Intrawell stochastic resonance versus interwell stochastic resonance in underdamped bistable systems

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We show that, for periodically driven noisy underdamped bistable systems, an intrawell stochastic resonance can exist, together with the conventional interwell stochastic resonance, resulting in a double maximum in the power spectral amplitude at the forcing frequency as a function of the noise intensity. The locations of the maxima correspond to matchings of deterministic and stochastic time scales in the system. In this paper we present experimental evidence of these phenomena and a phemonological nonadiabatic description in terms of a noise-controlled nonlinear dynamic resonance.

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The occurrence of the stochastic resonance (SR) phenomenon in physical systems is, by now, well documented [1]. SR is a noise-induced cooperative phenomenon, manifest in bistable systems, wherein the response to a subthreshold time-periodic force can be enhanced in the presence of an optimal amount of noise. Since its inception [2], when it was proposed as a possible explanation for the periodicity of the ice ages, SR has been studied in a variety of nonlinear dynamic models, and experimentally observed in a number of physical systems (see, e.g., [1]), including lasers [3], electron paramagnetic resonance spectrometers [4], superconducting quantum interference devices [5], various electronic and magnetic devices [6], and neurophysiology [7].

Due to the intrinsic complexity of the nonlinear dynamics in the presence of deterministic (periodic driving) and stochastic (noise) forces, theoretically modeling the SR phenomenon usually requires approximations, the most relevant being the assumption of strong damping and a driving frequency that is well below the system characteristic frequencies (the so-called adiabatic approximation). For large damping, the full inertial dynamics of the relevant observable can be simplified through an adiabatic elimination of the ''velocity'' variable, thus reducing the description to the overdamped motion of a point particle in a bistable potential. The need for a driving frequency that is much smaller than any other internal characteristic frequency arises from the necessity of describing the stochastic dynamics under a clear-cut separation between time scales. In a generic overdamped bistable system one confronts the escape time scale (characterizing interwell jumps) together with the relaxation time in a single well, the former being, for small noise intensity, much larger than the latter. For these systems, the spectral amplitude at the drive frequency exhibits a maximum at a critical noise intensity, the traditional hallmark of SR; the maximum corresponds to a matching of the driving period to twice the escape time [1].

In this work we take a different approach, dispensing with both the above approximations, and studying the complete inertial dynamics under broadband noise and a time-periodic driving signal. We show that in this case a resonant phenomenon (intrawell SR) appears, for a range of parameters, in addition to the classical (i.e., interwell) SR. SR in monostable systems (specifically, the underdamped single-well Duffing oscillator) was first reported in [8], while ''conventional'' SR in an underdamped bistable system was first addressed in [9]. In this paper we show the coexistence/competition between these effects, over a certain range of system and forcing parameters, in an underdamped bistable system.

We consider the Brownian motion of a point particle of unit mass in a bistable potential $V(x)$, subjected to a time-periodic force $A(t)=A_0\cos\Omega t$ and external noise $\xi(t)$, described by the following Langevin equation [1] (the overdot denotes time differentiation):

$$\ddot{x}=-\gamma\dot{x}-\frac{\partial V}{\partial x}+A_0\cos\Omega t+\xi(t). \quad (1)$$

Here, $\xi(t)$ denotes an exponentially correlated Gaussian noise with zero mean and autocorrelation $\langle\xi(t)\xi(0)\rangle=\sigma^2\exp(-|t|/\tau_c)$, τ_c and σ being the noise correlation time and standard deviation, respectively (in the following we use the noise intensity D, which is related to σ and τ by $D=\tau_c\sigma^2$). $V(x)=-ax^2/2+bx^4/4$ is the standard quartic bistable potential $(a,b>0)$ having minima at $\pm x_m=\pm\sqrt{a/b}$ and separated by the potential barrier $\Delta V=a^2/4b$. $\gamma>0$ is the viscous damping constant. The SR in Eq. (1) was first discussed in Refs. [9,10]. Since then, only a few papers [11] have addressed the study of SR in the underdamped bistable case, an important exception being the study of SR in quantum systems [12].

