String ratchets: ac driven asymmetric kinks

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We simulated numerically the time evolution of a one-kink bearing, damped elastic string sitting on noiseless periodic substrates of two types: (I) asymmetric, time independent, (II) symmetric, periodically deformable. An asymmetric kink subjected to an ac drive is shown to drift steadily with finite average speed independent of its initial kinetic conditions. In the overdamped regime the resulting net kink transport can be attributed to the rectification of the Brownian motion of a pointlike particle with oscillating mass. For intermediate to low damping completely different features show up, due to the finite size of the objects being transported; in particular, the kink current hits a maximum for an optimal value of the damping constant, resonates at the kink internal-mode frequency and, finally, reverses sign within a certain range of the drive parameters.

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I. INTRODUCTION

A transport mechanism of potential relevance both to applied physics and nanobiology is the so-called ratchet effect [1]. In its simplest instance a ratchet device can be assimilated to a Brownian particle with coordinate $x(t)$ moving in an asymmetric periodic potential $V(x)$, with $V(x+a)=V(x)$, subjected to viscous damping and ac drive (*rocked ratchet* [2]): The natural direction and the intensity of the ratchet current $\langle \dot{x} \rangle$ results from a rather intricate interplay of particle inertia, spatial asymmetry, and time correlation of the forcing terms (including fluctuations when present). A similar mechanism is expected to operate also when instead of a pointlike Brownian particle one considers a fluctuating (one-dimensional) elastic string $\phi(x,t)$ [3] and replaces, accordingly, $V(x)$ with a periodic substrate potential $V[\phi]$, such that $V[\phi+a]=V[\phi]$. The kinks and antikinks born by the string as it connects adjacent substrate valleys tend to glide apart, so that the string center of mass advances effectively in the natural ratchet direction [4]. In rocked ratchets noise is required either to aid the escape of the Brownian particles over the potential barriers or to nucleate kink-antikink pairs along the string, directed parallel to the substrate valleys. In both cases a sufficiently large drive amplitude can activate a net ratchet current even in the absence of fluctuations.

In the present paper we address the ratchet dynamics of an elastic string diffusing on a periodically tilted substrate. In order to catch the essence of the mechanism at work we ignore the spatiotemporal fluctuations responsible for the thermalization of unperturbed string [3]; instead, we impose that the string always bears at least one kink, that is $\phi(\infty,t)-\phi(-\infty,t)=a$. The initial problem is thus reduced to the question as how each individual *damped* kink (antikink) responds to an external periodic drive. In the case of a symmetric substrate ($V[-\phi]=V[\phi]$, like in the sine-Gordon theory) and a sinusoidal tilt, it has been noticed [5] that an isolated *frictionless* kink may travel in either direction depending on the tilt parameters and, most importantly, on its initial conditions (momentum modulus and phase). Such an effect follows from the spontaneous *symmetry breaking* induced by the external tilt; indeed, on averaging over all initial conditions the kink current vanishes, as one would expect on the ground of simple symmetry arguments. Furthermore, adding damping, no matter how small, makes this effect vanish completely.

As stated in the earlier ratchet literature [1], the onset of a kink ratchet current requires a sufficient amount of asymmetry in the system. In this paper we address two classes of asymmetric kink dynamics: (I) *Ratchet Potential (RP)*: $V[\phi]$ is intrinsically asymmetric and time independent; the kink is asymmetric also in the absence of a tilt (unperturbed kink) due to the asymmetry of the barrier separating any two adjacent potential valleys; (II) *Deformable Potentials (DP)*: $V[\phi]$ is symmetric at all times; its shape is modulated so that its valleys broaden and shrink (with constant height) periodically in time [6]; the tilt is phase locked to the substrate modulation, thus providing for an effective cycle asymmetry. To avoid further complications, we assume that the strings of both classes bear one species of kink, only; this implies that all $V[\phi]$ valleys are degenerate and have the same curvature. Entropic rectification effects like those described in Refs. [7,8] are thus ruled out.

The main conclusion of our study is that strings of classes I and II possess sufficient asymmetric coupling to the ac drive to sustain steady kink transport even in the presence of finite damping. The rather complicated dependence of the kink ratchet current on the damping constant and the tilt parameters makes a detailed analysis of the system at hand worthy our extensive numerical simulation effort reported on here. Our presentation is organized as follows. In Sec. II we introduce an example for each class of asymmetric strings RP and DP; we then estimate the effect of a small periodic tilt on the relevant kink dynamics in the adiabatic limit, thus explaining why we expect a finite kink current. In Sec. III we present the outcome of our simulation work and focus on the kink current dependence on the damping constant and the tilt frequency; intriguing effects like non-Smoluchowski overdamped laws, optimal ratchet damping and parametric resonant current inversions are thus revealed. Finally, in Sec. IV we outline a summary of the results and conclusions, as well as an outlook of potential extensions of this work.

