Ergod. Th. & Dynam. Sys. (2000), **20**, 963–990
Printed in the United Kingdom

Limit at resonances of linearizations of some complex analytic dynamical systems

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(*Received* 15 *May* 1998 *and accepted in revised form* 5 *May* 1999)

Abstract. We consider the behaviour near resonances of linearizations of germs of holomorphic diffeomorphisms of $(\mathbb{C}, 0)$ and of the semi-standard map.

We prove that for each resonance there exists a suitable blow-up of the Taylor series of the linearization under which it converges uniformly to an analytic function as the multiplier, or rotation number, tends non-tangentially to the resonance. This limit function is explicitly computed and related to questions of formal classification, both for the case of germs and for the case of the semi-standard map.

1. *Introduction*

It is well known since Poincaré's work on normal forms and celestial mechanics that resonances are responsible for the divergence of the series expansions of quasi-periodic motions in the theory of dynamical systems. In the simplest non-trivial case of one-dimensional holomorphic systems this difficulty already appears when one tries to conjugate a given system to its linear part $z \mapsto \lambda z$ in a neighbourhood of a fixed point $z = 0$.

For a fixed value of the multiplier λ, the linearization (when it exists) has a complicated analytic structure with respect to its argument. Almost nothing is known, except for some numerical studies (e.g. [**1**], Figures 6 and 7 of [**2**] and references quoted therein, [**3, 4**]).

If one considers the dependence of the linearization on λ, resonances appear as formal poles of the Taylor series of the conjugation: it is the famous phenomenon of 'small denominators'. In some sense (see §2.2), the unit circle $\mathbb{T}^1$ is a natural boundary of analyticity for the conjugation with respect to the multiplier λ. Nevertheless the series has non-tangential limits at almost all points of $\mathbb{T}^1$ (see §2.1 for more information and [**5, 6**] for a detailed study in the case of analytic diffeomorphisms of the circle). The relation between the formal poles (associated to the resonances) of the linearization and its natural boundary of analyticity calls for some enlightenment, especially if one places the problem in the perspective of Borel's theory of uniform monogenic functions [**7**].

In this paper we prove some results concerning the behaviour at resonances of the linearizations of local diffeomorphisms of $\mathbb{C}$ (see e.g. [**8**]) and of the so-called 'semi-standard map' [**1, 10**]. These two models are actually tightly related (compare §2.6 and §3.1).

In particular, for each of the systems we take into account, we prove that the linearizations have well defined limits as the multiplier (or the rotation number) tends non-tangentially to a root of unity (or a rational value p/q) *provided the Taylor series is suitably blown-up*, and we compute these limits. In the case of germs, this generalizes a result of Yoccoz [**8**, §3.3 and §3.4, pp. 73–78], as he computes this limit only for the quadratic family $P_\lambda(z) = \lambda(z - z^2)$.

These limit functions have a finite radius of convergence and ramification points. By reversal of the blow-up, they provide an approximation of the linearization when the multiplier is very close to a resonance.

Of course it would be quite interesting to generalize these results to general analytic small divisor problems (Hamiltonian systems, real analytic area-preserving twist maps, etc). The standard map case was studied in [**2**]: by a suitable blow-up the Lindstedt series has an explicitly computable limit at the resonances $0/1$ and $1/2$. The same is true for all resonances p/q, as recently proved in [**9**]. We also hope to be able to apply the method presented in §3 in order to improve these studies.

2. *Germs of holomorphic diffeomorphisms of* $(\mathbb{C}, 0)$

2.1. *The linearization of a germ.* Let G denote the group of germs of holomorphic diffeomorphisms of $(\mathbb{C}, 0)$. An important problem in complex dynamics is to describe the conjugacy classes of this group. Since the origin is a fixed point one has some natural conjugacy invariants such as the multiplier $\lambda = f'(0)$ of the germ f at 0 and the holomorphic index:

$$i(f, 0) = \frac{1}{2\pi i} \oint \frac{dz}{z - f(z)}$$

where we integrate on a small loop in the positive direction around zero. If $\lambda \neq 1$, the origin is a simple fixed point and one clearly has

$$i(f, 0) = \frac{1}{1 - \lambda}.$$

Let G_λ denote the set of $f \in G$ such that $f'(0) = \lambda$. Such a germ f is said to be linearizable if it belongs to the conjugacy class of the rotation $R_\lambda(z) = \lambda z$. This is always the case if $|\lambda| \neq 1$: by the Poincaré–Königs linearization theorem [**11, 12**], one knows that there exists a germ $h_f \in G_1$ such that

$$f \circ h_f = h_f \circ R_\lambda. \tag{2.1}$$

The condition $h_f \in G_1$ ensures the unicity of the solution of this conjugacy equation. From now on, when speaking of conjugacy classes of G without other specification, we shall always refer to the adjoint action of the subgroup G_1—not the whole group G.

The function h_f is called the *linearization* of f: attracting ($|\lambda| < 1$) and repelling ($|\lambda| > 1$) fixed points of the holomorphic diffeomorphisms that are linearizable. If one keeps $f - R_\lambda$ fixed as λ varies, or more generally if f depends analytically on the parameters, then by uniform convergence the dependence of the linearization h_f is also analytic. If f is an entire function and $|\lambda| > 1$ then h_f is an entire function.

When $|\lambda| = 1$ the fixed point at the origin is *indifferent* and one must distinguish three different kinds of multiplier:

(1) *parabolic* or *resonant point*: $\lambda = \Lambda = \exp(2\pi i(p/q))$, where $p \in \mathbb{N}$, $q \in \mathbb{N}^*$, $(p|q) = 1$;

(2) *Brjuno point*: $\lambda = B = \exp(2\pi i\omega)$, $\omega \in \mathbb{R} \setminus \mathbb{Q}$ and ω is a *Brjuno number* [**13, 14**]: $\sum_{k=0}^{\infty} \log q_{k+1}/q_k < +\infty$, where $\{p_k/q_k\}$ denotes the sequence of partial fractions of the continued fraction expansion of ω;

(3) *Cremer point*: $\lambda = \exp(2\pi i\omega)$, $\omega \in \mathbb{R} \setminus \mathbb{Q}$ and ω is *not* a Brjuno number.

Actually, Cremer proved [**15**] that G_λ is not a conjugacy class if

$$\sup_n \frac{\log q_{n+1}}{q_n} = \infty;$$

however, we think that it is quite fair to give his name to the complement of the Brjuno set.

In case (1) Écalle [**16, 17**] and Voronin [**18**] gave a complete classification of the conjugacy classes of G contained in G_Λ; among them the class of the rotation R_Λ consists of all elements of order q belonging to G_Λ, but there are other conjugacy classes in G_Λ.

In case (2), G_λ is a conjugacy class of G [**13**] (as in the hyperbolic case $|\lambda| \neq 1$): for all f in G_λ there exists a unique analytic linearization h_f. In 1987 Yoccoz [**8**] proved that in case (3) G_λ is not a conjugacy class and there exists at least one nonlinearizable germ $f \in G_\lambda$. A remarkable example is given by the quadratic polynomial

$$P_\lambda(z) = e^{2\pi i\omega}(z - z^2)$$

which is linearizable if and only if ω is a Brjuno number.

Our main motivation in this study is to understand how the formal poles due to resonances give rise to the complicated analytic structure of the linearization. For that reason we shall let λ vary in a non-tangential cone† with vertex at any root of unity Λ, and we shall treat it as a bifurcation parameter. Consequently, the germ f itself will vary—we thus denote it by f_λ henceforth—and its linearization h_{f_λ} will be studied as a function of

† By 'non-tangential' we mean: non-tangential to the unit circle; a non-tangential cone with vertex at Λ is any set $V = C(\Lambda, \alpha)$ with the notation of formula (2.10).

λ, singular at the resonance Λ. We shall provide all of the details for the situation where the nonlinear part $f_\lambda - R_\lambda$ is kept fixed as λ varies (this assumption is quite natural: see [**5, 6, 19**, ch. III] for the related problem of the dependence of the linearization in one-parameter families of analytic circle diffeomorphisms), but we shall also give (at the end of the next section) the corresponding statements for a more general dependence of f_λ on the parameter λ.

Note that if, instead of a sectorial neighbourhood of a resonance, one considers a non-tangential cone V with vertex at a Brjuno point $B = e^{2\pi i\omega}$, an easy† but interesting property of the linearization is its continuity with respect to λ (i.e. h_{f_λ} tends to h_{f_B} as λ tends to B inside V).

More generally if one assumes that f_{λ_0} is linearizable for a certain λ_0 satisfying $|\lambda_0| = 1$, when λ tends to λ_0 non-tangentially the linearizing maps h_λ are univalent on a small uniform disk‡ (they thus form a compact family) and any limit of these maps when λ tends to λ_0 is a linearizing map for f_{λ_0}. These limits clearly coincide if λ_0 is not resonant, otherwise this is a consequence of Theorem 2.1 below.

In what follows, putting the vertex of V at a resonance Λ, we shall prove the existence of a suitable scaling under which h_{f_λ} has a non-tangential limit. These scalings are a slight generalization of the notion of blow-up of a formal power series which is standard in algebraic geometry and which has already been applied to the study of complex analytic differential equations [**20**, §III]. We shall also compute these limits and study their relationship to the classification of *formal* conjugacy classes of G_Λ.

2.2. *Existence of non-tangential limits.* Let us consider the one-parameter family of germs of G_λ:

$$f_\lambda(z) = \lambda z + \sum_{n=2}^{\infty} f_n z^n,$$

where we keep fixed the coefficients $\{f_n\}_{n=2}^{\infty}$, i.e. the nonlinear part $f_\lambda - R_\lambda$ is constant. We denote simply by h_λ the corresponding linearization. We have the following theorem (compare with §3.3 and §3.4 of [**8**]).

† Let us consider the intersection V_ρ of V with a disk of centre B and of small radius ρ, i.e. $V = C(B, \alpha, \rho)$ with the notation of formula (2.10). It is immediate to check from the relation (2.1) which defines the linearization, that $h_{f_\lambda} = z + \sum h_n(\lambda) z^n$, where each coefficient h_n ($n \geq 2$) is a finite sum of rational functions of the form $(\lambda^{j_1} - \lambda)^{-1} \cdots (\lambda^{j_{n-1}} - \lambda)^{-1}$ with $2 \leq j_1, \ldots, j_{n-1} \leq n$. Clearly h_n is analytic in V_ρ and continuous on its closure $\bar{V}_\rho$; the uniform convergence follows from the standard Brjuno's argument, since there exists a positive constant C such that $\forall \lambda \in \bar{V}_\rho, \forall j \geq 2, |\lambda^j - \lambda|^{-1} \leq C|B^j - B|^{-1}$ (because there exists $c > 0$ such that any point of $\bar{V}_\rho$ can be written $e^{2\pi i\theta}$ with $\forall q \in \mathbb{N}^*$, $\mathrm{dist}(q\theta, \mathbb{Z}) \geq c\ \mathrm{dist}(q\omega, \mathbb{Z})$, and there exist $c', c'' > 0$ such that $\forall x \in \mathbb{C}$, $|e^{2\pi i x} - 1| \geq c'\,\mathrm{dist}(x, \mathbb{Z})$ and $\forall x \in \mathbb{R}$, $|e^{2\pi i x} - 1| \leq c''\,\mathrm{dist}(x, \mathbb{Z})$, thus $\forall q \in \mathbb{N}^*, \forall \lambda \in \bar{V}_\rho, |\lambda^q - 1| \geq (c'c/c'')|B^q - 1|$).

