

## Lagrangian velocity structure functions in Bolgiano turbulence

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Single-particle Lagrangian velocity statistics in the Bolgiano–Obukhov regime of two-dimensional turbulent convection is investigated. At variance with flows displaying the classical K41 phenomenology, here the leading contribution to the Lagrangian velocity fluctuations is given by the largest eddies. This implies a linear behavior in time for a typical velocity fluctuation in the time interval  $t$ . The contribution to the Lagrangian velocity fluctuations of local eddies (i.e., with a characteristic time of order  $t$ ), whose space/time scalings are ruled by the Bolgiano–Obukhov theory, is thus not detectable by standard Lagrangian statistical observables. To disentangle contributions arising from the large eddies from those of local eddies, a strategy based on exit-time statistics has successfully been exploited. Lagrangian velocity increments in Bolgiano convection thus provide a physically relevant example of a signal with *more than smooth* fluctuations.

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Understanding the statistical properties of particle tracers advected by turbulent flows is a fundamental problem in turbulence research and a key ingredient for the development of stochastic models of Lagrangian dispersion.<sup>1–3</sup> In recent years, there has been great improvement in theoretical,<sup>4</sup> experimental,<sup>5–7</sup> and numerical<sup>8,9</sup> understanding of Lagrangian turbulence. Most of the studies have been concentrated on the so-called single-particle dispersion in which the statistical objects are the Lagrangian velocity differences following a single trajectory  $\delta\mathbf{v}(t) = \mathbf{v}(t) - \mathbf{v}(0)$ . For this quantity, dimensional analysis in statistically stationary, homogeneous, and isotropic turbulence predicts  $\langle \delta v^2 \rangle \approx \varepsilon t$  (where  $\varepsilon$  is the mean energy dissipation). The remarkable coincidence with diffusive-like behavior is at the basis of stochastic models of turbulent dispersion.

In this Letter, we investigate on the basis of direct numerical simulations (DNS) the statistics of single-particle dispersion in two-dimensional Boussinesq convective turbulence. This system is characterized by an inverse cascade with scaling exponent in agreement with the Bolgiano–Obukhov theory of turbulent convection.<sup>10</sup> As a consequence, we expect a “nondiffusive” behavior for Lagrangian velocity variance. We show that a careful statistical analysis, based on exit time statistics, is necessary in order to observe the expected scaling.

The two-dimensional Boussinesq turbulent convection is described by the following set of partial differential equations:<sup>11</sup>

$$\partial_t \omega + \mathbf{v} \cdot \nabla \omega = \nu \Delta \omega - \beta \nabla T \times \mathbf{g}, \quad (1)$$

$$\partial_t T + \mathbf{v} \cdot \nabla T = \kappa \Delta T,$$

where  $T$  is the temperature field,  $\omega$  is the vorticity,  $\mathbf{g}$  is the gravitational acceleration,  $\beta$  is the thermal expansion coefficient,

and  $\kappa$  and  $\nu$  are molecular diffusivity and viscosity, respectively. Energy in (1) is injected by maintaining a mean temperature profile  $\langle T(\mathbf{r}, t) \rangle = \mathbf{G} \cdot \mathbf{r}$ , with a constant gradient  $\mathbf{G}$  pointing in the direction of gravity.

In Eq. (1), the temperature field affects the vorticity through the buoyancy force, thus providing an example of *active* tracers. In particular, buoyancy force equilibrates the inertial terms in the velocity dynamics, thus providing the mechanism on the basis of the Bolgiano–Obukhov theory of turbulent convection.

Let us briefly recall the dimensional argument leading to Bolgiano scaling in turbulent convection. Temperature fluctuations injected at large scales by the mean gradient are transported to small scale at a constant rate  $\varepsilon_T \sim \langle \delta_r T^2 \delta_r v \rangle / r$ . Assuming a balance between buoyancy and inertial terms in Eq. (1),  $(\delta_r v)^2 \sim \beta g r \delta_r T$ , one ends up with<sup>12</sup>

$$\delta_r T \approx \frac{\varepsilon_T^{2/5}}{(\beta g)^{1/5}} r^{1/5}, \quad \delta_r v \approx \varepsilon_T^{1/5} (\beta g)^{2/5} r^{3/5}, \quad (2)$$

which leads to the following prediction for the velocity structure functions:

$$S_p(r) = \langle (\delta_r v)^p \rangle \sim (\varepsilon_T r)^{p/3} \sim r^{\zeta_p} \quad (3)$$

with  $\zeta_p = 3p/5$ . The exponent of the second-order structure function ( $\zeta_2 = 6/5$ ) gives the scaling exponent for the energy spectrum  $E(k) \approx k^{-11/5}$ , which is shown in Fig. 1. The Bolgiano spectrum has been recently observed in two-dimensional laboratory experiments<sup>13</sup> while extensive numerical simulations have shown that velocity fluctuations follow the dimensional scaling (3) without intermittency corrections.<sup>11</sup> We remark that the above argument does not involve the so-called Bolgiano scale,<sup>12</sup> which is the scale below which inertial terms in (1) dominate and temperature