II. ASYMMETRIC SUBSTRATES

A damped elastic string $\phi(x,t)$ moving on a periodic substrate is described by the classical field equation [9–11]

$$\phi_{tt}-c_0^2\phi_{xx}+\omega_0^2 V'[\phi]=-\alpha\phi_t+F(t). \quad (1)$$

Here, c_0 and ω_0 are the parameters of the unperturbed string equation; the potential $V[\phi]$ is periodic in ϕ, i.e., $V[\phi+a]=V[\phi]$, and its amplitude was set to one following an appropriate choice of ω_0; the potential minima (valleys) are located at $\phi=0(\mathrm{mod}\ a)$ and separated by barriers centered at $\phi=\phi_0(\mathrm{mod}\ a)$ with $0<\phi_0\leqslant a$ depending on the degree of asymmetry of the substrate; α denotes the string damping constant and the periodic tilt $F(t)$ is taken sinusoidal, i.e., $F(t)=F_0\sin(\Omega t)$, for simplicity.

The unperturbed string $[F(t)=0,\alpha=0]$ bears both extended (phonons) and localized solutions (solitons) [9–11]. Localized solutions can be conveniently approximated to linear superpositions of moving kinks $\phi_+^{(0)}$ and antikinks $\phi_-^{(0)}$,

$$x-X(t)=\pm\left(1-\frac{u^2}{c_0^2}\right)^{1/2}\frac{c_0}{\omega_0}\int_{\phi_0}^{\phi_\pm^{(0)}(x-X(t))}\frac{d\phi}{\sqrt{2V[\phi]}}, \quad (2)$$

provided that the separation between their centers of mass $X(t)=x_0+ut(x_0+$ and u are integration constants) is very large compared with their size $c_0/\omega_{\mathrm{min}}$ with $\omega_{\mathrm{min}}^2=\omega_0^2V''[0]$ (dilute gas approximation). As anticipated in Sec. I we ignore the kink-antikink pair nucleation mechanism, where thermal fluctuations would play a central role [3], and focus on the response of a single preexisting (geometrical) kink $\phi_+^{(0)}$ subjected to the perturbation on the right-hand side of Eq. (1), i.e., in the absence of noise. Note that the ratchet current of $\phi_-^{(0)}$ is necessarily opposite to that of $\phi_+^{(0)}$; moreover, the string center of mass advances by one full step a to the right as a kink travels all the way from $X=\infty$ to $X=-\infty$ (or vice versa for an antikink).

A static $(u=0)$ perturbed kink $\phi_+(x,t)$ can be regarded as an extended quasiparticle [12] of radius $c_0/\omega_{\mathrm{min}}$ and mass

$$M(t)=\int_{-\infty}^{\infty}[\phi_+(x,t)]_x^2dx \quad (3)$$

(throughout the present paper $[\ \dots\ ]_x$ denotes a spatial derivative; here the t dependence accounts for the kink shape modulation). The time dependence of $M(t)$ is induced by the periodic tilt $F(t)$; however, it takes an intrinsic asymmetry or a tilt-dependent deformation of the substrate $\omega_0^2V[\phi]$ for the kink mass modulation to cause a rectification of the kink dynamics as shown in the following section.

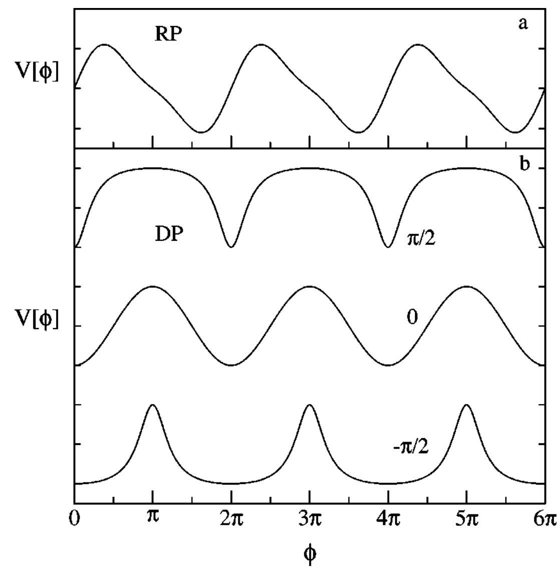

FIG. 1. Substrates (a) RP of Eq. (4); (b) DP of Eq. (7) in the absence of tilt, $F_0=0$. In the snapshots of (b) the deformation parameter $s(t)$, Eq. (11), has constant amplitude $s_0=0.5$ and decreasing phase, corresponding to $s=0.5,0,-0.5$, respectively. Note that the natural ratchet direction of the double-sine potential plotted in (a) is *positive* (see text).

A. Ratchet potentials

The RP model used in our simulations is the benchmark potential of the current literature on rocked ratchets [1,4], namely,

$$V[\phi]=k\sin\left[\frac{2\pi}{a}(\phi-\bar{\phi})\right]+\frac{k}{4}\sin\left[\frac{4\pi}{a}(\phi-\bar{\phi})\right], \quad (4)$$

with $k^{-1}=(3+\sqrt{3})(\sqrt{3}/2)^{1/2}a/8\pi$ and $\bar{\phi}=(a/2\pi)\times\cos^{-1}[(-1+\sqrt{3})/2]$. Substrate (4) can be regarded as the most straightforward variation of the sine-Gordon (SG) theory that allows for spatial anisotropy. Such an occurrence has been considered, for instance, in the context of dislocation theory [4,13] and long Josephson junction array design [14,15].