The dynamics (1) have been studied here via analog simulation (see Gammaitoni *et al.* [1], p. 252, for details of the electronic simulator). In Fig. 1 we show the amplitude of the

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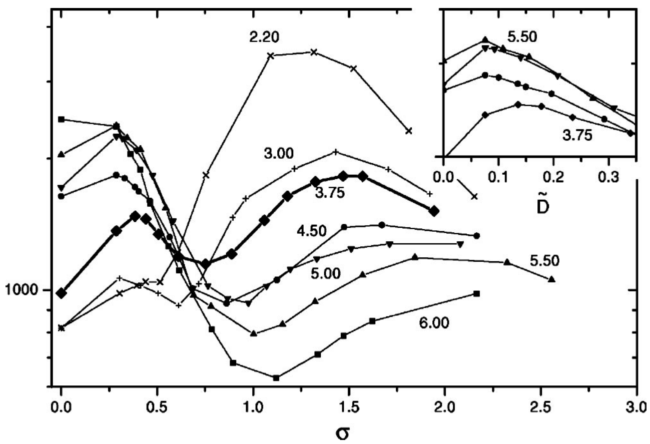

FIG. 1. Power spectral density amplitude $S(\omega)$ (a.u.) of the system response $x(t)$ at the forcing frequency $\omega=\Omega$ as a function of the noise standard deviation σ (a.u.), for various values of $\nu=\Omega/2\pi$ (in kHz). The characteristic frequency $\nu_0(D)=\omega_0(D)/2\pi$ in the absence of noise is $\nu_0(0)=6.00\pm0.15$ kHz. Other parameter values: $A_0x_m/\Delta V=0.452,\gamma=0.47,\tau_c=4.5\pm0.2$ μs. Inset: corresponding curves (from top 5.50 kHz, 5.00 kHz, 4.50 kHz and 3.75 kHz) plotted versus $\tilde{D}=D/(\gamma\Delta V)$.

power spectral density $S(\Omega)$ of the system response $x(t)$ at the forcing frequency Ω, as a function of the noise intensity. For a certain range of frequency values, a *double resonant* peak structure is evident: the peak at higher noise can be interpreted (see below) as the usual SR peak, due to the synchronization of the jumps between the potential wells with the external forcing, while the peak at smaller noise intensity values is due to an intrawell resonance characterized by the matching of the external forcing frequency Ω with the (noise-dependent) characteristic frequency of the nonlinear oscillations in a single well of the potential. When a single peak occurs (see, e.g., the 2.20 kHz curve), hopping dominates the dynamics. Clearly, both resonances coexist (at different noise intensities) over only a narrow range of the drive frequency Ω, for a certain choice of system parameters. A plot (Fig. 2) of the spectral amplitude $S(\Omega)$ vs the noise intensity for different damping parameters elucidates some of the phenomena alluded to above; most importantly, we observe that the double resonance appears to occur only for damping parameters and signal frequencies in a certain range.

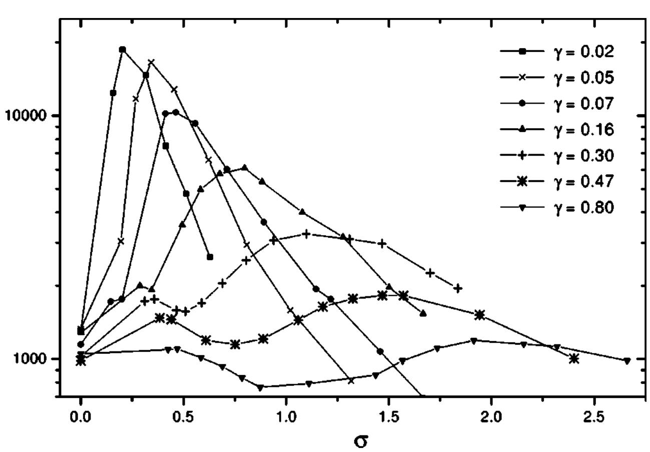

FIG. 2. $S(\Omega)$ (a.u.) vs σ (a.u.) as in Fig. 1, for various damping parameters. $\nu=3.75$ kHz. Other parameters as in Fig. 1.

A phenomenological description of the behavior detailed in Figs. 1 and 2 can readily be provided. At extremely low damping (see Fig. 2), even a small (broadband) perturbation can lead to noise-assisted hopping; the interwell motion clearly dominates the dynamics at small noise intensity also and the synchronization of the hopping rate to (one-half) the external signal frequency characterizes the (single) peak in the spectral amplitude, thus resulting in the (conventional) SR phenomenon. With increasing damping, one observes two peaks in $S(\Omega)$ but only when the forcing frequency takes values around the unperturbed characteristic frequency at the bottom of the wells. In this case, the motion is a combination of interwell hopping and small oscillations in the bottom of the wells. The latter motion dominates for low noise and is approximately harmonic: the system response amplitude changes as a function of the forcing frequency Ω, following the forced harmonic oscillator law. When the noise intensity is increased, for a fixed value of Ω, the system motion becomes more complex with nonlinear features (specifically, the profile of the single well) coming into play. For such a nonlinear oscillator we can define a *characteristic frequency* $\omega_0(D)$ which is a function of the amplitude of the motion and, thus, of the noise intensity $D=\tau_c\sigma^2$; the low-noise peak corresponds to $\Omega\approx\omega_0(D)$. As the noise increases even further, the hopping motion begins to dominate, and one recovers the usual (single-peaked) SR behavior.

