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A NUMERICAL METHOD FOR GENERALIZED EXPONENTIAL INTEGRALS

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Abstract—We present a method for evaluation of the exponential integral, $E_s(x)$, generalized to an arbitrary order $s > 0$. The algorithm is valid whatever $s > 0$ and $x > 0$. In the region $x \geqslant 1$, we start from a proper initial value, obtained by asymptotic calculation, and then compute the required $E_s(x)$ by means of a suitable combination of Taylor's expansions and recurrences, whatever $s > 0$. When $x < 1$ the starting element used is $E_{s_0}(x)$ $(0 < s_0 \leqslant 1)$, which is obtained by the means of suitable expansions. A forward recursion finally yields the required $E_s(x)$. Numerical stability and accuracy of the proposed algorithm are discussed and some results given.

1. INTRODUCTION

In many fields of applied sciences, the problem arises of evaluating the following function [1]:

$$E_s(x) = \int_1^\infty e^{-xt} t^{-s}\, dt \quad (x > 0, s > 0), \tag{1}$$

which is a generalization of the usual exponential integral

$$E_n(x) = \int_1^\infty e^{-xt} t^{-n}\, dt \quad (x > 0, \quad n = 0, 1, 2, \ldots) \tag{2}$$

to the order $n \doteq s$.

The function $E_s(x)$ can also be defined in terms of the incomplete gamma function $\Gamma(a, x)$ as follows [1]:

$$E_s(x) = x^{s-1}\Gamma(1-s, x) \tag{3}$$

and in terms of the incomplete gamma function $\gamma(a, x)$ as

$$E_s(x) = x^{s-1}[\Gamma(1-s) - \gamma(1-s, x)] \quad (s \neq 1, 2, \ldots), \tag{3'}$$

where $\Gamma(a)$ is the Euler gamma function.

As for a practical evaluation of $E_s(x)$, since specific numerical methods for exponential integrals $E_n(x)$ (see Ref. [2] for an overview), have not yet been extended to the calculation of $E_s(x)$, algorithms for incomplete gamma functions [3, 4] appear to be the only procedures generally available to obtain function $E_s(x)$.

Therefore, it is useful to develop computational methods for the above generalized exponential integral. Here we present an efficient algorithm for evaluating $E_s(x)$, which is valid whatever $s > 0$ and $x > 0$.

Formulas leading to evaluation of $E_s(x)$ for $x \geqslant 1$, and the related computational scheme, are presented in Section 2, while the analytical background for the case $x < 1$ is described in Section 3. Numerical aspects are treated in Section 4 and the Appendix presents examples of tables for $E_{n+1/2}(x)$, $0 \leqslant n \leqslant 4$, $0 \leqslant x \leqslant 100$, obtained by means of a FORTRAN routine, ESA, implementing the present method.

2. BACKGROUND OF THE METHOD IN THE CASE $x \geqslant 1$

The evaluation procedure for the exponential integral $E_s(x)$ in the region $x \geqslant 1$ makes use of a proper asymptotic formula, Taylor series expansions and forward or backward recursions.

The involved asymptotic expansion is a generalization of that used in Ref. [5] for the $E_n(x)$ and reads as follows [6]:

$$E_s(x) = [e^{-x}/(x+s)]\left[\sum_{i=0}^{k-1} h_i(x/s)(1+x/s)^{-2i}s^{-i} + R_k(x,s)\right]. \tag{4}$$

Here $h_i(u)$ are polynomials recursively defined by

$$\begin{cases} h_0(u) = 1, \\ h_{i+1}(u) = (1-2iu)h_i(u) + u(1+u)h_i'(u) \quad (i = 0, 1, 2, \ldots), \end{cases} \tag{5}$$

where $h_i'(u)$ denotes the first derivative of $h_i(u)$.

Considering the above polynomials $h_n(u)$ in the usual form

$$h_n(x) = \sum_{j=0}^{n} c_{j,n}u^j \quad (n = 0, 1, 2, \ldots), \tag{6}$$

analogous to Ref. [7], we find that, apart from $c_{0,0} = 1$, the coefficients $c_{j,n}$ can be generated columnwise in an upper triangular matrix by the formulas

$$c_{0,n} = 1, \quad c_{n,n} = 0 \quad (n = 1, 2, \ldots)$$

and (7)

$$c_{m,n+1} = (m+1)c_{m,n} + (m-2n-1)c_{m-1,n} \quad (m = 1, 2, \ldots, n),$$

which are derived by putting equation (6) into equation (5) and equating like powers of u.

