Light cones and repulsive gravity

Giovanni Preti[a)] and Fernando de Felice[b)]
Dipartimento di Fisica "Galileo Galilei," Università degli Studi di Padova, via Marzolo 8, 35131 Padova, Italy and INFN - Sezione di Padova, Dipartimento di Fisica "Galileo Galilei," Via Marzolo 8, 35131 Padova, Italy

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Contrary to common belief, gravitation can also be repulsive. Examples of repulsive gravity are provided by the naked singularity solutions of the Einstein equations corresponding to the negative mass Schwarzschild, the Reissner–Nordström, and the Kerr spacetimes. We show that their repulsive gravity regions can be identified by a particular behavior of the light cones when use is made of symmetry-adapted coordinate systems.
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I. INTRODUCTION

Despite the controversial nature of "naked" singularities as real astrophysical objects,[1] solutions to the Einstein equations containing such singularities exist. Contrary to what happens with black holes, the physical spacetime singularities, which are characterized[2] by a divergence of the Kretschmann invariant $I=R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}$, where $R_{\alpha\beta\gamma\delta}$ is the Riemann curvature tensor, are in the naked case accessible to external investigation, and information can be brought back from their close surroundings due to the absence of event horizons.

Singularities are commonly thought to be intrinsically attractive and infinitely deep potential wells. A remarkable property of many singularities is their repulsive nature. This property is true not only for those *ad hoc* spacetimes such as the negative mass Schwarzschild singularity (Sec. II) where the repulsive effects are generated by an unphysical source. The Reissner–Nordström solution (Sec. III), which describes a spherically symmetric spacetime with a charged mass as the source, provides an example in which the singularity is not only avoidable, but can also represent an infinite potential barrier against electrically neutral incoming particles.

The Kerr solution, which describes an axially symmetric spacetime with a rotating mass as the source, has a ring-shaped singularity which allows smooth tunneling between two non-isometric universes, usually identified as the $r>0$ and $r<0$ Kerr universes. The ring singularity turns out to be attractive only in the equatorial plane, and is repulsive otherwise. Tunneling between the two universes is made possible by "vortical" orbits,[3] which spiral around the axis of symmetry without crossing the equatorial plane. We shall focus on a subclass of vortical geodesics which represent the null principal directions of the Kerr metric.[2] In Sec. IV the radial behavior of these geodesics is analyzed; Sec. V illustrates the ergospheric structure of a naked Kerr singularity.

A general invariant criterion for determining the presence of repulsive domains is lacking. Hence, this information has to be derived by other means such as from an analysis of particle trajectories or observer-dependent physical measurements.[4] The characterization method that we shall illustrate is the direct examination of the light cone behavior. Light cones provide a valuable probe for investigating the causal structure of spacetime. The distortion, tilting, and twisting of their generators gives information about the physics in a curved background.[2] Their visualization depends on the particular choice of coordinates which describe the spacetime; an appropriate choice of coordinates is essential for the discovery of spacetime properties which otherwise would have been hidden. (For example, only oblate spheroidal coordinates can lead to the separability of the Hamilton–Jacobi equation in the Kerr spacetime,[5] and provide the physical interpretation of the corresponding fourth integral of motion.[6]) Because we will explore naked singularity spacetimes, the absence of horizons allows us to employ the physically most advantageous sets of coordinates, namely, those adapted to the metric symmetries, without fear of running into coordinate-induced pathologies. The possibility of employing these coordinates throughout our analysis yields simple and direct insight into the characteristics of the naked singularity solutions.