In Fig. 1(a) and throughout the present paper we set $a=2\pi$. The asymmetry of the substrate (4) is apparent as the potential barriers are located at $\phi_0(\mathrm{mod}\ 2\pi)$ with $\phi_0=2\bar{\phi}$ and $0<\phi_0<\pi$. The resulting unperturbed kink solution $\phi_+^{(0)}$ is stable under small perturbations due to the degeneracy of the $V[\phi]$ minima; its mass M_0 is a constant inverse proportional to the kink size $c_0/\omega_{\mathrm{min}}$.

In the presence of the tilt $F(t)$, the minima of the effective potential $\omega_0^2V[\phi]-F(t)\phi$ shift back and forth around their unperturbed position $0(\mathrm{mod}\ 2\pi)$; as a consequence, the curvature of the tilted valleys $\omega_{\mathrm{min}}^2(t)$ oscillates in time with amplitude proportional to F_0. An explicit calculation carried out in the adiabatic limit $\Omega\to 0$ and for small tilt amplitudes $F_0\ll\omega_0$ yields

$$\omega_{\mathrm{min}}^2(t)=\omega_{\mathrm{min}}^2\left[1+\left(1-\frac{1}{\sqrt{3}}\right)\frac{F_0}{\omega_0^2k}+\cdots\right], \quad (5)$$

with $\omega_{\min}^2 = \omega_0^2 k(3\sqrt{3}/2)^{1/2}$. Accordingly, being $M(t) \propto \omega_{\min}(t)$, one sets the maximum amplitude of the kink mass oscillation to

$$\frac{\delta M}{M_0} = \frac{1}{4}\left(\sqrt{\frac{3}{2}}\right)^{1/2} \frac{F_0}{\omega_0^2}. \tag{6}$$

[Notice that for a regular SG potential the time modulation of $\omega_{\min}^2(t)$ would be quadratic in $F(t)$.] Our derivation of Eq. (6) clearly hints at a coupling mechanism between kink center of mass and kink internal mode, as invoked in Ref. [15].

B. Deformable potentials

The class of the soliton-bearing DP strings was introduced by Peyrard and Remoissenet [6] as an improved model of real one-dimensional atomic chains where typical potential wells and barriers have different curvature, as opposed to the oversimplified Frenkel-Kontorova (or discrete SG) model [11], which is invariant under "inversion," i.e., $V[\phi + a/2] = -V[\phi]$. Following Ref. [6] with minor notation changes, we define [Fig. 1(b)]:

$$V[\phi;s] = \frac{1}{2}\left[\frac{(1+s)^2(1-\cos\phi)}{(1-s)^2 + 2s(1-\cos\phi)} - 1\right], \tag{7}$$

with $|s|<1$ and string constant $a = 2\pi$. The curvature of the substrate valleys and barriers are, respectively,

$$\omega_{\min}^2(s) = \omega_0^2 V''[0] = \frac{\omega_0^2}{2}\left(\frac{1+s}{1-s}\right)^2 \tag{8}$$

and $\omega_{\max}^2(s) = \omega_0^2 |V''[\pi]| = \omega_{\min}^2(-s)$. Note that the DP (7) coincides with a SG potential for $s=0$ and is $\phi \to -\phi$ symmetric for any allowed value of s.

The mass of the DP kink $\phi_+(x;s)$ has been derived in the original papers [6]; in our notation

$$\frac{M(s)}{M_0} = \frac{\mu(s)}{\sqrt{|\mu(s)^2 - 1|}} \begin{cases} \tanh^{-1}\sqrt{|\mu(s)^2-1|}, & s \leqslant 0, \\ \tan^{-1}\sqrt{|\mu(s)^2-1|}, & s \geqslant 0, \end{cases} \tag{9}$$

where $\mu(s) \equiv \omega_{\min}(s)/\omega_{\min}$, $\omega_{\min} \equiv \omega_{\min}(0)$ and M_0 is the mass of the SG kink $\phi_+^{(0)}(x) \equiv \phi_+(x;0)$, i.e., $M_0 = M(0) = 8\omega_{\min}/c_0$. On assuming that $|s| \ll 1$ the cumbersome expression (9) can be approximated to

$$M(s) = M_0\left(1 + \frac{2}{3}s + \cdots\right). \tag{10}$$

In our simulations—we depart here from the modelization of Ref. [6]—it was assumed that the deformation parameter s oscillates in time, $s \to s(t)$, with amplitude $s_0 < 1$ and angular frequency Ω. Moreover, the relative phase of the potential modulation $s(t)$ and the external tilt $F(t)$ was set arbitrarily in such a way that, when $V[\phi;t] \equiv V[\phi;s(t)]$ is tilted to the right $[F(t)>0]$, its valleys (barriers) get narrower (broader) and vive versa, namely,

$$s(t) = s_0 \sin(\Omega t), \qquad F(t) = F_0 \sin(\Omega t). \tag{11}$$

Accordingly, for $s_0 \ll 1$ the kink mass $M(t) = M(s(t))$ oscillates around its unperturbed value M_0 with maximal amplitude $\delta M/M_0 = 2s_0/3$.

