



Constructing quantum measurement processes via classical stochastic calculus

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Abstract

A class of linear stochastic differential equations in Hilbert spaces is studied, which allows to construct probability densities and to generate changes in the probability measure one started with. Related linear equations for trace-class operators are discussed. Moreover, some analogue of filtering theory gives rise to related non-linear stochastic differential equations in Hilbert spaces and in the space of trace-class operators. Finally, it is shown how all these equations represent a new formulation and a generalization of the theory of measurements continuous in time in quantum mechanics.

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1. Introduction

In the last few years there was an increasing interest in *non-linear stochastic Schrödinger equations* in quantum mechanics. These are stochastic differential equations in a Hilbert space, generalizing the linear Schrödinger equation and having additional interesting properties of reproducing non-linear phenomena such as collapse of the wave function.

There are essentially three ways to come to such equations. They can be postulated, and the consequences of such a postulate are then to be investigated, as is done by Gisin (1984), Diósi (1988b), Ghirardi et al. (1990). On the other hand, similar equations are deduced in the framework of quantum filtering theory (Belavkin, 1988, 1989a, b, 1990a, b; Belavkin and Staszewski, 1989) as equations for a posteriori states in a *quantum non-demolition measurement process*. Finally, they can be deduced from

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the formalism of *continuous quantum measurements* in the sense of Barchielli et al. (1983) through the classical stochastic representation for such a process (Diósi, 1988a; Holevo, 1991a, b; Barchielli and Belavkin, 1991). This last approach is related to the second one in the way that every continuous measurement process can be dilated to a quantum non-demolition measurement (Barchielli and Lupieri, 1985, 1989), the condition of non-demolition being nothing but the commutation relations between “in” and “out” processes in quantum stochastic evolutions (Gardiner and Collet, 1985; Barchielli, 1991).

The resulting non-linear equations are of independent mathematical interest and they can be studied by using Itô’s stochastic calculus in Hilbert spaces. Moreover, the analytic power of this calculus can be used to construct a stochastic representation for quantum measurement processes much more general than those treated up to now, e.g. by means of quantum stochastic calculus. Namely, the operator coefficients of the basic equations are allowed to be more or less arbitrary predictable operator-valued processes, and the class of driving noises can be also substantially enlarged. While the possibility of such a kind of extensions was declared elsewhere (Belavkin, 1990a, our aim here is to present definite mathematical results in the study of non-linear stochastic Schrödinger equations and related notions. In this paper we restrict to the case of coefficients whose values are bounded operators; the case of unbounded operators poses additional interesting problems and will be discussed in a separate paper. Simple examples involving unbounded operators are treated by Gatarek and Gisin (1991).

2. Some stochastic differential equations in Hilbert space

2.1. The conservative stochastic evolution equation

Let $(\Omega, \{\mathcal{F}_t, t \geq 0\}, P)$ be a stochastic basis satisfying “usual hypotheses” (Métivier, 1982, Definition 1.1), i.e. the filtration $\{\mathcal{F}_t, t \geq 0\}$ is right continuous ($\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$), for every $F \in \mathcal{F} \equiv \bigvee_{t \geq 0} \mathcal{F}_t$ such that $P(F) = 0$ one has $F \in \mathcal{F}_0$ and (Ω, \mathcal{F}, P) is complete. In this probability space we shall introduce many processes which we assume, if not otherwise stated, to be *regular right continuous* (RRC) (Métivier, 1982, Definition 1.5), i.e. adapted and with right continuous paths with left limits. Sometimes we shall have also *regular left continuous* (RLC) processes (adapted, with left continuous paths with right limits).

In this probability space let an \mathbb{R}^r -valued continuous martingale M and a σ -finite adapted point process Π on $\mathcal{A} \times \mathbb{R}^+$ be defined, where $\mathcal{A} := \mathbb{R}^d \times \mathbb{N}$ and \mathbb{N} is the set of natural numbers. We assume $\mathbb{E}_p[M_{kt}] = 0$ and the elements of the matrix of quadratic variation of M to have the form $\int_0^t \gamma_{kk'}(s) ds$, with the $\gamma_{kk'}$ being RLC processes (formally: $dM_{kt} dM_{k't} = \gamma_{kk'}(t) dt$). We ask the *compensator* of Π (or stochastic intensity, or dual predictable projection) to be of the form $\nu_t(dy) dt$ with $\nu_t(dy)$ RLC.

