BOREL SUMMABILITY: APPLICATION TO THE ANHARMONIC OSCILLATOR

S. GRAFFI, V. GRECCHI
Istituto Nazionale di Fisica Nucleare, Sezione di Bologna, Italy
and
B. SIMON
Department of Mathematics, Princeton University, USA

Received 3 August 1970

We prove that the energy levels of an arbitrary anharmonic oscillator (x^{2m} and in any finite number of dimensions) are determined uniquely by their Rayleigh-Schrödinger series via a (generalized) Borel summability method. To use this method for computations, one must make an analytic continuation which we accomplish by (a rigorously unjustified) use of Padé approximants in the case of $p^2 + x^2 + \beta x^4$. The numerical results appear to be better than with the direct use of Padé approximants.

In physics, one is often faced with a divergent [e.g. 1] (but finite order-by-order) perturbation series; in such cases, one must decide (or prove) what the meaning of such a series should be. The usual answer is that the series is asymptotic, but it is well known [2,3] that an infinite number of analytic functions have any given asymptotic series, so that such an answer is not entirely satisfactory. Under certain conditions, however, it is possible to obtain uniqueness theorems by means of a regular summability method, such as the Stieltjes-Padé and Borel methods. For example, Loeffel et al. [4] have proven that the perturbation series for the energy levels of a one-dimensional x^4-oscillator: $p^2 + x^2 + \beta x^4$, sums under the Stieltjes-Padé method to the actual levels. Their proof is not known to extend either to multidimensional oscillators or to x^{2m} oscillators †. In this note, we wish to demonstrate that the Borel summability method †† can be applied to these cases.

† The reasons in these two cases are distinct. For x^{2m}, the perturbation coefficients grow too fast; we avoid this problem by utilizing some weak analyticity properties on sheets after the first. For multi-dimensional oscillators, the global analyticity properties of Loeffel and Martin [8] have not been proven. We sidestep this difficulty by only needing local analyticity properties which can be deduced by Hilbert space methods independent of dimension.

†† Ogievski [5] noted long ago that for the exactly solvable Dirac particle in a (non-quantized) state field the answer is Borel summable and recently Reeken [6], has studied Borel summability for regular perturbations.

To this end, we first quote a generalized version of Watson's theorem †††: Let D be the sector on an n-sheeted Riemann surface defined as: $0 < |z| < B$, $|\arg z| < \lambda$, $\frac{3}{2}\pi > \lambda > \frac{1}{2}\pi$. Let D_1 be the sector: $|\arg z| \leq \delta < \lambda - \frac{1}{2}\pi$ and $\bar{D}$ the sector $0 < |z| < B$, $|\arg z| \leq \delta$. Given the formal power series $\sum_0^\infty a_n z^n$, suppose that:

(i) $f(z)$ is a function regular in D with the formal series as asymptotic series uniformly in D:

$$f(z) = \sum_0^N a_n z^n + R_N(z) \tag{1}$$

(ii) There are σ, C so that:

$$|a_n| < C\sigma^n n\,! \; ; \; |R_N(z)| < C\sigma^{N+1}(N+1)!\,|z|^{N+1} \tag{2}$$

uniformly in D and N ‡. Then the series $\sum_0^\infty a_n z^n$ is Borel summable to $f(z)$ in $\bar{D}$, i.e. one has there:

$$f(z) = \int_0^\infty \mathrm{e}^{-a} F(za)\mathrm{d}a \tag{3}$$

where:

††† Hardy [2, p. 192] proves the theorem only for $\lambda < \pi$, but its generalization to an n-sheeted surface is simple.

‡ We remark that the second part of eq. (2) automatically implies the first part.

$$F(z) = \sum_0^\infty \alpha_n z^n ; \quad \alpha_n = a_n/n! \tag{4}$$

This series, called the Borel transform of $f(z)$, being convergent for $|z| < 1/\sigma$ can be continued to a function regular in D_1. We note the obvious fact that once $f(z)$ is known in D, it is known in principle throughout its domain of analyticity, if this is bigger than D.