II. THE NEGATIVE MASS SCHWARZSCHILD SOLUTION

The Schwarzschild metric[7] in spherical coordinates can be expressed as[2]

$$ds^2=-\left(1-\frac{2M}{r}\right)dt^2+\left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2(d\vartheta^2+\sin^2\vartheta d\varphi^2). \tag{1}$$

The parameter M appearing in Eq. (1) has the physical meaning of a source mass, which is a positive quantity,[8] and implies the existence of a critical surface at $r=2M$, where the g_{rr} coefficient of the metric diverges. We say that an event horizon is present. A real pointlike unavoidable infinitely attractive singularity resides at $r=0$, screened from external investigation by the presence of the horizon. Even if the procedure is *ad hoc*, there exists the possibility of removing such screening: the parameter M is physically required to be positive, but Eq. (1) does not formally require $M\geq 0$ to hold. The $M<0$ condition may therefore be imposed, thus obtaining the negative mass Schwarzschild solution with no horizons at all; that is, the $r=0$ singularity is naked. This solution is interesting and instructive because it provides an example of a repulsive spacetime. (The repulsive character of this solution is intuitive: in the Newtonian limit it corresponds to a positive gravity potential given by $V=+|M|/r$.)

Spherical symmetry ensures that the light cone structure is completely determined by the radial null trajectories. Hence, we can solve $ds^2=0$ with $d\vartheta=d\varphi=0$ and examine the behavior of the angular coefficient(s)

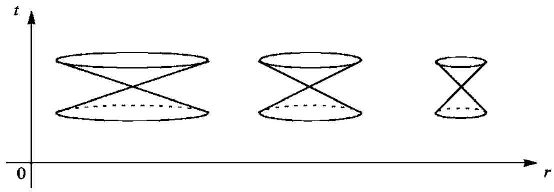

Fig. 1. Light cones in the $\{t,r\}$-plane of a naked (negative mass) Schwarzschild spacetime. The cones are centered on the coordinate time axis and their generators are null trajectories. Although we consider only one spatial dimension (r), we have depicted a 2d spatial section to make the rotational symmetry of the cones about their axes clear.

$$\tan \alpha_{\pm} \equiv \frac{dt}{dr} = \pm \left(1 - \frac{2M}{r}\right)^{-1}, \tag{2}$$

where $\alpha_{\pm}$ are the angles that the light cone generators form with the r axis in a $\{r,t\}$ diagram at any spacetime event.

If we keep in mind the well known light cone behavior in the usual $M>0$ Schwarzschild case[2] and refer to it for comparison, the $M<0$ case shows that (see Fig. 1) in the asymptotic ($r \to +\infty$) region, $\alpha_{\pm} = \pm 45°$, as in the $M>0$ case. As r is decreased the light cone monotonically *widens* instead of narrowing, until at $r=0$, the cone reaches its maximum ($180°$) aperture, being completely flattened. Such behavior agrees with the fact that the negative-mass Schwarzschild singularity not only can be avoided, but is also infinitely repulsive.

This example suggests that the attractive or repulsive character of the singularity leaves a clear imprint on the light cone behavior: when symmetry-adapted coordinates are employed, the light cones narrow as the attractive singularity is approached and widen when the singularity is repulsive. We can test this rule in the Reissner–Nordström case, characterized by a $M>0$ charged source.

III. THE REISSNER–NORDSTRÖM CASE

The Reissner–Nordström metric[9,10] in spherical coordinates can be expressed as[2]

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2), \tag{3}$$

where Q (in geometrized units, a length) represents the electric charge of the source. As in the Schwarzschild case, a real pointlike singularity exists at $r=0$. In addition, two event horizons are present at $r_{\pm} = M \pm \sqrt{M^2 - Q^2}$ if the condition $|Q| \leq M$ is satisfied; otherwise, a Reissner–Nordström naked singularity is present. The repulsive properties of the real singularity can be evidenced by analyzing test particle trajectories through effective potential techniques,[11] but can also be intuitively understood by considering the Newtonian limit, with its corresponding potential $V = -M/r + Q^2/2r^2$.