Furthermore, in equation (4) the remainder term $R_k(x,s)$ satisfies conditions ensuring that formula (4) is efficient for sufficiently large values of s. In particular, if $\{A_i, B_i\}$ are respectively, the lower and upper bounds of $h_i(u)$ in the interval $u \geqslant 0$, the following inequality holds [6]:

$$A_k s^{-k} \leqslant R_k(x,s) \leqslant B_k[1 + 1/(x+s-1)]s^{-k}. \tag{8}$$

For s sufficiently large, equation (8) provides the accuracy of expansion (4).

As far as the Taylor series expansion is concerned, we use the following expression:

$$E_s(x-y) = \sum_{k=0}^{\infty} (y^k/k!)E_{s-k}(x), \tag{9}$$

which generalizes a known result for the $E_n(x)$ and has been obtained from the Taylor series

$$E_s(x-y) = \sum_{k=0}^{\infty} [(-y)^k/k!][d^kE_s(x)/dx^k], \tag{10}$$

making use of the following relation [1]:

$$[d^kE_s(x)/dx^k] = (-1)^k E_{s-k}(x). \tag{11}$$

The $E_{s-k}(x)$ appearing in equation (9) are obtainable, for both positive and negative values of the index, from the following recurrence [1]:

$$E_{r-1}(x) = \{e^{-x} - (r-1)E_r(x)\}/x, \tag{12}$$

just as in the case $r = n$ (see Ref. [2]).

Moreover, the procedure for reaching $E_s(x)$ also involves proper use of the forward recursion

$$E_r(x) = \{e^{-x} - xE_{r-1}(x)\}/(r-1) \quad (r \neq 1). \tag{12'}$$

The above analytical expressions are then inserted into a suitable computational procedure in order to evaluate $E_s(x)$ in the case $x \geqslant 1$.

First, we are concerned with determination of a proper starting element, which is calculated asymptotically via equation (4), whatever the required $E_s(x)$.

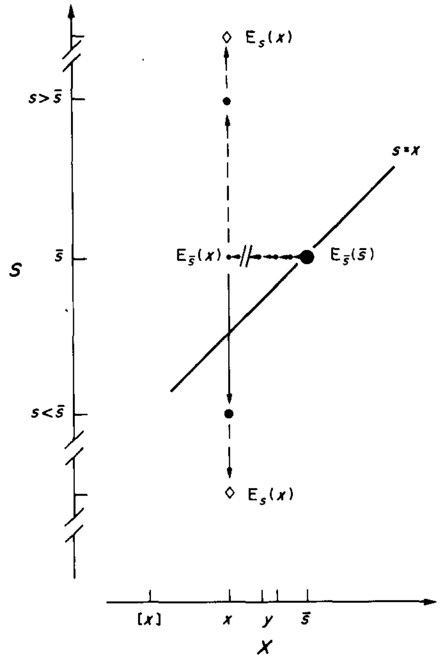

Fig. 1. Schematic representation of the two-stage algorithm for evaluation of the exponential integral when $x \geqslant 1$. See text for explanation of symbols.

The corresponding order s^* and argument x^* are related to the above s and x values by the relation

$$s^* \equiv x^* = \bar{s} + K. \tag{13}$$

Here K ($K < K_{\max}$) is a non-negative integer which is non-vanishing only when $\mathrm{E}_s(x)$ lies outside the region of asymptotic calculation, and $\bar{s}$ is expressed as

$$\bar{s} = [x] + D + (s - [s]) \quad (x \geqslant 1, s > 0) \tag{13'}$$

where, for numerical purposes, we assume $D = 1$ when $(s - [s]) < (x - [x])$, and $D = 0$ otherwise; $[w]$ denotes the integer part of w.

At this point if $s^* \neq \bar{s}$, i.e. $K \neq 0$, starting from the known value $\mathrm{E}_{s^*}(s^*)$, [see equation (4)], the use of expansion (9) allows evaluation of $\mathrm{E}_s(x - y)$ for $s = s^*$ and $x = x^*$. Since y is a part in a suitable partition of the interval $(s^* - 1, s^*)$, a repeated use of formula (9) leads to evaluation of $\mathrm{E}_{s^*}(s^* - 1)$. Then, by a backward recursion, equation (12), we get $\mathrm{E}_{s^*-1}(s^* - 1)$, which constitutes the new starting value for a "further" step if $(s^* - 1) \neq \bar{s}$.

Thus, proceeding step-by-step K times, we reach $\mathrm{E}_{s^*-K}(s^* - K) \equiv \mathrm{E}_{\bar{s}}(\bar{s})$.