Suitable hypotheses allow the generalization of this theorem to series diverging more rapidly than $n!$. In the former notation, suppose now that:

$$\tfrac{3}{2}m\pi > \lambda > \tfrac{1}{2}m\pi \ , \quad \delta = \lambda - \tfrac{1}{2}m\pi \ ; \ |a_n| < C\sigma^n (mn)! \ ;$$

$$|R_N(z)| < C\sigma^{N+1}\{m(N+1)\}! \, |z|^{N+1} \tag{5}$$

$m \geq 1$ being any integer, all other hypotheses remaining unchanged. The transformation $z^{1/m} = w$ reduces $\phi(w) = f(z^{1/m})$ to the conditions of Watson's theorem, so that one obtains easily:

$$f(z) = (1/m) \int_0^\infty e^{-a^{1/m}} F(za) a^{(1/m)-1} da \ , \ z \epsilon \bar{D} \ , \tag{6}$$

where:

$$F(z) = \sum_0^\infty a_n z^n/(mn)! \tag{7}$$

is convergent for $|z| < 1/\sigma$, defining a function with a regular continuation to D_1. This result is exactly the generalized Borel sum (B, m) of Leroy [2, p. 147].

Consider now the anharmonic oscillators defined by the Hamiltonians:

$$H_m = p^2 + x^2 + \beta x^{2m} \tag{8}$$

Simon [7] has proven for any m and p‡ :

(a) The functions $E^{(m)}(\beta)$ are analytic in the domain $D(\epsilon)$: $\{0 < |\beta| < B(\epsilon); \ |\arg \beta| < \frac{1}{2}(m+1)\pi - \epsilon\}$ for any ϵ (with $B(\epsilon) > 0$ dependent on ϵ, m, p).

(b) The perturbation series are asymptotic to $E^{(m)}(\beta)$, uniformly in each $D(\epsilon)$.

(c) The coefficients $a_n^{(m)}$ of the perturbation series for $E^{(m)}$ are $O(\sigma^n\{(m-1)n\}!)$.

Similar results also hold for an s-dimensional oscillator $H_0 + V$ where H_0 is an s-dimensional $\sum^s p_i^2 + x_i^2$ and V is a homogeneous $2m$-th degree polynomial which is everywhere positive on the unit sphere. In addition, it has been proven by Loeffel and Martin [8] in the one-dimensional case:

(d) $E^{(m)}(\beta)$ are analytic in the entire cut plane $|\arg \beta| < \pi$, and are Herglotz functions there.

By (b), (c), (d) one is allowed to apply for $m = 2$, 3 the Stieltjes-Padé summability method, which implies the determination of a unique Stieltjes function from the asymptotic series and the convergence of any $[N, N+j]$ (j fixed) Padé approximants sequence to the eigenvalues.

For $m > 3$ the Padé are known to convergence, but it is not known that the asymptotic series determines a unique Stieltjes function: thus the $[N, N+j]$ may not converge to the eigenvalues (or even to an answer independent of j!). For the multidimensional oscillator, (d) is not known so the Padé's are not proven to converge, let alone to converge to the eigenvalues.

What we will show is that the Borel method is applicable in all these cases. Given (a), one needs only prove the estimate:

$$|R_n^{(m)}(\beta)| \leq C\sigma^{n+1}\{(m-1)(n+1)\}! \beta^{n+1} \tag{9}$$

uniformly in $D(\epsilon)$ for ϵ small to conclude that the perturbation series are $(B, m-1)$ summable in the sense as defined above. Before turning to the proof of eq. (9), which is suggested by (c), we remark that (b)-(d) imply eq. (9) in the cut plane (thereby proving the summability by the ordinary Borel method for $m = 2$) and that one has summability in regions $\{|\beta| < B(\epsilon), |\arg \beta| < \pi - \epsilon\}$. We also note that for any m we have the Borel transform analytic in the whole β-plane, cut along the negative real axis from $-R_0$ to $-\infty$ ‡‡ .

