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Journal of Geometry and Physics 16 (1995) 99-119

JOURNAL OF GEOMETRY AND PHYSICS

Entropy estimates for Simplicial Quantum Gravity

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Received 21 April 1994

Abstract

Through techniques of controlled topology we determine the entropy function characterizing the distribution of combinatorially inequivalent metric ball coverings of n-dimensional manifolds of bounded geometry for every $n \geq 2$. Such functions control the asymptotic distribution of dynamical triangulations of the corresponding n-dimensional (pseudo)manifolds M of bounded geometry. They have an exponential leading behavior determined by the Reidemeister–Franz torsion associated with orthogonal representations of the fundamental group of the manifold. The subleading terms are instead controlled by the Euler characteristic of M. Such results are either consistent with the known asymptotics of dynamically triangulated two-dimensional surfaces, or with the numerical evidence supporting an exponential leading behavior for the number of inequivalent dynamical triangulations on three- and four-dimensional manifolds.

Keywords: Simplicial quantum gravity; Entropy;
1991 MSC: 83 C 27, 83 C 45

1. Introduction

The purpose of this paper is to count the isomorphism classes of triangulations, with fixed edge-length and given number of vertices, on Riemannian manifolds of arbitrary dimension. The search for such combinatorial *entropy* bounds in geometry is inspired by a long-standing conjecture [Fro] in (euclidean) simplicial quantum gravity and in

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SSDI 0393-0440(94)00022-V

Regge calculus [RG]: Does the number of triangulated manifolds of given dimension, volume and fixed topology grow with the volume at most at an exponential rate, possibly with a non-trivial subleading asymptotics?

In the case of surfaces, fine entropy bounds of this nature are provided either by direct counting arguments, or by quantum field theory techniques [ITZ,FR] as applied to graph enumeration, a technique that has found use in a number of far reaching applications in surface theory [Wt,Ko,Pe]. In higher dimensions, the natural generalizations of such approaches are not viable, and a systematic method for understanding and enforcing entropy bounds relating topology to Riemannian invariants still appears as a major issue for providing a deeper understanding of higher dimensional models of simplicial quantum gravity [Da].

Numerical as well as some (very limited) analytical evidence [Am,Ad,Ag] shows that exponential bounds do hold in simple situations (typically for manifolds with three-sphere topology). In such cases one can costruct a tentative statistical model of Euclidean quantum gravity through computer simulations of a representative set of inequivalent triangulations. It must also be stressed that this topic has an indipendent mathematical interest. For instance, by controlling the rate of growth of inequivalent triangulations one can provide an effective mathematical regularization of the formal functional integration over the space of Riemannian structures.

Without any restriction on topology, the above exponential growth conjecture is definitively false since it can be shown [Am] that the number of distinct (three-)manifolds, with given volume V and arbitrary topological type, grows at least factorially with V. Thus suitable constraints on the class of Riemannian manifolds considered are necessary for having exponential growth of the number of inequivalent triangulations. In this paper we implement such restrictions by considering Riemannian manifolds M of bounded geometries. Namely, for arbitrary $r \in \mathbb{R}$, $v \in \mathbb{R}^+$, $D \in \mathbb{R}^+$ and integers $n \geq 3$, we consider closed connected Riemannian n-manifolds, M whose sectional curvatures satisfy $sec(M) \geq r$, whose volume satisfies $Vol(M) \geq v$, and whose diameter is bounded above by D, $diam(M) \leq D$. We let $\mathcal{R}(n,r,D,V)$ denote the Gromov–Hausdorff closure of such a collection of manifolds [Gr]. This is an infinite-dimensional compact metric space whose utility in simplicial quantum gravity has been stressed in [CM1,CM2,CM3].

Our main result may be summarized as follows:

Theorem 1. *For any r, $D > 0$, $v > 0$, and integer $n \geq 3$, the number of isomorphism classes of triangulations, with λ n-simplices, of a manifold $M \in \mathcal{R}(n,r,D,V)$ of given Euler characteristic and Reidemeister–Franz torsion is exponentially bounded in λ.*

The explicit expression for the bound is provided in the proof of the theorem. It must be also stressed that this result can be extended to a class of metric spaces considerably more general than Riemannian manifolds. This is partially implicit in the remark that $\mathcal{R}(n,r,D,V)$ contains as limit points metric homology manifolds, but more generally, Theorem 1 can be formulated in the set of all metric spaces of Hausdorff dimension bounded above and for which Toponogov's comparison theorem locally holds (Aleksandrov spaces with curvature bounded below [Bu,Pe]). These are the spaces which arise naturally if one wishes to consider simplicial approximations to Riemannian manifolds, or more in general if one needs to avoid excessive assumptions of smoothness. Spaces of bounded geometries such as $\mathcal{R}(n, r, D, V)$ are easier to handle than Aleksandrov spaces, and afford a good compromise for what concerns a proper mathematical setting for simplicial quantum gravity.

For the proof of the theorem it is worth recalling that the field theoretic approach used by Brézin, Bessis, Itzykson, Zuber and Parisi [ITZ] does not seem to apply to higher dimensional manifolds ($n \geq 3$). Instead we develop a very elementary argument based on the properties of geodesic ball coverings. In particular we count the inequivalent ways of introducing, in a manifold $M \in \mathcal{R}(n, r, D, V)$, coverings with metric balls of a given radius. The basic observation here is that such coverings are naturally labelled (or coloured) by the fundamental groups of the balls (for spaces of bounded geometries, these balls need not be contractible). Their enumeration is thus relatively elementary, and the associated entropy function can be obtained through a rather direct argument. Finally, since the topology of such geodesic ball coverings dominates the topology of the underlying manifold, it follows that the number of inequivalent triangulations is controlled by the entropy function so obtained.

The paper is organized in three sections. After the introductory remarks, Section 2 provides the basic background to the properties of geodesic ball coverings on manifolds of bounded geometry. Also, we recall finiteness theorems concerning homotopy types and simple homotopy types, since these results provide the rationale for a geometrical understanding the entropy estimates. These latter are proved in Section 3.

We refer to Gromov's book [Gr] and to a very readable review paper of Cheeger [Ch] for basic tools and results in Riemannian geometry that will be used freely.

2. Review of geodesic ball coverings

The point in the introduction of $\mathcal{R}(n, r, D, V)$ or of a more general class of metric spaces with a lower bound on a suitably defined notion of curvature, is that for any manifold (or metric space) M in such a class one gets packing information which is most helpful in controlling the topology in terms of the metric geometry. In the case of $\mathcal{R}(n, r, D, V)$ this packing information is provided by suitable coverings with geodesic (metric) balls yielding a coarse classification of the Riemannian structures occurring in $\mathcal{R}(n, r, D, V)$.

In order to define such coverings [GP], let us parametrize geodesics on $M \in \mathcal{R}(n, r, D, V)$ by arc length, and for any $p \in M$ let us denote by $\sigma_p(x) \equiv d_M(x, p)$ the distance function of the generic point x from the chosen point p. Recall that $\sigma_p(x)$ is a smooth function away from $\{p \cup C_p\}$, where C_p, a closed nowhere dense set of measure zero, is the cut locus of p. Recall also that a point $y \neq p$ is a critical point of $\sigma_p(x)$ if for all vectors $v \in TM_y$, there is a minimal geodesic, γ, from y to p such that the angle between v and $\dot{\gamma}(0)$ is not greater than $\pi/2$.

Definition and Proposition 2. *For any manifold $M \in \mathcal{R}(n,r,D,V)$ and for any given $\epsilon > 0$, it is always possible [GP] to find an ordered set of points $\{p_1,\dots,p_N\}$ in M, such that*

(i) the open metric balls (the geodesic balls*) $B_M(p_i,\epsilon) = \{x \in M \mid d(x,p_i) < \epsilon\}$, $i = 1,\dots,N$, cover M; in other words the collection*

$$\{p_1,\dots,p_N\} \tag{1}$$

is an ϵ-net in M.