We focus our attention on the naked case and consider the angular coefficient(s) of the radial null geodesics, as done previously:

$$\frac{dt}{dr} = \pm \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1}. \tag{4}$$

Following the evolution of the light cones in going from the asymptotically flat $r \to +\infty$ region inward, we find that (see Fig. 2) starting from the $\pm 45°$ configuration, the light cone progressively narrows, but not monotonically as in the naked Schwarzschild case. At $r_* = Q^2/M$, the narrowing of the light cone stops. At this radius, we know[12–18] that the locally measured gravitational acceleration of a radially infalling particle goes to zero, changing sign inside; for $r < r_*$ gravitation acts repulsively.

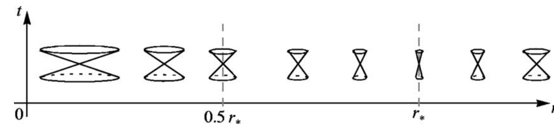

Fig. 2. Light cones in a naked Reissner–Nordström spacetime. As in Fig. 1, we have depicted a 2d section to make explicit the rotational symmetry of the cones about their axis.

As the singularity is approached closer, the cone progressively widens; it recovers the $\pm 45°$ configuration at $r = r_*/2$ (where the Newtonian potential vanishes) and reaches the maximal ($180°$) aperture at $r=0$. This behavior corresponds to the existence of an infinitely repulsive potential barrier for radial geodesics as $r \to 0$. Such behavior is related to the intrinsically repulsive character of the Reissner–Nordström source, which does not prevent the spacetime from manifesting an attractive character at some distance from it. However, attractive and repulsive effects coexist in the Reissner–Nordström case.

The Schwarzschild and Reissner–Nordström naked solutions, which have provided an introductory illustration of the efficiency of the light cone method for spacetime investigations, have thus established a framework for examination of the more complex naked Kerr case.

IV. KERR SPACETIME AND NAKED KERR SINGULARITIES

The Kerr metric[19] in the symmetry-adapted Boyer–Lindquist[20] coordinates is[2]

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right)dt^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma\, d\vartheta^2 + \frac{A}{\Sigma}\sin^2\vartheta d\varphi^2 - \frac{4aMr\sin^2\vartheta}{\Sigma}dt d\varphi, \tag{5}$$

where

$$\Sigma \equiv r^2 + a^2\cos^2\vartheta \tag{6a}$$

$$\Delta \equiv r^2 + a^2 - 2Mr \tag{6b}$$

$$A \equiv (r^2+a^2)^2 - a^2\Delta\sin^2\vartheta = (r^2+a^2)\Sigma + 2Mra^2\sin^2\vartheta, \tag{6c}$$

with a representing the specific angular momentum of the source ($a>0$ is understood). If $a \leq M$, we have a black hole solution with two event horizons located at

$$r_{h_{\pm}} = M \pm \sqrt{M^2 - a^2}, \tag{7}$$

where $r_{h_{\pm}}$ are solutions of $\Delta = 0$, corresponding to the divergence of the metric coefficient g_{rr}; if $a>M$, the $\{r=0, \vartheta = \pi/2\}$ ring singularity of the Kerr metric is naked.

The $r>0$ and $r<0$ Kerr universes, joined by the $r=0$ disc gateway, are causally connected; the most direct way to explore them is via null geodesics. The generic null geodesic in the Kerr metric is given by[2]

$$\dot{t} = \frac{A\gamma - 2Mar\lambda}{\Delta\Sigma} \tag{8a}$$

$$\Sigma^2 \dot{r}^2 = (2Mr\gamma - a\lambda)^2 + \Delta[\gamma^2 r(r+2M) - L] \tag{8b}$$

$$\Sigma^2 \dot{\vartheta}^2 = L - \frac{\lambda^2}{\sin^2\vartheta} + \gamma^2 a^2 \cos^2\vartheta \tag{8c}$$

$$\dot{\varphi} = \frac{1}{\Delta\Sigma}\left[(\Sigma - 2Mr)\frac{\lambda}{\sin^2\vartheta} + 2Mar\gamma\right], \tag{8d}$$