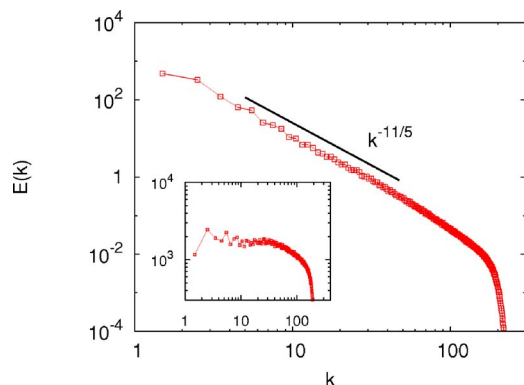


FIG. 1. Energy spectrum from direct numerical simulations of Boussinesq equations at resolution  $N=1024^2$  in stationary conditions. The line represents the dimensional prediction  $E(k) \approx k^{-11/5}$  and the inset shows the compensated spectrum.

behaves as a passive scalar. Indeed, in our setup energy is injected by the mean temperature gradient, therefore inertial terms never dominate and the Bolgiano scale is formally zero.

The dimensional prediction for the statistics of Lagrangian velocity increments  $\delta v = v(t) - v(0)$  is the following. Considering the velocity  $v$  as the superposition of the contributions from eddies of different sizes, the variation  $\delta v$  over a time  $t$  will be given by the superposition of the variations associated with eddies. The eddies at scale  $r$ , with a characteristic time  $\tau_r \ll t$ , will be decorrelated and thus give no contribution. The eddies for which  $\tau_r \approx t$  are the first ones to be considered. Assuming a scaling exponent  $h$  for the velocity, the characteristic turnover time is  $\tau_r \approx \tau_L (r/L)^{1-h}$  ( $\tau_L$  being the characteristic time of eddies at the large scale  $L$ ) and thus the scale of the eddies that decorrelates in a time  $t$  is given by  $r \approx L(t/\tau_L)^{1/(1-h)}$ . Their contribution to the Lagrangian velocity fluctuation is thus estimated as

$$\delta v \approx v_L (r/L)^h \approx v_L (t/\tau_L)^q, \quad (4)$$

where  $v_L$  represents velocity fluctuations at large scales and  $q \equiv h/(1-h)$ .

On top of this, the eddies at the large scales of the order of  $L$  have to be considered. At these scales  $t \ll \tau_L$  and the contribution to velocity fluctuations is differentiable, i.e.,  $\delta v \approx (\partial_t v_L)t$ . The typical velocity fluctuation on the time interval  $t$  is then given by the superposition of two contributions,

$$\delta v \approx \tau_L (\partial_t v_L)(t/\tau_L) + v_L (t/\tau_L)^q. \quad (5)$$

At small  $t/\tau_L$ , the dominant term will be the one with the minimum exponent,  $\delta v \sim \min(1, q)$ . In the framework of the classical K41 theory ( $h=1/3$ ,  $q=1/2$ ) the dominant contribution in (5) is the local one, which leads to the diffusive behavior  $\delta v \sim t^{1/2}$ . In the present case of Bolgiano convection, we have  $q=3/2$  ( $h=3/5$ ) and thus velocity increments in the inertial range are dominated by the infrared term  $\propto t$ . Therefore, a standard analysis of velocity fluctuations, i.e., Lagrangian structure functions  $S_p^L(t) = \langle (\delta v)^p \rangle$ , is unable to disentangle the Bolgiano–Obukhov scaling in the Lagrangian statistics (see Fig. 2).

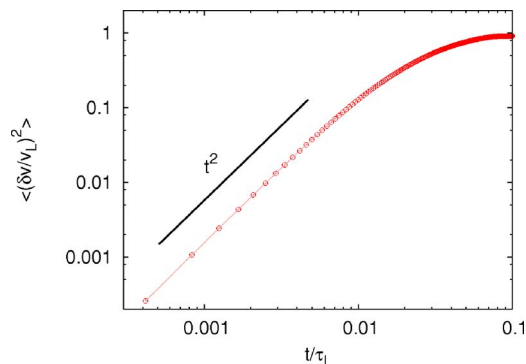


FIG. 2. Second-order Lagrangian structure function. The dominant ballistic regime ( $\delta v \approx t$ ) is evident.