In the absence of a signal, the Kramers rate or one of its generalizations [13] for low friction may be used to characterize the interwell hopping. While such a characterization forms the cornerstone of theoretical descriptions of SR under adiabatic conditions (periodic forcing with frequency much lower than the system characteristic frequencies), it fails to describe the dynamics in the presence of nonadiabatic conditions (i.e., high-frequency forcing as in the $\Omega=3.75$ kHz case of Fig. 2, to be compared to the 6.00 kHz of the unperturbed system characteristic frequency) which generate the lower-noise maximum in the spectral amplitude, when the double maximum exists. Hence a simple adiabatic theory [1] cannot be applied to the system in this case. The breakdown of the adiabatic conditions is also responsible for the absence of the well-known [13] crossover (as γ increases from 0) in the Kramers rate in the curves of Fig. 2 for fixed values of noise in the interwell SR regime (compare with the low-frequency curves in Fig. 6 of [10]).

We now compute the noise-dependent characteristic frequency $\omega_0(D)$ and investigate the approximate matching $\Omega\approx\omega_0(D)$. First we note that the characteristic frequency for the nonlinear oscillator with energy E can be estimated by computing the inverse of twice the deterministic transit time [14] from one extremum of the oscillation interval to the other, within the same well. Denoting these extrema (for the right well) by $x_\pm(x_-<x_+)$, we have for twice the transit time,

$$T(E)=2\int_{x_-}^{x_+}\frac{dx}{\sqrt{2(E-V(x))}}, \qquad (2)$$

with the extrema computed via the intersection $E=V(x)$ in the right well:

$$x_{\pm}(E)=\sqrt{\frac{a}{b}\left(1\pm\sqrt{1+\frac{E}{\Delta V}}\right)}. \qquad (3)$$

After some computation, we obtain $T(E)$ in terms of the complete elliptic integral ($\mathcal{K}$) of the first kind [15]

$$T(E)=\frac{2}{x_+}\sqrt{\frac{2}{b}}\mathcal{K}(q), \qquad (4)$$

where we set $q=\sqrt{x_+^2-x_-^2}/x_+$.

In the presence of noise, we replace E by $\overline{E}$, the average energy. To compute $\overline{E}$ we note that for weak damping and low noise the energy is approximately a constant of motion. Writing the dynamics (1) in terms of the position and energy variables (x,E), we may write down a two-dimensional (2D) Fokker-Planck equation (FPE) for the time evolution of the joint probability density function $W(x,E,t)$, whence the x variable may be integrated out [16], leaving us with a 1D FPE for $W(E,t)$, the energy density function,

$$\frac{\partial}{\partial t}W(E,t)=\gamma\left[\frac{\partial}{\partial E}\left(\frac{\phi(E)}{\phi'(E)}-\frac{D}{\gamma}\right)W(E,t)\right] + D\frac{\partial^2}{\partial E^2}\left[\frac{\phi(E)}{\phi'(E)}W(E,t)\right], \qquad (5)$$

with the identifications

$$\phi(E)=\int_{x_-}^{x_+}\sqrt{E-V(x)}dx,$$

$$\phi'(E)=\frac{1}{2}\int_{x_-}^{x_+}\frac{dx}{\sqrt{E-V(x)}}. \qquad (6)$$

We have taken the noise to be Gaussian δ correlated, having zero mean and correlation function $\langle\xi(t)\xi(t')\rangle=2D\delta(t-t')$ (the $\tau_c\rightarrow 0$ limit of the correlated noise defined earlier). Equation (5) has the steady state solution

$$W(E)=N^{-1}\frac{\phi(E)}{\phi'(E)}\exp\left(-\frac{\gamma E}{D}\right)\exp\left(\int^{E}\frac{\phi'(z)}{\phi(z)}dz\right) \qquad (7)$$

with N a normalization constant, and $\phi(E),\phi'(E)$ expressed in terms of the complete elliptic integrals of the first ($\mathcal{K}$) and second ($\mathcal{E}$) kind [15]:

$$\frac{\phi'(E)}{\phi(E)}=\frac{3}{4\Delta V(1+\sqrt{1+E/\Delta V})} \times\frac{\mathcal{K}(q)}{\mathcal{E}(q)-(1+\sqrt{1+E/\Delta V})\mathcal{K}(q)}. \qquad (8)$$

From the energy probability density function $W(E)$ we compute, numerically, the average energy as a function of the noise intensity $\overline{E}(D)$, which is then substituted into Eq. (4) to compute the characteristic frequency $\omega_0(D)$.