Of course the s-F phase may be changed thus altering the kink ratchet current in a predictable fashion (see, for instance, Sec. III C, bottom); some of the simulation results reported in Sec. III for our DP model may depend indeed on the choice (11), whereas the underlying physical interpretation does not. More importantly, one should notice that a time-dependent DP model may have practical applications, for instance, in dislocation theory. Equation (1) is known to describe the motion of a noiseless linear defect (or dislocation) ϕ gliding on a Peierls-Nabarro substrate $\omega_0^2 V[\phi]$ under the action of an external uniform periodic stress field $F(t)$ [13]. If one assumes that the applied stress can perturb periodically the lattice substrate, too, then compressions and dilatations are likely to affect the curvature of the Peierls-Nabarro barriers differently, as suggested in Ref. [6].

C. Kink ratcheting

We are now in the position to argue why we expect a nonvanishing kink ratchet current for both potentials (4) and (7). A simple perturbation approach [12,16] valid in the *adiabatic* limit $\Omega \to 0$, only, leads to state that a single kink (antikink) moving along the elastic string (1) obeys the noiseless Langevin equation (LE)

$$\dot{p} = -\alpha p \mp aF(t), \tag{12}$$

where $a = 2\pi$, $p = M(t)c_0/\sqrt{1-(u/c_0)^2}$ denotes the kinetic momentum of the kink and $u = \dot{X}$ is the velocity of its center of mass. For $F_0 \ll \omega_0$ and $\alpha \gg \omega_0$ the modulus of $u(t)$ is much smaller than c_0 and, therefore, the LE (12) can be rewritten in nonrelativistic form, $p \simeq M(t)u$ and

$$M(t)\ddot{X} = -\alpha M(t)\dot{X} \mp aF(t). \tag{13}$$

It follows immediately that kinks and antikinks are pulled apart with stationary speed $u(t) = \mp aF(t)/\alpha M(t)$; in particular, when the substrate is tilted to the right, $F(t)>0$, the kinks move to the left with negative velocity and vice versa.

In the same parameter regime, $F_0 \ll \omega_0$, $\alpha \gg \omega_0$, and $\Omega \to 0$, the mass $M(t)$ of both the RP and DP kinks oscillates with time according to Eqs. (5),(6), and (10),(11), respectively: A kink with a negative velocity, i.e., for $F(t)>0$, is more massive, or slower, than a kink with positive velocity, i.e., for $F(t)<0$; hence,

$$\overline{u} = \langle u(t) \rangle = \frac{1}{T_\Omega}\int_0^{T_\Omega} u(t)\,dt \simeq \frac{aF_0}{2\alpha M_0^2}\,\delta M, \tag{14}$$

with $T_\Omega = 2\pi/\Omega$ and $\delta M/M_0 \simeq 0.23(F_0/\omega_0^2)$ for the RP (4) and $\delta M/M_0 \simeq 0.67 s_0$ for the DP (7). In Fig. 2(a) we display the average ratchet velocity $\overline{u}\ F_0$ for a kink of either class: as predicted in Eq. (14), $\overline{u} \propto F_0^\beta$ with $\beta = 1$ (DP) and $\beta = 2$ (RP).

We make now an important remark. As mentioned earlier in this section, by pulling the kinks to the right, $X(t) \to \infty$, and the antikinks to the left, $X(t) \to -\infty$, the *string* center of

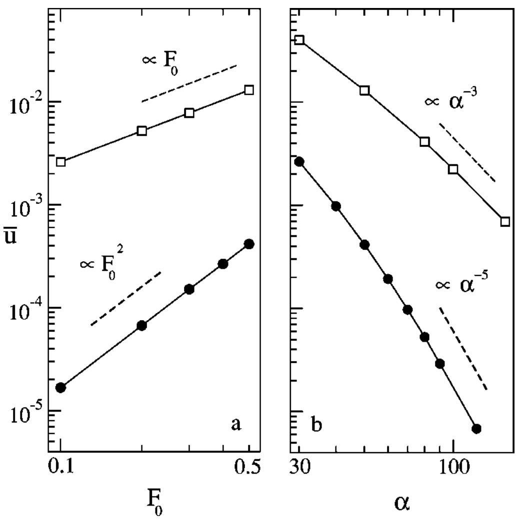

FIG. 2. Kink ratchet velocity $\overline{u}$ in the overdamped regime: (a) dependence on the tilt amplitude F_0 at $\alpha=50$; (b) dependence on the damping constant α at $F_0=0.5$. Other simulation parameters: $c_0=50$ and $\Omega=8\times10^{-3}$. The asymptotic laws (dashed lines) have been drawn for the reader's convenience.