Lagrangian velocity increments in Bolgiano convection provide a physically relevant example of a signal with *more than smooth* fluctuations (i.e., with scaling exponent  $q > 1$ ). The statistical analysis of this kind of signal has been recently addressed on the basis of an *exit-time statistics*.<sup>14</sup> In the following, we will discuss the application of this approach to the present case and the resulting bifractal distribution of scaling exponents.

The exit-time statistics is based on the time increments  $T(\delta v)$  needed for a tracer to observe a change of  $\delta v$  in its velocity.<sup>15</sup> Given the set of thresholds  $\delta v_n = \rho^n \delta v(0)$  (with  $\rho > 1$ ), one computes the exit times  $T_n$  corresponding to each threshold following the tracer trajectories. Now, let us assume we have a signal  $\delta v$  composed of two contributions as in (5). In the limit of small  $\delta v$ , the differentiable part  $\propto t$  will dominate except when the derivative  $\partial_t v_L$  vanishes and the local part thus becomes the leading one. For a signal with  $1 \leq q \leq 2$ , its first derivative is a one-dimensional self-similar signal with scaling exponent  $\xi = q - 1$ , which thus vanishes on a fractal set of dimension  $D = 1 - \xi = 2 - q$ . Therefore, the probability to observe the component  $O(t^q)$  is equal to the probability to pick a point on the fractal set of dimension  $D$ , i.e.,<sup>16</sup>

$$P(T \sim \delta v^{1/q}) \sim T^{1-D} \sim (\delta v)^{1-1/q}. \quad (6)$$

By using this probability for computing the average  $p$ -order moments of exit-time statistics, one obtains the following bifractal prediction:<sup>14</sup>

$$\langle T^p(\delta v) \rangle \sim \delta v^{\chi(p)}, \quad \text{with } \chi(p) = \min\left(p, \frac{p}{q} + 1 - \frac{1}{q}\right). \quad (7)$$

According to prediction (7), low-order moments ( $p \leq 1$ ) of the inverse statistics only see the differentiable part of the signal, while high-order moments ( $p \geq 1$ ) are dominated by the local fluctuations  $O(t^q)$ .

We now turn to the numerical procedure. The turbulent velocity field is obtained by direct integration of the convective Navier–Stokes equation (1) with a standard, fully dealiased pseudospectral method in a doubly periodic square domain with a second-order Runge–Kutta scheme in time. A linear friction term  $-\alpha \omega$  is added to the vorticity equation in

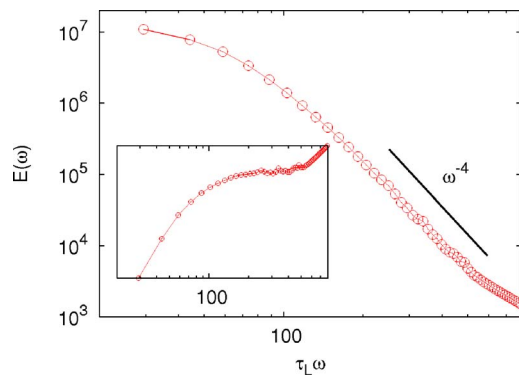


FIG. 3. The spectrum  $E(\omega)$  of the Lagrangian velocity increments  $\delta v$ . The line represents the prediction  $\omega^{-4}$ . The inset shows the compensated plot  $E(\omega)/\omega^{-4}$ . The  $\omega^{-4}$  contribution is due to the superimposed *more than smooth* fluctuations, while the tail of the spectrum is dominated by the contributions due to the nonperiodicity of the signal.

order to remove energy at large scale. As is customary, the viscous term is replaced by hyperviscosity for numerical convenience.<sup>17</sup>

Equations are integrated for about 100 large-scale eddy turnover times to reach a stationary state. Lagrangian trajectories are then obtained by integrating  $\dot{\mathbf{x}}(t)=\mathbf{v}(\mathbf{x}(t),t)$  with the velocity at particle position obtained by linear interpolation from the nearest grid points. A single run follows 64000 particles, homogeneously distributed on the integration domain. Velocity variations  $\delta v$  and exit times  $T(\delta v)$  of single particles are recorded for times comparable to the integral scale. The average is performed over about 150 large-scale eddy turnover times.