where a dot indicates the derivative with respect to a real parameter on the null geodesics; γ and λ are the constants of motions associated with the ignorable coordinates t and φ, respectively ($\gamma \equiv -p_t$ and $\lambda \equiv p_\varphi$); the parameter L is the separation constant[6] of the Hamilton–Jacobi equation. Only a restricted class of these trajectories allows crossing the $r=0$ gateway. If we set $\dot{\vartheta}=0$ in Eq. (8c) and study the resulting function $\lambda = \lambda(\vartheta;\gamma,L)$, considered as a "potential barrier," we easily single out a class of geodesics that spiral around the axis of symmetry with the coordinate angle θ ranging from a maximum to a minimum value that is both smaller or larger than $\pi/2$. These "vortical" geodesics[3] are bounded by two ϑ=constant hyperboloids and never cross the equatorial plane because their coordinate θ never equals $\pi/2$, as stated. For this reason they can penetrate into the $r<0$ Kerr universe, threading the ring singularity.

We now focus our attention on the subclass of these geodesics which lie entirely on a single θ=constant hyperboloid and are stable, having their parameter λ equal to a maximum with $\lambda>0$ (see Ref. 3). Orbits of this subclass described by

$$\dot{t} = \frac{(r^2+a^2)\gamma}{\Delta}, \quad \dot{r} = \pm\gamma, \quad \dot{\vartheta} = 0, \quad \dot{\varphi} = \frac{a\gamma}{\Delta}, \tag{9}$$

are null geodesics and provide the principal null directions of the Kerr spacetime;[2] a typical orbit of this kind is visualized in Fig. 3.

We put aside for the moment the inertial dragging effects (which affect the φ behavior of the cone) and now examine the angular coefficient(s):

$$\tan\alpha_\pm = \frac{dt}{dr} = \frac{\dot{t}}{\dot{r}} = \pm\frac{r^2+a^2}{\Delta}, \tag{10}$$

which determine the light cone aperture. Because we are employing symmetry-adapted coordinates, previous experience with the Schwarzschild and Reissner–Nordström metrics suggests that Eq. (10) provides direct information on the attractive or repulsive properties of the spacetime singularity.

An analysis of Eq. (10) begins with several preliminary observations. We see that Eq. (10) is independent of ϑ:

$$\tan\alpha_\pm \equiv \pm f(r). \tag{11}$$

In the naked singularity case $\Delta>0$, and hence $f(r)>0$ always. The extremal points for $f(r)$ are given by $r_{\mathrm{ex}_\pm} = \pm a$, in which case

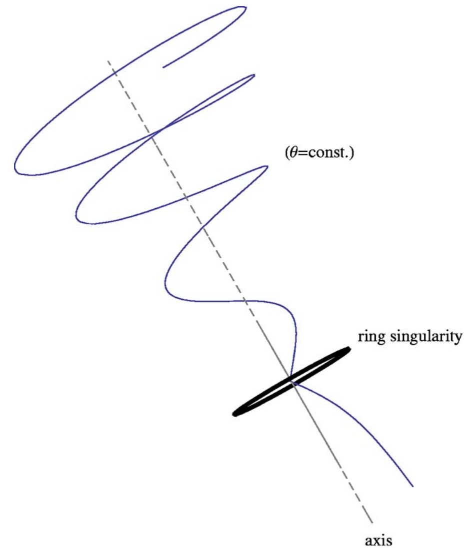

Fig. 3. The shape of a vortical orbit representing a null principal direction which is also a null geodesic of the Kerr metric: it extends into both the Kerr universes, spiraling around the axis of symmetry and crossing the $r=0$ disc.