‡ This is very easy to see. Denote by h_{λ_0} a linearizing map for f_{λ_0}. Then $g_\lambda(z) = (h_{\lambda_0}^{-1} \circ f_\lambda \circ h_{\lambda_0})(z)$ satisfies $g_\lambda(0) = 0$, $g'_\lambda(0) = \lambda$, $g_{\lambda_0}(z) = \lambda_0 z$. Thus $z^{-1} g_\lambda(z) - \lambda_0 = (\lambda - \lambda_0)(1 + zS(\lambda, z))$ (for some analytic function S), and for any non-tangential neighbourhood V_ρ of λ_0 there exists $r_0 > 0$ such that for any $\lambda \in V_\rho \setminus \{\lambda_0\}$ and for any z, $|z| < r_0$, one has $|\lambda| > 1 \Rightarrow |g_\lambda(z)| > |z|$ and $|\lambda| < 1 \Rightarrow |g_\lambda(z)| < |z|$. One deduces from this that, as λ varies in V_ρ, the radius of injectivity of h_λ is uniformly bounded from below by a strictly positive constant (independent of λ).

THEOREM 2.1. *Let us fix a resonance $\Lambda = \exp(2\pi ip/q)$ where $p \in \mathbb{N}$, $q \in \mathbb{N}^*$, $(p|q) = 1$. There are two possibilities, according to the qth iterate $f_\Lambda^{\circ q}$ of f_Λ.*

- *If $f_\Lambda^{\circ q}$ is the identity, the formulae*

$$T(z) = \frac{1}{q\Lambda^{q-1}} \frac{\partial}{\partial\lambda}(f_\lambda^{\circ q}(z))|_{\lambda=\Lambda}, \quad \mathcal{U}(z) = z \exp\left(\int_0^z \left(\frac{1}{T(z_1)} - \frac{1}{z_1} \right) dz_1 \right) \tag{2.2}$$

  define a germ $\mathcal{U} \in G_1$ whose reciprocal h_Λ linearizes f_Λ and has the following property. For all non-tangential cones V of $\mathbb{C}$ with vertex Λ (non-tangential to the unit circle), the linearization h_λ tends uniformly on some small open disk around zero to h_Λ as λ tends to Λ inside V.

- *If $f_\Lambda^{\circ q}$ is the not the identity, there exists a positive integer k and a non-zero complex number A such that*

$$f_\Lambda^{\circ q}(z) = z + Az^{kq+1} + O(z^{kq+2}),$$

  and for any non-tangential cone V of $\mathbb{C}$ with vertex Λ, the rescaled linearization

$$\tilde{h}_\lambda(z) = \frac{1}{(\lambda - \Lambda)^{1/kq}} h_\lambda((\lambda - \Lambda)^{1/kq} z) \tag{2.3}$$

  tends uniformly on some small open disk around zero to the function

$$\tilde{h}_\Lambda(z) = z \left(1 - \frac{A}{q\Lambda^{q-1}} z^{kq} \right)^{-1/kq} \tag{2.4}$$

  as λ tends to Λ inside V. (One can choose any of the kq determinations of $(\lambda - \Lambda)^{1/kq}$.)

Remark 2.1. Note that in the case $f_\Lambda^{\circ q} = \mathrm{Id}$ the linearization of f_Λ is not unique. What Theorem 2.1 asserts is that, among all the linearizations of f_Λ, one of them is the limit of h_λ as λ tends to Λ non-tangentially. The formulae (2.2) make sense and define $\mathcal{U} \in G_1$ as claimed in the theorem, since $T(z) = z(1 + O(z))$; in fact $\mathcal{U}$ is the unique solution in G_1 of the ordinary differential equation $\mathcal{U}' = \mathcal{U}/T$.

Remark 2.2. In the next section, we give some details about the numbers k and A which appear in the theorem. They are classically introduced as *formal* invariants of f_Λ: given two germs f_Λ and g_Λ in G_Λ, the existence of a formal series $\varphi(z) = z + O(z^2)$ such that

$$g_\Lambda = \varphi^{-1} \circ f_\Lambda \circ \varphi,$$

implies

$$k(f_\Lambda) = k(g_\Lambda) \quad \text{and} \quad A(f_\Lambda) = A(g_\Lambda),$$

but the converse is not true: a third formal invariant is necessary in order to describe all of the formal conjugacy classes. Thus, the limit $\tilde{h}_\Lambda$ depends only on the formal conjugacy class of f_Λ, but does not determine it completely.

Remark 2.3. Note that

$$\tilde{h}_\Lambda = z\left(1 - \frac{A\Lambda}{q}z^{kq}\right)^{-1/kq}, \quad \tilde{h}_\Lambda^{-1} = z\left(1 + \frac{A\Lambda}{q}z^{kq}\right)^{-1/kq}$$

and these maps commute with R_Λ.

If we now conjugate $\tilde{h}_\Lambda$ with the inverse scaling we find that, when λ approaches Λ non-tangentially and z approaches the origin in such a way that the quantity $z(\lambda - \Lambda)^{-1/kq}$ remains constant, then

$$h_\lambda(z) \approx z\left(1 - \frac{A\Lambda}{q}\frac{z^{kq}}{\lambda - \Lambda}\right)^{-1/kq}.$$

Therefore, for λ very close to a resonance, the linearization behaves as an analytic function with respect to z with *kq ramification points that collapse at the origin when λ tends to Λ*. We note that this has also been numerically observed (compare with figures 4 and 5 of [**4**]), with a very good quantitative agreement (compare with the discussion in the last section of [**2**] for a related problem concerning the standard map).

Remark 2.4. Note that one may use any scaling $z \to s(\lambda)^{1/kq}z$ instead of $z \to (\lambda - \Lambda)^{1/kq}z$, provided s is analytic, $s(\Lambda) = 0$ and $s'(\Lambda) \neq 0$. In this case the limit is

$$\tilde{h}_\Lambda = z\left(1 - \frac{As'(\Lambda)}{q\Lambda^{q-1}}z^{kq}\right)^{-1/kq}.$$

Remark 2.5. Let us envisage briefly a more general dependence of the germ f on the parameter and indicate the corresponding statements. Changing the notations, we now consider a family f_σ of local analytic diffeomorphisms which depends analytically on a complex parameter σ. The multiplier of f_σ will be denoted by $\lambda(\sigma)$. The linearization of f_σ, if it exists as an analytic germ of G_1, will be denoted by h_σ.

Let σ_* be a point of the parameter-space such that:

(1) the multiplier $\lambda(\sigma_*)$ is not a Cremer point;

(2) its derivative $\lambda'(\sigma_*)$ is non-zero.

If $|\lambda(\sigma_*)| \neq 1$, the linearization h_σ is analytic at σ_* (i.e. $h_\sigma(z)$ is analytic for σ close enough to σ_* and z close enough to the origin).

If $|\lambda(\sigma_*)| = 1$, we call 'non-tangential' a cone in the parameter-space of the form:

$$\left\{\sigma \;\middle|\; \arg\left(\pm\frac{\lambda'(\sigma_*)}{\lambda(\sigma_*)}(\sigma - \sigma_*)\right) \in]-\alpha, \alpha[\right\}$$

for some $\alpha \in]0, \pi/2[$. So, when σ 'tends to σ_* non-tangentially', this means that the multiplier $\lambda(\sigma)$ tends to $\lambda(\sigma_*)$ transversally to the unit circle. There are only two possibilities for the asymptotic behaviour of h_σ.

- f_{σ_*} is linearizable; $\lambda(\sigma_*)$ is a Brjuno point, or a resonant point of order q but then $f_{\sigma_*}^{\circ q} = \mathrm{Id}$ (where by Id, here and elsewhere, we denote the identity in whatever category we are dealing with); then the linearization h_σ tends to some $h_{\sigma_*} \in G_1$ uniformly on some small open disk around zero as σ tends to σ_* non-tangentially, and the limit germ h_{σ_*} is a linearization of f_{σ_*} (which is thus uniquely determined by f_{σ_*} in the Brjuno case, but which is determined by $(\partial/\partial\sigma)(f_\sigma^{\circ q})|_{\sigma=\sigma_*}$ in the resonant case†).

- $\lambda(\sigma_*)$ is resonant of order q and $f_{\sigma_*}^{\circ q} \neq \mathrm{Id}$: then there exist a positive integer k and a non-zero complex number A such that:

$$f_{\sigma_*}^{\circ q}(z) = z + Az^{kq+1} + O(z^{kq+2}),$$

and the rescaled linearization:

$$\tilde{h}_\sigma(z) = (\sigma - \sigma_*)^{-1/kq} h_\sigma((\sigma - \sigma_*)^{1/kq} z)$$

tends uniformly on some small open disk around 0 to the function:

$$\tilde{h}_{\sigma_*}(z) = z\left(1 - \frac{A}{q\lambda'(\sigma_*)\Lambda^{q-1}} z^{kq}\right)^{-1/kq}$$

as σ tends to σ_* non-tangentially.

All this is proved by adapting the proof of Theorem 2.1, to which the next two sections are devoted. The easier case where $f_\Lambda^{\circ q}$ is the identity is examined at the end of §2.4 only.

Formalizing a little more, we could say that we are studying the regularity properties of the 'linearization' mapping $\mathcal{L}$:

$$\mathcal{L} : f \in G \mapsto h_f \in G_1,$$

with a particular attention to the singular set

$$\hat{G} = \{f \in G \mid |f'(0)| = 1\},$$

through its restrictions to some *analytic paths*. We call path a map from some open connected subset U of $\mathbb{C}$ in G

$$\sigma \mapsto f_\sigma,$$

and we call it analytic if for all $\sigma_* \in U$, there exists $r > 0$ such that the path induces an analytic function $(\sigma, z) \mapsto f_\sigma(z)$ on the polydisk $\{(\sigma, z) \in \mathbb{C} \times \mathbb{C} \mid |\sigma - \sigma_*| < r$ and $|z| < r\}$.

If an analytic path satisfies $(\partial/\partial\sigma) f'_\sigma(0) \neq 0$ for some parameter σ_*, we can change (at least locally) the parametrization in order to obtain a *straightened* path: an analytic path $\lambda \mapsto f_\lambda$ such that $f'_\lambda(0) = \lambda$.

