Differential forms on vector bundles[1]

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Abstract. A bigraded algebra of polynomial forms on a vector bundle is introduced and a bidegree to each multivector valued form is assigned. Numerous examples of polynomial forms appearing in various constructions in differential geometry are given. The Poincaré Lemma for polynomial forms is proved.

1. Introduction.

Differential forms on vector bundles may posess particular properties in relation to the linear structure of the fibres. The most frequently encountered type of a form on a vector bundle is a linear form. Quadratic forms have also been considered. We introduce the bigraded algebra of polynomial forms on a vector bundle and assign a bidegree to each multivector valued form which acts as a differential operator in the algebra of polynomial forms. Our definition of the bidegree is compatible with the usual terminology for differential forms. Our definition of the bidegrees of a multivector valued form differs from the usual conventions. The "linear" Poisson structures are polynomial bivector fields of bidegree $(-2,-1)$ in our terminology. The "quadratic" Poisson structures are assigned the bidegree $(-2,0)$. We give numerous examples of polynomial forms appearing in various constructions in differential geometry. The Poincaré Lemma for polynomial forms is proved.

The analysis of polynomial forms presented in this note is far from complete. The importance of some of the examples suggests that a more extensive study should be undertaken. The concept of a polynomial form is related to the concept of a double vector bundle [1].

2. Differential forms and differential operators.

Let N be a differential manifold. The tangent bundle is denoted by $\mathsf{T}N$, the tangent fibration is a mapping $\tau_N\colon \mathsf{T}N \to N$. Let $\times^p_N \mathsf{T}N$ denote the p-fold fibred product of the tangent bundle $\mathsf{T}N$ for $p > 0$. A *differential p-form* is a differentiable mapping

$$\mu\colon \times^p_N \mathsf{T}N \to \mathbb{R}. \tag{1}$$

Restricted to a fibre $\times^p \mathsf{T}_b N$ a p-form is p-linear and totally antisymmetric. A 0-form is a differentiable function on N. The space of p-forms on N will be denoted by $\Phi^p(N)$.

The *exterior product* of a p-form μ with a p'-form μ' is the $(p+p')$-form

$$\begin{aligned}\mu\wedge\mu'\colon &\times^{p+p'}_N \mathsf{T}N \to \mathbb{R}\\ &:(v_1,\ldots,v_{p+p'}) \mapsto \sum_{\sigma\in S(p+p')} \frac{\mathrm{sgn}(\sigma)}{p!p'!}\mu(v_{\sigma(1)},\ldots,v_{\sigma(p)})\mu'(v_{\sigma(p+1)},\ldots,v_{\sigma(p+p')}),\end{aligned} \tag{2}$$

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where $S(p+p')$ denotes the group of permutations of the set $\{1,\dots,p+p'\}$ of integers.

The product of a function f on N with a p-form μ is the p-form

$$\begin{aligned} f\mu\colon & \times_N^p \mathsf{T}N \to \mathbb{R} \\ & : (v_1,\dots,v_p) \mapsto f(\tau_N(v_1))\mu(v_1,\dots,v_p)). \end{aligned} \tag{3}$$

This product makes the space $\Phi^p(N)$ a module over the algebra $\Phi^0(N)$ of differentiable functions. The exterior product $f\wedge\mu$ of a 0-form f with a p-form μ is identified with the product $f\mu$.

The tangent vector of a curve $\gamma\colon\mathbb{R}\to N$ will be denoted by $\mathsf{t}\gamma(0)$. It is well known that for each $(v_1,\dots,v_{p+1})\in\times_N^{p+1}\mathsf{T}N$ there is a differentiable mapping $\chi\colon\mathbb{R}^{p+1}\to N$ such that $v_i=\mathsf{t}\gamma_i(0)$, where γ_i is the curve

$$\begin{aligned} \gamma_i\colon & \mathbb{R}\to N \\ & : s\mapsto \chi(\delta^1{}_i s,\dots,\delta^{p+1}{}_i s) \end{aligned} \tag{4}$$

for $i=1,\dots,p+1$. We introduce mappings

$$\begin{aligned} \chi_{i,j}\colon & \mathbb{R}^2\to N \\ & : (s,t)\mapsto \chi(\delta^1{}_i s+\delta^1{}_j t,\dots,\delta^{p+1}{}_i s+\delta^{p+1}{}_j t) \end{aligned} \tag{5}$$

and curves

$$\begin{aligned} \rho_{i,j}\colon & \mathbb{R}\to \mathsf{T}N \\ & : s\mapsto \mathsf{t}\chi_{i,j}(s,\cdot)(0) \end{aligned} \tag{6}$$

for $i=1,\dots,p+1$ and $j=1,\dots,p+1$. The *exterior differential* of a p-form is the $(p+1)$-form

$$\begin{aligned} \mathrm{d}\mu\colon & \times_N^{p+1}\mathsf{T}N\to\mathbb{R} \\ & : (v_1,\dots,v_{p+1})\mapsto \sum_{i=1}^{p+1}(-1)^{i+1}\frac{d}{ds}\mu(\rho_{i,1}(s),\dots,\widehat{\rho_{i,i}(s)},\dots,\rho_{i,p+1}(s))\bigg|_{s=0}. \end{aligned} \tag{7}$$

The differential of a 0-form $f\colon N\to\mathbb{R}$ is the 1-form

$$\begin{aligned} \mathrm{d}f\colon & \mathsf{T}N\to\mathbb{R} \\ & : \mathsf{t}\gamma(0)\mapsto \frac{d}{ds}f(\gamma(s))\bigg|_{s=0}. \end{aligned} \tag{8}$$

The space

$$\Phi(N)=\oplus_{p=0}^{\infty}\Phi^p(N) \tag{9}$$

with the exterior product $\wedge$ and the exterior differential d is a differential graded algebra.