(ii) the open balls $B_M(p_i,\epsilon/2)$, $i = 1,\dots,N$, are disjoint, i.e., $\{p_1,\dots,p_N\}$ is a minimal *ϵ-net in M.*

Similarly, upon considering the higher order intersection patterns of the set of balls $\{B_M(p_i,\epsilon)\}$, we can define the two-skeleton $\Gamma^{(2)}(M)$, and eventually the nerve $\mathcal{N}\{B_i\}$ of the geodesic ball covering of the manifold M:

Definition 3. Let $\{B_i(\epsilon)\}$ denote a minimal ϵ-net in M. The geodesic ball nerve $\mathcal{N}\{B_i\}$ associated with $\{B_i(\epsilon)\}$ is the combinatorial complex whose k-symplices $p^{(k)}_{i_1 i_2 \cdots i_{k+1}}$, $k = 0,1,\dots,$ are defined by the collections of $k+1$ geodesic balls such that $B_1 \cap B_2 \cap \cdots \cap B_{k+1} \neq \emptyset$.

Thus, for instance, the vertices $p^{(0)}_i$ of $\mathcal{N}\{B_i\}$ correspond to the balls $B_i(\epsilon)$; the edges $p^{(1)}_{ij}$ correspond to pairs of geodesic balls $\{B_i(\epsilon), B_j(\epsilon)\}$ having a non-empty intersection $B_i(\epsilon) \cap B_j(\epsilon) \neq \emptyset$; and the faces $p^{(2)}_{ijk}$ correspond to triples of geodesic balls with non-empty intersection $B_i(\epsilon) \cap B_j(\epsilon) \cap B_k(\epsilon) \neq \emptyset$.

Notice that, in general, this polytope has a dimension which is greater than the dimension n of the underlying manifold. However, as $\epsilon \to 0$, such a dimension cannot grow arbitrarily large being bounded above by a constant depending only on r, n, and D (see below).

According to the properties of the distance function (see, for instance, the paper of Cheeger [ChJ]), given $\epsilon_1 < \epsilon_2 \leq \infty$, if in $\bar{B}_i(\epsilon_2) \backslash B_i(\epsilon_1)$ there are no critical points of the distance function σ_i, then this region is homeomorphic to $\partial B_i(\epsilon_1) \times [\epsilon_1, \epsilon_2]$, and $\partial B_i(\epsilon_1)$ is a topological submanifold without boundary. One defines a *criticality radius*, ϵ_i, for each ball $B_i(\epsilon)$, as the largest ϵ such that $B_i(\epsilon)$ is free of critical points. Corresponding to such a value of the radius ϵ, the ball $B_i(\epsilon)$ is homeomorphic to an arbitrarily small open ball with center p_i, and thus it is homeomorphic to a standard open ball.

The point of the above remarks is that it can be easily checked, through direct examples, that the criticality radius of geodesic balls of manifolds in $\mathcal{R}(n,r,D,V)$ can be arbitrarily small (think of the geodesic balls drawn near the rounded tip of a cone), thus arbitrarily small metric balls in manifolds of bounded geometry are not necessarily contractible, and therefore, in general, the $B_i(\epsilon)$ are not homeomorphic to a standard open ball.

The above remarks suggest that one should be careful in understanding in what sense, for ϵ sufficiently small, the geodesic ball nerve gives rise to a polytope whose topology approximates the topology of the manifold $M \in \mathcal{R}(n,r,D,V)$ (the metric structure of the polytope is instead a rather good approximation to the metric structure of M, in the sense of Gromov–Hausdorff topology).

Actually the topology of the nerve dominates the topology of the underlying manifolds, a natural consequence of the fact that the criticality radius for the geodesic balls is not bounded below. In particular, it can be proven that the inclusion of sufficiently small geodesic balls into suitably larger balls is homotopically trivial [PW], and the geodesic ball nerve is thus a polytope which is *homotopically dominating* the underlying manifold, viz., there exist maps $f : M \to \mathcal{N}(B_i)$, and $g : \mathcal{N}(B_i) \to M$, with $g \cdot f$ homotopic to the identity mapping in M.

Even if the geodesic ball nerve is topologically more complicated than the underlying Riemannian manifold, in particular being of higher dimension, the above homotopical approximation (yielding the homotopy finiteness theorem recalled below) is really all that is needed for the analysis of the entropic estimates that follow.

Remark 4. Notice that the triangulation of the geodesic ball nerve generated by the $p^{(k)}_{ijl\cdots}$ is coloured by the non-trivial fundamental groups of the balls: to each vertex of the polytope there corresponds an intrisic *label* defined by (a representation of) the fundamental group $\pi_1(B_i(\epsilon))$ of the corresponding ball, and each k-simplex $(k \geq 1)$ gets a corresponding labelling according to Van Kampen's theorem.

On the geometrical side, there are a wealth of good properties of geodesic ball coverings which make them particularly appealing for applications in simplicial quantum gravity. As a good start, we can notice that the equivalence relation defined by manifolds with (combinatorially) isomorphic geodesic ball one-skeletons partitions $\mathcal{R}(n,r,D,V)$ into disjoint equivalence classes whose finite number can be estimated in terms of the parameters n, k, D. Each equivalence class of manifolds is characterized by the abstract (unlabelled) graph $\Gamma_{(\epsilon)}$ defined by the 1-skeleton of the $L(\epsilon)$-covering. The order of any such graph (i.e., the number of vertices) defines the *filling function* $N^{(0)}_{(\epsilon)}$, while the structure of the edge set of $\Gamma_{(\epsilon)}$ defines the (first order) intersection pattern $I_{(\epsilon)}(M)$ of $(M, \{B_i(\epsilon)\})$.

It is important to remark that on $\mathcal{R}(n,r,D,V)$ neither the filling function nor the intersection pattern can be arbitrary. The filling function is always bounded above for each given ϵ, and the best filling, with geodesic balls of radius ϵ, of a Riemannian manifold of diameter $diam(M)$, and Ricci curvature $Ric(M) \geq (n-1)H$, is controlled by the corresponding filling of the geodesic ball of radius $diam(M)$ on the space form of constant curvature given by H, the bound being of the form [Gr] $N^{(0)}_{\epsilon} \leq N(n, H(diam(M))^2, (diam(M))/\epsilon)$.

The multiplicity of the first intersection pattern is similarly controlled through the geometry of the manifold to the effect that the average degree, $d(\Gamma)$, of the graph $\Gamma_{(\epsilon)}$ (i.e., the average number of edges incident on a vertex of the graph) is bounded above by a constant as the radius of the balls defining the covering tend to zero (i.e., as $\epsilon \to 0$). Such a constant is independent of ϵ and can be estimated [Gp] in terms of the parameters n, and $H(diam(M))^2$ (it is this boundedness of the order of the geodesic ball coverings that allows for the control of the dimension of the geodesic ball nerve).

As expected, the filling function can be also related to the volume $v = Vol(M)$ of the underlying manifold M. This follows by noticing that [Zh] for any manifold $M \in \mathcal{R}(n, r, D, V)$ there exist constants C_1 and C_2, depending only on n, r, D, V, such that, for any $p \in M$, we have

$$C_1 \epsilon^n \leq Vol(B_\epsilon(p)) \leq C_2 \epsilon^n \tag{2}$$

with $0 \leq \epsilon \leq D$ (actually, here and in the previous statements a lower bound on the Ricci curvature suffices). Thus, if v is the given volume of the underlying manifold M, there exists a function $\rho_1(M)$, depending on n, r, D, V, and on the actual geometry of the manifold M, with $C_1 \leq (\rho_1(M))^{-1} \leq C_2$, and such that, for any $\epsilon \leq \epsilon_0$, we can write

$$N_\epsilon^{(0)}(M) = v \rho_1(M) \epsilon^{-n}. \tag{3}$$

We conclude this section by recalling the following basic finiteness results. They provide the topological rationale underlying the use of spaces of bounded geometries in simplicial quantum gravity. We start with a result expressing finiteness of homotopy types of manifolds of bounded geometry [PW].