When the image of an analytic path lies outside $\hat{G}$, the composition with $\mathcal{L}$ is well defined and yields an analytic path in G_1. However, if the path is straightened and defined in a neighbourhood U of $\lambda_* \in \mathbb{T}^1$, we get an analytic path h_{f_λ} on each side of the unit circle and generally no 'analytic continuation' to a neighbourhood of λ_*. It follows indeed

† In the resonant case, any linearization of f_{σ_*} can be reached at the limit: one checks that, if a germ F of order q is given with a particular linearization $H \in G_1$, the formula

$$f_\lambda = F + (\lambda - \Lambda)\frac{TF'}{\Lambda} \quad \text{with } T = H^{-1}(H' \circ H^{-1})$$

defines a family f_λ for which the linearization h_λ tends to H.

from the corollary stated in §2.5 that in the case where the germs f_λ or their inverses extend to entire functions of z, the resonant points† of U are true singularities of h_{f_λ}, unlike the Brjuno points, where continuous extension in non-tangential sectors is always possible.

This situation is quite reminiscent of a monogenic uniform function according to Borel's definition [**7**], even if we cannot go much farther than a simple analogy for the moment. This work is indeed a first step in the understanding of the behaviour of the linearizations at these singular points, the idea being that the interesting information is localized there.

2.3. *Germs with almost resonant linear part.* We now begin the proof of Theorem 2.1 and consider the one-parameter family of germs $f_\lambda \in G_\lambda$, with λ close to a fixed resonance $\Lambda = \exp(2\pi i p/q)$. We denote by $f_\lambda^{\circ q}$ the composition of f_λ with itself q times, and we suppose that $f_\Lambda^{\circ q}$ is *not* the identity.

Let us first focus on the germ at the resonance.

LEMMA 2.1. *There exists a positive integer k and a non-zero complex number A such that*

$$f_\Lambda^{\circ q}(z) = z + Az^{kq+1} + O(z^{kq+2}). \tag{2.5}$$

Proof. This is quite easily proved by comparing the Taylor expansions of both sides in the equation $f_\Lambda \circ f_\Lambda^{\circ q} = f_\Lambda^{\circ q} \circ f_\Lambda$. □

Remark 2.6. If $f_2 = \cdots = f_n = 0$, then $kq + 1 \geq n$, but if f_λ is a polynomial of degree d, then $1 \leq k \leq d - 1$ (see [**21**, Proposition 6, p. 9]).

Remark 2.7. The numbers k and A are invariant under formal conjugacy of f_Λ, but there also exists a unique $B \in \mathbb{C}$ such that $f_\Lambda^{\circ q}$ belongs to the analytic conjugacy class of a germ g of the form

$$g(z) = z + Az^{kq+1} + A^2 B z^{2kq+1} + O(z^{2kq+2});$$

the coefficient $B = i(f_\Lambda^{\circ q}, 0)$ is invariant under formal conjugacy of f_Λ, and $f_\Lambda^{\circ q}$ is *formally* and *topologically* conjugate to the polynomial $z + Az^{kq+1} + A^2Bz^{2kq+1}$. Conversely, if two germs of G_Λ have the same formal invariants (k, A, B), they must belong to the same formal conjugacy class [**16, 17, 18**].

We now let λ vary in a disk around the resonance:

$$|\lambda - \Lambda| < \rho$$

where ρ is some small positive constant, but we always impose $|\lambda| \neq 1$. The power series expansion

$$h_\lambda(z) = z + \sum_{j=2}^{\infty} h_j(\lambda) z^j \tag{2.6}$$

for the linearization h_λ of f_λ can be recursively determined by means of (2.1). However, the equation

$$f_\lambda^{\circ q} \circ h_\lambda = h_\lambda \circ R_\lambda^{\circ q} \tag{2.7}$$

† At least those $\Lambda = \exp(2\pi i p/q)$ such that f_Λ is not of order q.

can be used instead, and one then obtains

$$\begin{cases} h_1(\lambda) = 1, \\ h_j(\lambda) = \dfrac{1}{\lambda^{jq} - \lambda^q} \displaystyle\sum_{i=2}^{j} f_i^{\circ q}(\lambda) \sum_{j_1+\cdots+j_i=j} h_{j_1}(\lambda)\cdots h_{j_i}(\lambda) \quad \text{for } j \geq 2, \end{cases} \tag{2.8}$$

denoting $\{f_j^{\circ q}(\lambda)\}$ the Taylor coefficients of $f_\lambda^{\circ q}$:

$$f_\lambda^{\circ q}(z) = \lambda^q z + \sum_{j=2}^{\infty} f_j^{\circ q}(\lambda) z^j.$$

Let us choose a positive constant r (independent of λ) such that the Taylor coefficients of $f_\lambda^{\circ q}$ satisfy

$$\forall j \geq 2, \quad |f_j^{\circ q}(\lambda)| \leq r^{1-j}$$

(in order to do this, we use the Cauchy inequalities and then choose r small enough). Each coefficient $f_j^{\circ q}(\lambda)$ is a polynomial in λ, with a zero at Λ if $2 \leq j \leq kq$, thus there exists $c_1 > 0$ such that for all $|\lambda - \Lambda| < \rho$: for $j = 2, \ldots, kq$,

$$|f_j^{\circ q}(\lambda)| \leq c_1 |\lambda - \Lambda| r^{1-j}. \tag{2.9}$$

Let $\alpha \in]0, \pi/2[$. We introduce notations for a cone of vertex Λ and aperture 2α and for its intersection with a disk around Λ:

$$\begin{aligned} C(\Lambda, \alpha) &= \left\{ \lambda \;\middle|\; \arg\left(\pm \frac{\lambda - \Lambda}{\Lambda} \right) \in]-\alpha, \alpha[\right\}, \\ C(\Lambda, \alpha, \rho) &= C(\Lambda, \alpha) \cap \{|\lambda - \Lambda| < \rho\}. \end{aligned} \tag{2.10}$$

LEMMA 2.2. *For all $\alpha \in]0, \pi/2[$ and for all sufficiently small $\rho > 0$, there exists a positive constant $c_2 < c_1$ such that*

$$\forall j \geq 2,\ \forall \lambda \in C(\Lambda, \alpha, \rho): \quad |\lambda^{jq} - \lambda^q| \geq c_2 |\lambda - \Lambda|. \tag{2.11}$$

Proof. For all $\lambda \in C(\Lambda, \alpha, \rho)$ and $j \geq 2$ we can write

$$\frac{\lambda^{jq} - \lambda^q}{\lambda - \Lambda} = A(\lambda) B_j(\lambda),$$

where $A(\lambda) = \lambda^q \cdot [(\lambda^q - \Lambda^q)/(\lambda - \Lambda)]$ is bounded from below, and $B_j(\lambda) = [(\lambda^{(j-1)q} - 1)/(\lambda^q - 1)]$. There exist $\theta_0, c > 0$ such that, for all λ in $C(\lambda, \alpha, \rho)$, λ^q can be written $e^{2\pi i \theta}$ with $0 < |\theta| < \theta_0$ and $\forall t \in \mathbb{R}$, $|\theta - t| \geq c|t|$; in particular,

$$\forall j \geq 2, \quad \operatorname{dist}((j-1)\theta, \mathbb{Z}) \geq \min((j-1)|\theta|, c).$$

However there also exist $c', c'' > 0$ such that for all θ in $\mathbb{C}$:

$$\begin{cases} |\theta| \leq \theta_0 \Rightarrow |e^{2\pi i \theta} - 1| \leq c'|\theta| \leq c'\theta_0, \\ \forall j \geq 2, |e^{2\pi i (j-1)\theta} - 1| \geq c'' \operatorname{dist}((j-1)\theta, \mathbb{Z}). \end{cases}$$

Therefore, $\forall \lambda \in C(\Lambda, \alpha, \rho)$, $\forall j \geq 2$:

$$|B_j(\lambda)| \geq \frac{c''}{c'} \min\left(j - 1, \frac{c}{\theta_0} \right) \geq c''',$$

where $c''' > 0$ does not depend on j. □

Now let

$$\begin{cases} \sigma_1 = 1, \\ \sigma_j = \displaystyle\sum_{i=2}^{j} \sum_{j_1+\cdots+j_i=j} \sigma_{j_1}\cdots\sigma_{j_i} \quad \text{for } j \geq 2. \end{cases}$$

We have the following lemmas.

LEMMA 2.3. *If $\rho > 0$ is sufficiently small, $\alpha \in]0, \pi/2[$ and $j \in \mathbb{N}^*$, then, $\forall \lambda \in C(\Lambda, \alpha, \rho)$, we have:*

$$|h_j(\lambda)| \leq \sigma_j \left(\frac{c_1}{c_2 r}\right)^{j-1} \left(\frac{1}{c_2|\lambda - \Lambda|}\right)^{\lfloor (j-1)/kq \rfloor}, \tag{2.12}$$

where $\lfloor x \rfloor$ denotes the integer part of x.

Proof. We proceed by induction: (2.12) is clearly true if $j = 1$; assume that it holds at ranks $1, \ldots, j-1$ for some $j \geq 2$. Let $\lambda \in C(\Lambda, \alpha, \rho)$; thanks to the subadditivity of the integer part we get

$$|h_j(\lambda)| \leq \sum_{i=2}^{j} H_{i,j}(\lambda) \sum_{j_1+\cdots+j_i=j} \sigma_{j_1}\cdots\sigma_{j_i},$$

with

$$H_{i,j}(\lambda) = \left(\frac{c_1}{c_2 r}\right)^{j-i} \left|\frac{f_i^{\circ q}(\lambda)}{\lambda^{jq} - \lambda^q}\right| \left(\frac{1}{c_2|\lambda - \Lambda|}\right)^{\lfloor (j-i)/kq \rfloor}.$$

From the inequality (2.9) and Lemma 2.2, it follows that

$$\left|\frac{f_i^{\circ q}(\lambda)}{\lambda^{jq} - \lambda^q}\right| \leq \begin{cases} \dfrac{r^{1-i}}{c_2|\lambda - \Lambda|} & \text{for } i = 2, \ldots, j, \\ \dfrac{c_1}{c_2} r^{1-i} & \text{if, moreover, } i \leq kq. \end{cases}$$

On the other hand:

$$\left\lfloor \frac{j-i}{kq} \right\rfloor \leq \begin{cases} \left\lfloor \dfrac{j-1}{kq} \right\rfloor & \text{for } i = 2, \ldots, j, \\ \left\lfloor \dfrac{j-1}{kq} \right\rfloor - 1 & \text{if, moreover, } i \geq kq + 1. \end{cases}$$

Therefore, using the inequality $c_2 < c_1$, we get in all cases

$$H_{i,j}(\lambda) \leq \left(\frac{c_1}{c_2 r}\right)^{j-1} \left(\frac{1}{c_2|\lambda - \Lambda|}\right)^{\lfloor (j-1)/kq \rfloor}. \qquad \square$$

LEMMA 2.4. *There exists a positive constant c_3 such that*

$$\forall j \geq 1, \quad \sigma_j \leq c_3 (3 - 2\sqrt{2})^{1-j}. \tag{2.13}$$

Proof. The generating series $\sigma(z) = \sum_{i=1}^{\infty} \sigma_i z^i$ satisfies the functional equation:

$$\sigma(z) = z + \frac{\sigma(z)^2}{1 - \sigma(z)},$$

so that

$$\sigma(z) = \frac{1 + z - \sqrt{1 - 6z + z^2}}{4}$$

is analytic in the disk $|z| < 3 - 2\sqrt{2}$ and bounded and continuous on its closure; (2.13) then follows by Cauchy's estimate. □

As Lemma 2.3 clearly shows, the radius of convergence of h_λ cannot tend to zero faster than $|\lambda - \Lambda|^{1/kq}$ as λ tends to Λ non-tangentially. We shall now perform the rescaling (2.3) which will compensate this possible divergence.

