2$ rotation/pseudo-rotation operators. These rotation/pseudo-rotation operators may be split into $p(p-1)/2$ and $q(q-1)/2$ rotations on the spaces $\mathbb{R}^p$ and $\mathbb{R}^q$, respectively, and $p\times q$ pseudo-rotations.

A $(n+2)\times(n+2)$ matrix realization of $\overline{\mathcal{K}}_{p,q}$ is given by

$$\begin{aligned} X_\alpha &= \left.\frac{\partial K_-}{\partial a^\alpha}\right|_{I_d} = \begin{pmatrix} 0 & \boldsymbol{\alpha}^T & 0 \\ \mathbf{0} & \mathbb{O}_n & \mathbf{0} \\ 0 & \mathbf{0}^T & 1 \end{pmatrix}, \quad P_\alpha = \left.\frac{\partial K_-}{\partial b^\alpha}\right|_{I_d} = \begin{pmatrix} 0 & \mathbf{0}^T & 0 \\ \mathbf{0} & \mathbb{O}_n & \boldsymbol{\alpha} \\ 0 & \mathbf{0}^T & 1 \end{pmatrix}, \\ I &= \left.\frac{\partial K_-}{\partial c}\right|_{I_d} = \begin{pmatrix} 0 & \mathbf{0}^T & 1 \\ \mathbf{0} & \mathbb{O}_n & \mathbf{0} \\ 0 & \mathbf{0}^T & 1 \end{pmatrix}, \quad D = \left.\frac{\partial K_-}{\partial k}\right|_{I_d} = \begin{pmatrix} 0 & \mathbf{0}^T & 0 \\ \mathbf{0} & \mathbb{I}_n & \mathbf{0} \\ 0 & \mathbf{0}^T & 1 \end{pmatrix}, \end{aligned} \tag{73}$$

where $\boldsymbol{\alpha}$ is either the column vector $(1,0,\dots,0)^T$, $(0,1,0,\dots,0)^T$,$\dots$, or $(0,0,\dots,1)^T$ corresponding to the values $\alpha = 1,2,\dots,n$, respectively. Finally, $\mathbb{O}_n$ is the $n\times n$ zero matrix. The remaining $n(n-1)/2$ generators belong to the Lie algebra $so_o(p,q)$ and are given by

$$J_{\alpha\beta} = \left.\frac{\partial A}{\partial \varphi_{\alpha\beta}}\right|_{Id} = \begin{pmatrix} 0 & \mathbf{0}^T & 0 \\ \mathbf{0} & M_{\alpha\beta} & \mathbf{0} \\ 0 & \mathbf{0}^T & 0 \end{pmatrix}, \tag{74}$$

with $\alpha<\beta$, $\alpha,\beta = 1,2,\dots,n$. By $\varphi_{\alpha\beta}$, we mean the parameters associated with the pseudo-orthogonal transformations defined in (59). For the sake of completeness, let us give here the complete list of the generators:

$$\begin{aligned} &[X_\alpha, X_\beta] = 0\,, \qquad [P_\alpha, P_\beta] = 0\,, \qquad [D, J_{\alpha\beta}] = 0\,, \qquad [I, \bullet] = 0\,, \\ &[X_\alpha, P_\beta] = \delta_{\alpha\beta}\, I\,, \quad [D, X_\alpha] = -X_\alpha\,, \quad [D, P_\alpha] = P_\alpha\,, \\ &[J_{\alpha\beta}, J_{\gamma\delta}] = -\big(g_{\beta\gamma}\, J_{\alpha\delta} - g_{\alpha\gamma}\, J_{\beta\delta} - g_{\beta\delta}\, J_{\alpha\gamma} + g_{\alpha\delta}\, J_{\beta\gamma}\big)\,, \quad \alpha<\delta\,,\ \beta<\gamma \\ &[J_{\alpha\beta}, X_\gamma] = \varepsilon_{\alpha\beta}\big(g_{\alpha\gamma}\, g_{\beta\delta} - g_{\alpha\delta}\, g_{\beta\gamma}\big)\, X_\delta = g_{\alpha\gamma}\, X_\beta - g_{\beta\gamma}\, X_\alpha\,, \\ &[J_{\alpha\beta}, P_\gamma] = \big(g_{\alpha\gamma}\, g_{\beta\delta} - g_{\alpha\delta}\, g_{\beta\gamma}\big)\, P_\delta = g_{\alpha\gamma}\, P_\beta - g_{\beta\gamma}\, P_\alpha\,. \end{aligned} \tag{75}$$