Let us recall that ν is a predictable σ -finite measure on \mathcal{A} ; the quantity

$$\tilde{\Pi}(dy, dt) := \Pi(dy, dt) - \nu_t(dy) dt \tag{2.1}$$

is called a *white random measure* in Métivier (1982, p. 219), or a *martingale measure* in Liptser and Shiriyayev (1986, p. 172). Let us recall that the stochastic integral $\int_{(0,t]} \int_{\mathcal{A}} F_s(y) \tilde{M}(dy, ds)$ is defined and is a local martingale (Liptser and Shiriyayev, 1986, Section 3.5) if the following condition holds (although this stochastic integral could be defined in more general cases):

$$\int_0^t \int_{\mathcal{A}} \frac{|F_s(y)|^2}{1 + |F_s(y)|} \nu_s(dy) ds < +\infty; \tag{2.2}$$

the modulus has to be substituted by the norm in the case of processes with values in a Hilbert space.

Let us consider the following linear Itô’s stochastic differential equation (SDE) for a process ψ_t with values in a separable complex Hilbert space \mathcal{H} and with the initial condition $\psi_0 = \psi$ (ψ a random unit vector in \mathcal{H} , measurable with respect to the initial σ -algebra \mathcal{F}_0):

$$d\psi_t = \sum_{k=1}^r L_{kt} \psi_t - dM_{kt} + \int_{\mathcal{A}} L_t(y) \psi_t - \tilde{M}(dy, dt) - K_t \psi_t - dt. \tag{2.3}$$

As usual, the differential form is a short hand notation for a stochastic integral equation.

2.0.A. We take $L_{kt}, L_t(y), K_t$ to be RLC processes (continuity in the strong operator topology) with values in the space $\mathcal{L}(\mathcal{H})$ of bounded linear operators on \mathcal{H} ($\forall k \in \{1, \dots, r\}, \forall y \in \mathcal{A}$); in particular they are predictable processes. We also assume the function $(t, \omega, y) \rightarrow L_t(y; \omega)$ to be strongly measurable (when $\mathbb{R}^+ \times \Omega$ is equipped with the σ -algebra of predictable sets) and the integral

$$R_t(\omega) := \int_{\mathcal{A}} L_t(y; \omega)^* L_t(y; \omega) \nu_t(\omega; dy) \tag{2.4}$$

to be weakly convergent ($\forall t \in \mathbb{R}^+, \forall \omega \in \Omega$).

2.0.B. We also require ($\forall t > 0, \forall \omega \in \Omega$)

$$\int_0^t \left\{ \left[\|K_s(\omega)\|^2 + \sum_{k=1}^r \|L_{ks}(\omega)\|^2 \right] \left[1 + \sum_{k'=1}^r \gamma_{k'k'}(s; \omega) \right] + \|R_s(\omega)\| \right\} ds < \infty. \tag{2.5}$$

2.1. Proposition. Under conditions 2.0, Eq. (2.3) admits a unique (up to P -equivalence) solution $\psi_t, t \in \mathbb{R}^+$, which is an \mathcal{H} -valued semimartingale.

Proof. For every \mathcal{H} -valued RRC process η and every $t \in \mathbb{R}^+$, we have

$$\begin{aligned} & \int_0^t \left[\|K_s \eta_{s-}\|^2 + \sum_{k=1}^r \|L_{ks} \eta_{s-}\|^2 \right] \left[1 + \sum_{k'=1}^r \gamma_{k'k'}(s) \right] ds + \int_0^t \langle \eta_{s-} | R_s \eta_{s-} \rangle ds \\ & \leq \int_0^t \sup_{r < s} \|\eta_r\|^2 \left\{ \left[\|K_s\|^2 + \sum_{k=1}^r \|L_{ks}\|^2 \right] \left[1 + \sum_{k'=1}^r \gamma_{k'k'}(s) \right] + \|R_s\| \right\} ds. \end{aligned}$$