Theorem 5. *For any dimension $n \geq 2$, and for ϵ sufficiently small, manifolds in $\mathcal{R}(n, r, D, V)$ with the same geodesic ball 1-skeleton $\Gamma_{(\epsilon)}$ are homotopically equivalent, and the number of different homotopy-types of manifolds realized in $\mathcal{R}(n, r, D, V)$ is finite and is a function of n, $V^{-1}D^n$, and rD^2.*

(Two manifolds M_1 and M_2 are said to have the same homotopy type if there exists a continuous map ϕ of M_1 into M_2 and f of M_2 into M_1, such that both $f \cdot \phi$ and $\phi \cdot f$ are homotopic to the respective identity mappings, I_{M_1} and I_{M_2}. Obviously, two homeomorphic manifolds are of the same homotopy type, but the converse is not true.)

Notice that in dimension three one can replace the lower bound of the sectional curvatures with a lower bound on the Ricci curvature [Zh]. Actually, a more general topological finiteness theorem can be stated under a rather weak condition of local geometric contractibility, and one obtains the following [GrP]:

Theorem 6. *Let $\psi : [0, \alpha) \to \mathbb{R}^+$, $\alpha > 0$, be a continuous function with $\psi(\epsilon) \geq \epsilon$ for all $\epsilon \in [0, \alpha)$ and such that, for some constants C and $k \in (0, 1]$, we have the growth condition $\psi(\epsilon) \leq C\epsilon^k$, for all $\epsilon \in [0, \alpha)$. Then for each $V_0 > 0$ and $n \in \mathbb{R}^+$ the class $C(\psi, V_0, n)$ of all compact n-dimensional Riemannian manifolds with volume $\leq V_0$ and with ψ as a local geometric contractibility function contains*

(i) finitely many simple homotopy types (all n),
(ii) finitely many homeomorphism types if $n = 4$,
(iii) finitely many diffeomorphism types if $n = 2$ or $n \geq 5$.

[Recall that ψ is said to be a local geometric contractibility function for a Riemannian manifold M if, for each $x \in M$ and $\epsilon \in (0, \alpha)$, the open ball $B(x, \epsilon)$ is contractible in $B(x, \psi(\epsilon))$.] Actually the growth condition on ψ is necessary in order to control the dimension of the limit spaces resulting from Gromov–Hausdorff convergence of a sequence of manifolds in $\mathcal{C}(\psi, V_0, n)$. As far as homomorphism types are concerned, this condition can be removed [Fe]. Note moreover that infinite-dimensional limit spaces cannot occur in the presence of a lower bound on sectional curvature as for manifolds in $\mathcal{R}(n, r, D, V)$. Finiteness of the homeomorphism types cannot be proved in dimension $n = 3$ as long as the Poincaré conjecture is not proved. If there were a fake three-sphere then one could prove [Fe] that a statement such as (ii) above is false for $n = 3$. Finally, the statement on finiteness of *simple homotopy* types, in any dimension, is particularly important for the applications in quantum gravity we discuss in the sequel. The notion of simple homotopy will be explained in same detail later on; roughly speaking it is a refinement of the notion of homotopy equivalence, and it may be thought of as an intermediate step between homotopy equivalence and homeomorphism.

3. Entropy functions for geodesic ball coverings of manifolds of bounded geometry

In what follows we shall explicitly assume, mainly for the sake of definiteness, that we are dealing with a space of bounded geometries $\mathcal{R}(n, r, D, V)$, even if many of the statements which follow hold either in Aleksandrov's spaces with curvature bonded below or in spaces admitting a local geometric contractibility function, and as a matter of fact, one can develop a rather general theory along the lines discussed below.

Within the above geometrical setting we now provide the exact *entropy* function which estimates the number of combinatorially inequivalent one-skeleton graphs associated with geodesic ball coverings of manifolds of bounded geometry. To this end, we shall enumerate the number of isomorphism classes of geodesic balls one-skeleton graphs of manifolds of a given homotopy type.

Since the geodesic ball nerve is a finite dimensional polytope (with the dimension bounded above by a constant depending on the parameter n, r, D, V and *not* on the radius of the balls), it follows that through the entropy function associated with geodesic ball one-skeletons we can recover the corresponding entropy functions for the generic k-skeleton of the nerve. Moreover, since the topology of the nerve dominates the topology of the underlying manifold, it follows that the entropy function we are going to introduce dominates all the possible inequivalent (dynamical) triangulations of the underlying manifold.

Let $L(m) \equiv 1/m$, $0 < m < \infty$ denote a cut-off parameter to be interpreted as the radius $\epsilon/2$ of minimal geodesic ball coverings on manifolds of bounded geometry, and let us consider $L(m)$-geodesic balls coverings whose filling function $N^{(0)}_{(m)}$ takes on the running (integer) value λ. Correspondingly let us introduce the function $B_\lambda(V, m, \pi(M))$ which, at scale $L(m)$, counts the number of combinatorially inequivalent 1-skeletons $\Gamma_{(m)}$ with λ vertices which can be generated by minimal geodesic balls coverings on manifolds M, of given volume V, in the homotopy class $\{\pi(M)\}$. This latter topological specification is justified by the following:

Lemma 7. *There is an m_0 depending only on n, r, D, V, such that, for any $m > m_0$, $B_\lambda(V, m, \pi(M))$ is a well-defined function of the the volume, and of the parameters n, r, D, in each homotopy class $\pi(M)$ of manifolds $M \in \mathcal{R}(n, r, D, V)$.*

Proof. This is a trivial consequence of the properties of minimal geodesic ball coverings as far as homotopy is concerned (see the previous paragraph and Theorem 4.1 in [PW]): There is an ϵ_0, whose value depends only on n, r, D, V, such that any two manifolds in $\mathcal{R}(n, r, D, V)$ having minimal ϵ_0-geodesic ball coverings with the same intersection pattern are homotopy equivalent. In each such homotopy class, the number of inequivalent one-skeletons is finite (such a number depends on n, r, D, V and m), thus B_λ is a well defined function of each homotopy type, as claimed. □

As will be explained shortly, the function $B_\lambda(V, m, \pi(M))$ is too rough for utility in simplicial quantum gravity, and we will be forced to specialize it a little bit. The point of the above lemma is that it is a good starting point on which we wish to elaborate in order to understand how much of the topology of M we need to know for estimating the entropy function.

To make explicit the relevant topological dependence of $B_\lambda(V, m, \pi(M))$, let us start by noticing that the function $B_\lambda(V, m, \pi(M))$ has natural continuity properties under Gromov–Hausdorff convergence. In particular, by using the relation between Gromov–Hausdorff convergence of manifolds and Lipschitz convergence of corresponding minimal nets [Gr] (generated by geodesic ball coverings), we can prove the following:

Lemma 8. *If $\{M_{(i)}\}$ is a sequence of manifolds in $\mathcal{R}(n, r, D, V)$ d_G-converging to a (topological) manifold M then for every $\bar{m}$ sufficiently large giving rise to $L(\bar{m})$-geodesic ball coverings on $M_{(i)}$, there is a corresponding $L(m)$-geodesic ball covering in M, with $m > \bar{m}$, such that, as $i \to \infty$, both coverings have the same value of the filling function: $N^{(0)}_{(\bar{m})} = N^{(0)}_{(m)} = \lambda$, and the same intersection pattern, and*

$$\lim_{M_{(i)} \to M} B_\lambda(V(M_{(i)}), \bar{m}, \pi(M_{(i)})) = B_\lambda(V(M), m, \pi(M)) . \tag{4}$$

Proof. If a sequence $\{M_{(i)}\}$ of Riemannian manifolds converges, in the Gromov–Hausdorff sense, to a (topological) manifold M, then for every positive ϵ and $\bar{\epsilon} > \epsilon$, every ϵ-net of M is the limit, for the Lipschitz distance, of a sequence of an $\bar{\epsilon}$-net of $M_{(i)}$. By definition of Lipschitz distance [Gr], the net on M and the corresponding nets in $M_{(i)}$ have the same number of vertices, say λ. Restricting our attention to minimal geodesic ball nets, we can establish, in this way, a one-to-one correspondence between geodesic ball nets (with λ vertices) on $M_{(i)}$ and geodesic ball nets (also with λ vertices) on M. Thus the statement of the lemma follows. □

We can put this last remark at work by relaxing the volume constraint in $\mathcal{R}(n, r, D, V)$ so as to allow for a sequence of manifolds $\{M_{(i)}\}$ with three-dimensional volume going to zero. Under such conditions we may have $\{M_{(i)}\}$ collapsing to a lower dimensional manifold. The classical example (see the paper by K. Fukaya quoted in [Gr]) in this direction is afforded by the Berger sphere: let g_{can} denote the standard metric on S^3, and consider the Hopf fibration $\pi : S^3 \to S^2$. Define $g_\epsilon(v, v) = \epsilon \cdot g_{can}(v, v)$ if $\pi_* v = 0$, and $g_\epsilon(v, v) = g_{can}(v, v)$ if the vector $v \in TS^3$ is perpendicular to the fibre of π. It is easily checked that $(S^3, g_\epsilon) \in \mathcal{R}(n = 3, D = 1, V = 0)$ for any $\epsilon \leq 1$, and that

$$\lim_{\epsilon \to 0} d_G[(S^3, g_\epsilon), (S^2, \hat{g})] = 0, \tag{5}$$

where $\hat{g}$ is the round metric on the two-sphere with curvature 4.