### 4.4. The nD Weyl–Heisenberg Group and Its Extension

In this brief subsection, we want to discuss the relations of the $n$D Weyl–Heisenberg group with those groups previously introduced. Let us just recall that the $n$D Weyl–Heisenberg group is behind of the most common commutation relations in quantum physics, which are $[x_i, p_j] \equiv [x_i, -i\hbar\frac{\partial}{\partial x_j}] = i\,\delta_{ij}\,\hbar$. It admits a representation by real $(n+2)\times(n+2)$ upper unitriangular matrices [14] such as

$$H_{p,q}[\mathbf{a},\mathbf{b},c] = \begin{bmatrix} 1 & \mathbf{a}^T & c \\ \mathbf{0} & \mathbb{I}_n & \mathbf{b} \\ 0 & \mathbf{0}^T & 1 \end{bmatrix}, \qquad \mathbf{a},\mathbf{b} \in \mathbb{R}^{p,q}\,,\ c\in\mathbb{R}\,. \tag{76}$$

Under this representation, it is easy to obtain the group law just based on matrix multiplication. This is

$$H_{p,q}[\mathbf{a},\mathbf{b},c]\cdot H_{p,q}[\mathbf{a}',\mathbf{b}',c'] = H_{p,q}[\mathbf{a}+\mathbf{a}',\mathbf{b}+\mathbf{b}',c+c'+\mathbf{a}\cdot\mathbf{b}']\,. \tag{77}$$

It comes from (76) that the identity element of the Weyl–Heisenberg group on this matrix representation is the identity matrix, $I_d = H_{p,q}[\mathbf{0}, \mathbf{0}, 0]$, and that the inverse of an arbitrary group element in (76) is

$$H^{-1}_{p,q}[\mathbf{a}, \mathbf{b}, c] = H_{p,q}[-\mathbf{a}, -\mathbf{b}, -c + \mathbf{a} \cdot \mathbf{b}] \,. \tag{78}$$

This is a subgroup of the group of all upper triangular matrices $(n+2) \times (n+2)$, $M_{n+2}(\mathbb{R})$ [15] and a subgroup of $\overline{K}_{p,q}$; see (70).

### 4.5. Bases on $\mathbb{R}^{p,q}$ and on $L^2(\mathbb{R}^{p,q})$

We may define a generalized basis of the multicomponent operators $\mathbf{X}$ and $\mathbf{P}$ such that for any of their components $X_\alpha$ or $P_\beta$, we have respective generalized continuous basis $|\mathbf{x}\rangle$ and $|\mathbf{p}\rangle$, ($\mathbf{x}, \mathbf{p}$ in either $\mathbb{R}^n$ or $\mathbb{R}^{p,q}$), such that

$$X_\alpha \, |\mathbf{x}\rangle = x_\alpha |\mathbf{x}\rangle \qquad P_\alpha \, |\mathbf{p}\rangle = p_\alpha |\mathbf{p}\rangle \,, \qquad \alpha = 1, 2, \ldots, n = p + q \,. \tag{79}$$

Then, everything is as in the 2D case described in Section 3.5.

As a support of infinite dimensional UIR of $\overline{K}_{p,q}$, we use the space $L^2(\mathbb{R}^{p,q})$, where we have selected the following orthonormal basis:

$$\left\{ \Psi_{\mathbf{m}}(\mathbf{x}) \equiv \prod_{\alpha=1}^{n} \psi_{m_\alpha}(x_\alpha) \right\}_{\mathbf{m} \in \mathbb{N}^n} \,, \quad n = p + q \,, \tag{80}$$

Function (80) is the *generalized nD Hermite functions* [20]. Therefore, any $f(\mathbf{x}) \in L^2(\mathbb{R}^{p,q})$ admits the following span:

$$f(\mathbf{x}) = \sum_{\mathbf{m} \in \mathbb{Z}^n} c^{\mathbf{m}} \; \Psi_{\mathbf{m}}(\mathbf{x}) \,, \qquad c^{\mathbf{m}} := \prod_{\alpha=1}^{n} c^{m_\alpha} \,, \;\; c^{m_\alpha} \in \mathbb{C} \,. \tag{81}$$

These functions satisfy the properties of orthonormalization and completeness similar to those of (46), but now, $\mathbf{m}$ and $\mathbf{x}$ are $n$D instead 2D. Transformation laws with respect to the Fourier transform ($FT$) or its inverse ($IFT$) of the generalized Hermite functions are like those of the 2D case (48) but now taking into accont that we consider the $n$D case.