Let $(y^i)\colon V\to\mathbb{R}^n$ be a local chart of N. At each $b\in V$ there is a base of T_bN composed of vectors $\partial_j(b)=\mathsf{t}\gamma_{j,b}(0)$, with curves $\gamma_{j,b}\colon\mathbb{R}\to N$ characterized by $(y^i\circ\gamma_{j,b})(s)=y^i(b)+\delta^i{}_j s$ for s sufficiently close to 0. The base $(\partial_j(b))$ is dual to the base $(\mathrm{d}y^i(b))$ of T^*_bN. We have

$$\mu|V=\frac{1}{p!}\mu_{i_1\dots i_p}\mathrm{d}y^{i_1}\wedge\dots\wedge\mathrm{d}y^{i_p}, \tag{10}$$

where $\mu_{i_1\dots i_p}$ are the functions

$$\begin{aligned} \mu_{i_1\dots i_p}\colon & V\to\mathbb{R} \\ & : b\mapsto \mu(\partial_{i_1}(b),\dots,\partial_{i_p}(b)). \end{aligned} \tag{11}$$

Following Claudette Buttin [2] we associate with each k-vector valued $(k+q)$-form

$$X\colon \times_N^{k+q}\mathsf{T}N \to \wedge^k\mathsf{T}N \tag{12}$$

a differential operator

$$\mathrm{i}_X\colon \Phi(N) \to \Phi(N) \tag{13}$$

of degree q defined for a p-form μ with $p \geqslant k$ by

$$\begin{aligned}
&\mathrm{i}_X\mu(v_1,\ldots,v_{p+q}) \\
&\qquad = \sum_{\sigma\in S(p+q)} \frac{\operatorname{sgn}(\sigma)}{(k+q)!(p-k)!}\mu(X(v_{\sigma(1)},\ldots,v_{\sigma(k+q)}),v_{\sigma(k+q+1)},\ldots,v_{\sigma(p+q)})
\end{aligned} \tag{14}$$

with some abuse of notation. If μ is a differential form of degree $p < k$, then $\mathrm{i}_X\mu = 0$. A differential operator

$$\mathrm{d}_X\colon \Phi(N) \to \Phi(N) \tag{15}$$

of degree $q+1$ is defined by

$$\mathrm{d}_X = [\mathrm{i}_X,\mathrm{d}] = \mathrm{i}_X\mathrm{d} + (-1)^{q+1}\mathrm{d}\mathrm{i}_X. \tag{16}$$

These definitions include special cases of operators associated with differential forms ($k=0$) and with multivector fields ($k+q=0$). The exterior differential d is a special case of an operator of degree 1.

In terms of a local chart $(y^i)\colon V \to \mathbb{R}^n$ of N we have the representation

$$X|V = \frac{1}{k!(k+q)!}X_{i_1\ldots i_{k+q}}{}^{j_1\ldots j_k}(\mathrm{d}y^{i_1}\wedge\ldots\wedge\mathrm{d}y^{i_{k+q}})\otimes(\partial_{j_1}\wedge\ldots\wedge\partial_{j_k}) \tag{17}$$

of a k-vector valued $(k+q)$-form X with functions $X_{i_1\ldots i_{k+q}}{}^{j_1\ldots j_k}$ defined by

$$\begin{aligned}
X_{i_1\ldots i_{k+q}}{}^{j_1\ldots j_k}&\colon V \to \mathbb{R} \\
&\colon b \mapsto \langle \mathrm{d}y^{j_1}(b)\wedge\ldots\wedge\mathrm{d}y^{j_k}(b), X(\partial_{i_1}(b),\ldots,\partial_{i_{k+q}}(b))\rangle.
\end{aligned} \tag{18}$$

3. The algebra of polynomial functions on a vector bundle.

Let $\varphi\colon F \to N$ be a vector fibration. We denote by $\Phi^0(F)$ the algebra of differentiable functions on F. We denote by $\Pi^{(0,0)}(\varphi)$ the subalgebra of the algebra $\Phi^0(F)$ composed of mappings constant on fibres of φ. Each element of $\Pi^{(0,0)}(\varphi)$ is the pullback $f\circ\varphi$ of a differentiable function $f\colon N \to \mathbb{R}$. We denote by $\Pi^{(0,1)}(\varphi)$ the space of functions linear on fibres of φ. A power $(\Pi^{(0,1)}(\varphi))^r$ of the space $\Pi^{(0,1)}(\varphi)$ will be denoted by $\Pi^{(0,r)}(\varphi)$ for each $r \geqslant 0$. The graded algebra

$$\Pi^{(0,*)}(\varphi) = +_{r=0}^{\infty}\Pi^{(0,r)}(\varphi) \tag{19}$$

will be called the *algebra of polynomial functions* on the bundle F.

4. Polynomial forms on a vector bundle.

Let $\varepsilon\colon E \to M$ be a vector fibration. The tangent fibration $\mathsf{T}\varepsilon\colon \mathsf{T}E \to \mathsf{T}M$ is a vector fibration. We have operations

$$\begin{aligned}
\cdot'&\colon \mathbb{R}\times\mathsf{T}E \to \mathsf{T}E \\
&\colon (\lambda,\mathsf{t}\gamma(0)) \mapsto \mathsf{t}(\lambda\gamma)(0)
\end{aligned} \tag{20}$$

and

$$\begin{aligned} +' \colon \mathsf{T}E \times_{\mathsf{T}M} \mathsf{T}E &\to \mathsf{T}E \\ \colon (\mathsf{t}\gamma(0), \mathsf{t}\gamma'(0)) &\mapsto \mathsf{t}(\gamma + \gamma')(0), \end{aligned} \tag{21}$$

where $\gamma \colon \mathbb{R} \to E$ and $\gamma' \colon \mathbb{R} \to E$ are differentiable curves such that $\varepsilon \circ \gamma' = \varepsilon \circ \gamma$ and the symbols $\mathsf{t}\gamma(0)$ and $\mathsf{t}\gamma'(0)$ denote the tangent vectors of the curves γ and γ'. Relations

$$\tau_E(\lambda \cdot' v) = \lambda \tau_E(v), \tag{22}$$

$$\tau_E(v +' v') = \tau_E(v) + \tau_E(v'), \tag{23}$$

$$\mathsf{T}\varepsilon(\lambda \cdot' v) = \mathsf{T}\varepsilon(v), \tag{24}$$

$$\mathsf{T}\varepsilon(v +' v') = \mathsf{T}\varepsilon(v) = \mathsf{T}\varepsilon(v') \tag{25}$$

follow directly from the definitions.