2.4. *Rescaling the linearization.* Let us fix $\rho > 0$ sufficiently small and $\alpha \in]0, \pi/2[$. The jth coefficient of the Taylor series of $\tilde{h}_\lambda$ is

$$\tilde{h}_j(\lambda) = (\lambda - \Lambda)^{(j-1)/kq} h_j(\lambda).$$

According to (2.8) the coefficients h_j are rational functions of λ, with a pole at Λ of order $\lfloor (j-1)/kq \rfloor$ at most according to (2.12). Thus each coefficient $\tilde{h}_j(\lambda)$ tends to a limit $\tilde{h}_j(\Lambda)$ and the chosen determination of $(\lambda - \Lambda)^{1/kq}$ does not matter: $\tilde{h}_\lambda$ converges formally as λ tends to Λ in $C(\Lambda, \alpha, \rho)$. Moreover, using (2.12) and (2.13), we see that

$$|\tilde{h}_j(\lambda)| \le \sigma_j \left(\frac{c_1}{c_2 r}\right)^{j-1} |\lambda - \Lambda|^{(j-1)/kq} \left(\frac{1}{c_2|\lambda - \Lambda|}\right)^{\lfloor (j-1)/kq \rfloor} \le c_4 R^{-j},$$

where $R = (3 - 2\sqrt{2})(c_2 r/c_1) c_2^{1/kq}$ and c_4 is some positive constant. Thus, the formal limit $\tilde{h}_\Lambda$ has non-zero radius of convergence and $\tilde{h}_\lambda$ converges uniformly on any disk around zero of radius less than R, by a standard compactness argument for sets of holomorphic functions†.

To compute the limit function $\tilde{h}_\Lambda$, we introduce the germ

$$\tilde{f}_\lambda(z) = (\lambda - \Lambda)^{-1/kq} f_\lambda((\lambda - \Lambda)^{1/kq} z),$$

whose linearization is just $\tilde{h}_\lambda$, and we note that

$$\tilde{f}_\lambda^{\circ q}(z) = \lambda^q z + \sum_{j=2}^{kq} f_j^{\circ q}(\lambda)(\lambda - \Lambda)^{(j-1)/kq} z^j + f_{kq+1}^{\circ q}(\lambda)(\lambda - \Lambda) z^{kq+1} + O((\lambda - \Lambda)^{1+1/kq}).$$

† Indeed, if $R' \in]0, R[$ and $\varepsilon > 0$, we have for any $l \ge 3$:

$$|z| \le R' \Rightarrow |\tilde{h}_\lambda(z) - \tilde{h}_\Lambda(z)| \le \sum_{j=2}^{l-1} |\tilde{h}_j(\lambda) - \tilde{h}_j(\Lambda)| + 2c_4 \sum_{j \ge l} \left(\frac{R'}{R}\right)^j,$$

and we can fix l big enough to make the second sum on the right-hand side smaller than $\varepsilon/2$; then, for λ close enough to Λ, we can ensure that

$$|z| \le R' \Rightarrow |\tilde{h}_\lambda(z) - \tilde{h}_\Lambda(z)| \le \varepsilon,$$

which is the property of uniform convergence.

Because of the inequality (2.9) and by definition of A, we have:

$$\begin{cases} f_j^{\circ q}(\lambda) = O(\lambda - \Lambda) & \text{for } j = 2, \ldots, kq, \\ f_{kq+1}^{\circ q}(\lambda) = A + O(\lambda - \Lambda), \end{cases}$$

hence

$$\tilde{f}_\lambda^{\circ q}(z) = z + (\lambda - \Lambda)(q\Lambda^{q-1}z + Az^{kq+1} + O((\lambda - \Lambda)^{1/kq})). \tag{2.14}$$

However, the linearization $\tilde{h}_\lambda$ of $\tilde{f}_\lambda$ is the one of $\tilde{f}_\lambda^{\circ q}$ as well:

$$\tilde{f}_\lambda^{\circ q}(\tilde{h}_\lambda(z)) = \tilde{h}_\lambda(\lambda^q z), \tag{2.15}$$

and by the Taylor formula:

$$\tilde{h}_\lambda(\lambda^q z) - \tilde{h}_\lambda(z) = (\lambda - \Lambda)q\Lambda^{q-1}z\tilde{h}'_\lambda(z) + O((\lambda - \Lambda)^2),$$

which after substitution of (2.14) into (2.15) yields

$$z\tilde{h}'_\lambda(z) = \tilde{h}_\lambda(z) + \frac{A}{q\Lambda^{q-1}}(\tilde{h}_\lambda(z))^{kq+1} + O((\lambda - \Lambda)^{1/kq}).$$

Taking the uniform limit as λ tends to Λ, we obtain the equation

$$z\tilde{h}'_\Lambda(z) = \tilde{h}_\Lambda(z) + \frac{A}{q\Lambda^{q-1}}(\tilde{h}_\Lambda(z))^{kq+1}, \tag{2.16}$$

which can be rewritten as a regular ordinary differential equation

$$H(z) = z^{-1}\tilde{h}_\Lambda(z), \quad H'(z) = \frac{A}{q\Lambda^{q-1}}z^{kq-1}(H(z))^{kq+1}$$

whose unique solution with initial condition $H(0) = 1$ gives rise to the formula (2.4).

Lastly, we consider the case where $f_\Lambda^{\circ q}$ is the identity. The Taylor expansion of $f_\lambda^{\circ q}$ may be written as

$$f_\lambda^{\circ q}(z) = \lambda^q z + (\lambda - \Lambda)\sum_{j \geq 2} \tau_j(\lambda) z^j,$$

where the τ_j are some analytic functions (in fact polynomials). Let us choose $r > 0$ small enough so that

$$\forall j \geq 2, \quad |\tau_j(\lambda)| \leq r^{1-j}.$$

The induction formulae (2.8) become

$$\begin{cases} h_1(\lambda) = 1, \\ h_j(\lambda) = \dfrac{\lambda - \Lambda}{\lambda^{jq} - \lambda^q} \displaystyle\sum_{i=2}^{j} \tau_i(\lambda) \sum_{j_1 + \cdots + j_i = j} h_{j_1}(\lambda) \cdots h_{j_i}(\lambda) & \text{for } j \geq 2. \end{cases}$$

This allows us to prove inductively the existence of a non-tangential limit for each coefficient. We obtain that, for each $j \geq 1$, the coefficient $h_j(\lambda)$ converges to a number $h_j(\Lambda)$ as λ tends non-tangentially to Λ, with the following recursion formulae

$$\begin{cases} h_1(\Lambda) = 1, \\ h_j(\Lambda) = \dfrac{\Lambda}{(j-1)q} \displaystyle\sum_{i=2}^{j} \tau_i(\Lambda) \sum_{j_1 + \cdots + j_i = j} h_{j_1}(\Lambda) \cdots h_{j_i}(\Lambda) & \text{for } j \geq 2. \end{cases} \tag{2.17}$$

Using (2.11) and the same numbers σ_j as in §2.3, we find

$$|h_j(\lambda)| \leq \sigma_j (rc_2)^{1-j},$$

and we conclude, as before, that the formal limit h_Λ of the linearization h_λ is analytic and that h_λ converges uniformly to it on a disk around zero with small enough radius. Moreover, taking the uniform limit of the conjugacy equation for f_λ, we obtain $f_\Lambda \circ h_\Lambda = h_\Lambda \circ R_\Lambda$.

Finally, we introduce

$$T(z) = \frac{1}{q\Lambda^{q-1}} \frac{\partial}{\partial\lambda}(f_\lambda^{\circ q}(z))|_{\lambda=\Lambda} = z + \frac{\Lambda}{q} \sum_{j \geq 2} \tau_j(\Lambda) z^j$$

and we observe that the equation

$$zh'_\Lambda(z) = T(h_\Lambda(z))$$

holds (either directly from the formulae (2.17) or by a chain of reasoning analogous to the one that led to the equation (2.16)). One then easily identifies its unique solution in G_1.

2.5. *Materialization of resonances.* Using the notations of Theorem 2.1 and considering the second situation which it describes, we can prove the following corollary on the *radius of injectivity* $r(\lambda)$ of h_λ (i.e. the supremum of the radii of the closed disks on which this function is univalent).

COROLLARY 2.1. *For any non-tangential cone $C(\Lambda, \alpha)$ and for all $R > 0$:*

$$R > \left|\frac{q}{A}\right|^{1/kq} \Rightarrow \exists \rho > 0 : \forall \lambda \in C(\Lambda, \alpha, \rho),\ r(\lambda) < R|\lambda - \Lambda|^{1/kq}. \tag{2.18}$$

Proof. Let $\tilde{r}(\lambda)$ be the radius of injectivity of $\tilde{h}_\lambda$:

$$\tilde{r}(\lambda) = r(\lambda)|\lambda - \Lambda|^{-1/kq}.$$

Suppose $R > |q/A|^{1/kq}$, i.e. R exceeds the radius of convergence of $\tilde{h}_\Lambda$. We must check that $\tilde{r}(\lambda) < R$ for λ close enough to Λ inside $C(\Lambda, \alpha)$.

Suppose that this is not true. We could then find a sequence $\{\lambda_n\}$ in $C(\Lambda, \alpha)$ converging to Λ and such that $\tilde{r}(\lambda_n) \geq R$, but $\tilde{h}_{\lambda_n}$ converges towards $\tilde{h}_\Lambda$ uniformly in a small disk around zero and the set of the functions of G_1 univalent in $\{|z| < R\}$ is a compact normal family: $\tilde{h}_\Lambda$ should belong to it, which is not the case. □

Obviously, the asymptotic inequality (2.18) also holds for the radius of injectivity of $h_\lambda^{\circ(-1)}$ (since these functions, properly rescaled, converge uniformly to $\tilde{h}_\Lambda^{\circ(-1)}$ on some small disk).

When $|\lambda| < 1$ and f_λ is an entire function (respectively, when $|\lambda| > 1$ and the inverse function $f^{\circ(-1)}$ is entire), the linearization h_λ is univalent in its disk of convergence, as is easily checked by means of the functional equation $f \circ h_f = h_f \circ R_\lambda$ (respectively, the functional equation $f^{\circ(-1)} \circ h_f = h_f \circ R_{\lambda^{-1}}$) which is then satisfied in this whole disk, and not only in a neighbourhood of the origin. Thus under these more restrictive assumptions the previous corollary gives an information on the decrease of the *radius of convergence* of h_λ.