When the spaces $L^2(\mathbb{R}^{p,q})$ are used to support a UIR of the group $\overline{K}_{p,q}$, the action of this representation on the members of the generalized $n$D Hermite functions is given by

$$\left\{ U(g)[\Psi_{\mathbf{m}}(\mathbf{x})] = \Psi_{\mathbf{m}}(g^{-1}\mathbf{x}) \right\} , \qquad \{U(g)[\Psi_{\mathbf{m}}(\mathbf{p}) = \Psi_{\mathbf{m}}(g\mathbf{p})\} \,, \quad g \in \overline{K}_{p,q} \,. \tag{82}$$

Then, we proceed as in the 1D case in terms of the 1D generalized Hermite functions, $\chi(x, a, b, k)$ [10] defining $n$ functions

$$\chi_{m_\alpha}(x^\alpha, a^\alpha, b^\alpha, k, \Lambda) \; := \; \sqrt{|k|} \; e^{-i\,k\,a_\alpha\,(\Lambda x)^\alpha} \; \psi_{m_\alpha}(k(\Lambda \mathbf{x})^\alpha + b^\alpha) \,; \tag{83}$$

hence,

$$\mathfrak{X}_{\mathbf{m}}(\mathbf{x}, \mathbf{a}, \mathbf{b}, k, \Lambda) := \prod_{\alpha=1}^{n} \chi_{m_\alpha}(x^\alpha, a^\alpha, b^\alpha, k, \Lambda) = |k|^{n/2} \, e^{-i\,k\,\mathbf{a} \cdot \Lambda \mathbf{b}} \, \Psi_{\mathbf{m}}(k\Lambda \, \mathbf{x} + \mathbf{b}) \,, \tag{84}$$

where each fixed set $\{\mathbf{a}, \mathbf{b}, k, \Lambda\}$ determines one reference frame and its properties: scale, origin and unit vectors, so that it determines an orthonormal basis. Therefore, the elements of the sequence $\{\mathfrak{X}_{\mathbf{m}}(\mathbf{x}, \mathbf{a}, \mathbf{b}, k, \Lambda)\}$ must satisfy the orthonormality and completeness relations displayed in (50) but now for $n$D objects.

Finally, under the $FT$ and $IFT$ basis $\{\mathfrak{X}_{\mathbf{m}}(\mathbf{x}, \mathbf{a}, \mathbf{b}, k, \Lambda)\}$, we recover expressions similar those of in the 2D case (51) but now considering nD objects instead of 2D.

Here, we close our discussion on these generalized basis.

## 5. Representations on Rigged Hilbert Spaces

Rigged Hilbert spaces are structures that are needed in order to define continuous bases, to relate discrete and continuous bases, and to define continuity for the representations of the elements of Lie algebras as linear operators defined on dense subspaces of infinite dimensional separable Hilbert spaces. As is well known, a rigged Hilbert space (RHS) or Gelfand triplet is a triplet of spaces

$$\Phi \subset \mathcal{H} \subset \Phi^\times \,, \tag{85}$$

where $\mathcal{H}$ is an infinite dimensional separable Hilbert space. As a subspace of $\mathcal{H}$, $\Phi$ is dense on $\mathcal{H}$ and is endowed with its own locally convex topology, which is finer than the topology inherited by $\Phi$ from $\mathcal{H}$. The space $\Phi^\times$ is the linear space of all continuous antilinear functionals over $\Phi$. We represent the action of any $F \in \Phi^\times$ into $\varphi \in \Phi$ as $\langle\varphi|F\rangle$. We consider antilinear functionals instead of linear functionals in order to be consistent with the Dirac notation as usual in quantum mechanics. The antidual space $\Phi^\times$ is endowed with any topology compatible with the structure of dual pair $\{\Phi, \Phi^\times\}$, although we usually consider the weak topology. In this case, the canonical injections (i.e., mappings $i : \mathcal{B} \subset \mathcal{D} \longmapsto \mathcal{D}$ such that $i(\varphi) = \varphi, \forall\, \varphi \in \mathcal{B}$) $i : \Phi \longmapsto \mathcal{H}$ and $i : \mathcal{H} \longmapsto \Phi^\times$ are continuous mappings.

RHSs have been used for various purposes, such as in a correct mathematical description of the Dirac formulation of quantum mechanics, in the construction of states for unstable quantum systems, in the description of quantum irreversibility produced by the quantum decay, in providing an appropriate context for some spectral decompositions of the operators used in classical chaotic systems, or in defining some constituents of the axiomatic quantum field theory among others. Some references on RHS, very far from exhaustive, include [11,12,22–28].