Let $\times^p_{M,E}\mathsf{T}\varepsilon$ be the mapping

$$\begin{aligned} \times^p_{M,E}\mathsf{T}\varepsilon \colon \times^p_E \mathsf{T}E &\to \times^p_M \mathsf{T}M \\ \colon (v_1, \ldots, v_p) &\mapsto (\mathsf{T}\varepsilon(v_1), \ldots, \mathsf{T}\varepsilon(v_p)). \end{aligned} \tag{26}$$

This mapping is again a vector fibration with operations

$$\begin{aligned} \cdot^p \colon \mathbb{R} \times (\times^p_E \mathsf{T}E) &\to \times^p_E \mathsf{T}E \\ \colon (\lambda, (v_1, \ldots, v_p)) &\mapsto (\lambda \cdot' v_1, \ldots, \lambda \cdot' v_p) \end{aligned} \tag{27}$$

and

$$\begin{aligned} +^p \colon (\times^p_E \mathsf{T}E) \times_{\times^p_M \mathsf{T}M} (\times^p_E \mathsf{T}E) &\to \times^p_E \mathsf{T}E \\ \colon ((v_1, \ldots, v_p), (v'_1, \ldots, v'_p)) &\mapsto (v_1 +' v_1, \ldots, v_p +' v'_p). \end{aligned} \tag{28}$$

The space

$$\Pi^{(p,r)}(\varepsilon) = \Pi^{(0,r)}(\times^p_{M,E}\mathsf{T}\varepsilon) \cap \Phi^p(E) \tag{29}$$

will be called the space of *polynomial forms* on E of *bidegree* (p, r).

It is easily seen from formulae (2) and (3) that the exterior product $\mu \wedge \mu'$ of polynomial forms μ and μ' of bidegrees (p, r) and (p', r') respectively is a polynomial form of bidegree $(p + p', r + r')$. It follows from formulae (7) and (8) that the exterior differential $\mathrm{d}\mu$ of a polynomial form μ of bidegree (p, r) is a polynomial form of bidegree $(p + 1, r)$. It follows that

$$\Pi(\varepsilon) = \oplus^\infty_{p=0} +^\infty_{r=0} \Pi^{(p,r)}(\varepsilon) \tag{30}$$

is a subalgebra of the differential graded algebra $\Phi(E)$. This algebra will be called the *bigraded differential algebra of polynomial forms* on E

If

$$\begin{tikzcd} F \arrow[r, "\alpha"] \arrow[d, "\varphi"'] & E \arrow[d, "\varepsilon"'] \\ M \arrow[r, equal] & M \end{tikzcd} \tag{31}$$

is a vector fibration morphism and μ is a polynomial form on E of bidegree (p, k), then $\alpha^*\mu$ is a polynomial form on F of the same bidegree (p, k).

Let $(x^\kappa)\colon U \subset M \to \mathbb{R}^m$ be a local chart of M and let $(x^\kappa, e^A)\colon \varepsilon^{-1}(U) \subset E \to \mathbb{R}^{m+k}$ be an adapted chart of E. A p-form μ on E is locally represented by

$$\mu|\varepsilon^{-1}(U) = \sum_{p'+p''=p} \frac{1}{p'!p''!} \mu_{\kappa_1\dots\kappa_{p'}A_1\dots A_{p''}} \mathrm{d}x^{\kappa_1} \wedge \dots \wedge \mathrm{d}x^{\kappa_{p'}} \wedge \mathrm{d}e^{A_1} \wedge \dots \wedge \mathrm{d}e^{A_{p''}} \tag{32}$$

with the functions $\mu_{\kappa_1\dots\kappa_{p'}A_1\dots A_{p''}}$ defined by

$$\begin{aligned} \mu_{\kappa_1\dots\kappa_{p'}A_1\dots A_{p''}}&\colon \varepsilon^{-1}(U) \to \mathbb{R} \\ &\colon e \mapsto \mu(\partial_{\kappa_1}(e), \dots, \partial_{\kappa_{p'}}(e), \partial_{A_1}(e), \dots, \partial_{A_{p''}}(e)). \end{aligned} \tag{33}$$

The coordinates x^κ are functions constant on fibres of ε and coordinates e^A are linear. If μ is a polynomial form of bidegree (p,r), then each function $\mu_{\kappa_1\dots\kappa_{p'}A_1\dots A_{p''}}$ is a polynomial form of bidegree $(0, r-p'')$.

A *polynomial differential operator* is a differential operator of the algebra $\Phi(M)$ which can be restricted to the subalgebra $\Pi(\varepsilon)$. A polynomial operator K is said to be of *bidegree* (q,s) if $\mu \in \Pi^{(p,r)}(\varepsilon)$ implies $K\mu \in \Pi^{(p+q,r+s)}(\varepsilon)$. A multivector valued differential form X is said to be a polynomial form of bidegree (q,s) if the differential operator i_X is a polynomial operator of bidegree (q,s). If a differential form μ is a *polynomial form* of bidegree (p,r), it is a polynomial 0-vector valued form of the same bidegree (p,r).