Once the value of $\mathrm{E}_{\bar{s}}(\bar{s})$ has been obtained by this procedure we can evaluate the required $\mathrm{E}_s(x)$ according to the following two-stage composite algorithm, also sketched in Fig. 1:

(1) Iterate for $h = 0, 1, \ldots, (m - 1)$ [with m integer and $y = (\bar{s} - x)/m$ sufficiently small] the Taylor series expansion of $\mathrm{E}_{\bar{s}}(\bar{x} - y)$, with $\bar{x} = \bar{s} - hy$, starting from $\mathrm{E}_{\bar{s}}(\bar{s})$ until $\mathrm{E}_{\bar{s}}(x)$ is reached.

(2) Compute $\mathrm{E}_s(x)$ by backward (forward) recursion for $s < \bar{s}$ ($s > \bar{s}$).

Owing to the above computational technique the generalized exponential integral, $\mathrm{E}_s(x)$, can be evaluated in the region $x \geqslant 1$.

3. EVALUATION PROCEDURE WHEN $x < 1$

The exponential integral, $\mathrm{E}_s(x)$, in the case $x < 1$ and $s < 1$, is obtainable by means of a series expansion of the form

$$\mathrm{E}_s(x) = \Gamma(1 - s)\left[x^{s-1} - \mathrm{e}^{-x/2} \sum_{m=0}^{\infty} a_m (x/2)^m T_{m+1-s}(-sx/2)\right], \tag{14}$$

where the coefficients a_m are determined by recursion and $T_r(t)$ are the Tricomi functions defined as follows [8]:

$$T_r(t) = \sum_{k=0}^{\infty} [(-1)^k/\Gamma(r+k+1)](t^k/k!), \tag{14'}$$

which are entire for every value of r.

Inserting expansion (14′) into equation (14), we get the expression

$$\begin{aligned} E_s(x) &= \Gamma(1-s)\left[x^{s-1} - e^{-x/2} \sum_{m,k=0}^{\infty} a_m \frac{(x/2)^m (sx/2)^k}{k!\Gamma(1-s+m+k+1)}\right] \\ &= \Gamma(1-s)x^{s-1} - e^{-x/2} \sum_{m,k=0}^{\infty} a_m \frac{(x/2)^{m+k} s^k}{k!(1-s)_{m+k+1}}, \end{aligned} \tag{15}$$

where the known relation $\Gamma(z+h) = (z)_h \Gamma(z)$ has been used, $(z)_h$ being Pochhammer's symbol defined as

$$(z)_h = z(z+1)\cdots(z+h-1).$$

The coefficients a_m in equations (14) and (15) are defined recursively by

$$(n+1)a_{n+1} = (n-s+1)a_{n-1} + sa_{n-2} \quad (n = 2, 3, \ldots) \tag{15'}$$

with

$$a_0 = 1, \quad a_1 = 0 \quad \text{and} \quad a_2 = 1 - s/2.$$

Expansion (14) has been obtained by using, in equation (3′), a suitable series representation of the incomplete gamma function, $\gamma(a, x)$, defined as follows [9]:

$$\gamma(1-s, x) = (1-s)^{-1} x^{1-s} e^{-x} M(1, 2-s, x). \tag{16}$$

Here, $M(1, 2-s, x)$ is the Kummer function which, following Tricomi [8], can be expressed as

$$M(1, 2-s, x) = \Gamma(2-s)\, e^{x/2} \sum_{m=0}^{\infty} a_m (x/2)^m T_{m+1-s}(-sx/2). \tag{17}$$

Inserting expansion (17) into equation (16) we get the expression

$$\gamma(1-s, x) = \Gamma(1-s) x^{1-s} e^{-x/2} \sum_{m=0}^{\infty} a_m (x/2)^m T_{m+1-s}(-sx/2), \tag{18}$$

which, once used in equation (3′), yields the series representations of formulas (14) and (15) which, in the present method, is used to evaluate $E_s(x)$ on $0 < s \leqslant d < 1$ $(d = 0.9)$.

Otherwise, i.e. when $d < s < 1$, we make use of the relation

$$E_s(x) = \Gamma(1-s)x^{s-1} - 1/(1-s) - \sum_{m=1}^{\infty} \frac{(-x)^m}{(1-s+m)m!}, \tag{19}$$

obtained by substitution of a known series expansion for the incomplete γ function [9, p. 338] in equation (3′). It has been adopted due to the particular numerical devices that can be applied [3, 4] when $s \approx 1$.

Finally, for evaluation of $E_s(x)$ in the case $x < 1$, $s > 1$, we use the forward recurrence (12′) with starting value $E_{s_0}(x)$, $s_0 = s - [s]$, for $s \neq [s]$ and $s_0 = 1$ otherwise, calculated via equation (15) for $0 < s_0 \leqslant d$ and via equation (19) for $d < s_0 \leqslant 1$. For the usual case $s = 1$, we refer to the next section.