In the present discussion, we use explicit representations of functions such as $\mathfrak{X}_{\mathbf{m}}$ (84), where the Hilbert space is of the type $L^2(\mathbb{R}^n)$, with $n$ being any positive integer.

To begin with, let us go back to Section 2, where the Hilbert space under consideration is $L^2(\mathbb{R})$. Let us consider the Schwartz space $\mathcal{S}$ of all indefinitely differentiable functions such that they and all their derivatives vanish at infinity faster than the inverse of any polynomial. Its antidual, $\mathcal{S}^\times$, is the space of tempered distributions, considered antilinear continuous functionals on $\mathcal{S}$. Then,

$$\mathcal{S} \subset L^2(\mathbb{R}) \subset \mathcal{S}^\times \tag{86}$$

is a RHS. We recall that all Hermite functions belong to $\mathcal{S}$ and that the Fourier transform is a one-to-one onto and bicontinuous mapping (i.e., continuous with continuous inverse) from $\mathcal{S}$ onto itself. For any $a, b \in \mathbb{R}$ and and any $k \in \mathbb{R}^*$, $\chi_m(x, a, b, k) \in \mathcal{S}$.

The operator $X$ and $P$ in (8) may be represented by the multiplication $(\widetilde{X}f)(x) = xf(x)$ and derivation $(\widetilde{P}f)(x) = -idf(x)/dx$ on $\mathcal{S}$, respectively. It is well known that both are linear and continuous with the topology on $\mathcal{S}$. In addition, they are essentially self adjoint on $\mathcal{S}$.

Recall that, if $Q$ is a continuous operator operator on $\Phi$, it may be extended to a continuous operator on $\Phi^\times$, by means of the *duality formula*:

$$\langle Q\varphi|F\rangle = \langle\varphi|QF\rangle \,, \qquad \forall\, \varphi \in \Phi, \quad \forall\, F \in \Phi^\times \,, \tag{87}$$

where we also denote by $Q$ the continuous extension of $Q$ into $\Phi^\times$. As a consequence, we may extend by continuity $\widetilde{X}$ and $\widetilde{P}$ to $\mathcal{S}^\times$.

Although the construction yielding most of results in Section 2 has been given in [9], let us summarize it here, so that this paper is self-contained. The point of departure is an abstract infinite dimensional separable Hilbert space $\mathcal{H}$ and a unitary mapping $W : \mathcal{H} \longmapsto L^2(\mathbb{R})$, which is in principle arbitrary. If $\Phi := W\mathcal{S}$, where the topology on $\mathcal{S}$ is transported into $\Phi$ by $W$. The construction of $\Phi$ automatically gives the antidual

$\Phi^\times$. Then, $W$ may be extended to a bicontinuous mapping $W\mathcal{S}^\times = \Phi^\times$ by means of the duality formula

$$\langle W\varphi|WF\rangle = \langle\varphi|F\rangle\,, \qquad \forall\, F \in \Phi^\times\,, \quad \forall\varphi \in \Phi\,, \tag{88}$$

where we also denote by $W$ this extension. Let us call $U := W^{-1}$, so that after (88), $\langle U\varphi|UF\rangle = \langle\varphi|F\rangle$, $\forall\, F \in \Phi^\times$ and $\forall\, \varphi \in \Phi$. Let us summarize this scheme using the following diagram:

$$\begin{array}{ccccc} \Phi & \subset & \mathcal{H} & \subset & \Phi^\times \\ \downarrow U & & \downarrow U & & \downarrow U \\ \mathcal{S} & \subset & L^2(\mathbb{R}) & \subset & \mathcal{S}^\times\,. \end{array} \tag{89}$$

Then, $X = U^{-1}\,\widetilde{X}\,U$ and $P := U^{-1}\,\widetilde{P}\,U$ are continuous and essentially self adjoint on $\Phi$ and, therefore, continually extensible to $\Phi^\times$.