$$f_{\max} \equiv f(r=+a) = \frac{a}{a-M}, \tag{12a}$$

$$f_{\min} \equiv f(r=-a) = \frac{a}{a+M}. \tag{12b}$$

The aperture of the light cones along the principal null directions of a naked Kerr spacetime behaves as follows (see Fig. 4): In the asymptotic $r\to+\infty$ region, $\alpha_\pm = \pm 45°$ (cone aperture =90°); as the singularity is approached, the cone progressively narrows until it reaches its minimum aperture at $r=+a$ (where $f=f_{\max}$). The light cone begins to widen as the $r=+a$ surface is traversed, until at $r=0$ (within the $r=0$ disc) the cone is again at $\pm 45°$. As the disc is it traversed and enters the $r<0$ Kerr universe, the light cone keeps widening, until it reaches its maximum less than 180° aperture at $r=-a$ (where $f=f_{\min}$). The light cone begins to narrow again as it traverses the $r=-a$ surface, and the narrowing continues until the $\pm 45°$ structure is recovered once more in the asymptotic $r\to-\infty$ region.

The asymptotic behavior of the light cones is a consequence of the asymptotic flatness of both the Kerr universes; less obvious is the fact that the same behavior is found everywhere in the whole $r=0$ disc. The metric (5) in the disc reduces to

$$ds^2(r=0) = -dt^2 + a^2\cos^2\vartheta\, d\vartheta^2 + a^2\sin^2\vartheta\, d\varphi^2, \tag{13}$$

which is the metric of a flat spatially circular (radius $=a\sin\vartheta$) three-dimensional spacetime, spanned by the time

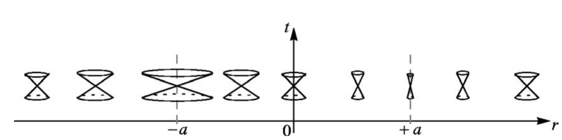

Fig. 4. Light cones along a null principal direction of a naked Kerr spacetime ($a>M$). They show a sequence of narrowing and widening behavior as the $r=0$ disc is approached. As in Figs. 1 and 2, the rotational symmetry of the cones about their axis is shown.

coordinate t and the two space coordinates ϑ and φ. (The M-dependent terms in Eqs. (5), (6b), and (6c) are proportional to r, and therefore vanish within the disc.) Does this fact mean the Kerr metric is flat there?

An explicit measure of the Kerr spacetime curvature in the disc region can be obtained by considering the Kretschmann invariant $I = R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}$; a lengthy calculation reveals that

$$I = I(r,\vartheta) = \frac{48M^2(r^2 - a^2\cos^2\vartheta)}{\Sigma^6}(a^4\cos^4\vartheta - 14a^2r^2\cos^2\vartheta + r^4), \tag{14}$$

which is negative at $r=0$. The fact that the Kretschmann invariant is negative does not necessarily imply repulsive gravitation, even though that is the case here, as we will see. Because a flat spacetime necessarily has a null Kretschmann invariant, we conclude that the Kerr spacetime is not flat in the disc region; the $r=0$ disc represents a flat three-dimensional submanifold of the four-dimensional Kerr manifold. There, the Kerr spacetime curvature (which does not vanish at all) is hidden in the residual fourth dimension, and is evident when we try to emerge either way from the disc.

To understand what is happening we need to have a closer look at its surroundings, which can be done by looking at light cones. Previous experience with the Schwarzschild and Reissner–Nordström metrics suggests that in a symmetry-adapted coordinate frame the light cones narrow as an attractive region of spacetime is approached and widen as a repulsive region is approached. In the present case we deduce that the Kerr spacetime is attractive for $r > +a$ and is repulsive for $+a > r > 0$, is repulsive for $r < -a$ and attractive for $-a < r < 0$. The $r = \pm a$ values, corresponding to the minimum/maximum aperture of the light cones, identify the loci where the gravitational effects, which are always finite and either attractive or repulsive, are at their maximum.[5,21,22] The $+a > r > -a$ belt across the two Kerr universes represents a sort of transition zone where each universe feels the counteracting effect of the other one. The structure at $\pm 45°$ of the light cones at the junction of the two Kerr universes (that is, in the $r=0$ disc) is due to the unstable equilibrium between these opposite effects; a particle residing in the $r=0$ disc will at the same time feel finite gravitational repulsion from "below" (away from the $r=-a$ surface) and finite gravitational attraction from "above" (toward the $r=+a$ surface), which will both push the particle away from the disc into the positive Kerr universe. This behavior shows that the situation in the surroundings of the $r=0$ disc is different from the one characterizing the asymptotically flat regions of the Kerr metric.