Therefore, considering also the results of Section 2, we may conclude that, according to the computational method here presented, evaluation of the generalized exponential integral, $E_s(x)$, can be performed in the whole region $(x > 0, s > 0)$.

4. NUMERICAL ANALYSIS

We first consider the numerical aspects related to the adopted series representations. As far as the asymptotic expansion (4) is concerned, we would point out that the choice $x = s$ (which has been performed in view of a correct successive recursive computation of $E_s(x)$ according to Gautschi's results in Refs [10, 11]) in practice simplifies the asymptotic computation too.

In fact, for $x = s$ we have $u = 1$ and the calculation of the polynomial $h_i(u)$ [equation (6)] reduces to the sum of the coefficients $c_{j,i}$.

In practice, since these $c_{j,i}$ are greatly increasing with order i, and considering the expression of the series in equation (4), it is more suitable to calculate the terms $(1+u)^{-2i}c_{j,i}$ directly since they are simply $4^{-i}c_{j,i}$ for $u = 1$.

Therefore, the calculations related to formula (4) are greatly simplified and the above expansion can be successfully applied in actual computation for $x \geqslant 20$. It follows that the maximum size of the step-by-step procedure inherent in the "non-asymptotic" case is $K_{max} = 20$.

As far as the other series representations are concerned, we note that, due to the above choice of s^*, $\bar{s}$ and D, expansion (9) has been properly used in the computational process in conditions involving $y > 0$, thus ensuring the positivity of the relevant series.

In fact, the coefficients $E_{s-k}(x)$, as obtained by equation (12), are positive functions also for negative values of the index since in this case the r.h.s. of equation (12) is a sum of positive terms.

Furthermore, as regards the stability of recursion (12), Gautschi's theory [10, 11] proves its stabilizing character for E_r with a positive index and the same result holds for E_r with a negative index since the recurrence is positive.†

Analogously it follows from Refs [10, 11] that all the recurrences, used in the procedure for the recursive computations of $E_s(x)$, are stable.

As regards the expansions adopted in the region $0 < x < 1$, the Tricomi series representation, equation (15), presents a positive series and does not involve numerical criticities when used, as in the present method, for starting values s_0 in the interval $0 < s_0 \leqslant d$.

Moreover, expansion (19), adopted in the remaining case $d < s_0 \leqslant 1$, can be easily put in a form suitable for numerical purposes by rewriting the term $Q = \Gamma(1-s)x^{s-1} - 1/(1-s)$ as follows [3]:

$$Q = -[\Gamma(2-s)Q_1 + Q_2]/x^{1-s} \tag{20}$$

with

$$Q_1 = [1/\Gamma(2-s) - 1]/(1-s)$$

and

$$Q_2 = [x^{1-s} - 1]/(1-s). \tag{21}$$

As for Q_1, we can obtain a suitable expression making use of the series expansion

$$\frac{1}{\Gamma(z)} = \sum_{m=1}^{\infty} b_m z^m, \quad |z| < 1, \tag{22}$$

whose coefficients b_m, which has been accurately calculated in Ref. [12], are defined recursively by‡

$$(i-1)b_i = \gamma b_{i-1} - \sum_{j=2}^{i-1} (-1)^j S_j b_{i-j} \quad (i = 2, 3, \ldots), \tag{23}$$

where γ is the Euler constant, $b_1 = 1$ and

$$S_k \equiv \zeta(k) = \sum_{n=1}^{\infty} n^{-k},$$

$\zeta(z)$ being the Riemann zeta function [9].

So Q_1 can be evaluated as

$$Q_1 = \gamma + \sum_{m=1}^{\infty} b_{m+2}(1-s)^m. \tag{24}$$

†It follows that, in principle, the present procedure can be easily extended to evaluation of $E_s(x)$, with $s < 0$, which, for integer values of s, are usually denoted as $\alpha_n(x)$ (see Ref. [2]).

‡The original formulas in Ref. [12] contain some misprints.

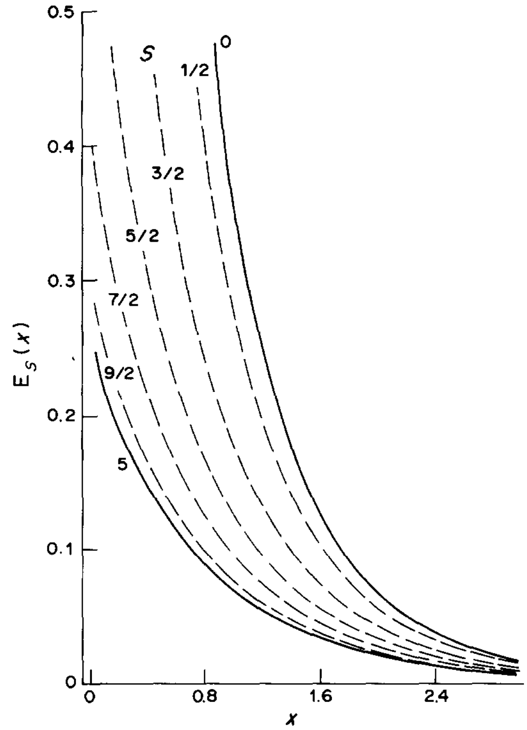

Fig. 2. Exponential integral functions of order $s = 0$, 1/2, 3/2, 5/2, 7/2, 9/2 and 5 in the region $0 < x < 3$.