We now turn to a sketch of the proof of eq. (9), the main technical point of this note. Since the extensions to arbitrary m and an arbitrary number of dimensions is trivial, we consider only the x^4 ground state in one dimension.

We first recall [7, § II.10,11] that to control eq. (9), we need only control the remainder for the asymptotic series to the ground state of $H(\gamma, |\beta|) = p^2 + \gamma x^2 + |\beta| x^4$ uniformly in $\{\gamma/|\gamma| = 1; \ |\arg \gamma| < \pi - \epsilon\}$. Let $P_\gamma(|\beta|)$ be the projection onto this ground state and let Ψ_0 be the unperturbed ground state normalized by #: $\langle \bar{\Psi}_0, \Psi_0 \rangle = 1$. Then

‡ $E_p^{(m)}(\beta)$ represents the p-th level of $p^2 + x^2 + \beta x^{2m}$ for $\beta > 0$, analytically continued in β. We suppress the p for convenience.

‡‡ R_0 depends on the behaviour of the perturbation coefficients. If the semi-heuristic arguments of Bender and Wu (to be published) are correct, $R_0^{(m)} = \{m\Gamma(\frac{1}{2} + m/(m-1))/\pi^{1/2}(m-1)\Gamma(1+m/(m-1))\}^{m-1}$. One of us (B.S.) should like to thank Dr. C. Bender for a discussion of these results before appearence of the preprint.

For γ complex, Ψ_0 is no longer real. The normalization $\int \Psi_0^2(x)dx = 1$ makes $\Psi_0(\gamma, x)$ analytic in γ.

$$E(\gamma, |\beta|) = \frac{\langle H^*(\gamma, |\beta|)\Psi_0, P_\gamma(|\beta|)\Psi_0\rangle}{\langle \Psi_0, P_\gamma(|\beta|)\Psi_0\rangle} \qquad (10)$$

It is not hard to prove that the quotient of functions $f(z)$, $g(z)$ obeying eq. (9) (with $g(0) \neq 0$ of course!) obeys eq. (9) perhaps with σ and C modified, so we need only prove a result of type eq. (9) for the numerator and denominator of eq. (10) separately. Since

$$P\,(|\beta|) = (-\tfrac{1}{2}\pi i) \int_{|\lambda - \frac{1}{2}\gamma^{1/2}| = \frac{1}{2}} d\lambda R(\lambda)\ ,$$

with $R(\lambda) = \{H(\gamma, |\beta|) - \lambda\}^{-1}$

it is enough to control the numerator and denominator of eq. (10) with P replaced with R (but uniformly in λ with $|\lambda - \frac{1}{2}\gamma^{1/2}| = \frac{1}{2}$). Finally since $\|H^*(\gamma, |\beta|)\Psi_0\|$ and $\|\Psi_0\|$ are bounded we need only control the norms of the remainder terms, i.e. $\|R(\lambda)\{x^4(p^2 + \gamma x^2 - \lambda)^{-1}\}^n \Psi_0\|$ or since $R(\lambda)$ is uniformly bounded $\|\{x^4(p^2 + \gamma x^2 - \lambda)^{-1}\}^n \Psi_0\|$.

Introduce "scaled" creation and annihilation operators † $b = 2^{-1/2}(\gamma^{1/4}x + i\gamma^{-1/4}p)$, $a = 2^{-1/2}(\gamma^{1/4}x - i\gamma^{-1/4}p)$. Let $\Psi_n = (n!)^{-1/2} b^n \Psi_0$ so Ψ_n is normalized by $\langle \bar{\Psi}_n, \Psi_n\rangle = 1$, i.e. Ψ_n is just the $\gamma = 1$ eigenfunction †† scaled by $\chi \to \gamma^{1/4}x$. It is not hard to prove ††† that $\|\Psi_n\| \leq B^{n+1}$ for some B which can be chosen independently of γ with $|\arg\gamma| < \pi - \epsilon$. To obtain the required bound ‡:

$$\|\{x^4(p^2 + \gamma x^2 - \lambda)^{-1}\}^n \Psi_0\| \leq D\sigma^n(n+1)!\ ,$$