Let $(x^\kappa, e^A)\colon \varepsilon^{-1}(U) \subset E \to \mathbb{R}^{m+k}$ be an adapted chart of E. A k-vector valued $(k+q)$-form is locally represented by

$$\begin{aligned} X|\varepsilon^{-1}(U) = &\sum_{q'+q''=k+q} \sum_{k'+k''=k} \frac{1}{q'!q''!k'!k''!} X_{\kappa_1\dots\kappa_{q'}A_1\dots A_{q''}}{}^{\lambda_1\dots\lambda_{k'}B_1\dots B_{k''}} \\ &\big(\mathrm{d}x^{\kappa_1} \wedge \dots \wedge \mathrm{d}x^{\kappa_{q'}} \wedge \mathrm{d}e^{A_1} \wedge \dots \wedge \mathrm{d}e^{A_{q''}}\big) \otimes \big(\partial_{\lambda_1} \wedge \dots \wedge \partial_{\lambda_{k'}} \wedge \partial_{B_1} \wedge \dots \partial_{B_{k''}}\big) \end{aligned} \tag{34}$$

with the functions $X_{\kappa_1\dots\kappa_{q'}A_1\dots A_{q''}}{}^{\lambda_1\dots\lambda_{k'}B_1\dots B_{k''}}$ defined by

$$\begin{aligned} &X_{\kappa_1\dots\kappa_{q'}A_1\dots A_{q''}}{}^{\lambda_1\dots\lambda_{k'}B_1\dots B_{k''}}(e) \\ &= \langle \mathrm{d}x^{\lambda_1} \wedge \dots \wedge \mathrm{d}x^{\lambda_{q'}} \wedge \mathrm{d}e^{B_1} \wedge \dots \wedge \mathrm{d}e^{B_{q''}}, X(\partial_{\kappa_1}(e), \dots, \partial_{\kappa_{p'}}(e), \partial_{A_1}(e), \dots, \partial_{A_{p''}}(e)) \rangle. \end{aligned} \tag{35}$$

If X is a polynomial form of bidegree (q,s), then each function $X_{\kappa_1\dots\kappa_{q'}A_1\dots A_{q''}}{}^{\lambda_1\dots\lambda_{k'}B_1\dots B_{k''}}$ is a polynomial form of bidegree $(0, r-q''+k'')$.

5. Examples.

EXAMPLE 1. Let $\varepsilon\colon E \to M$ be a vector fibration. We introduce the mapping

$$\begin{aligned} \chi(\varepsilon)&\colon E \times_M E \to \mathsf{T}E \\ &\colon (e, e') \mapsto \mathsf{t}\gamma(0), \end{aligned} \tag{36}$$

where γ is the curve

$$\begin{aligned} \gamma&\colon \mathbb{R} \to E \\ &\colon s \mapsto e + se'. \end{aligned} \tag{37}$$

The image of $\chi(\varepsilon)$ is the subbundle of vertical vectors

$$\mathsf{V}E = \{v \in \mathsf{T}E;\ \mathsf{T}\varepsilon(v) = 0\} \tag{38}$$

of the tangent bundle $\mathsf{T}E$.

The Liouville field on E is the vector field

$$\begin{aligned} L(\varepsilon) \colon E &\to \mathsf{T}E \\ \colon e &\mapsto \chi(e,e). \end{aligned} \tag{39}$$

The Liouville field is a vector field of bidegree $(-1,0)$. The flow of the Liouville field is the mapping

$$\begin{aligned} \lambda(\varepsilon) \colon \mathbb{R}\times E &\to E \\ \colon (s,e) &\mapsto \exp(s)e. \end{aligned} \tag{40}$$

Example 2. Let $\sigma\colon M \to E$ be a section of a vector fibration $\varepsilon\colon E \to M$. We define a vector field

$$\begin{aligned} S \colon E &\to \mathsf{T}E \\ \colon e &\mapsto \chi(e,\sigma(\varepsilon(e))). \end{aligned} \tag{41}$$

The flow of the field S is the mapping

$$\begin{aligned} \widetilde{\sigma} \colon \mathbb{R}\times E &\to E \\ \colon (s,e) &\mapsto e + s\sigma(\varepsilon(e)). \end{aligned} \tag{42}$$

The field S is a polynomial field of bidegree $(-1,-1)$.

Example 3. Let $\varepsilon\colon E \to M$ be a vector fibration. A *connection* on the fibration is vector valued 1-form $H\colon \mathsf{T}E \to \mathsf{T}E$ such that $H \circ H = H$ and the image of H is the vertical subbundle $\mathsf{V}E \subset \mathsf{T}E$. The connection is said to be *linear* if H is a polynomial vector valued form of bidegree $(0,0)$.

Example 4. Let M be a differential manifold. The pull back $\tau_M^*\mu$ of a p-form $\mu \in \Phi(M)$ is the p-form

$$\begin{aligned} \tau_M^*\mu \colon \times^p_{\mathsf{T}M}\mathsf{T}\mathsf{T}M &\to \mathbb{R} \\ \colon (w_1,\ldots,w_p) &\mapsto \mu(\mathsf{T}\tau_M(w_1),\ldots,\mathsf{T}\tau_M(w_p)). \end{aligned} \tag{43}$$

A derivation $\mathrm{i}_T\colon \Phi(M) \to \Phi(\mathsf{T}M)$ relative to the pull back monomorphism $\tau_M^*\colon \Phi(M) \to \Phi(\mathsf{T}M)$ [3] is defined by

$$\begin{aligned} \mathrm{i}_T\mu \colon \times^p_{\mathsf{T}M}\mathsf{T}\mathsf{T}M &\to \mathbb{R} \\ \colon (w_1,\ldots,w_p) &\mapsto \mu(\tau_{\mathsf{T}M}(w_1),\mathsf{T}\tau_M(w_1),\ldots,\mathsf{T}\tau_M(w_p)) \end{aligned} \tag{44}$$

for a $(p+1)$-form μ. Relations (22)–(25) with $\varepsilon\colon E \to M$ replaced by $\tau_M\colon \mathsf{T}M \to M$ imply that if $\mu \in \Phi^p(M)$, then $\tau_M^*\mu \in \Pi^{(p,0)}(\mathsf{T}M)$ and $\mathrm{i}_T\mu \in \Pi^{(p-1,1)}(\tau_M)$. A derivation $\mathrm{d}_T\colon \Phi(M) \to \Phi(\mathsf{T}M)$ relative to the tangent projection τ_M is defined by $\mathrm{d}_T = \mathrm{i}_T\mathrm{d} + \mathrm{d}\mathrm{i}_T$. If $\mu \in \Phi^p(M)$, then $\mathrm{d}_T\mu \in \Pi^{(p,1)}(\tau_M)$.