Arnol'd [**22**] raised the question of the origin of the divergence of the series of classical perturbation theory:

> 'L'idée de la matérialisation des résonances est de trouver des obstacles topologiques à la convergence des séries de la théorie des perturbations dans le comportement des orbites du système perturbé dans l'espace des phases complexe.'

We shall now give an elementary illustration of such a manifestation of the resonances in the singular behaviour of the linearizations. According to Remark 2.2, we may think of the function

$$z\left(1-\frac{A\Lambda}{q}.\frac{z^{kq}}{\Lambda-\lambda}\right)^{-1/kq}$$

as a first-order approximation of $h_\lambda^{\circ(-1)}$ for λ tending to Λ in a non-tangential cone. This function has ramification points at the boundary of its disk of injectivity (which coincides with its disk of convergence), and we can write them

$$e^{2\pi im/kq}\sigma, \quad m\in\mathbb{Z}/kq\mathbb{Z},$$

where σ denotes some determination of $[(q/A)(1-(\lambda/\Lambda))]^{1/kq}$.

Now, since $f_\lambda = h_\lambda\circ R_\Lambda\circ h_\lambda^{\circ(-1)}$, we may expect that something occurs near these points in the dynamic of f_λ, explaining the result of Corollary 2.1.

LEMMA 2.5. *Let us denote by σ a determination of $[(q/A)(1-(\lambda/\Lambda))]^{1/kq}$. There exists positive constants ε and ρ such that, for $0<|\lambda-\Lambda|<\rho$, the local diffeomorphism f_λ admits exactly k orbits of period q inside the pointed disk $D_\varepsilon^* = \{z\in\mathbb{C}\mid 0<|z|<\varepsilon\}$. Moreover, the kq fixed points of $f_\lambda^{\circ q}$ in D_ε^* are analytic functions of σ which can be written*

$$z_m(\lambda) = e^{2\pi im/kq}\sigma(1+O(\sigma)), \quad m\in\mathbb{Z}/kq\mathbb{Z}$$

(with $f_\lambda(z_m(\lambda)) = z_{m+kp}(\lambda)$); we thus get the following upper bound for the radius of injectivity $r'(\lambda)$ of $h_\lambda^{\circ(-1)}$:

$$r'(\lambda)\le\sup\{|z_m(\lambda)|,\ m\in\mathbb{Z}/kq\mathbb{Z}\}\sim\left|\frac{q}{A}(\lambda-\Lambda)\right|^{1/kq}.$$

Note that here the multiplier λ is not required to lie in a non-tangential cone with vertex at Λ.

Proof. The relation between the periodic orbits of f_λ and the radius of injectivity of $h_\lambda^{\circ(-1)}$ is due to the fact that a whole periodic orbit cannot be included in the disk of injectivity of $h_\lambda^{\circ(-1)}$, for in that disk the dynamic is conjugate to R_λ and $|\lambda|\neq 1$. More precisely, whenever $z\neq 0$ belongs to the disk of convergence of $f_\lambda^{\circ q}$ and both z and $f_\lambda^{\circ q}(z)$ belong to the disk of injectivity of $h_\lambda^{\circ(-1)}$, we have

$$h_\lambda^{\circ(-1)}(f_\lambda^{\circ q}(z)) = \lambda^q h_\lambda^{\circ(-1)}(z)\neq h_\lambda^{\circ(-1)}(z),$$

thus $f_\lambda^{\circ q}(z) \neq z$.

We now begin the proof of the first statement by writing the qth iterate of f_λ as

$$f_\lambda^{\circ q}(z) = \lambda^q z + Az^{kq+1}(1 + zB(z)) + (\lambda - \Lambda)z^2 C(\lambda, z)$$

where B and C are analytic for λ close to Λ and z close to the origin. On the other hand, $1 - \lambda^q = q\Lambda^{q-1}(\Lambda - \lambda)(1 + (\Lambda - \lambda)D(\lambda))$ with D analytic at Λ. Thus, the equation of the non-zero fixed points of $f_\lambda^{\circ q}$ is equivalent to the equation

$$z^{kq} = \frac{q}{A}\left(1 - \frac{\lambda}{\Lambda}\right)E(\lambda, z)$$

where $E(\lambda, z) = (1 + (\Lambda - \lambda)D(\lambda) + z(\Lambda/q)C(\lambda, z))(1 + zB(z))^{-1}$ is analytic at $(\Lambda, 0)$ and $E(\Lambda, 0) = 1$. This function can be written $E = F^{kq}$ where F has the same properties, and our equation amounts to

$$\exists m \in \mathbb{Z}/kq\mathbb{Z} \mid z = \sigma F\left(\Lambda\left(1 - \frac{A\sigma^{kq}}{q}\right), z\right) e^{2\pi i m/kq}$$

which is equivalent, by the implicit function theorem, to saying that z is the value of one of kq functions $z_m(\lambda)$ analytic in σ, with $z_m = e^{2\pi i m/kq}\sigma(1 + O(\sigma))$. Finally, since $f_\lambda(z_m(\lambda)) = \Lambda z_m(\lambda) + O(\sigma^2)$ is also fixed by $f_\lambda^{\circ q}$, we must identify it with $z_{m+kp}(\lambda)$. $\square$

2.6. *Two-variable version of Theorem 2.1.* In this section we consider λ as a variable—just as z—rather than a parameter, so we pretend we are dealing with mappings of $\mathbb{C} \times \mathbb{C}^*$:

$$\Phi(z, \lambda) = (\phi_\lambda(z), \lambda)$$

analytic on subsets of $\mathbb{C}\times\mathbb{C}^*$ and acting trivially on the second argument. Our purpose now is simply to rephrase the results of §2.2 in this two-variable context, in view of extending them to one case where the second variable is no longer inert (see the next section).

We start with a family of germs $f_\lambda \in G_\lambda$ as before, except that we do not assume the nonlinear part $f_\lambda - \mathrm{Id}$ to be independent of λ, and we translate the results of the end of §2.2 (with $\lambda = \sigma$) for the map

$$F(z, \lambda) = (f_\lambda(z), \lambda)$$

which is assumed to be analytic on $D_r \times U$, where $r > 0$ and U is some open subset of $\mathbb{C}^*$.

The rotations R_λ become the map:

$$R(z, \lambda) = (\lambda z, \lambda), \tag{2.19}$$

and the linearizations h_λ, available at least at the points λ of U which do not lie on the unit circle S^1, become the map

$$H(z, \lambda) = (h_\lambda(z), \lambda),$$

solution of the conjugacy equation:

$$F \circ H = H \circ R,$$

with the normalizing condition $H - \mathrm{Id} = (O(z^2), 0)$.

As a function of two variables, H is analytic at all the points of the form $(0, \lambda_*)$, where λ_* lies in U, but not on S^1.

For $\lambda_* \in U \cap S^1$, calling V the intersection of U with any non-tangential cone with vertex at $(0, \lambda_*)$, we can state the following facts about the behaviour of H in $D_r \times V$.

(1) If $\lambda_* = B$ is a Brjuno point, there exists $\rho > 0$ such that H admits a continuous extension to $D_\rho \times V$.

(2) If $\lambda_* = \Lambda$ is resonant of order q and if $F^{\circ q}(z, \Lambda) = (z, \Lambda)$, the same conclusion holds.

(3) If $\lambda_* = \Lambda$ is resonant of order q, but not all the points (z, Λ) are q-periodic, we can attach to Λ a positive integer k and a non-zero complex number A defined by

$$F^{\circ q}(z, \Lambda) = (z + Az^{kq+1} + O(z^{kq+2}), \Lambda)$$

and such that the same conclusion holds for the rescaled mapping:

$$\tilde{H} = S^{-1} \circ H \circ S,$$

with $S(z, \lambda) = ((\lambda - \Lambda)^{1/kq} z, \lambda)$ and with

$$\tilde{H}(z, \Lambda) = \left(z \left(1 - \frac{A}{q\Lambda^{q-1}} z^{kq} \right)^{-1/kq}, \Lambda \right).$$

3. *The semi-standard map*

3.1. *Introduction to the semi-standard map.*

We now extend the results of the previous section to the biholomorphic symplectic mapping F of $\mathbb{C}/2\pi\mathbb{Z} \times \mathbb{C}$ defined by:

$$\mathsf{F}(x, y) = (x_1, y_1) \quad \begin{cases} x_1 = x + y + e^{ix}, \\ y_1 = y + e^{ix}. \end{cases} \tag{3.1}$$

This is the so-called *semi-standard map*, which has been studied by many authors [**23–27**] as a model-problem of symplectic twist map.

In particular it provides a simple model for the study of invariant circles of symplectic twist maps, with power series involved instead of trigonometric series [**23**, §32, p.173]: indeed, for $\mathrm{Im}(x)$ large, we may see F as a perturbation of the rotation $\mathsf{R}(x, y) = (x+y, y)$ and ask whether the invariant curves $y =$ *constant* of R have any counterpart for F; i.e. we fix $\omega \in \mathbb{C}$ and we look for an invariant curve parametrized by θ:

$$\begin{cases} x = \theta + \varphi(e^{i\theta}), \\ y = 2\pi\omega + \psi(e^{i\theta}), \end{cases}$$

in such a way that $\mathsf{F}(x, y)$ corresponds to $\theta + 2\pi\omega$, with φ and ψ analytic and vanishing at the origin. If $\omega \in \mathbb{R}$, the above problem admits a solution $(\varphi, \psi) = (\varphi_{2\pi\omega}, \psi_{2\pi\omega})$ if and only if ω is a Brjuno number [**24, 25**].

We shall see that for $\omega \in \mathbb{C} \setminus \mathbb{R}$, there always exists a solution, which we still denote by $(\varphi_{2\pi\omega}, \psi_{2\pi\omega})$, and we shall study its behaviour as ω tends to a *resonance*, i.e. a rational number. For the very same reason as the one mentioned in the second footnote, the solution will depend continuously on ω in any cone non-tangential with respect to the real axis with vertex at a Brjuno number.

Since we consider now ω as a variable rather than a parameter, we can state the problem as a conjugacy problem: find H of the form $\mathsf{H}(x, y) = (x + \varphi_y(e^{ix}), y + \psi_y(e^{ix}))$ such that $\mathsf{F} \circ \mathsf{H} = \mathsf{H} \circ \mathsf{R}$.