Based on the results of Gelfand and Maurin [12,29] (see a summary in [9]), for all $x, p \in \mathbb{R}$ (the original result states "for all almost everywhere with respect to some measure", but in this case, the Lebesgue measure is on a straight line and since the representation space is the Schwartz space $\mathcal{S}$, the result is now valid for "all"), there exist $|x\rangle, |p\rangle \in \Phi^\times$ such that (i) $X|x\rangle = x\,|x\rangle$ and $P|p\rangle = p\,|p\rangle$; (ii) the unitary operator $U$ may be chosen such that $\forall\, \varphi \in \Phi$, $U\varphi = \langle\varphi|x\rangle^* =: [\varphi(x)]^* \in \mathcal{S}$ and $\mathcal{F}U\varphi = \mathcal{F}[\varphi(x)]^*(p) = \langle\varphi|p\rangle$, where $\mathcal{F}$ is the Fourier transform (this is the unitary operator $U$, which "diagonalizes" the operator $X$, i.e., $\widetilde{X}\varphi(x) = x\varphi(x)$, $\forall\, \varphi(x) \in \mathcal{S}$) and the star denotes complex conjugation; and (iii) for any $\varphi, \psi \in \Phi$, one has the following two decompositions valid for $n = 0, 1, 2, \ldots$:

$$\langle\varphi|X^n\psi\rangle = \int_{-\infty}^{\infty} x^n\,\langle\varphi|x\rangle\langle x|\psi\rangle\,dx\,, \quad \langle\varphi|P^n\psi\rangle = \int_{-\infty}^{\infty} p^n\,\langle\varphi|p\rangle\langle p|\psi\rangle\,dp\,, \tag{90}$$

where $\langle x|\psi\rangle = \langle\psi|x\rangle^*$ and $\langle p|\psi\rangle = \langle\psi|p\rangle^*$.

Thus, the Theorem by Gelfand and Maurin justifies the relations (8). Relations (9)–(11) are also discussed in [9]. Formula (12) are also easily justified. Sinc, for all $\varphi(x) \in L^2(\mathbb{R})$, we have that

$$e^{i\widetilde{P}b}\,\varphi(x) = \varphi(x+b)\,, \qquad \forall b \in \mathbb{R}. \tag{91}$$

This shows that $e^{i\widetilde{P}b}$ preserves $\mathcal{S}$. Note that $U^{-1}\,e^{i\widetilde{P}b}\,U = e^{iPb}$, so that using the duality formula, we have for all $\varphi \in \Phi$

$$\langle\varphi|e^{-iPb}|x\rangle = \langle e^{iPb}\,\varphi|x\rangle = \varphi^*(x+b) = \langle\varphi|x+b\rangle\,, \tag{92}$$

so that for all $x \in \mathbb{R}$, we have that $e^{-iPb}\,|x\rangle = |x+b\rangle$. We show that $e^{-iXa}\,|p\rangle = |p-a\rangle$, for any real number $a$, analogously.

Taking into account that $\mathcal{S}$ is invariant under the action of $e^{i\widetilde{P}b}$, we have that

$$\langle\varphi|e^{-iPb}\,|p\rangle = \langle e^{iPb}\,\varphi|p\rangle = \left[e^{i\widetilde{P}b}\,\varphi(p)\right]^* = \left[e^{ipb}\,\varphi(p)\right]^* = e^{-ipb}\,\langle\varphi|p\rangle\,, \tag{93}$$

so that $e^{-iPb}\,|p\rangle = e^{-ipb}\,|p\rangle$, for any $p \in \mathbb{R}$ and arbitrary real number $b$. Analogously, $e^{-iXa}\,|x\rangle = e^{-ixa}\,|x\rangle$ for all $x \in \mathbb{R}$ and arbitrary real number $a$. From these considerations, relations (14)–(18) also follow.

In the case from Section 4.5, the Hilbert space is represented by $L^2(\mathbb{R}^n)$. We replace the space $\mathcal{S}$ by the $n$D Schwartz space, $\mathcal{S}(\mathbb{R}^n)$. This is the space of all indefinitely differentiable functions $f(\mathbf{x}) : \mathbb{R}^n \longmapsto \mathbb{C}$, such that for any non-negative integers $m_1, m_2, \ldots, m_n$ and $p_1, p_2, \ldots, p_n$, one has

$$\lim_{||\mathbf{x}||\to\infty} \left| x_1^{p_1} x_2^{p_2} \ldots x_n^{p_n}\, \frac{\partial^{\widetilde{m}} f(x_1, x_2, \ldots, x_n)}{\partial^{m_1} x_1 \partial^{m_2} x_2 \ldots \partial^{m_n} x_n} \right| = 0\,, \tag{94}$$

with $\widetilde{m} = m_1 + m_2 + \cdots + m_n$ and $||\mathbf{x}|| := \sqrt{x_1^2 + x_2^2 + \ldots x_n^2}$. Here, the realization of the abstract RHS $\Phi \subset \mathcal{H} \subset \Phi^\times$ as in (89) is

$$\mathcal{S}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \subset \mathcal{S}^\times(\mathbb{R}^n) \,. \tag{95}$$