These considerations, based on the behavior of the light cones, are confirmed by direct analysis of redshift measurements. Take for instance a Kerr static on-axis observer $u^\alpha = \delta_t^\alpha(r^2+a^2)/\Delta$ located at a generic radial position r, who measures the frequency of a photon $\mathbf{k}$ propagating geodesically along the same axis. The measured frequency[11] $\nu = -u^\alpha k_\alpha$ is $\nu(r) = (r^2+a^2)\nu_\infty/\Delta$ (where ν_∞ is the conserved quantity associated with the cyclic variable t, coinciding with the frequency measured by the asymptotic observer). The frequency $\nu(r)$ has extremal values $\nu_{\max} = \nu_\infty/\sqrt{1-M/a} = \nu(+a)$ and $\nu_{\min} = \nu_\infty/\sqrt{1+M/a} = \nu(-a)$. Hence, a photon traveling, for example, from $r=0$ to $r=+a$ will undergo a blueshift (signaling a repulsive gravity region), while a trip from $r=0$ to $r=-a$ will give rise to a redshift (attractive gravity behavior). In contrast, if the photon originates outside the $r = \pm a$ surfaces and travels outward, a redshift would be observed in the positive Kerr universe, and a blueshift in the negative one. If the photon starts from $r=0$ and reaches infinity, no frequency shift would be observed there (consistent with the unusual metric characteristics of the $r=0$ disc we have discussed).

All of these coordinate invariant results of local physical measurements are in agreement with the conclusions based on the light cone analysis. Thus, the examples we have considered suggest that an examination of the light cone behavior gives a consistent characterization of the attractive and repulsive domains of the naked Kerr spacetime.

We have analyzed the radial behavior of the light cones along the null principal directions of a naked Kerr spacetime when a symmetry-adapted coordinate frame is employed, and we have obtained a picture of both the attractive and the repulsive gravity regions of this spacetime. At variance with the Schwarzschild and Reissner–Nordström cases, the reduced degree of symmetry of the Kerr metric implies that an analysis of the radial null geodesics is not sufficient to characterize the spacetime. Rotation introduces an azimuthal asymmetry in the structure of the light cones, which we can take into account by considering the function $dt/d\varphi$ for the trajectories of interest, as we have done for the function dt/dr. In so doing, we find two relevant differences. First, the function $dt/d\varphi$ is single valued:

$$\frac{dt}{d\varphi} = \frac{r^2+a^2}{a}, \tag{15}$$

and not double valued (providing two radial light cone generators, the ingoing and the outgoing ones) as is dt/dr. Second, Eq. (15) cannot represent the tangent of an angle, because it is not dimensionless as is Eq. (10). The first point poses the question of what Eq. (15) suggests about the behavior of the light cone. It directly implies that both the ingoing and the outgoing null principal directions have the same r-dependent tilt in the direction of increasing azimuthal coordinate; thus, we can figure the light cone to tilt a certain angle, β say, in the same direction (see Fig. 5). Although the