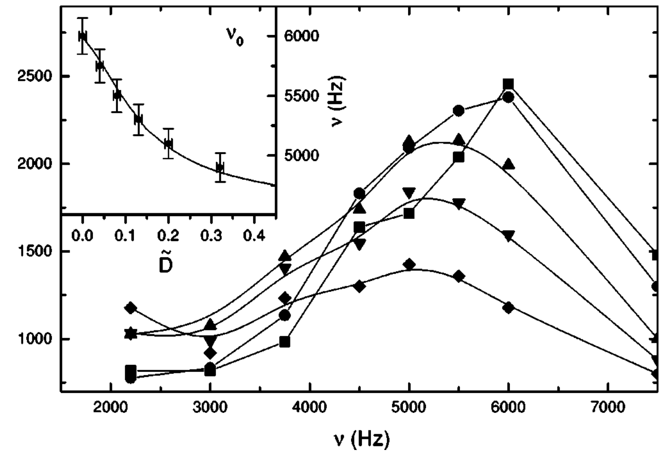

FIG. 3. Power spectral density $S(\omega)$ (a.u.) as a function of the frequency, for various noise intensities: squares $\widetilde{D}=0.00$, circles $\widetilde{D}=0.04$, up triangles $\widetilde{D}=0.13$, down triangles $\widetilde{D}=0.20$, diamonds $\widetilde{D}=0.32$, (same data as in Fig. 1). Inset: (data points) measured eigenfrequencies $\nu_0(D)=\omega_0(D)/2\pi$ as a function of the noise intensity, and (continuous line) theoretical prediction.

The above calculation has been carried out for white noise and in the absence of the periodic forcing. To take into account the additional energy contribution coming from the periodic forcing we added the energy due to the forcing itself by approximating the system response (at resonance) to the periodic signal by that of a harmonic oscillator of natural frequency $\omega_0(D)$. For the small amplitude value of the forcing used here, the additional contribution turns out to be smaller than 5% of the unperturbed mean energy $\overline{E}$. The white noise approximation is justified here by the choice of a simulated noise bandwidth larger than any other intrinsic frequency (>35 kHz).

To compare our theoretical prediction for $\omega_0(D)$ with experimental results, we plot the analog-simulated spectral amplitude curves of Fig. 1 as a function of the frequency, for various noise intensities (Fig. 3), and determine the characteristic frequency values at the maxima of the resonant curves. These characteristic frequencies are then plotted against the theoretical prediction in the inset of Fig. 3; the agreement, in the low-noise regime and for the weak signal amplitude considered here, is remarkable. The shift of the characteristic frequency to lower values with increasing noise intensity is apparent and is well reproduced by the theoretical prediction (continuous line). On comparing the values of the frequencies for which we have the presence of the first peak with theoretical predictions, it is apparent that the strength of the peak is greater when the matching condition is more closely satisfied. For example, for $\nu=5.50$ kHz the peak is near its maximum (inset in Fig. 1) and the position is located at $\widetilde{D}=0.075\pm0.020$[$\widetilde{D}=D/(\gamma\Delta V)$], to be compared with a prediction of $\widetilde{D}\approx 0.085$. On decreasing the frequency the position of the peak is supposed to move toward higher noise values. This behavior is contrasted with that of the low-noise tail of the interwell SR peak, which tends to move toward lower noise values and, for low-frequency values, dominates until the intrawell motion is no longer effective.

A large number of nonlinear dynamic systems and processes may be approximated by underdamped dynamics ofthe form (1), albeit with different (monostable or bistable) forms of the potential energy function $V(x)$; as examples we offer normal modes in hydrodynamic media and dynamic excitations in lattices. In this work we discussed the appearance of an additional reso-nance due to the intrawell oscillator dynamics and its relationship with the damping when $V(x)$ is bistable. A more detailed description ofdamping and noise color effects will appear in a future publication.

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