By (2.5), this implies immediately the “global Lipschitz condition” $[L_1]$ and the condition for “non-explosion” $[K]$ of Métivier (1982, pp. 244–245 and 252). Theorems 34.7 and 35.2 of Métivier (1982) give the existence of a unique solution ψ_t for all $t \in \mathbb{R}^+$, which is an RRC process. Then, by the definition of stochastic integrals, we have that ψ_t is a semimartingale: see Métivier (1982), Section 26.4, Theorem 24.4 ($1^\circ, 5^\circ$), Sections 18, 31.14. \square

It is convenient to set

$$S_t(\omega) := \sum_{k,k'=1}^r L_{k't}(\omega)^* \gamma_{k'k}(t; \omega) L_{kt}(\omega) \tag{2.6}$$

and

$$H_t := \frac{i}{2} (K_t^* - K_t). \tag{2.7}$$

Then, we assume

2.2. $K_t^* + K_t = R_t + S_t;$

so, we can write

$$K_t = iH_t + \frac{1}{2}(R_t + S_t). \tag{2.8}$$

The stochastic evolution equation (2.3) satisfying condition 2.2 will be called *conservative*.

We shall assume also that, $\forall t \in \mathbb{R}^+$,

2.3.A. $\text{ess sup}_{\omega \in \Omega} \int_0^t \|S_s(\omega)\| ds < +\infty,$

2.3.B. $\text{ess sup}_{\omega \in \Omega} \int_0^t \|R_s(\omega)\| ds < +\infty.$

The following property may be called *conservativity of the solution* of Eq. (2.3).

2.4. Theorem. *Under conditions 2.0, 2.2 and 2.3, the square norm $\|\psi_t\|^2 =: p_t$ of the solution of Eq. (2.3) is a positive martingale with*

$$\mathbb{E}_p[p_t] = \mathbb{E}_p[\|\psi\|^2] \equiv 1 \tag{2.9}$$

and satisfying the Doléans equation

$$dp_t = p_t - dZ_t, \tag{2.10}$$

where Z_t is a local martingale defined by

$$Z_t := \int_{(0,t]} \sum_{k=1}^r \hat{m}_{k_s} dM_{k_s} + \int_{(0,t]} \int_{\mathcal{A}} (\hat{I}_s(y) - 1) \tilde{\Pi}(dy, ds), \tag{2.11}$$

$$\hat{m}_{k_t} := 2 \operatorname{Re} \langle \hat{\psi}_{t-} | L_{k_t} \hat{\psi}_{t-} \rangle, \quad \hat{I}_t(y) := \|(L_t(y) + \mathbb{1}) \hat{\psi}_{t-}\|^2, \tag{2.12}$$

$$\hat{\psi}_t := \begin{cases} \psi_t / \|\psi_t\|, & \text{if } \|\psi_t\| \neq 0, \\ v \text{ (fixed unit vector)} & \text{if } \|\psi_t\| = 0. \end{cases} \tag{2.13}$$

Proof. By Remarks 3.9, p. 50, of Métivier and Pellaumail (1980), it is possible to apply Itô formula (Métivier, 1982, Theorem 27.2) to the function $f(\psi_t) = \|\psi_t\|^2 \equiv p_t$; under assumption 2.2 we obtain Eqs. (2.10)–(2.13).

Considered as a linear equation for p_t , (2.10) has a unique solution called the exponential of the semimartingale Z_t (Métivier, 1982, Theorem 29.2, Definition 29.3); an explicit expression for p_t is given by Eq. (29.2.3) of Métivier (1982).

Since Z_t turns out to be a local martingale, p_t is a local martingale (Liptser and Shirayev, 1986, Theorem 2, p. 124) and a nonnegative local martingale is a supermartingale (Liptser and Shirayev, 1986, p. 23); so, we have $\mathbb{E}_P[p_t] \leq \mathbb{E}_P[p_0] = 1$.