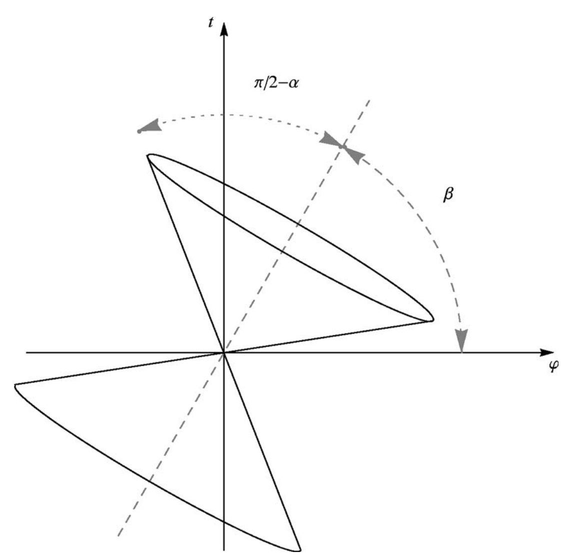

Fig. 5. A light cone along a null principal direction of the Kerr metric shows narrowing-widening behavior with respect to its axis, measured by the angle α [see Eq. (10)], and also an azimuthal tilting, measured by the angle β.

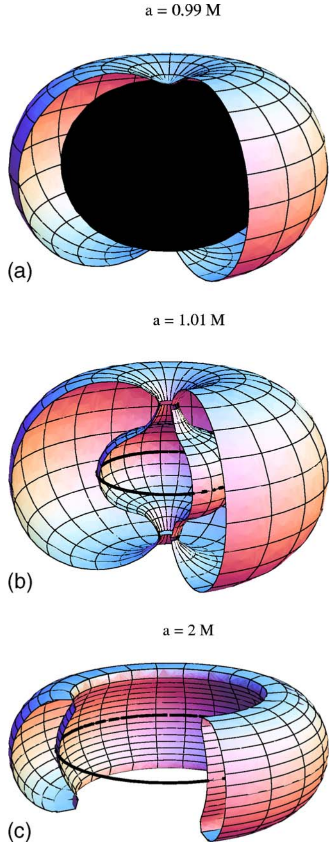

Fig. 6. The ergosphere around a (a) Kerr black hole $a=0.99M$ and around two Kerr naked singularities (b) $a=1.01M$ and (c) $a=2M$.

gravitational dragging of the light cones in the azimuthal direction is relevant for understanding the Kerr spacetime structure, it has no connection with the repulsive character of the solution, so we shall not pursue this analysis any further.

V. THE ERGOSPHERE AROUND A KERR NAKED SINGULARITY

One of the most distinguishing properties of the Kerr solution is the existence of an *ergosphere*,[2] a region where negative total energy states with respect to a distant observer are allowed. The boundary of this region is where the metric coefficient g_{tt} goes to zero, giving the two solutions

$$r_{\mathrm{erg}_\pm} = M \pm \sqrt{M^2 - a^2 \cos^2\vartheta}. \tag{16}$$

From Eq. (7) we see that $0 \le r_{\mathrm{erg}_-} \le r_{h_-} \le r_{h_+} \le r_{\mathrm{erg}_+} \le 2M$. In the black hole case, only the r_{h_+} and r_{erg_+} solutions matter for an external observer, and the ergosphere extends from the event horizon located at r_{h_+} outward to the radius r_{erg_+}. For a naked singularity $(a > M)$, neither of the horizons in Eq. (7) exists; the ergosphere still exists, though, even if its shape changes remarkably (see Fig. 6). The requirement that r be real implies that $\cos^2\vartheta \le M^2/a^2$. Hence, both the inner and the outer spheroidal boundaries in Eq. (16) of the ergosphere are pierced by an hyperboloidal channel at $\vartheta = \cos^{-1}(M/a)$, centered on the rotation axis. The resulting shape of the ergosphere is therefore toroidal-like, and the more squeezed toward the equatorial plane, the more the ratio a/M increases.

The existence of an ergosphere signals the presence of a spacetime region where gravitational effects are so extreme that they force everything to rotate with the source.[2] The presence of a hyperboloidal hole piercing through the ergosphere is a witness of the fact that the strength of gravity is weakened by the nakedness of the source. This effect can be enhanced by increasing the distance of the naked spacetime from a black hole condition (that is, by increasing the ratio a/M). We have already met a weakening of the attractive character of gravity when examining the light cone behavior: it originated from the gravitationally repulsive character of the Kerr solution near the ring singularity, away from the equatorial plane. Because the residual ergosphere around the Kerr naked singularity is concentrated toward this same plane (in which gravitation remains always attractive), we see that the ergosphere clearly shows by its changing shape the presence of a repulsive gravity region. Actually, the information provided by the light cones is much richer because, for instance, no information about the repulsive character of negative Kerr universe can come from analysis of the ergospheric structure—for the very simple reason that the ergosphere is confined to the positive Kerr universe.