For each $\alpha = 1, 2, \ldots, n$, we define the following operators on $\mathcal{S}(\mathbb{R}^n)$:

$$\widetilde{X}_\alpha f(\mathbf{x}) = x_\alpha\, f(\mathbf{x}_) \,, \qquad \widetilde{P}_\alpha f(\mathbf{x}) = \partial\, f(\mathbf{x})/\partial x^\alpha \,. \tag{96}$$

Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space and $U : \mathcal{H} \longmapsto L^2(\mathbb{R}^n)$ by unitary. Define $\Phi := U^{-1}\,\mathcal{S}(\mathbb{R}^n)$ and transport the Frèchet topology from $S(\mathbb{R}^n)$ to $\Phi$ via $U$. Then, for any $\alpha = 1, 2, \ldots, n$, define $X_\alpha := U\,\widetilde{X}_\alpha\,U^{-1}$ and $P_\alpha := U\,\widetilde{P}_\alpha\,U^{-1}$. Now, the Theorem by Gelfand and Maurin states that, for any $\mathbf{x} \in \mathbb{R}^n$, there is a $|\mathbf{x}\rangle \in \Phi^\times$ so that

(i) $X_\alpha\,|\mathbf{x}\rangle = x_\alpha\,|\mathbf{x}\rangle$ and a similar result holds for $P_\alpha$ (recall that all $X_\alpha$ commute with each other and that the same is true with the $P_\alpha$).
(ii) For any $\varphi \in \Phi$, we have that $U\varphi = \langle\varphi|\mathbf{x}\rangle^* \in S(\mathbb{R}^n)$ and $\mathcal{F}[U\varphi](\mathbf{p}) = \mathcal{F}(\langle\varphi|\mathbf{x}\rangle^*)(\mathbf{p}) = \langle\varphi|\mathbf{p}\rangle^*$, where $\mathcal{F}$ is the $n$D Fourier transform.
(iii) Similar spectral decompositions to those of (90) hold for $X_\alpha$ and $P_\alpha$.

With all these ideas in mind, we straightforwardly generalize the formulas in Section 2 to achieve similar formulas to those in Section 4.5.

Thus, in the continuous representation, the elements of the Lie algebras $\mathcal{K}_{p,q}$ may be represented by continuous linear operators on $\mathcal{S}(\mathbb{R}^n)$. By duality, these operators may be extended to continuous operators on the dual $\mathcal{S}^\times(\mathbb{R}^n)$, for any topology compatible with the dual pair. Hence, these algebras are also represented by algebras of continuous operators on the dual.

## 6. Discussion and Concluding Remarks

We carefully reviewed the properties of some generalizations of Euclidean and pseudo-Euclidean groups with interest in physics. In a previous paper, [10], we studies the geometric transformations of symmetry on the real line as groups represented by UIR on $L^2(\mathbb{R})$. These groups included translations and dilatations, the Euclidean group $E(1)$, as well as dilations on $\mathbb{R}$. We also analyzed the role of the Fourier transform that is in close connection with the Heisenberg–Weyl group $H_1$ with well-known implications in ordinary non-relativistic quantum mechanics. This gives a new Lie group, $K_1$, which has dimension four, as the spatial reversion has been enlarged with a discrete symmetry. The four infinitesimal generators of $K_1$, when represented as operators acting on $L^2(\mathbb{R})$ are $X = x$, $P = -ih^{-1}\,d/dx$, $D = -i(2h)^{-1}(x\,d/dx + d/dx)$ and $I = h$. This group has two connected components, and it may also be represented as a subgroup of the group of $(1+2)\times(1+2)$ matrices (4).

Here, we provide a generalization of the above results to $n$ dimensions ($n$D). This generalization produces the group $K_{p,q}$, which admits a UIR on $L^2(\mathbb{R}^{p,q})$. This group contains the pseudo-Euclidean groups $E_{p,q}$ and $H_{p,q}$ plus the discrete symmetry acting on $\mathbb{R}^{p,q}$ as $\mathbf{x} \to -\mathbf{x}$ when $p$ and $q$ are not both even. We show that $K_{p,q}$ may be represented by a subgroup of the group of $(n+2)\times(n+2)$ matrices (2), where $n = p+q$.

We also introduce an orthonormal basis on $L^2(\mathbb{R}^{p,q})$ formed by $n$D Hermite functions on the variable $\mathbf{x} \in \mathbb{R}^{p,q}$ and analyze their properties. These are given by products of Hermite functions on a single variable $x^\alpha$ (80). We have shown that these orthonormal basis are suitable for a UIR of $K_{p,q}$. These $n$D Hermite functions are also eigenfunctions of the Fourier transform (48).