To establish the equality, we check the condition of Kabanov et al. (1979, Lemma 7), ensuring that p_t is a (locally) uniformly integrable martingale. Because of $\|\hat{\psi}_t\| = 1$, we have

$$\sum_{k,k'=1}^r \hat{m}_{k'_t} \gamma_{k',k}(t) \hat{m}_{k_t} \leq 4 \sum_{k,k'=1}^r \gamma_{k',k}(t) \langle L_{k'_t} \hat{\psi}_{t-} | L_{k_t} \hat{\psi}_{t-} \rangle \leq 4 \|S_t\|, \tag{2.14}$$

$$(\sqrt{\hat{I}_t(y)} - 1)^2 = (\|(L_t(y) + \mathbb{1}) \hat{\psi}_{t-}\| - \|\hat{\psi}_{t-}\|)^2 \leq \|L_t(y) \hat{\psi}_{t-}\|^2. \tag{2.15}$$

Then, by conditions 2.3, we have

$$\begin{aligned} & \operatorname{ess\,sup}_{\omega \in \Omega} \int_0^t \left[\sum_{k,k'=1}^r \hat{m}_{k'_s}(\omega) \gamma_{k',k}(s; \omega) \hat{m}_{k_s}(\omega) \right. \\ & \left. + \int_{\mathcal{A}} \left(\sqrt{\hat{I}_s(y; \omega)} - 1 \right)^2 \nu_s(\omega; dy) \right] ds < +\infty \end{aligned}$$

for all $t > 0$. This is essentially the sufficient condition of Kabanov et al. (1979).

Going back to the existence of the stochastic integral with respect to the white random measure $\tilde{\Pi}$ in Eq. (2.11), by recalling Eq. (2.2) and that, for $x \geq -1$, $x^2/(1 + |x|) \asymp (\sqrt{1+x} - 1)^2$, we have that the integral in consideration converges by the estimate (2.15) and the condition 2.3.B. \square

If the jump part is absent ($L_t(y) \equiv 0$), then assumption 2.3.A can be essentially relaxed by requiring only Novikov's type condition $\mathbb{E}_P[\exp \frac{1}{2} \int_0^t \|S_s\|^2 ds] < +\infty$

(Ikeda and Watanabe, 1981); there are some extensions of this condition to the general case (Lépingle and Mémin, 1978), however, they seem not to be easily applicable to our problem. We mention also that without such auxiliary conditions conservative stochastic evolution equations may have non-conservative solutions.

2.2. The non-linear stochastic Schrödinger equation

By the previous theorem, if ψ is a unit vector, then p_t is a local probability density and one can define the new probability measure \hat{P} , as in Métivier (1982, Section 30.2) by

$$\forall F \in \mathcal{F}_t, \quad \hat{P}(F) := E_P[p_t 1_F]. \tag{2.16}$$

Next proposition says what the processes M and Π become under the new law \hat{P} .

2.5. Proposition. Under the law \hat{P} , Π is a point process with compensator $\hat{I}_t(y)v_t(dy)dt$ and the process \hat{M} , defined by

$$\hat{M}_{kt} := M_{kt} - \int_0^t \sum_{k'=1}^r \hat{m}_{k's} \gamma_{k'k}(s) ds, \quad k = 1, \dots, r, \tag{2.17}$$

is a continuous martingale with vanishing \hat{P} -expectation and matrix of quadratic variation $\int_0^t \gamma_{k'k}(s) ds$.

Proof. The two statements are essentially Theorem 1, p. 223 and Theorem 3, p. 232 of Liptser and Shiriyayev (1986). One has only to make the computations implied by Eqs. (5.2) p. 222, (5.25) p. 231 and (5.26) p. 232 of Liptser and Shiriyayev (1986). \square

2.6. Remark. If under the law P the M_k are independent standard Wiener processes (in this case $\gamma_{k'k}(t) = \delta_{k'k}$), then under the law \hat{P} the \hat{M}_k are again independent standard Wiener processes. This statement is related to Girsanov transformation and follows from the computation of the compensators of M_{kt} under the new law and from the martingale characterization of Wiener process (Liptser and Shiriyayev, 1986, Theorem 3, p. 232, Ikeda and Watanabe, 1981, Theorem 6.1, p. 74).

Let us set

$$\hat{\Pi}(dy, dt) := \Pi(dy, dt) - \hat{I}_t(y)v_t(dy)dt; \tag{2.18}$$

according to the previous proposition $\hat{\Pi}$ is a white random measure under the law \hat{P} .

It is interesting and useful to have an equation for the normalized vectors $\hat{\psi}_t$ defined by Eq. (2.13).