mass drifts with negative average velocity $\langle\langle\dot{\phi}\rangle\rangle<0$, $\langle\langle\cdots\rangle\rangle$ denoting the double average of $\phi(x,t)$ with respect to time and space. A negative string current is to be expected for the RP of Fig. 1(a): On replacing ϕ with x in Eq. (4) one obtains the standard double-sine potential of Ref. [2], which in the overdamped regime corresponds to a rocked ratchet with negative natural direction. The case of the DP in Fig. 1(b) is more intriguing. A particle with coordinate x moving along the periodic potential $V(x;s)$, obtained by setting $\phi\to x$ in Eq. (7), and subject to the modulation of Eq. (11), would obey the equation of motion

$$\ddot{x}=-\alpha\dot{x}-V'(x;s)+(s/|s|)F_0. \tag{15}$$

As $V(x;-s)=-V(x-\pi;s)$, one can easily prove that in the stationary regime $\dot{x}(t;s)\coloneqq-\dot{x}(t;-s)$ (*i.e.*, $x(t;s)$ and $-x(t;-s)$ obey the same equation of motion), for any constant value of s, and therefore, after averaging over an entire deformation cycle of $s(t)$, $\overline{u}=\langle\dot{x}(s(t))\rangle=0$. Note that this property of Eq. (15) depends crucially on the choice (11) of the $s-F$ phase. The observed net string ratchet current $\langle\langle\dot{\phi}\rangle\rangle\neq0$ should remind us that in our DP model asymmetry comes into play only because of the finite extension of the ratchetet objects, namely, the kinks and antikinks born by the string.

III. NUMERICAL ANALYSIS

We simulated [17] a *one-kink* bearing chain ($x\to i\Delta x$)

$$\ddot{\phi}_i-c_0^2\Delta_2\phi_i+V'[\phi_i]=F-\alpha_0\dot{\phi}_i+\zeta_i(t), \tag{16}$$

with $i=1,2,\ldots,N$, where $\Delta x=1$, $\Delta_2\phi_i=\phi_{i+1}+\phi_{i-1}-2\phi_i$, and $V[\phi]$ denotes either on-site potential (4) or (7). The chain $\{\phi_i\}$ is free-end ($\phi_0=\phi_1, \phi_{N+1}=\phi_N$) with $\phi_1=0$ and $\phi_N=2\pi$ to accommodate one kink and long enough for the moving kink not to experience boundary forces [3]. Details of the integration code employed in our numerical study are reported in Refs. [16,17]. Here we limit ourselves to noticing that, due to the presence of the viscous damping term $-\alpha\dot{\phi}_t$, the kink ratchet velocity $\overline{u}$ does not depend appreciably on its initial conditions. This might be an issue at extremely small α values, where the string dynamics is expected to turn chaotic. In such a limit, however, the string configuration would be no longer separable into a linear superposition of stable kinks and antikinks and the present approach would become untenable (see Sec. III B).

A. The overdamped regime

The numerical results of Fig. 2(b) already show that in the limit $\alpha\to\infty$ the kink ratchet velocity $\overline{u}$ decays like $\alpha^{-(1+2\beta)}$ with $\beta=1$ (DP) and $\beta=2$ (RP), as opposed to the standard Smoluchowski law $\beta=0$ applicable to stationary dynamics.

A qualitative explanation of such an asymptotic behavior lies beyond the reach of the adiabatic limit $\Omega\to0$ adopted so far. For small amplitude modulations the kink mass $M(t)$ oscillations are expected to conform to the linear response theory, i.e.,

$$M(t)\simeq M_0+\delta M(\Omega)\sin[\Omega t-\varphi(\Omega)], \tag{17}$$

where $\varphi(\Omega\to0)=0+$ is a phase lag and

$$\frac{\delta M(\Omega)}{\delta M(0)}\sim\left[\frac{1}{1+(\Omega\tau)^2}\right]^{\beta}, \tag{18}$$

with $\tau\equiv\alpha/\omega_{\min}^2$ and $\delta M(0)\equiv\delta M$ given in Eq. (6) for the RP kink and in Eq. (11) for the DP kink.

The result of Eq. (18) was derived through the following perturbation argument [9,18]. Let $\delta\phi(x,t)=\mathcal{O}(F_0)$ quantify the amount of deformation the static kink $\phi_+^{(0)}(x)$ underwent because of the perturbation $F(t)$, i.e., $\phi_+(x,t)=\phi_+^{(0)}(x,t)+\delta\phi(x,t)$. On employing definition (3) the perturbed kink mass reads

$$M(t)\simeq M_0+2\int_{-\infty}^{\infty}[\phi_+^{(0)}(x)]_x[\delta\phi(x,t)]_x dx + \int_{-\infty}^{\infty}[\delta\phi(x,t)]_x^2 dx. \tag{19}$$

The $\mathcal{O}[\delta\phi]$ term on the rhs of Eq. (19) is of the same order as the potential deformation factor $2(\omega_0/c_0)^2\int(V[\phi_+^{(0)}]-V[\phi_+])dx$. Recalling that for the *nondeformable* RP (4) this quantity is $\mathcal{O}(F_0^2)$ [18], we conclude that $\delta M=\mathcal{O}(|\delta\phi|^{\beta})$.