Using the action of the group $K_{p,q}$ on $L^2(\mathbb{R}^{p,q})$ and the $n$D Hermite functions, we construct the $n$D generalized Hermite functions. These are also orthonormal basis for $L^2(\mathbb{R}^{p,q})$, although they fail to be eigenfunctions of the Fourier transform.

As usual, the generators of the group $K_{p,q}$ may be represented by self-adjoint operators on the representation space $L^2(\mathbb{R}^{p,q})$, which are unbounded. Nevertheless, the Lie algebra of $K_{p,q}$ admits a representation on suitable rigged Hilbert spaces (Gelfand triplets) such that all its elements as well as the elements of its envolving algebra may be represented as bounded (continuous) operators on two different locally convex topologies.

The $n$D Hermite functions appear in many quantum systems for which their respective Hamiltonians in $n$D are quadratic [30,31]. Typical examples include the following: (i) in Quantum Optics, the study of the photon distribution on multimodes mixed states [32]; (ii) in multidimensional signals analysis, the $n$D Hermite functions play a role in the decomposition of signals in terms of wavelets involves Fourier transform or Gabor transform [3,33,34]; and (iii) some further applications in optics of the $n$D Hermite functions, such us in vision studies are discussed in [35–37].

A commutator of the Lie algebra of the groups of the form $K_{p,q}$ suggests the possibility of a definition of a time–energy commutator. This commutator is well defined for finite as well as for infinite dimensional representations, although such an interpretation is not clear for well-known reasons. Nevertheless, in relation to the groups $K_{p,1}$, with $q = 1$ and particularly $K_{3,1}$, $K_{2,1}$, and $K_{1,1}$, one may interpret the dimension represented by $q$ as a *time*. This introduces, along its canonical conjugate variable, a *time–energy commutator*, for their corresponding Lie algebra generators. However, this interpretation is untenable on the whole space and only acquires a meaning locally, which means that this interpretation could have a sense when dealing with the algebra and not when considering the group.

Finally, the study of the effects of the other discrete symmetries that currently are associated with the group $O_{p,q}$ will be the object of future research.

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## References

1. Folland, G.B. *Harmonic Analysis in Phase Space*; Princeton University Press: Princeton, NJ, USA, 1989.
2. Howe, R.B. On the role of the Heisenberg group in harmonic analysis. *Bull. Am. Math. Soc.* **1980**, *3*, 821–843. [CrossRef]
3. Gabor, D. Theory of communication. *J. Inst. Elect. Eng. Part III Radio Commun.* **1946**, *93*, 429–457. [CrossRef]
4. Czaja, W.; Zienkiewicz, J. Uncertainty principle for Gabor systems and the Zak transform. *J. Math. Phys.* **2006**, *47*, 123507. [CrossRef]
5. Hogan, J.A.; Lakey, J.D. Extensions of the Heisenberg group by dilations and frames. *Appl. Comp. Harm. Anal.* **1995**, *2*, 174–199. [CrossRef]
6. Torresani, B. Time-frequency representations: Wavelet packets and optimal decomposition. *Ann. I.H.P. Sect. A* **1992**, *56*, 215–234.
7. Kalisa, C.; Torresani, B. N-dimensional affineWeyl-Heisenberg wavelets. *Ann. I.H.P. Sect. A* **1993**, *59*, 201–236.
8. Cohen-Tannoudji, C.; Diu, B.; Laloë, F. *Quantum Mechanics: Basic Concepts, Tools, and Applications*, 2nd ed.; Wiley: Hoboken, NJ, USA, 2019; Volume 1.
9. Celeghini, E.; Gadella, M.; del Olmo, M.A. Applications of rigged Hilbert spaces in quantum mechanics and signal processing. *J. Math. Phys.* **2016**, *57*, 072105. [CrossRef]
10. Celeghini, E.; Gadella, M.; del Olmo, M.A. Heisenberg-Weyl Groups and Generalized Hermite Functions. *Symmetry* **2021**, *13*, 1060. [CrossRef]
11. Bohm, A. *Boulder Lectures in Theoretical Physics*; Gordon and Breach: New York, NY, USA, 1967; Volume 9a.
12. Gelfand, I.M.; Vilenkin, N.Y. *Generalized Functions*; Academic: New York, NY, USA, 1964; Volume 4.