In Fig. 2, we show the variance of Lagrangian velocity fluctuations as a function of time. The smooth behavior  $\langle(\delta v)^2\rangle\sim t^2$  is clearly recognizable, making impossible the observation of the Bolgiano–Obukhov scaling. We remark that the same result is expected for any system with large-scale domination over the local contribution, i.e., for any  $q\leq 1$ . Therefore, for these systems (with  $h\leq 1/2$ ) the direct computation of Lagrangian structure functions (3) is unable to disentangle the turbulent components in the velocity field.

A first check of Bolgiano scaling in Lagrangian statistics is obtained by computing the temporal energy spectrum

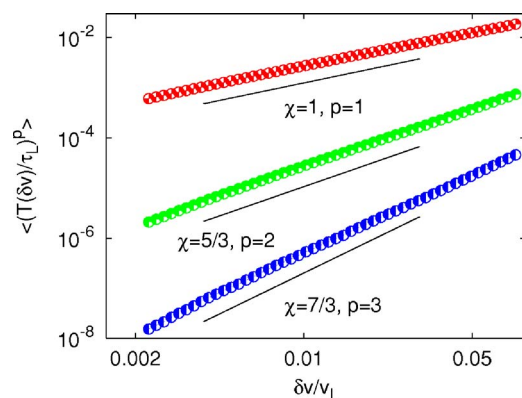


FIG. 4. Exit times  $\langle T^p(\delta v) \rangle$  of order  $p=1,2,3$  vs  $\delta v$ . Lines represent the predicted power-law behavior (7) with exponents  $\chi(p)$ .

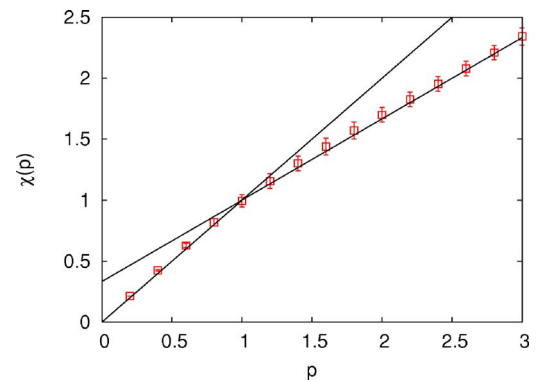


FIG. 5. Scaling exponents  $\chi(p)$  for the moments of the exit times  $\langle T^p(\delta v) \rangle$ . The lines represent the bifractal prediction (7). The error bars on the exponents have been estimated by evaluating differences in  $\chi(p)$  changing the fitting interval.

$E(\omega)$ . We remind the reader that for an infinitely differentiable signal, the smooth contribution gives an exponential spectrum, while the  $\delta_i v \sim t^{3/2}$  contribution gives a spectrum proportional to  $\omega^{-4}$ . Figure 3 clearly shows the expected  $\omega^{-4}$  behavior for an intermediate range of frequencies. The high frequencies are dominated by the effects of nonperiodicity of the signal.

We now turn to the exit-time analysis. Figure 4 shows the first few moments of exit times  $\langle T^p(\delta v) \rangle$ , which display a clear power-law scaling in the range  $2 \times 10^{-3} v_L \leq \delta v \leq 5 \times 10^{-2} v_L$ . We were able to compute the moments up to  $p=3$  with statistical significance. The set of scaling exponents obtained from a best fit in this range is shown in Fig. 5. The bifractal spectrum predicted by (7) is clearly reproduced. We remark that the fact that for  $p>1$  exponents follow the linear behavior  $\chi(p)=(2p+1)/3$  indicates the absence of intermittency in Lagrangian statistics in this flow. This feature is a consequence of the self-similarity of the inverse cascade in two-dimensional Bolgiano convection.

In conclusion, we have analyzed single-particle dispersion in the two-dimensional Bolgiano turbulent convection. A remarkable property of this turbulent regime is that single-particle dispersion turns out to be dominated by the large-scale eddies, whose contribution to velocity fluctuations is linear in time. A completely different scenario thus emerges with respect to the classical single-particle dispersion in a flow that displays the standard K41 phenomenology. The Bolgiano–Obukhov contribution to the scaling of Lagrangian velocity increments turns out to be undetectable by standard Lagrangian structure functions. To disentangle its effect from the (trivial) one played by large scales, we have exploited a method of analysis based on exit-time statistics for signals having more than smooth fluctuations. The obtained results clearly permit us to identify the contribution of “local” eddies to the Lagrangian velocity fluctuations.

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