In such a case, and more in general when three-dimensional manifolds collapse to surfaces, the counting function $B_\lambda(V_\epsilon, \Gamma_{(m)}, \pi(M_\epsilon))$ approaches, as $\epsilon \to 0$, the corresponding function on the surface Σ resulting from the collapse, namely

$$B_\lambda(m, \pi(\Sigma)) \stackrel{\lambda \to \infty}{\longrightarrow} (\Lambda)^\lambda \lambda^{\chi(\Sigma)(\gamma - 2)/2 - 1} \cdot \rho(1 + O(1/\lambda)), \tag{6}$$

where Λ, γ, and ρ are suitable constants (as stressed in the introductory remarks, these asymptotics can be obtained in a number of inequivalent ways [ITZ]). We caution the reader that in this particular example the homotopy type is obviously not preserved. This does not contradict the homotopical characterization of the counting function since we are allowing for a sequence of Riemannian manifolds along which the three-dimensional volume goes to zero. Under such circumstances geodesic ball one-skeletons with the same intersection pattern do not necessarily correspond to manifolds of the same homotopy type.

It is also clear that the above result seems somehow at variance with the remarks following Lemma 7. The fact is that the topological dependence in the graph counting function $B_\lambda(m, \pi(\Sigma))$ comes by (in the subleading asymptotics) through the Euler characteristic, a topological invariant which basically classifies surfaces. Thus, it seems that a detailed knowledge of the topology of M is after all needed in providing the asymptotics of the counting function. However, notice that the role of Euler characteristic, in the above expression, rather than classifying surfaces, is only that of providing the *homotopy cardinality* of the complex determining the surface Σ (this *dimensional* point of view on the Euler characteristic and a similar cardinality interpretation of the Reidemeister torsion, which we exploit below, has been suggested to us by the paper of D. Fried on dynamical systems quoted in [RS]). In a sense it is only an accident (due to the particular use of $\chi(\Sigma)$ in surface theory) that the role of χ as a homotopic dimension is traded for a topology labelling role.

The *homotopical dimensional* meaning of the Euler characteristic mentioned above comes about by noticing that the Euler number of a finite complex is the only homotopy invariant that satisfies $\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$ for any two sub-complexes A, B of Σ, with the normalization $\chi(point) = 1$.

This latter remark confirms that homotopy is a natural topological label for $B_\lambda(V, m, \pi(M))$. However a good counting function of utility for simplicial quantum gravity should provide the number of inequivalent geodesic ball one-skeletons in nerves which are piecewise-linearly (PL) equivalent, and not just of the same homotopy type. But according to the finiteness theorems recalled in Section 2, asking for such a counting function is too much. In dimension three we have not yet control on the enumeration of the homeomorphism types while in dimension four no enumeration is possible for the PL types (otherwise by Cerf's theorem we could count the differentiable structures). Thus in the physically significant dimensions there is no enumerative criterion for PL structures.

The necessary compromise between what can be counted and what is of utility for quantum gravity brings into evidence a particular equivalence relation in homotopy known as *simple homotopy equivalence* [Co]. The homotopy equivalence associated with geodesic ball nerves with the same one-skeleton is a simple homotopy equivalence if the induced homotopy equivalence between the associated closed regular neighborhood in some $\mathbb{R}^n$ is homotopic to a piecewise-linear (PL) homeomorphism. Moreover, two nerves (one may think of them as polyhedra) are simple-homotopy equivalent if they have PL homeomorphic closed regular neighborhoods in some $\mathbb{R}^n$ (here and in the following remarks on simple homotopy theory we follow the particularly clear discussion of the argument due to S. Ferry [Fe]).

Thus, roughly speaking, resolving the homotopy types of the geodesic ball nerves into the corresponding simple homotopy types comes as close as possible to providing an enumeration of geodesic ball one-skeletons in manifolds which are PL homeomorphic.

Intuitively one should view simple homotopy as a sort of generalized holonomy representation which with any geodesic ball nerve associates as a label an element of an (infinite dimensional) abelian group: the *Whitehead group* of the (fundamental group of the) nerve $Wh(\mathcal{N})$. More explicitly, let $Z\pi_1(\mathcal{N})$ denote the integral group ring generated by the fundamental group of the nerve $\pi_1(\mathcal{N}) \simeq \pi_1(M)$, i.e., the set of all finite formal sums $\sum n_i g_i$, $n_i \in Z$, $g_i \in \pi_1(\mathcal{N})$, with the natural definition of addition and multiplication. Then the generic element of the Whitehead group $Wh(\mathcal{N}) \simeq Wh(Z\pi_1(M))$ can be represented as a non-singular $Z\pi_1(M)$ matrix w_{ik} which basically tells us how the nerve in question is generated, as a CW-complex, by adjoining cells to the underlying two-skeleton $\Gamma^{(2)}_{(m)}$.

The matrix w_{ik} is naturally acted upon by a set of operations which consists in: (i) multiplying on the left the ith row of the matrix by (plus or minus) an element of the fundamental group $\pi_1(\Gamma_2, e_0)$; (ii) adding a left group-ring multiple of one row to another; (iii) expanding the matrix by adding a corner identity matrix.

The equivalence class, $\tau(w_{ik})$, under the operations (i), (ii), (iii), generated by the non-singular incidence matrix w_{ik}, is the *Whitehead torsion* associated with the given nerve: a topological invariant which distinguishes between nerves having the same homotopy type (thus in particular having isomorphic fundamental groups), but with different simple-homotopy types. Thus, given two manifolds M_1 and M_2 in $\mathcal{R}(n,r,D,V)$ with the same geodesic ball one-skeleton and hence in the same homotopy type $h : M_1 \rightarrow M_2$, it follows that h is homotopic to a simple homotopy only if M_1 and M_2 have the same Whitehead torsion [Co].

For a given fundamental group the Whitehead group is, in general, an infinite-dimensional (abelian) group; however, as recalled in the section dealing with the properties of geodesic ball coverings, similarly to what happens to the number of distinct homotopy types, the number of inequivalent simple homotopy types realized by manifolds in $\mathcal{R}(n,r,D,V)$ is *finite*. Thus, independently of m, there are only a finite number of inequivalent Whitehead torsions $\tau(w_{ik})$ realized as we consider the totality of finer and finer geodesic balls coverings of manifolds in $\mathcal{R}(n,r,D,V)$, and the following holds [Pt,GrP,Fe]:

Theorem 9. *There is an $\epsilon_0 > 0$ which can be estimated in terms of the parameters n, r, D, V such that any two manifolds in $\mathcal{R}(n,r,D,V)$ whose Gromov–Hausdorff distance is $< \epsilon_0$ have the same Whitehead torsion, (i.e., they are simple-homotopy equivalent), and the number of distinct Whitehead torsions is finite.*

Notice that the original formulation of the above theorem is more general, see e.g., the paper of S. Ferry [Fe].