2.7. Theorem. Under assumptions 2.0, 2.2, 2.3, the normalized process $\hat{\psi}_t$, defined by Eq. (2.13), is a \hat{P} -semimartingale satisfying the non-linear SDE

$$d\hat{\psi}_t = \sum_{k=1}^r \hat{L}_{kt} \hat{\psi}_t - d\hat{M}_{kt} + \int_{E_t^c} \hat{L}_t(y) \hat{\psi}_t - \hat{\Pi}(dy, dt) - \hat{K}_t \hat{\psi}_t - dt, \tag{2.19}$$

where E_t^c is the complement of the random set $E_t := \{y: \hat{I}_t(y) = 0\}$ and

$$\hat{L}_{kt} := L_{kt} - \frac{1}{2} \hat{m}_{kt} \mathbb{1}, \quad \hat{L}_t(y) := \frac{L_t(y) + \mathbb{1}}{\sqrt{\hat{I}_t(y)}} - \mathbb{1}, \tag{2.20}$$

$$\begin{aligned} \hat{K}_t := & K_t + \frac{1}{2} \sum_{k,k'=1}^r \hat{m}_{k't} \gamma_{k'k}(t) (\frac{1}{4} \hat{m}_{kt} \mathbb{1} - L_{kt}) \\ & + \int_{\mathcal{A}} [(1 - \sqrt{\hat{I}_t(y)}) L_t(y) + \frac{1}{2} (1 - \sqrt{\hat{I}_t(y)})^2 \mathbb{1}] v_t(dy), \end{aligned} \tag{2.21}$$

with \hat{m}_{kt} and $\hat{I}_t(y)$ given by Eq. (2.12).

Proof. Let us consider the linear SDE

$$d\hat{p}_t = \hat{p}_t - d\hat{Z}_t, \tag{2.22}$$

with initial condition $\hat{p}_0 = 1$, where \hat{Z}_t is a local \hat{P} -supermartingale defined by

$$\begin{aligned} \hat{Z}_t := & - \int_{(0,t]} \sum_{k=1}^r \hat{m}_{ks} d\hat{M}_{ks} + \int_{(0,t]} \int_{E_t^c} \left(\frac{1}{\hat{I}_s(y)} - 1 \right) \hat{\Pi}(dy, ds) \\ & - \int_{(0,t]} \int_{E_s} v_s(dy) ds. \end{aligned} \tag{2.23}$$

The convergence of $\int_{E_t} v_t(dy)$ follows from the fact that on the set E_t we have $1 = \|\hat{\psi}_t - \mathbb{1}\|^2 = \|L_t(y)\hat{\psi}_t - \mathbb{1}\|^2 = \langle \hat{\psi}_t - |L_t(y)|^* L_t(y)\hat{\psi}_t \rangle$ and from condition 2.0.B (see Eq. (2.4)).

The existence of the stochastic integral with respect to the white random measure $\hat{\Pi}$ is proved exactly as the existence of the similar integral in Eq. (2.11), by recalling that now the compensator of $\hat{\Pi}$ is given by the expression in Proposition 2.5 and that, for $x \geq -1$, $x^2/(1+x+|x|) \asymp (\sqrt{1+x}-1)^2$.

As for Eq. (2.10), Eq. (2.22) has a unique solution which is a positive supermartingale; in particular,

$$\mathbb{E}_{\hat{P}}[\hat{p}_t] \leq \mathbb{E}_{\hat{P}}[\hat{p}_0] = 1. \tag{2.24}$$

Moreover, we have

$$\hat{p}_t = \frac{1}{p_t} \pmod{\hat{P}}. \tag{2.25}$$

To show this we use Eqs. (2.10), (2.11), (2.17), (2.18), (2.22), (2.23) and Itô's product formula to obtain $d(p_t \hat{p}_t) = - \int_{E_t} \hat{\Pi}(dy, dt) = 0 \pmod{\hat{P}}$. The equality $\mathbb{E}_{\hat{P}}[\hat{p}_t] = 1$ takes place if and only if P and \hat{P} are equivalent on \mathcal{F}_t .