If the algebra $\Phi(M)$ is interpreted as a trivial case

$$\Pi(id_M) = \oplus_{p=0}^{\infty}\Pi(p,0)(id_M) \tag{45}$$

of an algebra of polynomial forms, then the pull back monomorphisms is of bidegree $(0,0)$ and the derivations i_T and d_T ore of bidegrees $(-1,1)$ and $(0,1)$ respectively.

EXAMPLE 5. Let $X: M \rightarrow \mathsf{T}M$ be a vector field. Let

$$\kappa_M: \mathsf{TT}M \rightarrow \mathsf{TT}M. \tag{46}$$

be the canonical involution appearing in the vector fibration isomorphism

$$\begin{tikzcd} \mathsf{TT}M \arrow[r, "\kappa_M"] \arrow[d, "\mathsf{T}\tau_M"'] & \mathsf{TT}M \arrow[d, "\tau_{\mathsf{T}M}"] \\ \mathsf{T}M \arrow[r, equal] & \mathsf{T}M \end{tikzcd} . \tag{47}$$

The mapping $\mathrm{d}_T X = \kappa_M \circ \mathsf{T}X: \mathsf{T}M \rightarrow \mathsf{TT}M$ is a vector field on $\mathsf{T}M$. It is the *tangent lift* of X. The tangent lift is a polynomial field of bidegree $(-1, 0)$.

EXAMPLE 6. Let M be a differential manifold. The Liouville 1-form on T^*M is the mapping

$$\begin{aligned} \vartheta_M &: \mathsf{TT}^*M \rightarrow \mathbb{R} \\ &: w \mapsto \langle \tau_{\mathsf{T}^*M}(w), \mathsf{T}\pi_M(w) \rangle. \end{aligned} \tag{48}$$

Relations (22)–(25) with $\varepsilon: E \rightarrow M$ replaced by the cotangent projection $\pi_M: \mathsf{T}^*M \rightarrow M$ imply that ϑ_M is of bidegree $(1, 1)$. Consequently the symplectic form $\omega_M = \mathrm{d}\vartheta_M$ is of bidegree $(2, 1)$ and the Poisson bivector field Λ is of bidegree $(-2, -1)$.

EXAMPLE 7. Let $X: M \rightarrow \mathsf{T}M$ be a vector field. The function

$$\begin{aligned} \widetilde{X} &: \mathsf{T}^*M \rightarrow \mathbb{R} \\ &: p \mapsto \langle p, X(\pi_M(p)) \rangle \end{aligned} \tag{49}$$

is form of bidegree $(0, 1)$. Consequently the Hamiltonian vector field Y on T^*M defined by

$$\mathrm{i}_Y \mathrm{d}\vartheta_M = -\mathrm{d}\widetilde{X} \tag{50}$$

is of bidegree $(-1, 0)$. The field Y is the *canonical lift* of X.

EXAMPLE 8. Let $X: \times_M^q \mathsf{T}M \rightarrow \mathsf{T}M$ be a vector valued differential form. The q-form

$$\begin{aligned} \widetilde{X} &: \times_{\mathsf{T}^*M}^q \mathsf{TT}^*M \rightarrow \mathbb{R} \\ &: (w_1, \ldots, w_q) \mapsto \langle \tau_{\mathsf{T}^*M}(w_1), X(\mathsf{T}\pi_M(w_1), \ldots, \mathsf{T}\pi_M(w_q)) \rangle \end{aligned} \tag{51}$$

is of bidegree $(q, 1)$. The equation

$$\mathrm{i}_Y \mathrm{d}\vartheta_M = -\mathrm{d}\widetilde{X} \tag{52}$$

defines a vector valued q-form on T^*M of bidegree $(q-1, 0)$. The vector valued form Y can be considered the *canonical lift* of X.

EXAMPLE 9. Let $\varepsilon: E \rightarrow M$ be a vector fibration and let Λ be a Poisson bivector field on E. If Λ is a polynomial field of bidegree $(-2, -1)$, then the associated Poisson bracket

$$\begin{aligned} \{\ ,\ \}_\Lambda &: \Phi^0(E) \times \Phi^0(E) \rightarrow \Phi^0(E) \\ &: (f, g) \mapsto \langle \mathrm{d}f \wedge \mathrm{d}g, \Lambda \rangle \end{aligned} \tag{53}$$

is usually said to be linear [4].

A polynomial Poisson bivector field Λ on E of bidegree $(-2,-1)$ induces a Lie algebroid structure [5] on the dual fibration $\tilde{\varepsilon}\colon E^* \to M$. The space $\Pi^{(0,0)}(\varepsilon)$ is identified with the algebra $\Phi^0(M)$ and the space $\Pi^{(0,1)}(\varepsilon)$ is identified with the module $\Gamma(\tilde{\varepsilon})$ of sections of the fibration $\tilde{\varepsilon}$. With a function $f \in \Pi^{(0,0)}(\varepsilon)$ we associate a function $\bar{f} \in \Phi^0(M)$ such that $f = \bar{f} \circ \varepsilon$. With a function $g \in \Pi^{(0,1)}(\varepsilon)$ we associate a section $\bar{g}\colon M \to E^*$ such that $\langle \bar{f}(a), e\rangle = f(e)$ for each $a \in M$ and each $e \in E$ such that $\varepsilon(e) = a$. If $g \in \Pi^{(0,1)}(\varepsilon)$ and $h \in \Pi^{(0,1)}(\varepsilon)$, then $\{g,h\}_\Lambda \in \Pi^{(0,1)}(\varepsilon)$. A Lie bracket in $\Gamma(\tilde{\varepsilon})$ is defined by

$$\begin{aligned} [\ ,\ ]\colon & \Gamma(\tilde{\varepsilon}) \times \Gamma(\tilde{\varepsilon}) \to \Gamma(\tilde{\varepsilon}) \\ \colon & (\bar{g}, \bar{h}) \mapsto \overline{\{g,h\}}_\Lambda. \end{aligned} \tag{54}$$