we expand $\chi = 2^{-1/2}\gamma^{-1/4}(a+b)$ and majorize the norm by 4^n terms of the form

$$\|a_1^+ a_2^+ a_3^+ a_4^+ (p^2 + \gamma x^2 - \lambda)^{-1} \dots a_{4n}^+ (p^2 + \gamma x^2 - \lambda)^{-1} \Psi_0\|$$

with each $a^+ = a$ or b. Using:

$$a\Psi_n = n^{1/2}\Psi_{n-1}\ ; \qquad b\Psi_n = (n+1)^{1/2}\Psi_{N+1}\ ,$$

$$(p^2 + \gamma x^2 - \lambda)^{-1}\Psi_m = (\gamma^{1/2}m - \lambda)^{-1}\Psi_m\ , \qquad \text{and}$$

$$\|\Psi_m\| \leq B^m$$

we obtain the required bound.

While the Borel method is ideal for proving that the perturbation series determines the eigenvalues, it differs from the Stieltjes-Padé method in that it is not convenient for direct computations. This is because an analytic continuation is involved in the Borel method. One can try to perform this continuation of the Borel transform by

† Since γ is not real, the creation and annihilation operators are not adjoint to each other, so we indicate them a, b.

†† That is the usual Hermite function normalized, for $\gamma = 1$, by $\|\Psi_n\| = 1$. For γ non-real, $\|\Psi_n(\gamma)\|^2 \geq \int \Psi_n^2(\gamma, x)dx = 1$.

††† It is only necessary to employ the integral relation

$$\Psi_n(\gamma, x) = (n!)^{-1/2}\, 2^n \pi^{-1} \exp(-\tfrac{1}{2}\gamma^{1/2}x^2) \times \int_{-\infty}^{\infty} (\gamma^{1/4}x + it)^n\, e^{-t^2} dt\ .$$

‡ Our proof of this bound provides an alternative proof of the results of appendix V of ref. [7]; see footnote .

Table 1
Rate of convergence of $f_B^{[N,N]}(\beta)$ for β small.

N	$\beta = 0.1$	$\beta = 1.0$
1	1.064 466 870 450 69	1.324 091 073 785 0
2	1.065 280 566 374 99	1.385 465 009 227 7
4	1.065 285 501 926 18	1.391 918 130 550 1
5	1.065 285 509 194 08	1.392 245 893 189 4
6	1.065 285 509 531 32	1.392 326 613 576 2
8	1.065 285 509 543 65	1.392 349 165 747 9
9	1.065 285 509 543 70	1.392 350 518 885 0
10	1.065 285 509 543 70	1.392 350 653 679 1

Table 2
Comparison of $f_B^{[N,N]}$ with the Padé approximants $f^{[N,N]}$.

β	$f^{[20,20]}$	$f_B^{[10,10]}$
0.1	1.065 285 509 543	1.065 285 509 543 70
0.2	1.118 292 654 3(57)	1.118 292 654 35(8 5)
0.3	1.164 047 156 (234)	1.164 047 157 0(75 4)
0.4	1.204 810 31(0 603)	1.204 810 324 (767 4)
0.5	1.241 853 9(48 135)	1.241 854 04(6 678 2)
0.6	1.275 983 (105 974)	1.275 983 5(21 854 5)
0.7	1.307 747 (246 301)	1.307 748 5(31 549 3)
0.8	1.337 54(1 726 579)	1.337 544 9(37 046 5)
0.9	1.365 66(2 398 911)	1.365 669 2(83 162 3)
1.0	1.392 3(37 481 861)	1.392 350 (653 679 1)

Table 3
Comparison for β intermediate.