Let us consider now the linear SDE

$$\begin{aligned} d\hat{q}_t = & \hat{q}_t - \left\{ -\frac{1}{2} \sum_{k=1}^r \hat{m}_{kt} d\hat{M}_{kt} - \frac{1}{8} \sum_{k,k'=1}^r \hat{m}_{k't} \gamma_{k'k}(t) \hat{m}_{kt} dt \right. \\ & \left. + \int_{E_t^c} \left(\frac{1}{\sqrt{\hat{I}_t(y)}} - 1 \right) \hat{\Pi}(dy, dt) - \frac{1}{2} \int_{\mathcal{A}} (\sqrt{\hat{I}_t(y)} - 1)^2 v_t(dy) dt \right\}, \end{aligned} \tag{2.26}$$

which is obtained by formal application of Itô’s formula to $f(\hat{p}_t) = \sqrt{\hat{p}_t}$. Conditions 2.3 and inequalities (2.14), (2.15) imply the “global Lipschitz condition” [L₁] and the condition for “non-explosion” [K] of Métivier (1982); as in Proposition 2.1, we have that Eq. (2.26) has a unique solution \hat{q}_t . By applying Itô’s formula to $f(\hat{q}_t) = \hat{q}_t^2$, we see that \hat{q}_t^2 satisfies the Eq. (2.22) as \hat{p}_t . By uniqueness, $\hat{p}_t = \hat{q}_t^2$, so that

$$\hat{q}_t = \frac{1}{\|\psi_t\|} \pmod{\hat{P}} \tag{2.27}$$

and, from Eq. (2.24),

$$\mathbb{E}_{\hat{P}}[\hat{q}_t^2] \leq 1. \tag{2.28}$$

Let us re-express Eq. (2.3) by means of the new processes (2.17) and (2.18); we obtain

$$\begin{aligned} d\psi_t &= \sum_{k=1}^r L_{kt}\psi_t - d\hat{M}_{kt} + \int_{\hat{I}_t(y) \leq \kappa} L_t(y)\psi_t - \hat{\Pi}(dy, dt) \\ &+ \int_{\hat{I}_t(y) > \kappa} L_t(y)\psi_t - \Pi(dy, dt) \\ &+ \int_{\hat{I}_t(y) \leq \kappa} (\hat{I}_t(y) - 1)L_t(y)\psi_t - \nu_t(dy) dt \\ &- \int_{\hat{I}_t(y) > \kappa} L_t(y)\psi_t - \nu_t(dy) dt \\ &+ \sum_{k,k'=1}^r \hat{m}_{k't} \gamma_{k'k}(t) L_{kt}\psi_t - dt - K_t\psi_t - dt, \end{aligned} \tag{2.29}$$

where $\kappa > 1$. The integrals over the domain $\hat{I}_t(y) \leq \kappa$ converge obviously due to assumption 2.0.B. The integrals over $\hat{I}_t(y) > \kappa$ converge by assumption 2.3.B and the estimate

$$\begin{aligned} &\left\| \int_0^t \int_{\hat{I}_s(y) > \kappa} L_s(y)\psi_s - \nu_s(dy) ds \right\|^2 \\ &\leq \int_0^t \int_{\hat{I}_s(y) > \kappa} \nu_s(dy) ds \int_0^t \int_{\mathcal{A}} \|L_s(y)\psi_s - \nu_s(dy)\|^2 \nu_s(dy) ds \\ &\leq \int_0^t \int_{\mathcal{A}} \left(\frac{\sqrt{\hat{I}_s(y)} - 1}{\sqrt{\kappa} - 1} \right)^2 \nu_s(dy) ds \int_0^t \|R_s\| ds \sup_{r < t} \|\psi_r\|^2; \end{aligned}$$

$\sup_{r < t} \|\psi_r\|^2$ is a.s. finite since ψ_t is RRC and hence locally bounded.

By applying Itô’s product formula to $\hat{\psi}_t = \hat{q}_t\psi_t$, we obtain Eqs. (2.19)–(2.21). \square

One may call Eq. (2.19) the *non-linear stochastic Schrödinger equation*; it turns out into the usual Schrödinger wave equation in the case of a non-random Hamiltonian H_t and $L_{kt} \equiv 0$, $L_t(y) \equiv 0$. In the context of “quantum filtering theory” the first examples of this stochastic equation appeared in Belavkin (1988).