If $f \in \Pi^{(0,0)}(\varepsilon)$ and $g \in \Pi^{(0,1)}(\varepsilon)$, then $\{f,g\}_\Lambda \in \Pi^{(0,0)}(\varepsilon)$. The anchor $\rho\colon E^* \to \mathsf{T}M$ is characterized by

$$\langle \mathrm{d}\bar{f}(a), \rho(\bar{g}(a))\rangle = \{f,g\}_\Lambda(a). \tag{55}$$

Example 10. Let $\varepsilon\colon E \to M$ be a vector fibration and let μ be a $(p+1)$-form on E. A vector fibration morphism

$$\begin{tikzcd} E \arrow[r, "\tilde{\mu}"] \arrow[d, "\varepsilon"'] & \wedge^p \mathsf{T}^* M \arrow[d, "\pi^p_M"'] \\ M \arrow[r, equal] & M \end{tikzcd} \tag{56}$$

is characterized by

$$\langle v_1 \wedge \ldots \wedge v_p, \tilde{\mu}(e)\rangle = \mu(\chi(\varepsilon)(O(a), e)), \mathsf{T}O(v_1), \ldots, \mathsf{T}O(v_p)), \tag{57}$$

where $O\colon M \to E$ denotes the zero section of ε and $a = \varepsilon(e) = \tau_M(v_1) = \ldots = \tau_M(v_p)$. Let ϑ^p_M be the Liouville p-form on $\wedge^p \mathsf{T}^* M$ and let $\omega^p_M = \mathrm{d}\vartheta^p_M$ be the canonical $(p+1)$-form on $\wedge^p \mathsf{T}^* M$. For each vector fibration morphism

$$\begin{tikzcd} E \arrow[r, "\rho"] \arrow[d, "\varepsilon"'] & \wedge^p \mathsf{T}^* M \arrow[d, "\pi^p_M"'] \\ M \arrow[r, equal] & M \end{tikzcd} \tag{58}$$

the form $\rho^*\omega^p_M$ is an exact polynomial form of bidegree $(p+1,1)$. In particular the form $\tilde{\mu}^*\omega^p_M$ is an exact polynomial form of bidegree $(p+1,1)$.

Let μ be a polynomial form of bidegree $(p+1,1)$ and let $(x^\kappa, e^A)\colon \varepsilon^{-1}(U) \subset E \to \mathbb{R}^{m+k}$ be an adapted chart of E. The form μ is locally represented by

$$\begin{aligned} \mu|\varepsilon^{-1}(U) = {} & \frac{1}{(p+1)!}\mu_{A;\kappa_1\ldots\kappa_{p+1}} e^A \mathrm{d}x^{\kappa_1} \wedge \ldots \wedge \mathrm{d}x^{\kappa_{p+1}} \\ & + \frac{1}{p!}\mu_{\kappa_1\ldots\kappa_p A}\mathrm{d}x^{\kappa_1} \wedge \ldots \wedge \mathrm{d}x^{\kappa_p} \wedge \mathrm{d}e^A \end{aligned} \tag{59}$$

with $\mu_{A;\kappa_1\ldots\kappa_{p+1}}$ and $\mu_{\kappa_1\ldots\kappa_p A}$ in $\Pi(0,0)(\varepsilon)$.

Let $(x^\kappa, p_{\kappa_1\ldots\kappa_p})\colon (\pi^p_M)^{-1}(U) \to \mathbb{R}^{m+\binom{m}{p}}$ be an adapted local chart of $\wedge^p \mathsf{T}^* M$. The mapping $\tilde{\mu}$ defined above is locally characterized by

$$p_{\kappa_1\ldots\kappa_p} \circ \tilde{\mu} = (-1)^p \mu_{\kappa_1\ldots\kappa_p A} e^A. \tag{60}$$

The canonical $(p+1)$-form ω^p_M on $\wedge^p \mathsf{T}^*M$ is expressed locally by

$$\omega^p_M|(\pi^p_M)^{-1}(U) = \frac{1}{p!}\mathrm{d}p_{\kappa_1\ldots\kappa_p} \wedge \mathrm{d}x^{\kappa_1} \wedge \ldots \wedge \mathrm{d}x^{\kappa_p}. \tag{61}$$

The local expression of the pull back $\tilde{\mu}^*\omega^p_M$ is the form

$$\begin{aligned}\tilde{\mu}^*\omega^p_M|\varepsilon^{-1}(U) = \frac{1}{p!}\mu_{\kappa_1\ldots\kappa_p A}\mathrm{d}x^{\kappa_1} \wedge \ldots \wedge \mathrm{d}x^{\kappa_p} \wedge \mathrm{d}e^A \\ + \frac{(-1)^p}{p!}\partial_\kappa \mu_{\kappa_1\ldots\kappa_p A} e^A \mathrm{d}x^\kappa \wedge \mathrm{d}x^{\kappa_1} \wedge \ldots \wedge \mathrm{d}x^{\kappa_p}.\end{aligned} \tag{62}$$

Using these local expressions it is easy to verify that the equality $\tilde{\mu}^*\omega^p_M = \mu$ holds if and only if $\mathrm{d}\mu = 0$. It is also clear that if $\mu = \rho^*\omega^p_M$, then $\tilde{\mu} = \rho$.

If μ is a closed polynomial form on E of bidegree $(p+1,1)$, then

$$\mu = \tilde{\mu}^*\omega^p_M = \tilde{\mu}^*\mathrm{d}\vartheta^p_M = \mathrm{d}\tilde{\mu}^*\vartheta^p_M. \tag{63}$$

It follows that the form μ is exact. The Liouville form is a polynomial form of bidegree $(p,1)$. Hence, the form $\tilde{\mu}^*\vartheta^p_M$ is a polynomial form of bidegree $(p,1)$. It follows that a closed polynomial form on E of bidegree $(p+1,1)$ is the differential of a polynomial form of bidegree $(p,1)$.