In order to estimate the magnitude of $\delta\phi$ one can replace ϕ in Eq. (1) with $\phi_+^{(0)}+\delta\phi$ and then solve the ensuing phonon equation [9]

$$\delta\phi_{tt}-c_0^2\delta\phi_{xx}=-\omega_0^2V''[\phi_+^{(0)}(x)]\delta\phi-\alpha\delta\phi_t+F(t). \tag{20}$$

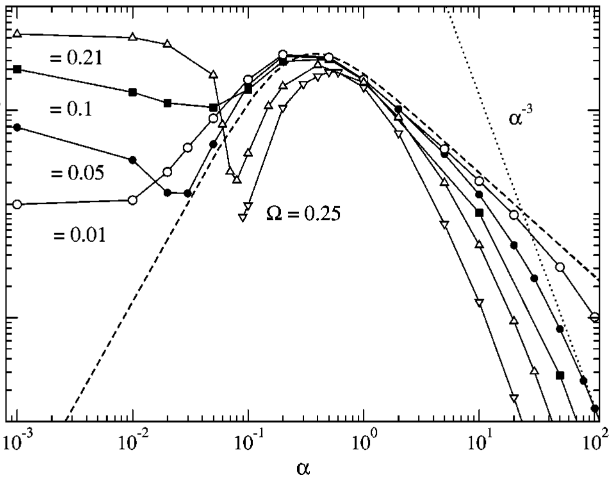

FIG. 3. Kink ratchet velocity $\overline{u}$ vs α (logarithmic scale) on the DP substrate (7) for different values of the ac frequency Ω. Dashed curve: adiabatic approximation (21) with $M(t)$ given by Eq. (10) with $s(t)$ of Eq. (11). Other simulation parameters: $s_0=0.5$, $c_0=50$, and $F_0=0.3$.

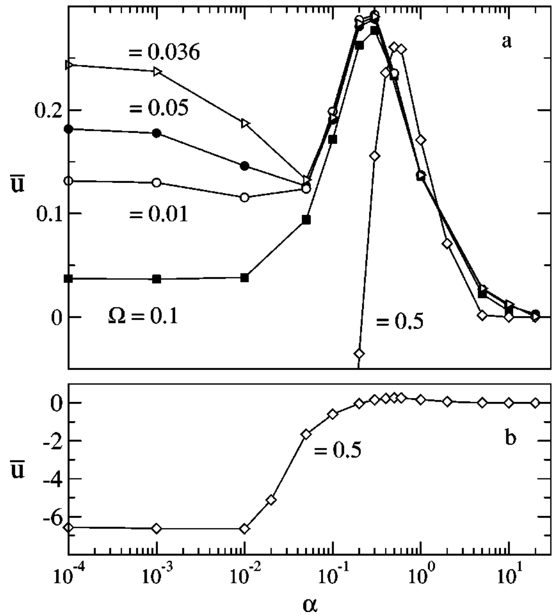

FIG. 4. Kink ratchet velocity $\overline{u}$ vs α (linear scale) on the RP substrate (4) for different values of the ac frequency Ω. Other simulation parameters: $c_0=50$ and $F_0=0.3$.

The kink deformations that contribute the most to the actual value of $M(t)$ are the so-called kink internal mode(s) [15], whose characteristic frequency is typically close to $\omega_{\min}$. In the overdamped limit, $\delta\phi_{tt}\simeq 0$, and for $\Omega \ll \omega_0$, one immediately recovers the linear response formula $|\delta\phi(\Omega)|/|\delta\phi(0)| \sim [1+(\Omega\tau)^2]^{-1}$ and, eventually, Eq. (18) for $\delta M(\Omega)$.

Finally, inserting Eq. (18) into the velocity expression (14) [with $\delta M \rightarrow \delta M(\Omega)$] leads to the asymptotic power law $\overline{u} \propto \alpha^{-(1+2\beta)}$ that fits the simulation data of Fig. 2(b) for $\alpha \gg \omega_0^2/\Omega$ (or $\Omega\tau \gg 1$).

B. Damping constant dependence

Our analysis of the kink ratchet velocity dependence on the damping constant is summarized in Fig. 3 for a DP substrate and in Fig. 4 for a RP substrate, respectively. Curves of $\overline{u}$ α have been drawn at increasing values of the tilt frequency in order to highlight a few important properties of kink ratchets:

(i) *The $\overline{u}(t)$ peak*. The curves $\overline{u}(\alpha)$ peak for an optimal value $\overline{\alpha}$ of the damping constant. At variance with Ref. [15]), we relate this effect to the relativistic nature of the kink dynamics, as on lowering α the nonrelativistic approximation $p \simeq M(t)u$ is no longer tenable. In the parameter regime $F_0 \ll \omega_0$ and $\Omega \rightarrow 0$, but with no restrictions on α, LE (12) yields immediately the relativistic version of Eq. (14),

$$\frac{\overline{u}(\alpha)}{c_0} = \left\langle \frac{aF(t)/\alpha M(t) c_0}{\sqrt{1+[aF(t)/\alpha M(t) c_0]^2}} \right\rangle, \tag{21}$$