The relationship with the previous section is best seen using the following variables:

$$z = e^{ix}, \quad \lambda = e^{iy},$$

which give to F the form†:

$$F(z, \lambda) = (z_1, \lambda_1), \quad \begin{cases} z_1 = \lambda z e^{iz}, \\ \lambda_1 = \lambda e^{iz}. \end{cases}$$

The points $(0, \lambda)$ are fixed by F, and our problem consists in finding a map H fixing these points and satisfying the conjugacy equation:

$$F \circ H = H \circ R, \tag{3.2}$$

where R corresponds to R and coincides with the previously defined rotation (2.19). Thanks to the relation $z_1 = \lambda_1 z$, it is easy to check that H can be written as

$$H(z, \lambda) = \left(h(z, \lambda), \frac{h(z, \lambda)}{h(\lambda^{-1} z, \lambda)} \right).$$

We choose the normalization

$$h(z, \lambda) = i z e^{\phi_\lambda(z)}, \quad \phi_\lambda(0) = 0,$$

so that $\varphi_y = \pi/2 - i\phi_{e^{iy}}$. This leads us to the equation

$$\phi_\lambda(\lambda z) - 2\phi_\lambda(z) + \phi_\lambda(\lambda^{-1} z) = -z e^{\phi_\lambda(z)}, \tag{3.3}$$

and to the formula

$$H(z, \lambda) = (i z e^{\phi_\lambda(z)}, \lambda e^{i(\phi_\lambda(z) - \phi_\lambda(\lambda^{-1} z))}). \tag{3.4}$$

The existence and the analyticity of H for $\lambda \in \mathbb{C}^* \backslash S^1$ and $|z|$ small enough is guaranteed by the stable manifold theorem applied in the intermediate set of variables (z, y). Indeed, in these variables the semi-standard map takes the form:

$$\begin{cases} z_1 = z e^{i(y+z)}, \\ y_1 = y + z; \end{cases}$$

the points $(0, y)$ in these variables are fixed, and the Jacobian at such points has eigenvalues 1 and $\lambda = e^{iy}$. We thus get, for each y of positive imaginary part ($0 < |\lambda| < 1$), a local stable manifold, and for each y of negative imaginary part ($|\lambda| > 1$), a local unstable manifold; these are analytic imbedded disks tangent to the eigenvectors $(\lambda - 1, 1)$, which we can parametrize by z, so that the dynamic on them is conjugate to the rotation $z \mapsto \lambda z$. By glueing these parametrizations and using the appropriate set of variables, we get the maps H or H. Note that when $y = 2\pi\omega$, where ω is Brjuno number (i.e. when λ is a Brjuno point), we may consider the image of $z \mapsto (z e^{i\varphi_y(z)}, y + \psi_y(z))$ as a centre manifold [**23, 28**].

† Instead of the semi-standard map F itself, we are thus considering the quotient map from $\mathbb{C}/2\pi\mathbb{Z} \times \mathbb{C}/2\pi\mathbb{Z}$ to itself, but this does not change anything for the kind of invariant circles which we are interested in; observe that φ_y and ψ_y must be 2π-periodic in y.

3.2. *Existence of non-tangential limits.* The functions under study are the unique solution ϕ of (3.3) such that $\phi_\lambda(0) = 0$ and the normalized solution H of (3.2) which is determined by ϕ through (3.4). With reference to §2.6, we (improperly) call them *linearizations*.

THEOREM 3.1. *Let us fix a resonance* $\Lambda = \exp(2\pi i p/q)$ *where* $p \in \mathbb{N}$, $q \in \mathbb{N}^*$, $(p|q) = 1$, *and let us define the maps*

$$s_\lambda(z) = (\lambda - \Lambda)^{2/q} z, \quad S(z, \lambda) = (s_\lambda(z), \lambda)$$

(how we choose the determination of $(\lambda - \Lambda)^{2/q}$ *does not matter), and the rescaled linearizations*

$$\tilde{\phi}_\lambda = \phi_\lambda \circ s_\lambda, \quad \tilde{H} = S^{-1} \circ H \circ S. \tag{3.5}$$

For any non-tangential cone V *with vertex* Λ*, the function* $\tilde{\phi}_\lambda$ *converges to some analytic function* $\tilde{\phi}_\Lambda$*, uniformly on some small open disk* D_r *around zero, as* λ *tends to* Λ *in* V*; and for* $\rho > 0$ *small enough, if we denote by* U *the disk around* Λ *of radius* ρ*, the function* $\tilde{H}(z, \lambda)$ *extends continuously in* $D_r \times (\bar{V} \cap U)$ *by*

$$\tilde{H}(z, \Lambda) = (i z e^{\tilde{\phi}_\Lambda(z)}, \Lambda).$$

Proof. Expanding ϕ_λ into powers of z:

$$\phi_\lambda(z) = \sum_{j=1}^{\infty} \phi_j(\lambda) z^j$$

and substituting into (3.3) one finds

$$\begin{cases} \phi_1(\lambda) = \dfrac{1}{D_1(\lambda)}, \\[2ex] \phi_j(\lambda) = \dfrac{1}{D_j(\lambda)} \displaystyle\sum_{k=1}^{j-1} \frac{1}{k!} \sum_{j_1+\cdots+j_k=j-1} \phi_{j_1}(\lambda) \cdots \phi_{j_k}(\lambda) \quad \text{for } j \geq 2, \end{cases}$$

where $D_j(\lambda) = -(\lambda^{j/2} - \lambda^{-j/2})^2$.

Since λ tends to Λ in V, there exists a positive constant $c_5 < 1$ such that

$$|D_j(\lambda)| \geq \begin{cases} c_5 & \text{if } j \neq 0 \pmod{q}, \\ c_5 |\lambda - \Lambda|^2 & \text{if } j = 0 \pmod{q}. \end{cases}$$

We now check by induction that, for all $j \geq 1$,

$$|\phi_j(\lambda)| \leq c_5^{-j} \sigma_j^* \left(\frac{1}{c_5 |\lambda - \Lambda|^2} \right)^{\lfloor j/q \rfloor}, \tag{3.6}$$

where the numbers σ_j^* are defined according to the formulae

$$\sigma_1^* = 1, \quad \sigma_j^* = \sum_{k=1}^{j-1} \sum_{j_1+\cdots+j_k=j-1} \sigma_{j_1}^* \cdots \sigma_{j_k}^* \quad (j \geq 2).$$

Indeed, the inequality (3.6) is satisfied for $j = 1$ (whether $q = 1$ or $q \geq 2$). Let us suppose that it holds at ranks $1, \dots, j-1$ for some $j \geq 2$: thanks to the subadditivity of the integer part we get

$$|\phi_j(\lambda)| \leq c_5^{-j+1} \sigma_j^* A_j, \quad \text{with } A_j = \left| \frac{1}{D_j(\lambda)} \right| \left(\frac{1}{c_5 |\lambda - \Lambda|^2} \right)^{\lfloor (j-1)/q \rfloor}.$$

However, we have always $\lfloor (j-1)/q \rfloor \leq \lfloor j/q \rfloor$, and also $\lfloor (j-1)/q \rfloor \leq \lfloor j/q \rfloor - 1$ in the case where q divides j, thus

$$A_j \leq c_5^{-1} \left(\frac{1}{c_5 |\lambda - \Lambda|^2} \right)^{\lfloor j/q \rfloor}$$

in all cases, which proves (3.6) at rank j.

The generating series for the numbers σ_j^* is easily computed: $\sum_{j \geq 1} \sigma_j^* z^j = \frac{1}{2}[1 - (1 - 4z)^{1/2}]$, it defines a holomorphic function which extends continuously to the closure of the disk of radius $\frac{1}{4}$ with center at the origin, thus $\sigma_j^* \leq \text{const}\, 4^j$ and the first statement of the theorem clearly follows by the arguments already used in §2.4. We note that $\tilde{\phi}_\Lambda$ is in fact a function of z^q, and this gives the second statement relative to $\tilde{H}$. □

Remark 3.1. If λ tends to one non-tangentially, one can easily compute the limit $\tilde{\phi}_1$. Note that $\tilde{\phi}_\lambda$ is the solution of

$$\tilde{\phi}_\lambda(\lambda z) - 2\tilde{\phi}_\lambda(z) + \tilde{\phi}_\lambda(\lambda^{-1} z) = -(\lambda - 1)^2 z e^{\tilde{\phi}_\lambda(z)};$$

since:

$$\tilde{\phi}_\lambda(\lambda z) = \tilde{\phi}_\lambda(z) + (\lambda - 1) z \tilde{\phi}_\lambda'(z) + \tfrac{1}{2}(\lambda - 1)^2 z^2 \tilde{\phi}_\lambda''(z) + \cdots,$$
$$\tilde{\phi}_\lambda(\lambda^{-1} z) = \tilde{\phi}_\lambda(z) + \frac{1 - \lambda}{\lambda} z \tilde{\phi}_\lambda'(z) + \frac{1}{2} \frac{(1 - \lambda)^2}{\lambda^2} z^2 \tilde{\phi}_\lambda''(z) + \cdots,$$

one finds at the limit:

$$z\tilde{\phi}_1'(z) + z^2 \tilde{\phi}_1''(z) = -z e^{\tilde{\phi}_1(z)}, \tag{3.7}$$

with the initial condition $\tilde{\phi}_1(0) = 0$. Thus,

$$\tilde{\phi}_1(z) = -2 \log \left(1 + \frac{z}{2} \right). \tag{3.8}$$

Remark 3.2. If the term e^{ix} in (3.1) is replaced by $\gamma(e^{ix})$ where γ is any analytic function vanishing at the origin (even with finite radius of convergence), the conjugacy equation (3.3) becomes

$$\phi_\lambda(\lambda z) - 2\phi_\lambda(z) + \phi_\lambda(\lambda^{-1} z) = i\gamma(ize^{\phi_\lambda(z)})$$

and Theorem 3.1 can also be proved in this more general case. However, the differential equation for the non-tangential limit at $\Lambda = 1$ is then

$$z\tilde{\phi}_1'(z) + z^2 \tilde{\phi}_1''(z) = i\gamma(ize^{\tilde{\phi}_1(z)});$$

the solution with initial condition $\tilde{\phi}_1(0) = 0$ might be very different, and its analytic continuation much more savage in that case.

TABLE 1. The first few values of $C(\Lambda)$.

p	q	$C(e^{2\pi i p/q})$
0	1	1
1	2	1/4
1	3	1/6
1	4	5/24
1	5	$\dfrac{63-11\sqrt{5}}{120(3-\sqrt{5})}$
2	5	$\dfrac{63+11\sqrt{5}}{120(3+\sqrt{5})}$
1	6	99/80

3.3. *Explicit formulae for the limits at resonances.* We now compute the limits $\tilde{\phi}_\Lambda$ for all $\Lambda = \exp(2\pi i(p/q))$ with $(p|q) = 1$.

We attach to such a resonance Λ the $q-1$ positive numbers:

$$D_r(\Lambda) = 2 - \Lambda^r - \Lambda^{-r} = 4\sin^2\left(\frac{rp}{q}\pi\right) \quad r = 1, \ldots, q-1$$

whose product is q^2 (note that $D_{q-r}(\Lambda) = D_r(\Lambda)$).