Moreover, as for Q_2, it can be expressed as

$$Q_2 = \left[\frac{e^{(1-s)\ln(x)} - 1}{(1-s)\ln(x)}\right]\ln(x) \tag{25}$$

$$= \left\{1 + \sum_{m=1}^{\infty} \frac{[(1-s)\ln(x)]^m}{(m+1)!}\right\}\ln(x), \tag{25'}$$

and evaluated according to equation (25) when $|(1-s)\ln(x)| \geqslant 1$, and via equation (25′) otherwise.

It follows that, thanks to the above procedure, we can properly evaluate the starting value $E_{s_0}(x)$ when s_0 approaches 1 and even if $s_0 = 1$. In fact, in this case, quantity Q reduces to $-\gamma - \ln(x)$ and, added to the series in equation (19), yields a known formula [9] suitable for evaluation of the exponential integral $E_1(x)$.

As for the validity of the present numerical method, significant checks are described in the next section.

5. RESULTS AND COMPARISONS

A first test for the efficiency of the present computational algorithm is given by the results obtained for $E_s(x)$, $0 \leqslant x \leqslant 100$, $s = n + 1/2$ $(0 \leqslant n \leqslant 4)$, and reported in the Appendix.†

The above functions have been computed by using the double precision arithmetic [$\epsilon \sim 10(-16)$] of an IBM 370/168 computer and after checking against the corresponding double precision values calculated on a CRAY-X/MP system [$\epsilon \sim 10(-29)$], show disagreement at most in the last significant digit.

These exponential integral functions are illustrated in Fig. 2, which shows the characteristic monotonically decreasing behaviour of $E_s(x)$ for increasing s and fixed x.

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†The limit values as x approaches zero for fixed s are

$$E_s(0) = \begin{cases} \infty, & \text{for } s \leqslant 1 \\ 1/(s-1), & \text{for } s > 1. \end{cases}$$

As for comparisons with different methods, we have checked values of $E_s(x)$ computed via the ESA routine with the corresponding ones obtained by means of the GAMMA routine [4] (taken as a reference), using IBM double precision.

The results of this test, carried out for values $s = n/2$, with $n = 1, 3, 5, 7, 9, 15, 25, 35, 45$ and 75, and x randomly distributed in equally spaced subintervals of the considered region $(0 < x < 100)$, are in agreement, apart from slight last figure discrepancies.

This analysis confirms the efficiency and accuracy of the present algorithm and of the related implementation, ESA.

As for the region of very large x values, considering that physical applications often require evaluation of exponentially-scaled sequences of generalized exponential integrals, in practice the routine ESA calculates directly the sequence $\{\exp(x)[E_{s_0+p}(x)]\}$ with $p = 0, 1, 2, \ldots, N$, and $N = [s]$ for non-integer values of s and $N = s - 1$ otherwise.

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APPENDIX

Values of the Generalized Exponential Integral, $E_s(x)$, for $s = n + 1/2$, $0 \leqslant n \leqslant 4$ and $0 \leqslant x \leqslant 100$