In Sec. IV we have seen that the $r=+a$ surface represents the locus where the attractive effects of the Kerr source reach their maximum. Any real particle in the ergosphere of a Kerr black hole feels increasing gravitational effects as it moves in the direction of decreasing r toward the horizon, because in this case the whole ergosphere lies outside the $r=+a$ surface $(r_{h_+} \ge M \ge +a)$. The ergosphere of a Kerr naked singularity can lie, at least partially, in the region where the repulsive effects of the $r<0$ Kerr universe mitigate the attractive character of the $r>0$ region. The inner ergospheric surface (radius r_{erg_-}) always lies inside the $r=+a$ boundary. Hence, the ergospheric toroid lies at least partially inside this same boundary. If we use the expression for r_{erg_+} in Eq. (16), we see that for the naked singularity case the following behavior is found:

$$a < 2M, \begin{cases} \cos^2\vartheta < 2M/a - 1 \Rightarrow r_{\mathrm{erg}_+} > +a \\ \cos^2\vartheta > 2M/a - 1 \Rightarrow r_{\mathrm{erg}_+} < +a \end{cases} \tag{17}$$

$$a \ge 2M \Rightarrow r_{\mathrm{erg}_+} < +a. \tag{18}$$

The existence condition $\cos^2\vartheta \le M^2/a^2$ for the ergosphere does not set any restrictions on the case $r_{\mathrm{erg}_+} > +a$ of Eq. (17), because $2M/a - 1 < M^2/a^2$. It does set a restriction on the case $r_{\mathrm{erg}_+} > +a$ of Eq. (17), namely $2M/a - 1 < \cos^2\vartheta \le M^2/a^2$. In short, we see that around a naked Kerr singularity the ergosphere lies entirely inside the $r=+a$ surface if the condition $a \ge 2M$ is satisfied; the situation is more complex for $a < 2M$, with the ergosphere partly extending across the $r=+a$ surface.

VI. SUMMARY

We have analyzed the light cone behavior in some interesting cases of naked spacetimes by employing symmetry-adapted coordinate frames. We used these frames to examine the negative-mass Schwarzschild case and found that the light cones widen instead of narrowing as the singularity is approached. This behavior suggests a simple and direct way of detecting the presence of repulsive domains. The usefulness of this criterion when symmetry-adapted coordinates are employed has been confirmed by the analysis of the other naked spacetimes examined here, starting from the Reissner–Nordström spacetime, where the intrinsically repulsive character of the central pointlike singularity has emerged, and for the naked Kerr solution, where rotation of the source modifies the physics significantly compared to the previous spherically symmetric cases. To explore both the Kerr universes, we jumped across the ring singularity following the null principal directions of the Kerr metric. By analyzing the radial behavior of the light cones along these trajectories, we were able to characterize the attractive versus repulsive properties of the Kerr spacetime, with particular attention to the surroundings of the naked singularity.

We also commented on the spacetime peculiarities of the $r=0$ disc gateway connecting the two Kerr universes, fully exploiting information coming from the analysis of the light cone behavior. We then examined the modifications of the ergosphere due to the nakedness of the Kerr source, which shows its repulsive nature.

We also showed the possibility of characterizing repulsive domains via by analyzing the light cone behavior when a symmetry-adapted coordinate frame is used. This method allows deep physical insight at little calculation expense, and can be usefully employed in the study of the spacetime properties.

[a]Electronic mail: giovanni.preti@pd.infn.it
[b]Electronic mail: fernando.defelice@pd.infn.it

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