For our particular purposes, the knowledge of the full Whitehead torsion associated with a given geodesic ball nerve (i.e., of the $Z\pi_1$-incidence matrix w_{ik}) is not necessary, since we do not need to reconstruct (in a simple homotopical sense) the nerve from its two-skeleton. We need only to be able to tell if two nerves have the same simple homotopy type. To this end it is sufficient to evaluate the torsion of the nerves in correspondence to an orthogonal representation θ, of the fundamental group $\pi_1(\Gamma^{(2)})$, say by orthogonal $p \times p$ matrices, turning the p-dimensional Euclidean space $\mathbb{R}^p$ into a right $\mathbb{R}(\pi_1(\Gamma^{(2)}))$-module.

The point is that the orthogonal representation $\theta : \pi_1(\mathcal{N}) \rightarrow O(p)$ induces a ring homomorphism $Z\pi_1(\mathcal{N}) \rightarrow Z(O(p))$, which allows us to represent the $Z\pi_1$-incidence matrix w_{ik} via a non-singular real matrix $\theta_*(w_{ik})$. Since orthogonal representations capture in an essential way the structure of the fundamental group, we may use $\theta_*(w_{ik})$ in place of w_{ik} without losing much. Moreover, in so doing we have a further advantage, since the determinant of $\theta_*(w_{ik})$ can be used to keep track of the represented torsion, for the operations which take the represented incidence matrix $\theta_*(w_{ik})$ to another matrix of the same torsion, can only change the determinant by a factor of the form (up to a sign) $\det\theta(\pi_1)$. As a matter of fact, one can define the Reidemeister–Franz torsion associated with $\theta_*(w_{ik})$ according to [DR]

Definition 10. Let us denote by $\theta_*[w_{jl}(\pi_1(\Gamma^{(2)}_{(m)}))]$ (if no confusion arises we shall write $\theta_*(w_{il})$) the image under θ of the incidence matrix w_{jl}; then

$$\Delta^{\theta}(M) = |\det(\theta_*(w_{jl})| \tag{7}$$

defines the associated Reidemeister-Franz (RF) torsion in the represention θ.

Notice that $\theta_*(w)$ is a $a_{(m)} \times a_{(m)}$ matrix with entries in $Mat_p(\mathbb{R})$, namely a matrix of order $pa_{(m)}$ with real entries (where $Mat_p(\mathbb{R})$ denotes the ring of all $p \times p$ matrices with real entries). Notice also that, strictly speaking, the above definition of Reidemeister-Franz torsion is valid if the CW-pair $(\mathcal{N}, \Gamma^{(2)})$ is homotopically trivial, i.e., if the two-skeleton $\Gamma^{(2)}$ is a strong deformation retract of the nerve $\mathcal{N}$. Without the assumption of homotopical triviality of the pair $(\mathcal{N}, \Gamma^{(2)})$, the definition of the Reidemeister-Franz torsion takes on the more familiar aspect of a product of ratios of determinants [RS], and can be extended also to representations which are not acyclic. In such a case, we let $\mathcal{E}_{\theta} \equiv \tilde{\mathcal{N}} \times \mathbb{R}^p/\pi$ denote the flat orthogonal bundle associated with the representation θ, where $\tilde{\mathcal{N}}$ is the universal cover of $\mathcal{N}$ and π acts on $\tilde{\mathcal{N}}$ by deck transformations and on $\mathbb{R}^p$ by θ. The cochain complex $C^*(\mathcal{N}, \mathcal{E}_{\theta})$ and the cohomology $H^*(\mathcal{N}, \mathcal{E}_{\theta})$ are then defined. The torsion associated to this twisted cochain complex, still denoted by $\Delta^{\theta}(M)$, can be computed provided that we choose volume elements ν_i for $C^i(\mathcal{N}, \mathcal{E}_{\theta})$ and μ_i for the cohomology groups $H^i(\mathcal{N}, \mathcal{E}_{\theta})$ (see [Co,RS] for details). If $\bar{\mu}_i = k_i\mu_i$, $\bar{\nu}_i = h_i\nu_i$ is another choice of volume elements, then the corresponding torsion, $\Delta^{\theta}(M; \bar{\mu}, \bar{\nu})$ is related to $\Delta^{\theta}(M)$ by

$$\Delta^{\theta}(M; \bar{\mu}, \bar{\nu}) = \prod_j (k_j/h_j)^{(-1)^j} \Delta^{\theta}(M) . \tag{8}$$

Let $Hom(\pi_1(X), O(p))$ denote the set of all orthogonal representations,$\pi_1(X) \to O(p)$, of the fundamental group of the generic space X. Then, for what concerns the possible inequivalent Reidemeister-Franz torsions realized we have the following obvious

Lemma 11. *For each given orthogonal representation of the fundamental group, $\theta \in Hom(\pi_1(M), O(p))$, there are only a finite number of distinct representation torsions Δ^{θ} that can be realized for the manifolds in $\mathcal{R}(n, r, D, V)$.*

Proof. The homomorphism $\theta : \pi_1(M) \to O(p)$ induces a homomorphism $\theta_* : Wh(Z\pi_1(M)) \to Wh(ZO(p))$. Since the distinct simple homotopy types realized in $\mathcal{R}(n, r, D, V)$ are finite in number, it follows that for each given orthogonal representation of the fundamental group there are only a finite number of possibilities for the values of the corresponding representation torsions. □

Notice also that, given an orthogonal representation, $\theta : \pi_1(M) \to O(p)$, the associated Reidemeister-Franz representation torsion of the manifold M, $\Delta^{\theta}(M)$, is a topological invariant which satisfies a cardinality law which is analogous to the one satisfied by the Euler characteristic. Let A and B denote subcomplexes of the geodesic ball nerve with $\mathcal{N} = A \cup B$, and for a representation $\theta \in Hom(\pi_1(A), O(p)) \cap Hom(\pi_1(B), O(p)) \cap Hom(\pi_1(\mathcal{N}), O(p))$ denote by $\Delta^{\theta}(M|A)$, $\Delta^{\theta}(M|B)$, and $\Delta^{\theta}(M|A \cap B)$ the Reidemeister–Franz torsions associated with the subcomplexes A, B, and $A \cap B$ respectively. Then

$$\Delta^{\theta}(\mathcal{H}_{A,B})\Delta^{\theta}(M|A \cup B)\Delta^{\theta}(M|A \cap B) = \Delta^{\theta}(M|A)\Delta^{\theta}(M|B)\,, \tag{9}$$

where $\mathcal{H}_{A,B}$ is the long exact cohomology sequence associated with the short exact sequence generated by the complexes $C^*(A \cup B)$, $C^*(A) \oplus C^*(B)$, and $C^*(A \cap B)$ (the correction term $\Delta^{\theta}(\mathcal{H}_{A,B})$, associated with the twisted cohomology groups of the above three cochain complexes, disappears when the representation is acyclic).

The cardinality laws which hold either for the Euler characteristic or for the Reidemeister–Franz torsion suggest that we can try to determine the asymptotics of $B_{\lambda}(V, m, \pi(M))$ for higher dimensional manifolds by direct counting.

3.1. Determination of the entropy function

According to the foregoing remarks we can specialize the rough counting function $B_{\lambda}(\Gamma_{(m)}(M), \pi(M))$ according to the more specific

Definition 12. $B_{\lambda}(v, m, \chi, \Delta^{\theta})(M)$ is the entropy function which counts the number of combinatorially inequivalent one-skeletons $\Gamma^{(1)}_{(m)}$ with λ vertices which can be generated by minimal geodesic ball coverings on manifolds M, of given volume v, of given Euler characteristic χ, and given Reidemeister–Franz torsion in the given representation $\theta : \pi_1(M) \to O(p)$.