THEOREM 3.2. *The limit at the resonance Λ is given by the formulae:*

$$\tilde{\phi}_\Lambda(z) = -\frac{2}{q}\log\left(1 + \frac{\Lambda^2 C(\Lambda)}{2q} z^q\right),$$

$$\tilde{H}(z,\Lambda) = \left(iz\left(1 + \frac{\Lambda^2 C(\Lambda)}{2q} z^q\right)^{-2/q}, \Lambda\right),$$

where the number $C(\Lambda)$ is obtained from auxiliary coefficients $C_1, \ldots, C_{q-1}$ according to the formulae:

$$C_1 = \frac{1}{D_1(\Lambda)},$$

$$C_r = \frac{1}{D_r(\Lambda)} \sum_{n=1}^{r-1} \frac{1}{n!} \sum_{r_1+\cdots+r_n = r-1} C_{r_1}\cdots C_{r_n} \quad \textit{for } r = 2, \ldots, q-1,$$

$$C(\Lambda) = \sum_{n=1}^{q-1} \frac{1}{n!} \sum_{r_1+\cdots+r_n = q-1} C_{r_1}\cdots C_{r_n}.$$

The constant $C(\Lambda)$ is always an algebraic positive number. Obviously $C(\Lambda^{-1}) = C(\Lambda)$ (since $D_r(\Lambda^{-1}) = D_r(\Lambda)$); the first few values of $C(\Lambda)$ are given in Table 1.

This result agrees with that computed for $\Lambda = 1$ in the previous section, and in the proof below we shall suppose $q \geq 2$. We shall use a slightly different scaling and new variables

s and η:

$$\varphi(s,\eta) = \phi_{\Lambda e^{\eta}}(\eta^{2/q} e^{s}),$$

so that

$$\lim_{\eta \xrightarrow{\text{n.t.}} 0} \varphi(s,\eta) = \varphi_\Lambda(s) = \tilde{\phi}_\Lambda(\Lambda^{-2/q} e^{s}),$$

where the notation $\eta \xrightarrow{\text{n.t.}} 0$ means that the limit exists for η tending to zero in any non-tangential cone of the half-plane $\{\operatorname{Re}\eta < 0\}$ with vertex zero, uniformly in some half-plane $\{\operatorname{Re} s \leq s_0\}$.

It is sufficient to show that

$$\varphi_\Lambda(s) = -\frac{2}{q}\log\left(1 + \frac{C(\Lambda)}{2q} e^{qs}\right),$$

and, similarly to the end of the proof of Theorem 2.1, this formula will derive from a differential equation; we need only to show that

$$\varphi''_\Lambda(s) = -C(\Lambda) e^{qs + q\varphi_\Lambda(s)}. \tag{3.9}$$

We shall also prove the following.

THEOREM 3.3. *The following expansion holds for $\varphi(s,\eta)$:*

$$\varphi(s,\eta) = \varphi_\Lambda(s) + \sum_{1 \leq k < q/2} \frac{\eta^{2k/q} C_k e^{ks}}{(1 + (C(\Lambda)/2q) e^{qs})^{2k}} + O(\eta) \tag{3.10}$$

where the C_k are the ones defined in Theorem 3.2.

We believe that (3.10) provides the beginning of an infinite asymptotic expansion in powers of $\eta^{2/q}$; moreover the method that we use leads to a system of equations involving the same operators as in [**29**], where these operators are shown to produce *resurgence* in the variable η.

3.4. *Proof of Theorems 3.2 and 3.3.*

In this section we shall prove the two theorems stated above. We shall prove them through a series of lemmas.

Proof of Theorem 3.2. In the variables (s,η), the conjugacy equation (3.3) becomes

$$\varphi(s + \Omega + \eta, \eta) - 2\varphi(s,\eta) + \varphi(s - \Omega - \eta, \eta) = -\eta^{2/q} e^{s + \varphi(s,\eta)}, \tag{3.11}$$

where $\Omega = 2\pi i p/q$. We introduce the following linear combinations of the Ω-translations of φ:

$$\sigma_r(s,\eta) = \sum_{k=0}^{q-1} \frac{\Lambda^{-kr}}{q} \varphi(s + k\Omega, \eta) \quad \text{for } r = 0, 1, \ldots, q-1.$$

We also introduce the Kronecker symbol on $\mathbb{Z}/q\mathbb{Z}$:

$$\forall a, b \in \mathbb{Z}, \quad \tilde{\delta}_{a,b} = \begin{cases} 1 & \text{if } a = b \pmod{q}, \\ 0 & \text{otherwise.} \end{cases}$$

The following identity:

$$\forall a, b \in \mathbb{Z}, \quad \tilde{\delta}_{a,b} = \sum_{r=0}^{q-1} \frac{(\Lambda^{a-b})^r}{q} \tag{3.12}$$

allows us to write the inverse formulae:

$$\varphi(s + k\Omega, \eta) = \sum_{r=0}^{q-1} \Lambda^{kr} \sigma_r(s, \eta) \quad \text{for } k = 0, 1, \ldots, q-1,$$

and by combining the Ω-translations of equation (3.11), we obtain the system of equations

$$(\Delta_r \sigma_r)(s, \eta) = -\eta^{2/q} e^{s+\sigma_0(s,\eta)} S_r \quad \text{for } r = 0, 1, \ldots, q-1, \tag{3.13}$$

where the operator Δ_r is acting on a function $\psi(s, \eta)$ according to:

$$(\Delta_r \psi)(s, \eta) = \Lambda^r \psi(s + \eta, \eta) - 2\psi(s, \eta) + \Lambda^{-r} \psi(s - \eta, \eta)$$

and

$$S_r = \sum_{k=0}^{q-1} \frac{\Lambda^{-k(r-1)}}{q} \exp\left(\sum_{r'=0}^{q-1} \Lambda^{kr'} \sigma_{r'}(s, \eta) \right).$$

Developing the exponentials and using again the identity (3.12), we find:

$$S_r = \tilde{\delta}_{r-1,0} + \sum_{n \geq 1} \frac{1}{n!} \sum_{1 \leq r_1, \ldots, r_n \leq q-1} \tilde{\delta}_{r_1 + \cdots + r_n, r-1} \sigma_{r_1} \cdots \sigma_{r_n}. \tag{3.14}$$

We know that the functions σ_r and S_r tend to some analytic $2\pi i$-periodic functions of s as $\eta \xrightarrow{\text{n.t.}} 0$. We may add, thanks to the uniformness statement and because these are analytic functions of $z = e^s$, that the same is true for the partial derivatives with respect to s of the functions σ_r; this allows us to give the following approximation for the operator appearing in the left-hand side of equation (3.13):

$$(\Delta_r \sigma_r)(s, \eta) = \begin{cases} \eta^2 (\partial_s^2 \sigma_0(s, \eta) + O(\eta^2)) & \text{if } r = 0, \\ -D_r(\Lambda) \sigma_r(s, \eta) + O(\eta) & \text{if } r = 1, \ldots, q-1, \end{cases} \tag{3.15}$$

thanks to the Taylor formula. This was indeed the purpose of using these functions σ_r: to get rid of the difference operator in the approximation of our equation, replacing it by a differential operator (one might consider that the operator Δ_r is a 'differential operator of infinite order'; some series to which the inversion of a closely related operator gives rise are studied in [**29**]).

This implies immediately that $\sigma_r = O(\eta^{2/q})$ for $r = 1, \ldots, q-1$, but we shall see better estimates in the following lemma.

LEMMA 3.1.

$$\sigma_r(s, \eta) = O(\eta^{2r/q}) \quad \textit{for } r = 0, 1, \ldots, q-1.$$

Before proving Lemma 3.1, we introduce a notation for the intervals of integers: for $a, b \in \mathbb{Z}$ such that $a \leq b$, $[\![a, b]\!] = \{r \in \mathbb{Z} \mid a \leq r \leq b\}$, and we state an easy combinatorial lemma.

LEMMA 3.2.

(a) *If $n \in \mathbb{N}^*$ and $r, r_1, \dots, r_n \in [\![1, q-1]\!]$,*

$$\tilde{\delta}_{r_1+\dots+r_n, r-1} \neq 0 \Rightarrow r_1 + \dots + r_n \geq r - 1.$$

(b) *Suppose that $q \geq 3$ and $r_0 \in [\![1, q-2]\!]$. Define N_r for $r \in [\![1, q-1]\!]$ as $N_r = \min(r, r_0)$. Then, if $n \in \mathbb{N}^*$, $r \in [\![r_0+1, q-1]\!]$ and $r_1, \dots, r_n \in [\![1, q-1]\!]$,*

$$\tilde{\delta}_{r_1+\dots+r_n, r-1} \neq 0 \Rightarrow N_{r_1} + \dots + N_{r_n} \geq r_0.$$

Proof of Lemma 3.2.

(a) Suppose that $\tilde{\delta}_{r_1+\dots+r_n, r-1} \neq 0$. There exists $m \in \mathbb{Z}$ such that $r_1 + \dots + r_n = r - 1 + mq$. The positiveness of all r_j implies that $mq \geq n + 1 - r > -q$, so $m > -1$, thus $m \geq 0$.

(b) Suppose that we are given q, r_0, r and $r_1, \dots, r_n$ as in the second part of the lemma. Since $r - 1 \geq r_0$ and because of (a), it is sufficient to prove that

$$r_1 + \dots + r_n \geq r_0 \Rightarrow N_{r_1} + \dots + N_{r_n} \geq r_0.$$

However, if we suppose $N_{r_1} + \dots + N_{r_n} \leq r_0 - 1$, the positiveness of all N_{r_j} implies that each one satisfies $N_{r_j} \leq r_0 - 1$, and thus $N_{r_j} = r_j$ by definition of N. Therefore $r_1 + \dots + r_n = N_{r_1} + \dots + N_{r_n} \leq r_0 - 1$ and we are done. □

Proof of Lemma 3.1. The property to be proved was already checked for $r = 0$ and $r = 1$; this settles the case of $q = 2$. We shall suppose $q \geq 3$ and prove by induction that for all r_0 in $[\![1, q-1]\!]$,

$$\forall r \in [\![1, q-1]\!], \quad \sigma_r = O(\eta^{(2/q)\min(r, r_0)}).$$

This property at rank $r_0 = q - 1$ is nothing but the desired estimate. It was already checked for $r_0 = 1$, so let us suppose it to be true for some $r_0 \in [\![1, q-2]\!]$: in order to establish it at order $r_0 + 1$ we only need to show that

$$\forall r \in [\![r_0+1, q-1]\!], \quad \sigma_r = O(\eta^{2(r_0+1)/q}).$$

Now, if $r_0 + 1 \leq r \leq q - 1$, the property at rank r_0 and Lemma 3.2 imply that

$$\forall r_1, \dots, r_n \in [\![1, q-1]\!], \quad \tilde{\delta}_{r_1+\dots+r_n, r-1}\sigma_{r_1} \dots \sigma_{r_n} = O(\eta^{2r_0/q}),$$

thus $S_r = O(\eta^{2r_0/q})$; because of the equation (3.13), $\Delta_r \sigma_r = O(\eta^{2(r_0+1)/q})$ and the estimate (3.15) implies that $\sigma_r = O(\eta^{\min(1, 2(r_0+1)/q)})$, hence the result in the case where $2(r_0+1)/q \leq 1$.