We conclude with the observation that there is a one to one correspondence between closed (exact) polynomial forms on E of bidegree $(p+1,1)$ and vector fibration morphisms (58).

Example 11. Let $\pi\colon P \to M$ be a vector fibration and let ω be a closed polynomial form of bidegree $(2,1)$ on P. Let $\beta_{(P,\omega)}\colon \mathsf{T}P \to \mathsf{T}^*P$ be the mapping characterized by

$$\langle \beta_{(P,\omega)}(u), v\rangle = \omega(u,v) \tag{64}$$

for u and v in $\mathsf{T}P$ such that $\tau_P(v) = \tau_P(u)$. The vector fibration morphism

$$\begin{tikzcd} P \arrow[r, "\tilde{\omega}"] \arrow[d, "\pi"'] & \mathsf{T}^*M \arrow[d, "\pi_M"'] \\ M \arrow[r, equal] & M \end{tikzcd} \tag{65}$$

defined in the preceding example is an isomorphism if and only if ω is nondegenerate in the sense that the diagram

$$\begin{tikzcd} \mathsf{T}P \arrow[r, "\beta_{(P,\omega)}"] \arrow[d, "\tau_P"'] & \mathsf{T}^*P \arrow[d, "\pi_P"'] \\ P \arrow[r, equal] & P \end{tikzcd} \tag{66}$$

is a vector fibration isomorphism. The correspondence between closed nondegenerate polynomial forms of bidegree $(2,1)$ on P and vector fibration isomorphisms (65) is one to one.

The closed nondegenerate polynomial form of bidegree $(2,1)$ on T^*M associated with the identity morphism

$$\begin{tikzcd} \mathsf{T}^*M \arrow[r, "1_{\mathsf{T}^*M}"] \arrow[d, "\pi_M"'] & \mathsf{T}^*M \arrow[d, "\pi_M"'] \\ M \arrow[r, equal] & M \end{tikzcd} \tag{67}$$

is the canonical symplectic form ω_M. This property can be used to characterize the canonical form ω_M.

Let (P, ω) be a symplectic manifold. A vector fibration $\pi\colon P \to M$ is called a *special symplectic structure* for (P, ω) if ω is a polynomial form of bidegree $(2, 1)$ on P in relation to this fibration. The objects ω and π establish a symplectomorphism between (P, ω) and $(\mathsf{T}^*M, \omega_M)$. The concept of a special symplectic structure is useful for constructing generating functions of Lagrangian submanifolds of (P, ω). Two important examples of special symplectic structures are known. The form $\mathrm{d}_T\omega_M$ is a symplectic form on $\mathsf{T}\mathsf{T}^*M$. This form is a polynomial form of bidegree $(2, 1)$ in relation to the fibration $\tau_{\mathsf{T}^*M}\colon \mathsf{T}\mathsf{T}^*M \to \mathsf{T}^*M$ and also in relation to the fibration $\mathsf{T}\pi_M\colon \mathsf{T}\mathsf{T}^*M \to \mathsf{T}M$. Consequently we have vector fibration isomorphisms

$$\begin{array}{ccc} \mathsf{T}\mathsf{T}^*M & \xrightarrow{\beta_{(\mathsf{T}^*M,\omega_M)}} & \mathsf{T}^*\mathsf{T}^*M \\ {\scriptstyle \tau_{\mathsf{T}^*M}}\big\downarrow & & \big\downarrow{\scriptstyle \pi_{\mathsf{T}^*M}} \\ \mathsf{T}^*M & = & \mathsf{T}^*M \end{array} \tag{68}$$

and

$$\begin{array}{ccc} \mathsf{T}\mathsf{T}^*M & \xrightarrow{\alpha_M} & \mathsf{T}^*\mathsf{T}M \\ {\scriptstyle \mathsf{T}\pi_M}\big\downarrow & & \big\downarrow{\scriptstyle \pi_{\mathsf{T}M}} \\ \mathsf{T}M & = & \mathsf{T}M \end{array} \quad . \tag{69}$$

Example 12. The "quadratic" Poisson structures classified by Xu [6] are polynomial bivector fields of bidegree $(-2, 0)$ in our terminology.

6. Poincaré Lemma for polynomial forms.

We define a polynomial differential operator $\mathcal{K}$ of bidegree $(-1, 0)$ by the formula

$$\mathcal{K}(\mu) = \frac{1}{r}\mathrm{i}_{L(\varepsilon)}(\mu), \tag{70}$$

where $L(\varepsilon)$ is the Liouville field on E and μ is of bidegree $(p + 1, r)$, $r > 0$, i.e.,

$$\mathcal{K}(\mu)(v_1, \ldots, v_p) = \frac{1}{r}\mu(L(\varepsilon)(\tau_E v_i), v_1, \ldots, v_p) \tag{71}$$

and $\mathcal{K}(\mu) = 0$ for μ of bidegree $(p, 0)$.