We shall obtain the asymptotic estimate for this counting function by a direct argument exploiting geodesic ball coverings and the cardinality laws for the Euler characteristic and the Reidemeister–Franz torsion so as to obtain

Theorem 13. *There is a value m_0 of the parameter m, depending only on n, r, D, and V, such that, for any $m \geq m_0$ and for any manifold $M \in \mathcal{R}(n, r, D, V)$ with RF-torsion Δ^{θ}, Euler number χ, and volume v, the entropy function $B_{\lambda}(v, m, \chi, \Delta^{\theta})(M)$ is given by*

$$\begin{aligned} B_{\lambda}(v, m, \chi, \Delta^{\theta})(M) &= \sqrt{2\pi} m^{n} \rho_1(M) \frac{A_1^{\lambda}}{\Delta^{\theta}(M)} \\ &\quad \times (\lambda)^{\chi(M)(\gamma_1 - 2)/2 + 1/2} \cdot \exp(a/12\lambda)\,, \end{aligned} \tag{10}$$

where A_1 and γ_1 are suitable constants, a is a function of λ with $0 < a < 1$, while ρ_1 is a function, depending on the metric geometry of M. ρ_1 is uniformly bounded below and above in terms of the parameters n, r, D, V. The costants A_1 and γ_1 can be explicitly computed and measure the average topological complexity of the generic geodesic ball of the covering.

Proof. As recalled in the section on the properties of geodesic ball coverings, each of the metric balls $B_{\epsilon}(p)$ has, in general, a non-trivial fundamental group $\pi_1(B_{\epsilon}(p))$. In order to exploit the argument described in the sequel, we need to consider the ball $B_\epsilon(p)$ included in larger balls, say $B_{C\epsilon}(p)$, with $C > 1$ (and in the underlying manifold M), and we let $I : B_\epsilon(p) \to B_{C\epsilon}(p)$ denote the inclusion. Thus, the question naturally arises of the size of the induced group $I_*(\pi_1(B_\epsilon(p)))$. Since arbitrarily small metric balls can be topologically complicated, it cannot be excluded a priori that $I_*(\pi_1(B_\epsilon(p)))$ has an infinite number of generators, and this may spoil the possibility of defining the RF torsion associated with a reconstruction of the manifold out of a minimal geodesic balls covering. (This possibility, as well as the indications of how to take care of it was kindly pointed out to us by P. Petersen.) In order to avoid such troubles, it is sufficient to choose ϵ small enough as follows from [Zh]

Lemma 14. *There are constants $R_0 > 1$, and ϵ_0 depending only on n, r, D, V, such that for any manifold $M \in \mathcal{R}(n,r,D,V)$, $p \in M$, $\epsilon \le \epsilon_0$, if $I : B_\epsilon(p) \to B_{R_0\epsilon}(p)$ is the inclusion, then there is no element of infinite order in $I_*(\pi_1(B_\epsilon(p)))$ whenever $\epsilon \le \epsilon_0$.*

Actually, this is a part of a more general result proved by S. Zhu [Zh], which also provides, under the same hypotheses, a uniform upper bound to the order of any subgroup of $I_*(\pi_1(B_\epsilon(p)))$.

With the notation of the above lemma, for any given $\epsilon \le \epsilon_0$, let us consider a collection of λ metric balls $\{B_\epsilon(p_i)\}_{i=1,\dots,\lambda}$ in the generic $M \in \mathcal{R}(n,r,D,V)$, with $B_\epsilon(p_i) \cap B_\epsilon(p_j) = \emptyset$, for every i,j.

Let us denote by $M_{(1)} \equiv \bigcup_i^\lambda B_{R_0\epsilon}(p_i)$, and, more generally, for any integer $k \ge 1$ we set

$$M_{(k)} \equiv \bigcup_i^\lambda B_{(2-1/k)R_0\epsilon}(p_i)\,. \tag{11}$$

Note that we assume that for $k = 1$ the balls $\{B_{R_0\epsilon}(p_i)\}$ are pairwise disjoint in M, while for $k \to \infty$ the balls $\{B_{2R_0\epsilon}(p_i)\}$ cover M, in such a way that $\{B_{R_0\epsilon}(p_i), B_{2R_0\epsilon}(p_i)\}$ defines a minimal net with λ vertices in M, and we correspondingly define $1/m_0 \equiv R_0\epsilon_0/2$ and $1/m = R_0\epsilon/2$, for every $\epsilon \le \epsilon_0$.

The family of metric spaces $\{M_{(k)}\}_{k=1}^\infty$ interpolates (continuously in the Gromov–Hausdorff topology), between the collection of disjoint balls $\{B_{R_0\epsilon}(p_i)\}$ and the covering $\{B_{2R_0\epsilon}(p_i)\}$. The idea is to use such an interpolating family to extend the asymptotics of the function which counts the inequivalent ways of introducing disjoint balls into M, to the function counting the corresponding inequivalent coverings of M.

The counting of the possible inequivalent ways of introducing pairwise disjoint geodesic balls in M is related to few elementary considerations exploiting the topology of the metric space $M_{(1)}$. To begin with, let us note that if the balls $\{B_{R_0\epsilon}(p_i)\}$ were contractible then $\chi(B_\epsilon(p_i)) = 1$ and $\lambda = \chi(M_{(1)})$ (since the balls are disjoint). However, for manifolds $M \in \mathcal{R}(n,r,D,V)$, arbitrarily small geodesic balls need not be contractible and consequently $\chi(B_{R_0\epsilon}(p_i))$ is not necessarily unity. Thus, it is natural to introduce a parameter γ_1, which measures to what extent the local ϵ-balls fail to be contractible and which is defined according to

$$\chi(M_{(1)}) = \sum_{i}^{\lambda} \chi(B_{R_0\epsilon}(p_i)) = 2\lambda/(\gamma_1 - 2)\,. \tag{12}$$

(The particular choice of the ratio $2/(\gamma_1 - 2)$ is for later convenience.) More explicitly, $2/(\gamma_1 - 2)$ is the average over $M_{(1)}$ of the Euler characteristic of the balls $B_{R_0\epsilon}(p_i)$, viz.,

$$\frac{2}{\gamma_1 - 2} = \langle \chi(B_{R_0\epsilon}(p_i))\rangle_\lambda \equiv \frac{1}{\lambda}\sum_{i}^{\lambda} \chi(B_{R_0\epsilon}(p_i))\,. \tag{13}$$

For what concerns the characterization of the RF-torsion of $M_{(1)}$, let us remark that if we denote by $Q_i(k) \equiv Hom(\pi_1(B_{(2-1/k)R_0\epsilon}(p_i)), O(p))$ the space of all representations of $\pi_1(B_{(2-1/k)R_0\epsilon}(p_i))$ into the orthogonal group $O(p)$, then $Hom(\pi_1(M_{(k)}), O(p)) = \bigcap_{i=1}^{\lambda} Q_i(k)$. Since $I_*(\pi_1(B_\epsilon(p_i)))$ are of finite order, it follows that, given an orthogonal representation $\theta_k \in \bigcap Hom(\pi_1(M), O(p)) \bigcap_{i=1}^{\lambda} Q_i(k)$, we can evaluate the corresponding RF-torsion of $M_{(k)}$, $\Delta^\theta(M_{(k)})$, if we provide volume elements for the twisted cochain complex $C^*(\mathcal{N}, \mathcal{E}_{\theta_k})$. In particular for $k = 1$, since the balls are disjoint, we get

$$\Delta^\theta(M_{(1)}) = \prod_{i}^{\lambda} \Delta^{\theta_1}(B_{R_0\epsilon}(p_i))\,. \tag{14}$$

If we introduce the average value of $\log[\Delta^{\theta_1}(B_{R_0\epsilon}(p_i))]$ over the set λ balls $B_{R_0\epsilon}(p_i)$, viz.,

$$\langle \log[\Delta^{\theta_1}(B_{R_0\epsilon}(p_i))]\rangle_\lambda \equiv \frac{1}{\lambda}\sum_{i=1}^{\lambda} \log[\Delta^{\theta_1}(B_{R_0\epsilon}(p_i))]\,, \tag{15}$$

then we can write

$$\Delta^\theta(M_{(1)}) = \exp[\lambda\langle \log[\Delta^{\theta_1}(B_{R_0\epsilon}(p_i))]\rangle_\lambda]\,, \tag{16}$$

which, in general, depends on a choice of volume elements ν_k and μ_k in the twisted cochain complex $C^k(M_{(1)}, \mathcal{E}_\theta)$ and in the cohomology groups $H^k(M_{(1)}, \mathcal{E}_\theta)$ associated with the given representation θ (we stress that the volume elements must be the same for all twisted cochain complexes associated with the balls). To keep track of such a choice, we can introduce a parameter, Λ_1, defined according to

$$\Lambda_1 = \exp[\langle \log[\Delta^{\theta_1}(B_{R_0\epsilon}(p_i))]\rangle_\lambda - 1] \tag{17}$$

(the normalization of Λ_1 to e^{-1} is again for later convenience), and in terms of which we can write

$$\Delta^{\theta}(M_{(1)}) = \exp[\lambda + \lambda \ln \Lambda_1] . \tag{18}$$

Having established these preliminary results, we can introduce a natural counting function for the possible inequivalent ways of introducing the set of disjoint balls $M_{(1)}$ into M. We exploit the fact that each of the λ induced fundamental groups, $I_*(\pi_1(B_\epsilon(p_i)))$, can be thought of as providing a colouring of the corresponding ball $B_{R_0\epsilon}(p_i)$.