plotted in Fig. 3 for the reader's convenience. In the overdamped regime $\alpha \gg \omega_{\min}$, Eq. (21) tends to the Smoluchowski limit (14); as a matter of fact, we know from Sec. III A that this law only applies for $\omega_{\min} \ll \alpha \ll \omega_{\min}^2/\Omega$, that is as long as the adiabatic approximation $\Omega\tau \ll 1$ holds. In the underdamped regime $\alpha \ll \omega_{\min}$, the kink velocity $\overline{u}$ grows quadratically with α, $\overline{u}(\alpha)/c_0 \simeq \underline{\alpha}^2 \delta M(c_0/aM_0F_0)^2$; hence, the appearance of a peak in the $\overline{u}(\alpha)$ curve at an intermediate value $\overline{\alpha}$ of the damping constant.

(ii) *The $\overline{u}(\alpha)$ plateau*. The quadratic branch of $\overline{u}(\alpha)$ predicted by Eq. (21) is actually observable only at fairly small values of the tilt frequency Ω; instead, the curves of Figs. 3 and 4 approach a plateau as α is decreased below a characteristic value of the order of Ω.

The explanation of this property touches upon the mechanism of *phonon damping* [19,20], viz. the interplay of discreteness and relativistic effects. At extremely low α values, the kink responds to an external tilt by raising the modulus of its speed up close to c_0; as a result its effective size, see Eq. (2), contracts until it becomes of the order of the discretization constant Δx introduced in Eq. (16), no matter how large the *static* kink size $c_0/\omega_{\min}$. At this point the kink starts radiating phonons, thus dissipating the excess kinetic energy being pumped into it by the external forcing term; as a consequence, the kink speed levels off, growing insensitive to any further decrease of α; this means that for $\alpha \rightarrow 0$ the effective damping α_{ph} acting on the moving kink is caused mostly by phonon radiation, that is $\alpha_{ph} \gg \alpha$ [17]. As phonon radiation is essentially a resonant process taking place at the bottom of the substrate valleys, we expect the underdamped limit of the phononless law (21) to fail us for $\alpha \lesssim \Omega$ [17], in agreement with our numerics.

One might wonder why in Figs. 3 and 4 the $\overline{u}(\alpha)$ plateaus increase with Ω. In the limit $\alpha \rightarrow 0$, the effective kink damping constant boils down to the phonon damping constant α_{ph}; accordingly, each plateau value $\overline{u}(0)$ is expected to rest close to the corresponding ideal value $\overline{u}(\alpha_{ph})$, obtained from Eq. (21) by replacing α with α_{ph} in the absence of phonon radiation. Moreover, α_{ph} increases with Ω (stimulated phonon radiation) and so does the relevant kink velocity plateau $\overline{u}(0)$, as long as $\alpha_{ph} \lesssim \overline{\alpha}$.

One might also argue that the properties of $\overline{u}(\alpha)$ due to discreteness are a mere numerical artifact, as we actually integrated Eq. (1) instead of Eq. (16) (namely, the dynamics of a chain instead of a string [16]). Although more sophisticated simulation algorithms may be especially devised to eliminate such a difficulty, most of the physical systems we can model by means of an elastic string, are indeed discrete on the microscopic scale [17], such as like dislocations in crystals, magnetic flux lines through layered type II superconducting films, etc. Therefore, the discreteness effects reported here have a physical interest of their own, beyond the moot question of their numerical explanation.

(iii) *$\overline{u}(\alpha)$ inversions.* At higher Ω values instability effects start playing an important role. In Fig. 3 for the DP substrate we notice that an abrupt dynamics change occurs around $\Omega \simeq 0.2$: When on increasing Ω the corresponding plateau $\overline{u}(0)$ attains the maximum of the ideal $\overline{u}(\alpha_{ph})$ curve (21), all of a sudden the actual $\overline{u}(\alpha)$ peak shifts to higher α values. The same behavior was detected for the RP substrate, too [see Fig. 4(a)]. In the case of a DP kink we could not increase Ω any further (over the entire α domain), as a fast kink-antikink nucleation phenomenon sets in, thus making us unable to monitor the time evolution of the tagged kink; in other words, kink and antikink become unstable, thus signaling a route to the chaotic string dynamics. The RP kink is more stable towards periodic tilting (see Sec. III A) and, therefore, we did manage to see what happens on increasing Ω: In the limit $\alpha \to 0$, $\overline{u}(\alpha)$ drops fast but continuously towards negative values, namely, a kink current inversion takes place when $\alpha_{ph} \lesssim \overline{\alpha}$ (not an unusual occurrence in the underdamped particle rocked ratchets [1,2,21]). The Ω dependence of $\overline{u}(\alpha \simeq 0)$ in the adiabatic regime is shown in the inset of Fig. 5.