β	$f^{[20,20]}$	$f_B^{[10,10]}$
1	1.392 34	1.392 350 (6)
2	1.607 1	1.607 50(9 3)
3	1.767	1.769 4(14 1)
4	1.897	1.902 (624 1)
5	2.00(5)	2.017 (235 0)
6	2.10(0)	2.118 (543 6)
7	2.18(-2)	2.209 (732 0)
8	2.25(0)	2.292 (886 7)
9	2.31(3)	2.369 (463 7)
10	2.37(0)	2.440 (527 3)
11	2.4(21)	2.506 (883 4)
12	2.4(68)	2.56(9 160 3)
13	2.5(11)	2.62(7 859 3)
14	2.5(57)	2.68(3 387 9)

means of the familiar Padé approximants, since it is known [9] that if there is a sequence of $[N,M]$, $N+M \to \infty$, Padé approximants free of poles and zeroes in a region A strictly containing the origin, then this sequence converges uniformly to a function analytic in A, which is exactly the analytic continuation of the Taylor series.

Let us apply these considerations to the ground state eigenvalue $E_0(\beta)$ of the x^4 perturbed anharmonic oscillator. In the notation of ref. [4] and [7], $f^{[N,N]}(\beta)$ are the diagonal Padé approximants formed from the perturbation series whose first 75 coefficients have been computed by Bender and Wu [10]. Indicating the Borel transform of the perturbation series by $F(\beta)$, we have investigated its diagonal Padé approximants $F^{[N,N]}(\beta)$ and found that, up to $N = 10$ the poles and zeroes lie away from the real positive axis, with the exception of the [3,3] and [7,7]. For $N > 10$ the computations cannot be done because the input coefficients are only given to 12 figures and cancellations in the determinants become significant.

In any event, one can assume ‡‡ the existence of an infinite subsequence, formed by the diagonal approximants with the exception of the $[4N-1, 4N-1]$ ones, free of zeroes and poles in a neighborhood of the positive real axis and moreover that the nonuniformity of convergence at ∞ is sufficiently mild to justify the interchange of the limit and the integral needed to have:

$$f_B^{[N,N]}(\beta) = \int_0^\infty e^{-a} F^{[N,N]}(\beta a)\, da$$

convergent as $N \to \infty$ to $f(\beta)$.

‡‡ We emphasize that unlike the results in section II, the assumptions made for purposes of computation are not rigorous.

Numerically, one finds first that the $f_B^{[N,N]}$ appear to be monotonically increasing which is not to be a priori expected since F is not Stieltjes and one also finds convergence to the correct eigenvalues more rapidly than for the usual Padé's (compare our table 1 with table 2 of ref. [7] or look at our table 2 where more figures are convergent for $f_B^{[10,10]}$ than $f^{[20,20]}$). This more rapid convergence ‡‡‡ is even more evident for intermediate β # (table 3) where $f_B^{[5,5]}$ (14) compares with $f^{[20,20]}$ (14) in degree of precision.

It is a pleasure to thank Dr. G. Turchetti for useful discussions, and Dr. D. Bessis for the hospitality of the Cargèse Summer School where our collaboration was begun.

‡‡‡ One possible explanation of this rapid convergence is that the logarithmic cuts of $f_B^{[N,N]}$ are better able to mock up the singularities of f than the poles of $f^{[N,N]}$.

For very large β, both $f_B^{[N,N]}$ and $f^{[N,N]}$ suffer the same defect of going to constants while $f(\beta)/\beta^{1/3}$ has a finite limit for the actual function $f(\beta)$.

References

[1] A. Jaffe, Commun. Math. Phys. 1 (1965) 127.
[2] G. H. Hardy, Divergent series, (Oxford, 1949).
[3] E. Borel, Leçons sur les séries divergents, (Paris, 1928).
[4] J. J. Loeffel et al., Phys. Letters 30B (1969) 656.
[5] V. I. Ogievski, Dokl. Akad. Nauk SSR 109 (1956) 919.
[6] Reeken, J. Math. Phys. 11 (1970) 822.
[7] B. Simon, Ann. Phys. 58 (1970) 76-136.
[8] J. J. Loeffel and A. Martin, Proc. of the Programme 25 Conference, Strasbourg, May 1970, to be published.
[9] G. Baker, Adv. Theor. Phys. 1 (1965) 1.
[10] C. M. Bender and T. T. Wu, Phys. Rev. 184 (1969) 1231.

* * * * *