13. Emch, G.G. *Algebraic Methods in Statistical Mechanics and Quantum Field Theory*; Dover: New York, NY, USA, 2000.
14. Hall, B.C. *Lie Groups, Lie Algebras, and Representations*; Springer: Berlin/Heidelberg, Germany, 2015.
15. Schempp, W. *Harmonic Analysis on the Heisenberg Nilpotent Lie Group*; Wiley: New York, NY, USA, 1986.
16. Ali, S.T.; Antoine, J.P.; Gazeau, J.P. *Coherent States, Wavelets and Their Generalizations*; Springer: New York, NY, USA, 1999.
17. Folland, G.B. *A Course in Abstract Harmonic Analysis*; CRS Press: Boca Raton, FL, USA, 1995.
18. Szegö, G. *Orthogonal Polynomials*; American Mathematical Society: Providence, RI, USA, 2003.
19. Folland, G.B. *Fourier Analysis and Its Applications*; Wadsworth: Pacific Grove, CA, USA, 1992.
20. Grad, H. Note on $N$-Dimensional Hermite Polynomials. *Comp. Appl. Math.* **1949**, *133*, 325–330.
21. Wünsche, A. Hermite and Laguerre 2D polynomials. *J. Commun. Pure Appl. Math.* **2001**, *2*, 665–678. [CrossRef]
22. Roberts, J.E. Rigged Hilbert spaces in quantum mechanics. *Commun. Math. Phys.* **1966**, *3*, 98–119. [CrossRef]
23. Antoine, J.P. Dirac Formalism and Symmetry Problems in Quantum Mechanics. I. General Dirac Formalism. *J. Math. Phys.* **1969**, *15*, 902–916. [CrossRef]
24. Melsheimer, O. Rigged Hilbert space formalism as an extended mathematical formalism for quantum systems. I. General theory. *J. Math. Phys.* **1973**, *15*, 902–916. [CrossRef]
25. Gadella, M.; Gómez, F. On the mathematical basis of the Dirac formulation of Quantum Mechanics. *Int. J. Theor. Phys.* **2003**, *42*, 2225–2254. [CrossRef]
26. Gadella, M.; Gómez, F. Dirac formulation of Quantum Mechanics: Recent and new results. *Rep. Math. Phys.* **2007**, *59*, 127–143. [CrossRef]
27. Bellmonte, G.; di Bella, S.; Trapani, C. Operators in rigged Hilbert spaces: Some spectral properties. *J. Math. Appl. Math.* **2014**, *411*, 931–946. [CrossRef]
28. Chiba, H. A spectral theory of linear operators on rigged Hilbert spaces under analyticity conditions. *Adv. Math.* **2015**, *273*, 324–379. [CrossRef]
29. Maurin, K. *General Eigenfunction Expansions and Unitary Representations of Topological Groups*; Polish Scientific Publishers: Warszawa, Poland, 1968.
30. Dodonov, V.V.; Manko, O.V. The density matrix of the canonically transformed multidimensional Hamiltonian in the Fock basis. *Nouvo Cimento B* **1984**, *83*, 146–161. [CrossRef]
31. Dodonov, V.V.; Manko, O.V. New relations for two-dimensional Hermite polynomials. *J. Math. Phys.* **1994**, *35*, 4277–4294. [CrossRef]
32. Dodonov, V.V.; Manko, O.V. Multidimensional Hermite polynomials and photon distribution por polymode mixed light. *Phys. Rev. A* **1994**, *50*, 813–817. [CrossRef]
33. Feichtinger, H.G.; Zimmermann, G.A. Banach space of test functions for Gabor analysi. In *Gabor Analysis and Algorithms*; Birkhäuser Boston: Boston, MA, USA, 1998; pp. 123–170.
34. Gröcheniga, K.; Lyubarskii, Y. Gabor frames with Hermite functions. *C. R. Acad. Sci. Paris Ser.* **2007**, *344*, 157–162. [CrossRef]
35. Yang, J.; Reeves, A. Bottom-up visual image processing probed with weighted Hermite polynomials. *Neural Netw.* **1995**, *8*, 669–691. [CrossRef]
36. Yang, J.; Reeves, A. Visual pattern encoding with weighted Hermite polynomials. *Spat. Vis.* **2001**, *14*, 391–412. [PubMed]
37. Victor, J.D.; Mechler, F.; Repucci, M.A.; Purpura, K.P.; Sharpee, T. Responses of V1 Neurons to Two-Dimensional Hermite Functions. *J. Neurophysiol.* **2006**, *95*, 379–400. [CrossRef] [PubMed]