Table A1

X	$E_{1/2}(x)$	$E_{3/2}(x)$	$E_{5/2}(x)$	$E_{7/2}(x)$	$E_{9/2}(x)$
0.0000	∞	2.00000000000000D + 00	6.66666666666667D − 01	4.00000000000000D − 01	2.85714285714286D − 01
0.0500	5.95949382352600D + 00	1.30650946664883D + 00	5.90602634112182D − 01	3.68679717118042D − 01	2.66512982449946D − 01
0.1000	3.66970459112667D + 00	1.07573391784659D + 00	5.31509350834201D − 01	3.40674593181016D − 01	2.48791416776531D − 01
0.1500	2.67211230377189D + 00	9.19782261718550D − 01	4.81827091444850D − 01	3.15373565083332D − 01	2.32400554760731D − 01
0.2000	2.08902724001000D + 00	8.01850610151963D − 01	4.38907087365059D − 01	2.92379734241988D − 01	2.17215658922738D − 01
0.2500	1.69978367615981D + 00	7.07709728062902D − 01	4.01248900703786D − 01	2.71395423158183D − 01	2.03129122080531D − 01
0.3000	1.41925743352729D + 00	6.30081981247063D − 01	3.67862417538399D − 01	2.52183798168079D − 01	1.90046594637513D − 01
0.3500	1.20673746150031D + 00	5.64659956387212D − 01	3.38038069988793D − 01	2.34549906089054D − 01	1.77884463596441D − 01
0.4000	1.03998753831908D + 00	5.08650061416016D − 01	3.11240014312822D − 01	2.18329616124204D − 01	1.66568057024559D − 01
0.4500	9.05704075275042D − 01	4.60122635496009D − 01	2.87048643765713D − 01	2.03382504770881D − 01	1.56030292707108D − 01
0.5000	7.95379490846669D − 01	4.17681828578597D − 01	2.65126496948890D − 01	1.89586964495275D − 01	1.46210622132856D − 01
0.5500	7.03289010769898D − 01	3.80281708914086D − 01	2.45196580318493D − 01	1.76836676482126D − 01	1.37054182375805D − 01
0.6000	6.25422339731349D − 01	3.47116464510434D − 01	2.27027838258511D − 01	1.65037973255568D − 01	1.28511100611624D − 01
0.6500	5.58877204369977D − 01	3.17551187841062D − 01	2.10425003109551D − 01	1.54107809895923D − 01	1.20535914379619D − 01
0.7000	5.01495937501995D − 01	2.91076295080026D − 01	1.95221264823594D − 01	1.43972167365957D − 01	1.13087081895783D − 01
0.7500	4.51637796662982D − 01	2.67276410487556D − 01	1.81272829916899D − 01	1.34564772121336D − 01	1.06126563900004D − 01
0.8000	4.08030907879538D − 01	2.45808475627182D − 01	1.68454789076984D − 01	1.25826053142254D − 01	9.96194633152624D − 02
0.8500	3.69672891207247D − 01	2.26385948845133D − 01	1.56657916953576D − 01	1.17702281015275D − 01	9.35337123102123D − 02
0.9000	3.35762318424559D − 01	2.08767146316992D − 01	1.45786152036871D − 01	1.10144849162966D − 01	8.78397987125513D − 02
0.9500	3.05650302333402D − 01	1.92746472475539D − 01	1.35754583068493D − 01	1.03109667815773D − 01	8.25105254370048D − 02
1.0000	2.78805585280662D − 01	1.78147711781561D − 01	1.26487819593254D − 01	9.65566486312752D − 02	7.75207978686192D − 02
1.1000	2.33233790596507D − 01	1.52627828083843D − 01	1.09986981870568D − 01	8.47541614561818D − 02	6.84690017417942D − 02
1.2000	1.96323055168718D − 01	1.31213091419480D − 01	9.58256681392175D − 02	7.44813640580565D − 02	6.05190214407241D − 02
1.3000	1.66124439268296D − 01	1.13140043970456D − 01	8.36331572482795D − 02	6.55234754444997D − 02	5.35289357017609D − 02
1.4000	1.41207640692182D − 01	9.78125339451037D − 02	7.31062776123075D − 02	5.76992701137504D − 02	4.73765673663874D − 02
1.5000	1.20500617435866D − 01	8.47584679892625D − 02	6.39949721096907D − 02	5.08550807935575D − 02	4.19564397023124D − 02
1.6000	1.03185464303759D − 01	7.35995502172833D − 02	5.60914917646681D − 02	4.48600524684746D − 02	3.71772668700275D − 02
1.7000	8.86286671879222D − 02	6.40295796665340D − 02	4.92221590797513D − 02	3.96023414468630D − 02	3.29598695980193D − 02
1.8000	7.63331171593737D − 02	5.57985546694279D − 02	4.32409932110775D − 02	3.49860401766588D − 02	2.92354331153145D − 02
1.9000	6.59043213559609D − 02	4.87008172926186D − 02	3.80247109111065D − 02	3.09286673966131D − 02	2.59440431911629D − 02