Proposition 1. *For each polynomial form μ of bidegree (p, r), where $r > 0$, we have the formula*

$$\mu = \mathcal{K}(\mathrm{d}\mu) + \mathrm{d}\mathcal{K}(\mu). \tag{72}$$

Proof: Let $(v_1, \ldots, v_p) \in \times^p_E \mathsf{T}E$ and let χ be a differentiable mapping $\chi\colon \mathbb{R}^{p+1} \to E$ such that $v_i = \mathsf{t}\gamma_i(0)$, where γ_i is the curve

$$\begin{aligned} \gamma_i &\colon \mathbb{R} \to E \\ &\colon s \mapsto \chi(\delta^1{}_i s, \ldots, \delta^p{}_i s) \end{aligned} \tag{73}$$

for $i = 1, \ldots, p$. As in Section 2 we introduce mappings $\chi_{i,j}\colon \mathbb{R}^2 \to E$ and curves $\rho_{i,j}\colon \mathbb{R} \to \mathsf{T}E$ by the formulae (5) and (6). The exterior differential $\mathrm{d}\mathcal{K}(\mu)$ is given by

$$\begin{aligned} \mathrm{d}\mathcal{K}(\mu)(v_1, \ldots, v_p) &= \sum_{i=1}^{p}(-1)^{i+1}\frac{d}{ds}\mathcal{K}(\mu)(\rho_{i,1}(s), \ldots, \widehat{\rho_{i,i}(s)}, \ldots, \rho_{i,p}(s))\bigg|_{s=0} \\ &= \frac{1}{r}\sum_{i=1}^{p}(-1)^{i+1}\frac{d}{ds}\mu(L(\varepsilon)(\tau_E(\rho_{i,1}(s))), \rho_{i,1}(s), \ldots, \widehat{\rho_{i,i}(s)}, \ldots, \rho_{i,p}(s))\bigg|_{s=0} \quad . \end{aligned} \tag{74}$$

On the other hand,

$$\mathcal{K}(\mathrm{d}\mu)(v_1,\ldots,v_p) = \frac{1}{r}\mathrm{d}\mu(L(\varepsilon)(\tau_E(v_1)),v_1,\ldots,v_p). \tag{75}$$

Let us define a mapping $\widetilde{\chi}\colon \mathbb{R}^{p+1} \to E$ by the formula

$$\widetilde{\chi}(t,t_1,\ldots,t_p) = (1+t)\chi(t_1,\ldots,t_p). \tag{76}$$

We introduce $\widetilde{\chi}_{i,j}$ and $\widetilde{\rho_{ij}}$ by using the formulae (5) and (6). We obtain

$$\widetilde{\chi}(s,t) = \begin{cases} \chi_{i-1,j-1}(s,t), & \text{for } i,j \neq 1 \\ (1+s)\chi(\delta_j^2 t,\ldots,\delta_j^{p+1}t), & \text{for } i=1, j\neq 1 \\ (1+t)\chi(\delta_j^2 s,\ldots,\delta_j^{p+1}s), & \text{for } i\neq 1, j=1 \end{cases} \tag{77}$$

It follows that

$$\widetilde{\rho}_{i,j}(s) = \begin{cases} \rho_{i-1,j-1}, & \text{for } i,j \neq 1 \\ (1+s)\cdot' v_{j-1}, & \text{for } i=1, j\neq 1 \\ L(\varepsilon)(\chi(\delta_i^2 s,\ldots,\delta_i^{p+1}s)) = L(\varepsilon)(\tau_E(\rho_{i,1}(s))) & \text{for } i\neq 1, j=1 \end{cases} \tag{78}$$

Consequently, the formula (75) assumes the form

$$\begin{aligned} \mathcal{K}(\mathrm{d}\mu)(v_1,\ldots,v_p) = {} & \frac{1}{r}\sum_{i=1}^{p+1}(-1)^{i+1}\frac{d}{ds}\mu(L(\varepsilon)(\widetilde{\rho}_{i,1},\widetilde{\rho}_{i,2}(s),\ldots,\widehat{\widetilde{\rho}_{i,i}(s)},\ldots,\widetilde{\rho}_{i,p+1}(s))\Big|_{s=0} \\ = {} & -\frac{1}{r}\sum_{i=1}^{p}(-1)^{i+1}\frac{d}{ds}\mu(L(\varepsilon)(\tau_E(\rho_{i,1}(s))),\rho_{i,1}(s),\ldots,\widehat{\rho_{i,i}(s)},\ldots,\rho_{i,p}(s))\Big|_{s=0} \\ & + \frac{d}{ds}\mu((1+s)\cdot' v_1,\ldots,(1+s)\cdot' v_p)\Big|_{s=0} \end{aligned} \tag{79}$$

From this formula and from (74), we get

$$\begin{aligned} (\mathcal{K}(\mathrm{d}\mu)+\mathrm{d}\mathcal{K}(\mu))(v_1,\ldots,v_p) &= \frac{1}{r}\,\frac{d}{ds}\mu((1+s)\cdot' v_1,\ldots,(1+s)\cdot' v_p)\Big|_{s=0} \\ &= \frac{1}{r}\,\frac{d}{ds}(1+s)^r\Big|_{s=0}\mu(v_1,\ldots,v_p) = \mu(v_1,\ldots,v_p). \end{aligned} \tag{80}$$

■

The formula (72) is the classical homotopy formula applied to the contraction

$$I\times E \ni (t,e)\mapsto te \in E$$

and to polynomial forms.

Corollary 1. *Let μ be a closed polynomial form of bidegree $(p+1,r)$, $r>0$. There exists a polynomial form ν of bidegree (p,r) such that $\mu = \mathrm{d}\nu$.*

Proof: It is enough to take $\nu = \mathcal{K}(\mu)$. From the formula (72) $\mathrm{d}\mu = 0$ implies $\mu = \mathrm{d}\mathcal{K}(\mu) = \mathrm{d}\nu$. Since $\mathcal{K}$ is a polynomal operator of bidegree $(-1,0)$, we have that ν is of bidegree (p,r). ■

The formula (63) of Example 10

$$\mu = \tilde{\mu}^*\omega_M^p = \tilde{\mu}^*\mathrm{d}\vartheta_M^p = \mathrm{d}\tilde{\mu}^*\vartheta_M^p$$

gave us a similar result for polynomial forms of bidegree $(p,1)$. Since $\tilde{\mu}\colon E \to \wedge^p \mathsf{T}^* M$ is linear in fibers, it sends the Liouville vector field $L(\varepsilon)$ into the Liouville vector field $L(\pi^p_M)$. Hence

$$\mathrm{i}_{L(\varepsilon)}(\mu) = \tilde{\mu}^*(\mathrm{i}_{L(\pi^p_M)}\omega^p_M) = \tilde{\mu}^*\vartheta^p_M. \tag{81}$$

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