More explicitly, given an orthogonal representation $\theta \in \bigcap_{i=1}^{\lambda} Q_i(1)$, let $H^*(B_{R_0\epsilon}(p_i), \mathcal{E}_\theta)$ be the twisted cohomology of the generic ball $B_{R_0\epsilon}(p_i)$. Since the balls are disjoint we can write $H^*(M_{(1)}, \mathcal{E}_\theta) \simeq \bigoplus_i^{\lambda} H^*(B_{R_0\epsilon}(p_i), \mathcal{E}_\theta)$ and we can define *inequivalent* any two $M_{(1)}$ if the corresponding balls are labelled with distinct permutations of the cohomology groups $H^*(B_{R_0\epsilon}(p_i), \mathcal{E}_\theta)$. Therefore there are $\lambda!$ ways of distributing the *labels* $H^*(B_{R_0\epsilon}(p_i), \mathcal{E}_\theta)$ over the unlabelled balls $\{B_{R_0\epsilon}(p_i)\}$ in $M_{(1)}$ (the *coordinate* labelling of the balls arising from the centers $\{p_i\}$ are factored out to the effect that $M_{(1)}$ is considered as a collection of λ empty boxes on which we are distributing the λ labellings $H^*(B_{R_0\epsilon}(p_i), \mathcal{E}_\theta)$). Consequently we can define a counting function for $M_{(1)}$ according to

$$B_\lambda(v, m, \chi, \Delta^\theta)(M_{(1)}) \equiv m^n \rho_1(M) \lambda! , \tag{19}$$

where the m^n-dependent factor which multiplies $\lambda!$ connects the volume of the balls $B_\epsilon(p_i)$ in $M_{(1)}$ to the volume v of the underlying manifold M (it is the inverse of the average volume of the balls, see (3)).

In order to extend to the generic $M_{(k)}$ the above argument notice that, as soon as the balls in $M_{(k)}$ start intersecting each other, the cohomologies $H^*(M_{(k)}, \mathcal{E}_\theta)$, $\bigoplus H^*(B_{(2-1/k)R_0\epsilon}(p_i), \mathcal{E}_\theta)$, and $H^*(\bigcap B_{(2-1/k)R_0\epsilon}(p_i), \mathcal{E}_\theta)$ are related through a (twisted) Mayer–Vietoris exact sequence.

For instance, given any two intersecting balls $B_{(2-1/k)R_0\epsilon}(p_i)$ and $B_{(2-1/k)R_0\epsilon}(p_h)$ in $M_{(k)}$, there is a corresponding twisted Mayer–Vietoris short exact sequence

$$\begin{aligned} 0 &\to C^j(B(p_i) \cup B(p_h)) \to C^j(B(p_i)) \oplus C^j(B(p_h)) \\ &\to C^j(B(p_i) \cap B(p_h)) \to 0 \end{aligned} \tag{20}$$

(with obvious notations), with an associated cohomology long exact sequence. We can define inequivalent any two $M_{(k)}$ if the corresponding balls and their associated intersections are labelled with distinct permutations of the corresponding cohomology groups, according to the constraints expressed by the Mayer–Vietoris sequence. Namely, inequivalent labellings are generated only by those geodesic ball groups whose elements are in the kernel of the map $\bigoplus_i H^j(B(p_i)) \to H^j(\cap_i B(p_i))$. (Think of colouring intersecting boxes in such a way that on the intersections the colours blend according to a given pattern.) We can easily count the inequivalent ways of performing such a constrained labelling by exploiting the cardinality properties (9) of the RF-torsion (and of the Euler characteristic), which indeed are a consequence of the constraints associated with (20).

We start by noticing that the information we have gathered on $M_{(1)}$ allows us to rewrite (19) in a geometrical way in terms of the topological parameters $\chi(M_{(1)})$ and $\Delta^\theta(M_{(1)})$ introduced above (see Eqs. (12) and (17)).

First we apply Stirling's formula to (19) so as to get

$$B_\lambda(v,m,\chi,\Delta^\theta)(M_{(1)}) = m^n \rho_1(M)\sqrt{2\pi}\exp\left(-\lambda + \frac{a}{12\lambda}\right)\lambda^{\lambda+1/2}, \tag{21}$$

where a depends on λ but satisfying $0 < a < 1$.

It is easily checked that, in terms of the parameters γ_1 and Λ_1, defined by (12) and (17), we can rewrite the counting function for $M_{(1)}$ as a function of the Euler characteristic and of the RF-torsion according to

$$\begin{aligned} B_\lambda(v,m,\chi,\Delta^\theta)(M_{(1)}) = \sqrt{2\pi} m^n \rho_1(M) \frac{\Lambda_1^\lambda}{\Delta^\theta(M_{(1)})} \\ \times (\lambda)^{\chi(M_{(1)})(\gamma_1-2)/2+1/2} \cdot \exp(a/12\lambda) \end{aligned} \tag{22}$$

Note that we can rewrite $M_{(k)}$ as the disjoint union

$$M_{(k)} = M_{(1)} \bigcup_{j=2}^{k} (M_{(j)} \backslash M_{(j-1)}) \tag{23}$$

and if we apply the cardinality laws for the Euler characteristic and the torsion, then it is straightforward to verify that we can rewrite $B_\lambda(v,m,\chi,\Delta^\theta)(M_{(1)})$ as

$$\begin{aligned} & B_\lambda(v,m,\chi,\Delta^\theta)(M_{(1)}) \\ & \quad = \sqrt{2\pi} m^n \rho_1(M) \frac{\Lambda_1^\lambda}{\Delta^\theta(M_{(k)})} (\lambda)^{\chi(M_{(k)})(\gamma_1-2)/2+1/2} \cdot \exp(a/12\lambda) \\ & \quad \times \prod_{j=2}^{k} \frac{\Delta^\theta(M_{(j)})}{\Delta^\theta(M_{(j-1)})} (\lambda)^{-\chi(M_{(j)}-\chi(M_{(j-1)}))(\gamma_1-2)/2}. \end{aligned} \tag{24}$$

On the other hand, from a combinatorial point of view, given the possible inequivalent ways of inserting $M_{(k)}$ in M, $B_\lambda(v,m,\chi,\Delta^\theta)(M_{(k)})$, we can define $B_\lambda(v,m,\chi,\Delta^\theta)(M_{(1)})$ according to

$$B_\lambda(v,m,\chi,\Delta^\theta)(M_{(1)}) = B_\lambda(v,m,\chi,\Delta^\theta)(M_{(k)}) \prod_{j=2}^{k} B_\lambda(M_{(j-1)} \to M_{(j)}), \tag{25}$$

where $B_\lambda(M_{(j-1)} \to M_{(j)})$ denotes the number of inequivalent ways of introducing the balls of $M_{(j-1)}$ in the already inserted balls of $M_{(j)}$. Notice that such $B_\lambda(M_{(j-1)} \to M_{(j)})$ of necessity must satisfy the recursive relation

$$B_\lambda(M_{(k-p)} \to M_{(k-q)}) = \prod_{j=q}^{p-1} B_\lambda(M_{(k-j-1)} \to M_{(k-j)}) \tag{26}$$

for $k > p > q$, which says that the number of inequivalent ways of introducing the balls of $M_{(k-p)}$ into the (larger) balls of $M_{(k-q)}$ is equal to the number of ways of introducing the balls of $M_{(k-p)}$ into the balls of $M_{(k-p+1)}$ times the number of ways of introducing those of $M_{(k-p+1)}$ into those of $M_{(k-p+2)}$ and so on.