If $2(r_0+1)/q > 1$, we have $\sigma_r = O(\eta)$ and this allows for a refinement of the approximation (3.15):

$$\Delta_r \sigma_r = -D_r \sigma_r + O(\eta^2),$$

thus $\sigma_r = O(\eta^{2(r_0+1)/q})$ in that case too. □

We shall now study the behaviour of $\eta^{-2r/q}\sigma_r(s, \eta)$ as $\eta \xrightarrow{\text{n.t.}} 0$, again by induction on r. As for $\sigma_0(s, \eta)$, its non-tangential limit is nothing but $\varphi_\Lambda(s)$ since $\varphi = \sum_{r=0}^{q-1} \sigma_r$ and σ_r tends to zero for $1 \leq r \leq q - 1$. (We also see in this way that all the Ω-translations of φ coincide at the limit: φ_Λ must be $(2\pi i/q)$-periodic, and $\tilde{\phi}_\Lambda$ must be function of z^q.)

LEMMA 3.3. *For all r in $[\![1, q-1]\!]$,*

$$\sigma_r(s,\eta) = \eta^{2r/q}(C_r e^{r(s+\sigma_0(s,\eta))} + O(\eta)). \tag{3.16}$$

For all r in $[\![1, q]\!]$,

$$S_r = \eta^{2(r-1)/q}(E_r e^{(r-1)(s+\sigma_0(s,\eta))} + O(\eta)), \tag{3.17}$$

with $S_q = S_0$ and

$$E_1 = 1,$$
$$E_r = \sum_{n\geq 1} \frac{1}{n!} \sum_{r_1+\cdots+r_n=r-1} C_{r_1}\cdots C_{r_n} \quad \textit{for } 2 \leq r \leq q.$$

Proof. If $1 \leq r \leq q-1$ the estimate for σ_r in (3.16) follows from the estimate of S_r in (3.17), by virtue of equation (3.13) which yields

$$\Delta_r\sigma_r = -\eta^{2r/q}(E_r e^{r(s+\sigma_0)} + O(\eta))$$

and of the refinement of the approximation (3.15):

$$\Delta_r\sigma_r = -D_r\sigma_r + O(\eta^{1+2r/q}),$$

which is made possible by the fact that $\sigma_r = O(\eta^{2r/q})$. Indeed, $C_r = E_r/D_r$.

If $1 \leq r \leq q$, the estimate of S_r in (3.17) will be deduced from the estimates (3.16) at ranks $1, \ldots, r-1$, i.e. we proceed by induction on r.

For $r = 1$, we apply the first part of Lemma 3.2 inside equation (3.14) and get

$$S_1 = 1 + \sum_{\substack{n\geq 1\\ m\geq 0}} \frac{1}{n!} \sum_{r_1+\cdots+r_n=mq} \sigma_{r_1}\cdots\sigma_{r_n}.$$

However, in each term of the sum, since the r_j must be positive, the integer m must also be positive and the corresponding product $\sigma_{r_1}\cdots\sigma_{r_n}$ which is $O(\eta^{2(r_1+\cdots+r_n)/q})$, is also $O(\eta^2)$. Therefore $S_1 = 1 + O(\eta^2)$.

For $2 \leq r \leq q$, supposing (3.16) to be true at ranks $1, \ldots, r-1$, we rewrite (3.14) taking into account the first part of Lemma 3.2:

$$S_r = \sum_{n\geq 1} \frac{1}{n!} \sum_{r_1+\cdots+r_n=r-1} \sigma_{r_1}\cdots\sigma_{r_n} + \sum_{n,m\geq 1} \frac{1}{n!} \sum_{r_1+\cdots+r_n=r-1+mq} \sigma_{r_1}\cdots\sigma_{r_n}.$$

The second sum is $O(\eta^{2+2(r-1)/q})$ because of Lemma 3.1. The induction hypothesis applies to each term of the first sum, for this sum involves only values of the indices $r_1, \ldots, r_n$ ranging from 1 to $r-1$. Thus:

$$S_r = \sum_{n\geq 1} \frac{1}{n!} \sum_{r_1+\cdots+r_n=r-1} \eta^{2(r-1)/q} C_{r_1}\cdots C_{r_n} e^{(r-1)(s+\sigma_0)}(1 + O(\eta)) + O(\eta^{2+2(r-1)/q}),$$

hence the desired estimate. □

We can now finish the proof of the theorem. On one hand the estimate (3.17), when specialized to the case $r = q$, introduces the coefficient $C(\Lambda) = E_q$:

$$S_q = S_0 = \eta^{2(q-1)/q}(C(\Lambda)e^{(q-1)(s+\sigma_0)} + O(\eta)).$$

On the other hand the equation (3.13), together with the estimate (3.15), yields

$$\eta^2(\partial_s^2\sigma_0 + O(\eta^2)) = -\eta^{2/q}e^{s+\sigma_0}S_0.$$

Therefore,

$$\partial_s^2\sigma_0(s,\eta) = -C(\Lambda)e^{qs+q\sigma_0(s,\eta)} + O(\eta), \tag{3.18}$$

and the uniform limit φ_Λ of σ_0 must satisfy the limit equation (3.9), which characterizes it since all our functions are $2\pi i$-periodic and tend to zero as $\mathrm{Re}(s)$ tends to $-\infty$. □

Proof of Theorem 3.3. A kind of analytic Gronwall lemma may be applied to the estimate (3.18) in order to prove that

$$\sigma_0(s,\eta) = \varphi_\Lambda(s) + O(\eta). \tag{3.19}$$

This provides estimates of $\sigma_0, \sigma_1, \ldots, \sigma_{q-1}$ (with the help of (3.16)) which prove (3.10). More precisely, let us explain how (3.19) may be derived from (3.18).

Let $\chi(s,\eta) = \sigma_0(s,\eta) - \varphi_\Lambda(s)$. We can write the difference between (3.18) and (3.9):

$$\partial_s^2\chi = -C(\Lambda)e^{qs+q\varphi_\Lambda}(e^{q\chi} - 1) + O(\eta)$$

as

$$\partial_s^2\chi = f(s,\eta)\chi + g(s,\eta),$$

where

$$f = -C(\Lambda)\frac{e^{q\chi} - 1}{\chi}e^{qs+q\varphi_\Lambda(s)} = O(1)$$

(because χ is known to be $o(1)$) and $g = O(\eta)$.

Returning to the variable $z = e^s$, we get three functions $\chi^*(z,\eta) = \chi$, $f^*(z,\eta) = f$ and $g^*(z,\eta) = g$, which are analytic for z in some fixed disc D_ρ centred at the origin and η in some non-tangential cone V_0 with vertex zero. The three of them vanish when $z = 0$, so that their Taylor expansions are

$$\chi^* = \sum_{n\geq 1}\chi_n(\eta)z^n, \quad f^* = \sum_{n\geq 1} f_n(\eta)z^n, \quad g^* = \sum_{n\geq 1} g_n(\eta)z^n.$$

Because of the uniformity of our estimates and Cauchy inequalities, there exist a convergent series $\alpha = \sum_{n\geq 1}\alpha_n z^n$ with constant positive coefficients and a positive constant η_0 such that

$$\forall\eta \in V_0, \forall n \geq 1, \quad |\eta| \leq \eta_0 \Rightarrow |f_n(\eta)| \leq \alpha_n \quad \text{and} \quad |g_n(\eta)| \leq |\eta|\alpha_n.$$

When expanding the differential equation into powers of z,

$$(z\partial_z)^2\chi^* = g^* + f^*\chi^*,$$

we see that the coefficients of χ^* can be bounded by $|\chi_n| \le |\eta|\beta_n$ provided that:

$$\begin{cases} \beta_1 \ge \alpha_1, \\ \beta_n \ge \dfrac{1}{n^2}\Bigl(\alpha_n + \displaystyle\sum_{n_1+n_2=n} \alpha_{n_1}\beta_{n_2}\Bigr). \end{cases}$$

Such a requirement may be fulfilled by a convergent series $\beta = \sum_{n\ge 1} \beta_n z^n$, e.g. $\beta = \alpha/(1-\alpha)$, which is sufficient to conclude. □

3.5. *Invariance of the limits under formal conjugacy.* We have associated to the semi-standard map a set of numbers $\{C(\Lambda)\}$ which determine the non-tangential limit of the rescaled linearization at any resonance Λ, and we mentioned at the end of §3.2 the existence of non-tangential limits for the more general case of the map:

$$F_\gamma(z,\lambda) = (z_1,\lambda_1) \quad \begin{cases} z_1 = \lambda z e^{i\gamma(z)}, \\ \lambda_1 = \lambda e^{i\gamma(z)}, \end{cases}$$

for any function γ analytic and vanishing at the origin. We shall denote by H_γ and $\tilde H_\gamma = S^{-1} \circ H_\gamma \circ S$ the corresponding linearization and its rescaling.

We now prove the invariance of these limits under a suitable notion of formal conjugacy. All the mappings from $\mathbb{C} \times \mathbb{C}^*$ to itself that we consider leave the points $(0,\lambda)$ fixed: these points are the null section of the bundle $\mathbb{C} \times \mathbb{C}^* \mapsto \mathbb{C}^*$.

Let us fix a resonance Λ. The formal conjugating diffeomorphisms that we shall use are of the form

$$\xi(z,\lambda) = \Bigl(\sum_{n\ge 1} \alpha_n(\lambda) z^n, \lambda + \sum_{n\ge 1} \beta_n(\lambda) z^n\Bigr),$$

where the coefficients α_n and β_n are continuous functions defined in a neighbourhood of Λ. We may think of such a ξ as a local diffeomorphism of $(\mathbb{C}^2, (0,\Lambda))$, formal in z, continuous in λ and leaving the null section fixed.

Let F^* be some *analytic* local diffeomorphism of $(\mathbb{C}^2, (0,\Lambda))$ leaving the null section fixed, defined in $D_r \times U$, where r is a positive number and U is some open subset of $\mathbb{C}^*$. If we assume F^* to be *formally* conjugate to F_γ by some ξ, we find immediately the relation between its linearization and that of F_γ:

$$F^* \circ H^* = H^* \circ R \quad \text{with } H^* = \xi \circ H_\gamma.$$

Thus, applying the same rescaling S to the linearizations of F_γ and F^*, we obtain

$$\tilde H^* = S^{-1} \circ H^* \circ S = S^{-1} \circ \xi \circ S \circ \tilde H_\gamma.$$

However $(S^{-1} \circ \xi \circ S)(z,\lambda)$ tends to (z,Λ) as λ tends non-tangentially to Λ, thus we have proved that *for any non-tangential cone V with vertex Λ, and for $\rho > 0$ small enough, the rescaled linearization $\tilde H^*$ extends continuously in $D_\rho \times (\bar V \cap U)$ with the same value as $\tilde H_\gamma$ at the points (z,Λ).*

Acknowledgement. This work was supported by the CEE contract ERB-CHRX-CT94-0460.

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