continued overleaf

Table A1—*continued*

X	$E_{1/2}(x)$	$E_{3/2}(x)$	$E_{5/2}(x)$	$E_{7/2}(x)$	$E_{9/2}(x)$
2.0000	5.70261239928920D − 02	4.25660705016572D − 02	3.34687614888655D − 02	2.73591041035526D − 02	2.30334500084307D − 02
2.1000	4.94429474159307D − 02	3.72524773590547D − 02	2.94841505326447D − 02	2.42158848537712D − 02	2.04580200171607D − 02
2.2000	4.29465958901934D − 02	3.26412948078167D − 02	2.59948731900915D − 02	2.14457749376530D − 02	1.81778438569992D − 02
2.3000	3.73663112800502D − 02	2.86326555573764D − 02	2.29358239605586D − 02	1.90025794454076D − 02	1.61579745709618D − 02
2.4000	3.25611846113855D − 02	2.51422204441744D − 02	2.02510828155595D − 02	1.68461418127929D − 02	1.43677751253456D − 02
2.5000	2.84142997092898D − 02	2.20984987013488D − 02	1.78925012470178D − 02	1.49414982025417D − 02	1.27803580335842D − 02
2.6000	2.48281673424676D − 02	1.94406862478362D − 02	1.58185293133066D − 02	1.32581607998947D − 02	1.13721028956022D − 02
2.7000	2.17211325582492D − 02	1.71169096649537D − 02	1.39932377629165D − 02	1.17695083119500D − 02	1.01222400849956D − 02
2.8000	1.90245240274448D − 02	1.50827906967449D − 02	1.23854991162215D − 02	1.04522660399192D − 02	9.01249077526981D − 03
2.9000	1.6680374849222D − 02	1.33002660238655D − 02	1.09682990581316D − 02	9.28606111513023D − 03	8.02675509215130D − 03
3.0000	1.46395874836109D − 02	1.17366118340626D − 02	9.71815524378402D − 03	8.25304105460476D − 03	7.15084148687133D − 03
3.1000	1.28604469343092D − 02	1.03636337943983D − 02	8.61462508728204D − 03	7.33754584919339D − 03	6.37223150315951D − 03
3.2000	1.13074088324784D − 02	9.15699142887098D − 03	7.63988760398605D − 03	6.52582545824434D − 03	5.67987500342410D − 03
3.3000	9.95010643520129D − 03	8.09563233015149D − 03	6.77838714116005D − 03	5.80579593416474D − 03	5.06401166242754D − 03
3.4000	8.76253284236996D − 03	7.16131659253643D − 03	6.01652903046814D − 03	5.16682850269376D − 03	4.51601515747637D − 03
3.5000	7.72236444768757D − 03	6.33821571082401D − 03	5.34241895628964D − 03	4.59956683012191D − 03	4.02825700482623D − 03
3.6000	6.81039884457547D − 03	5.61257321364172D − 03	4.74563925212159D − 03	4.09576845586194D − 03	3.59398743033988D − 03
3.7000	6.01008598027514D − 03	4.97241668664272D − 03	4.21705648650756D − 03	3.64816698810457D − 03	3.20723103267214D − 03
3.8000	5.30713565459394D − 03	4.40731273741724D − 03	3.74865563598672D − 03	3.25035217576643D − 03	2.86269531092947D − 03
3.9000	4.68918781047500D − 03	3.90815796990376D − 03	3.33339690878647D − 03	2.89666540061486D − 03	2.55569039525898D − 03
4.0000	4.14553469033633D − 03	3.46700025477769D − 03	2.96509191308228D − 03	2.58210849456203D − 03	2.28205854585316D − 03
4.1000	3.66688600442362D − 03	3.07688556724879D − 03	2.63829638402747D − 03	2.30226409089945D − 03	2.03811217973528D − 03
4.2000	3.24516990205213D − 03	2.73172646371751D − 03	2.34821711524278D − 03	2.05322597458322D − 03	1.82057935063663D − 03
4.3000	2.87336384916413D − 03	2.42618892159036D − 03	2.09063109957492D − 03	1.83153811361152D − 03	1.62655574962040D − 03
4.4000	2.54535056869168D − 03	2.15559480165013D − 03	1.86181518387190D − 03	1.63414123761283D − 03	1.45346241644914D − 03
4.5000	2.25579504930182D − 03	1.91583763276825D − 03	1.65848479385680D − 03	1.45832598635469D − 03	1.29900845704177D − 03
4.6000	2.00003931594166D − 03	1.70330978260390D − 03	1.47774049643709D − 03	1.30169178440919D − 03	1.16115815324323D − 03
4.7000	1.77401221529087D − 03	1.51483937965750D − 03	1.31702134487037D − 03	1.16211071232203D − 03	1.03810192965208D − 03
4.8000	1.57415192655548D − 03	1.34763560310744D − 03	1.17406410273621D − 03	1.03769574235449D − 03	9.28230710205285D − 04
4.9000	1.39733928320447D − 03	1.19924116644488D − 03	1.04686757022963D − 03	9.26772790719664D − 04	8.30113256113711D − 04