A direct comparison shows that (24), the explicit expression for $B_\lambda(v, m, \chi, \Delta^\theta)(M_{(1)})$ in terms of χ, and $\Delta^\theta(M_{(k)})$, is consistent with the combinatorial definition (25) and the recursive relation (26) if and only if

$$B_\lambda(v, m, \chi, \Delta^\theta)(M_{(k)}) = \sqrt{2\pi} m^n \rho_1(M) \frac{\Lambda_1^\lambda}{\Delta^\theta(M_{(k)})} \times (\lambda)^{\chi(M_{(k)})(\gamma_1 - 2)/2 + 1/2} \cdot \exp(a/12\lambda) \tag{27}$$

with

$$B_\lambda(M_{(j-1)} \to M_{(j)}) = \left(\frac{\Delta^\theta(M_{(j)})}{\Delta^\theta(M_{(j-1)})} \right) (\lambda)^{-\chi(M_{(j)} - \chi(M_{(j-1)}))(\gamma_1 - 2)/2}. \tag{28}$$

Since as $k \to \infty$, $M_{(k)}$ converges, in the Gromov-Hausdorff sense, to M we immediately get from (27) the required asymptotics for the one-skeleton graphs counting function on a manifold $M \in \mathcal{R}(n, r, D, V)$, namely

$$B_\lambda(\Gamma_{(m)}^{(1)}(M), v, \Delta^\theta(M)) = \sqrt{2\pi} m^n \rho_1(M) \frac{\Lambda_1^\lambda}{\Delta^\theta(M)} \times (\lambda)^{\chi(M)(\gamma_1 - 2)/2 + 1/2} \cdot \exp(a/12\lambda), \tag{29}$$

and this completes the proof of the theorem. □

A few remarks are in order concerning the structure of (29). A first remark concerns the fact that given an orthogonal (in general not acyclic) representation $\theta : \pi_1 \to O(p)$, the counting function $B_\lambda(\Gamma_{(m)}^{(1)}(M), v, \Delta^\theta(M))$ depends on the choice of measures in the twisted cochain complex $C^*(\mathcal{N}, \mathcal{E}_\theta)$ and in the associated cohomology $H^*(\mathcal{N}, \mathcal{E}_\theta)$. Notice in particular that, if we rescale, by a factor $t = \exp[(2 - \gamma_1)/2p\lambda]$, the flat density in $\mathbb{R}^p$ yielding the volume form ν_k in $C^k(\mathcal{N}, \mathcal{E}_\theta)$, then

$$(\Delta^\theta(M)) \to (\Delta^\theta(M; \nu))(\lambda)^{\chi(M)(\gamma_1 - 2)/2}, \tag{30}$$

which shows that the entropy function $B_\lambda(\Gamma_{(m)}^{(1)}(M), v, \Delta^\theta(M))$ basically is a torsion evaluated for a suitable choice of the flat volume density in the bundle $\mathcal{E}_\theta$.

A second, related, remark concerns the role of the orthogonal representations of the fundamental group that have been introduced in order to compute the RF-torsion. It is obvious that $B_\lambda(\Gamma_{(m)}^{(1)}(M), v, \Delta^\theta(M))$ only counts the isomorphism classes of geodesic ball coverings coloured according to the chosen representation $\theta : \pi_1(M) \to O(p)$. Thus we are also interested in carrying out a suitable sum of $B_\lambda(\Gamma_{(m)}^{(1)}(M), v, \Delta^\theta(M))$ over the conjugacy classes of inequivalent representations having the same Whitehead torsion (i.e., summing over all possible inequivalent colourings within a given simple homotopy class). Recall that the space of inequivalent representations is $Hom(\pi_1(M), O(p))/O(p)$, namely the space of $O(p)$-orbits of $Hom(\pi_1(M), O(p))$ with the quotient topology. This space can be geometrically interpreted in various equivalent ways, for instance, as the moduli space of flat orthogonal bundles over M. Particularly suggestive is the interpretation, due to W. Goldman [Gl], which characterizes $Hom(\pi_1(M), O(p))/O(p)$ as the deformation space of locally flat Euclidean structures on M (M has a locally Euclidean structure if every point has a neighborhood isometric to an open subset of Euclidean space). This latter interpretation very strongly suggests that locally Euclidean spaces have, not surprisingly, a distinguished role in determining the configurational entropy of the space of Riemannian structures of use in quantum gravity. Roughly speaking, infinitesimal deformations around a given Euclidean structure are already accounted for in $B_\lambda(\Gamma^{(1)}_{(m)}(M), v, \Delta^\theta(M))$, owing to the presence of the RF-torsion. This latter remark can be more easily understood if we recall [Sc] that RF-torsion (or rather its smooth counterpart, the Ray–Singer analytic torsion), comes about as a stationary-phase approximation of a formal path integral over fluctuations around a given flat connection. The characterization of the sum of $B_\lambda(\Gamma^{(1)}_{(m)}(M), v, \Delta^\theta(M))$ over the representation variety $Hom(\pi_1(M), O(p))/O(p)$, for each given simple homotopy class, is quite not-trivial. Since $Hom(\pi_1(M), O(p))/O(p)$ is an algebraic variety, for each p there are only a finite number of connected components, which correspond to the finite number of distinct principal $O(p)$-bundles that have flat connections [Mor]. An example is afforded by the representations of surface groups in $SL(2, \mathbb{R})$ where it is possible to show [Gl] that $Hom(\pi_1(\Sigma), SL(2, \mathbb{R}))$ has $2^{2g+1} + 2g - 3$ components, where g is the genus of the surface Σ. We do not know of any similar result on the relation between the topology of M and the number, $|Hom(\pi_1(M), O(p))/O(p)|_{\tau(w)}$, of inequivalent connected components of $Hom(\pi_1(M), O(p))/O(p)$ for a given Whitehead torsion. Even less is known (at least to us) concerning the way of integrating over inequivalent representations in each connected component. We think that such a result would be very valuable for a deeper understanding of the structure of the entropy function (29).

A final remark calls for a comment on the application of (29) to four-dimensional simplicial quantum gravity (implicitly, (29) has already been applied to 3-D simplicial quantum gravity in [CM3]; actually, this particular application suggested the proof of the entropy estimate presented here). The delicate point is a correct formulation of the regularized version of the Einstein–Hilbert action to be associated with geodesic ball nerve over four-manifolds. Recall that, in general, such a nerve is a polytope of dimension larger than the dimension of the underlying manifold, and thus a simple transcription, in the spirit of dynamically triangulated models, as a weighted sum of the orders of the various subskeletons, is not, a priori, the most proper choice. In dimension three, one overcomes this problem for reasons connected with the nature of simple homotopy theory. Basically, given the non-singular $Z\pi_1$-incidence matrix describing the nerve, one can shift to a three-dimensional complex which is in the same simple homotopy class of the original nerve. This construction is not trivially extended to the four-dimensional case: we have some preliminary results on how to take care of the excess topology of the nerve, but they are not yet in a simple geometrical form and will not be discussed here.

At this point, a fairly civilized approach would be to use (29) for the asymptotics of the entropy function of standard dynamical triangulations of four dimensional manifolds, and then proceed with a Peierls *entropy-versus-action* argument to discuss the thermodynamical and possibly the continuum limit of the theory. Details of our results in this connection and some further developments will appear elsewhere.

Acknowledgement

We are very grateful to Peter Petersen and to Shun-hui Zhu for interesting and very stimulating discussions, and for pointing out to us some important recent results. We wish to express our gratitude to C. Bartocci, U. Bruzzo, C. Reina, L. Sbano, and P. Siniscalco and in particular to A. Vancheri for some extremely helpful conversations and for suggesting to us a more correct normalization for the RF-torsion. The authors are also much indebted to the referee, who carefully read a preliminary version of the manuscript and made many constructive suggestions.

This research was supported in part by the National Science Foundation under Grant No. PHY89-04035

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