C. Frequency dependence

The dependence of the kink ratchet velocity $\overline{u}$ on the modulation frequency Ω is also interesting. In Fig. 5 we display three curves $\overline{u}(\Omega)$ for the RP kink in the low damping regime. The α values were chosen in order to explore the Ω dependence of the $\overline{u}(\alpha)$ peak of Fig. 4(a). All curves in Fig. 5 exhibit a main (positive) resonance peak at $\Omega = \omega_b$ with $\omega_b \simeq 0.8\omega_{\min}$; the peak height increases with decreasing α. At the lowest α value, a secondary (negative) peak shows up at half the fundamental resonance frequency, i.e., for $\Omega \sim \omega_b/2$. The onset of such a negative peak is clearly related to the $\overline{u}$ inversion shown in Fig. 4(b).

The fundamental frequency ω_b lends itself to a simple interpretation in terms of the perturbation Eq. (20). By means of standard numerical integration (see also Ref. [15]), one can prove that the RP kink admits of one internal mode (or bound state) whose eigenfrequency coincides right with ω_b. In the presence of the periodic kink mass modulation $M(t)$, Eq. (17), such an internal mode gets easily excited, hence the main resonance peak of Fig. 5. When accounting for higher order corrections, Eq. (19), one soon recognizes that the time-dependent kink profile sustains $n\Omega$ harmonics of the drive frequency (with amplitude decreasing with increasing n); therefore, for sufficiently small α values, the kink deformation $\delta\phi$ (viz. the kink mass) may appear to resonate under the parametric condition $2\Omega \sim \omega_b$ [22]. In the limit of vanishingly small ac frequencies the ratchet current turns positive again as explained in item (iii) of Sec. III B.

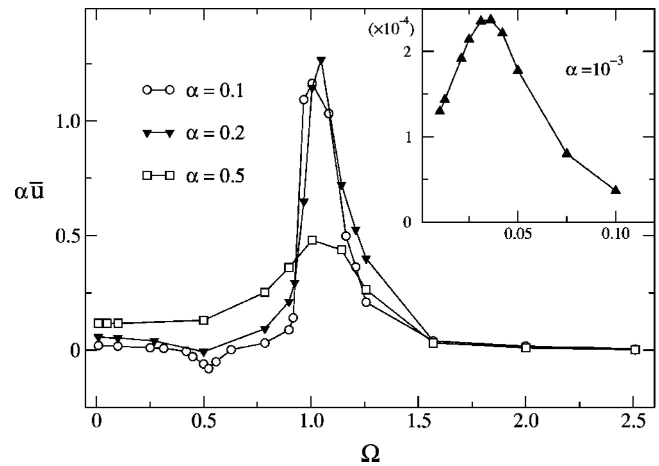

FIG. 5. Kink ratchet velocity $\overline{u}\ \Omega$ on the RP substrate (4) for different values of the damping constant α. Note that $\overline{u}$ has been rescaled by the relevant factor α. Inset: the extremely underdamped case $\alpha = 10^{-3}$ in the adiabatic regime of small Ω. Other simulation parameters: $c_0 = 50$ and $F_0 = 0.3$.

The argument above, while locating correctly the position of the resonance peaks of $\overline{u}(\Omega)$, fails to explain the current inversion corresponding to the onset of the parametric resonance $2\Omega \sim \omega_b$. This question would require a sophisticated perturbation analysis [23] capable of reaching beyond the leading order approximation adopted throughout the present work. On the other hand, current inversions in particle rocked ratchet at low damping and/or high frequency [1,2,21] have never been related to a resonant dynamics.

IV. CONCLUSIONS

In the present work we addressed the problem of ac driven asymmetric kinks. The kink ratchet current has been shown to exhibit nontrivial dependence on both the string damping constant and the drive frequency. More importantly, we have analyzed two classes of asymmetric kinks: (I) The RP kinks, whose asymmetry is built-in due to the anisotropy of the substrate. In such a case the kink shape (and mass) gets modulated in second order and, consequently, the relevant kink ratchet velocity decays like α^{-5} in the overdamped regime. The existence of a ratchet mechanism for this class of kinks is largely expected in view of the rocked ratchet theory for pointlike Brownian particles; (II) The DP kinks, supported by a deformable symmetric substrate, whose valleys and barriers change continuously in time; substrate modulation and external tilt are arbitrarily phase locked. The ensuing kink mass modulation is of the first order in the substrate perturbation, as proved by the asymptotic α^{-3} decay of the corresponding kink ratchet velocity. The ratchet mechanism for a string on a symmetric DP substrate has no counterpart in the theory of particle ratchets.

As stated in the Introduction, we neglected the presence of noise sources, otherwise required for the string to thermalize. The role of thermal fluctuations as the trigger of kink-antikink nucleation has been mimicked by simulating a *one-kink* bearing string. In our strategy a full understanding of the dynamics of an individual ac driven kink ought to allow us to reconstruct *a posteriori* the corresponding time evolution of the entire string. Noise, however, is capable of influencing the diffusion of a single kink, as well, a possibility ignored altogether in the present study. It is believed, though, that at low temperatures (i.e., in the presence of low intensity fluctuation sources), the ratchet efficiency is only marginally affected by noise. The diffusion of an ac driven asymmetric kink in equilibrium at finite temperature and, eventually, the *ab initio* simulation of a *thermalized* string diffusing on a rocked asymmetric substrate are certainly topics that deserve more accurate investigation.

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