5.0000	1.24084030017512D − 03	1.06749099641975D − 03	9.33661344657816D − 04	8.27856110318554D − 04	7.42476127855056D − 04
5.5000	6.88602565374547D − 04	5.98914657808122D − 04	5.28493880346263D − 04	4.72022038623849D − 04	4.25900064580828D − 04
6.0000	3.84959850950666D − 04	3.37986141924725D − 04	3.00556883412006D − 04	2.70164350477729D − 04	2.45076021085710D − 04
6.5000	2.16552762672076D − 04	1.91692471218163D − 04	1.71625420039676D − 04	1.55149585087872D − 04	1.41419111401830D − 04
7.0000	1.22469337560207D − 04	1.09193205266140D − 04	9.83530191276888D − 05	8.93643326642780D − 05	8.18090391155916D − 05
7.5000	6.95821622091179D − 05	6.24363071588992D − 05	5.65413776373930D − 05	5.16096151469546D − 05	4.74320732987641D − 05
8.0000	3.96940151357873D − 05	3.58210136324272D − 05	3.25963458953959D − 05	2.98767442957378D − 05	2.75567638676026D − 05
8.5000	2.27249355738348D − 05	2.06128332660964D − 05	1.88395241658830D − 05	1.73329654402555D − 05	1.60394750767064D − 05
9.0000	1.30514621578532D − 05	1.18932893320020D − 05	1.09134667324408D − 05	1.00754413978850D − 05	9.35166614448975D − 06
9.5000	7.51709494125523D − 06	6.87885589155178D − 06	6.33513261197245D − 06	5.86722802958493D − 06	5.46090388761249D − 06
10.0000	4.34062650738866D − 06	3.98732937719649D − 06	3.68442399367994D − 06	3.42227593027418D − 06	3.19347727421230D − 06
15.0000	1.97724097937065D − 08	1.86323471924567D − 08	1.76114084099837D − 08	1.66924777408283D − 08	1.58614726826859D − 08
20.0000	1.00653793144305D − 10	9.61555191049041D − 11	9.20288268936502D − 11	8.82308338262217D − 11	8.47248416897496D − 11
25.0000	5.45015306649275D − 13	5.25122397464307D − 13	5.06589285570894D − 13	4.89284690276664D − 12	4.73093316584980D − 13
30.0000	3.06962564461669D − 15	2.97492069980206D − 15	2.88573912955992D − 15	2.80162232064166D − 15	2.72216001975770D − 15
35.0000	1.77675791570513D − 17	1.72928110358058D − 17	1.68421931743302D − 17	1.64139659652562D − 17	1.60065334944945D − 17
40.0000	1.04928164758979D − 19	1.02455329864895D − 19	1.00094040463871D − 19	9.78370546947059D − 20	9.56777335723865D − 20
45.0000	6.29270729568988D − 22	6.16005949778986D − 22	6.03278710293050D − 22	5.91057536922678D − 22	5.79313326849549D − 22
50.0000	3.82002780777866D − 24	3.74718881491734D − 24	3.67702936701652D − 24	3.60940657822623D − 24	3.54418739573728D − 24
55.0000	2.34195540973522D − 26	2.30118993062650D − 26	2.26179754419531D − 26	2.22371102800337D − 26	2.18686741730420D − 26
60.0000	1.44754870506038D − 28	1.42437064668471D − 28	1.40191255058829D − 28	1.38014183666707D − 28	1.35902783626886D − 28
65.0000	9.00879127806448D − 31	8.87524217102964D − 31	8.74554202851949D − 31	8.61952892237569D − 31	8.49704977293887D − 31
70.0000	5.63948777663568D − 33	5.56165845273475D − 33	5.48592126628799D − 33	5.41219398028221D − 33	5.34039856317434D − 33
75.0000	3.54816687078818D − 35	3.50236174338905D − 35	3.45771028441959D − 35	3.41416993973535D − 35	3.37170020018748D − 35
80.0000	2.24222029633499D − 37	2.21503015548477D − 37	2.18848423050670D − 37	2.16256013760226D − 37	2.13723650753155D − 37
85.0000	1.42243346472885D − 39	1.40617084666103D − 39	1.39027197272979D − 39	1.37472489730019D − 39	1.35951818707613D − 39
90.0000	9.05469860257626D − 42	8.95677633437559D − 42	8.86092820349882D − 42	8.76708963366302D − 42	8.67519867696569D − 42
95.0000	5.78154854291105D − 44	5.72223225260663D − 44	5.66410913681586D − 44	5.60714388213852D − 44	5.55130254276973D − 44
100.0000	3.70174786040828D − 46	3.66562312251141D − 46	3.63019023396183D − 46	3.59542968236031D − 46	3.56132267601492D − 46
200.0000	6.90231207340425D − 90	6.86822411177384D − 90	6.83446958797978D − 90	6.80104365631241D − 90	6.76794156407302D − 90