2.8. Remark. As we shall see in the following sections, changes of phase in ψ_t and $\hat{\psi}_t$ do not matter. The following replacements in Eqs. (2.3), (2.4), (2.6), (2.8) give rise to a stochastic change of phase in ψ_t :

$$L_{kt} \rightarrow L_{kt} + i l_{kt} \mathbb{1}, \quad L_t(y) \rightarrow e^{i f_t(y)} (L_t(y) + \mathbb{1}) - \mathbb{1}, \tag{2.30}$$

$$H_t \rightarrow H_t + h_t - \frac{1}{2} \sum_{k, k'=1}^r l_{k't} \gamma_{k'k}(t) (L_{kt} + L_{kt}^*) + \frac{i}{2} \int_{\mathcal{A}} [(e^{i f_t(y)} - 1) L_t(y) - (e^{-i f_t(y)} - 1) L_t(y)^*] \nu_t(dy), \tag{2.31}$$

where $l_{kt}(\omega)$, $f_t(y; \omega)$, $h_t(\omega) \in \mathbb{R}$ and are such that all the assumptions on the coefficients of Eq. (2.3) are satisfied. About $\hat{\psi}_t$, a suitable stochastic change of phase allows to write Eq. (2.19) with the following more symmetric form of the coefficients: $\hat{L}_t(y)$ unchanged,

$$\hat{L}_{kt} = L_{kt} - \langle L_{kt} \rangle_t, \quad \langle L_{kt} \rangle_t := \langle \hat{\psi}_t - | L_{kt} \hat{\psi}_t - \rangle, \tag{2.32}$$

$$\hat{K}_t = i \hat{H}_t + \frac{1}{2} \sum_{k, k'=1}^r \hat{L}_{k't}^* \gamma_{k'k}(t) \hat{L}_{kt} + \frac{1}{2} \int_{\mathcal{A}} [L_t(y)^* + (1 - \sqrt{\hat{I}_t(y)}) \mathbb{1}] [L_t(y) + (1 - \sqrt{\hat{I}_t(y)}) \mathbb{1}] \nu_t(dy), \tag{2.33}$$

$$\hat{H}_t := H_t + \frac{i}{2} \sum_{k, k'=1}^r (\langle \hat{L}_{k't}^* \rangle_t \gamma_{k'k}(t) L_{kt} - L_{k't}^* \gamma_{k'k}(t) \langle L_{kt} \rangle_t) + \frac{i}{2} \int_{\mathcal{A}} (\sqrt{\hat{I}_t(y)} - 1) (L_t(y) - L_t(y)^*) \nu_t(dy). \tag{2.34}$$

3. Equations for quantum-mechanical means

3.1. Mixed states and quantum evolutions

Let us recall that quantum-mechanical (normal) states are represented by statistical operators (positive bounded operators on \mathcal{H} with trace equal to one) and that the linear span of statistical operators is called *trace class* $\mathcal{T}(\mathcal{H})$; let us denote by $\mathcal{S}(\mathcal{H})$ the set of statistical operators. Let us use the notation $\langle \varrho, a \rangle := \text{Tr}\{\varrho a\}$, $\varrho \in \mathcal{T}(\mathcal{H})$, $a \in \mathcal{L}(\mathcal{H})$. If ϱ is a statistical operator and a is self-adjoint, then $\langle \varrho, a \rangle$ is the “quantum mean” of the “observable” a in the state ϱ . By introducing the norm $\|\varrho\|_1 := \text{Tr} \sqrt{\varrho^* \varrho} \equiv \sup_{a \in \mathcal{L}(\mathcal{H}), \|a\|=1} |\langle \varrho, a \rangle|$, $\mathcal{T}(\mathcal{H})$ becomes a Banach space, whose dual is $\mathcal{L}(\mathcal{H})$ (Reed and Simon, 1972, Ch. VI).

We introduce a countable family of random vectors $\psi^\beta \in \mathcal{H}$, ψ^β measurable with respect to \mathcal{F}_0 , with $\sum_{\beta=1}^\infty \|\psi^\beta\|^2 = 1$. Then, the equation

$$\langle \varrho, a \rangle = \sum_{\beta=1}^\infty \langle \psi^\beta | a \psi^\beta \rangle, \quad \forall a \in \mathcal{L}(\mathcal{H}) \tag{3.1}$$

defines a random statistical operator ϱ : our initial state from now on.

Let ψ_t^β be a solution of the linear equation (2.3) with initial condition $\psi_0^\beta = \psi^\beta$. We introduce the random positive trace-class operators σ_t by

$$\langle \sigma_t, a \rangle = \sum_{\beta=1}^{\infty} \langle \psi_t^\beta | a \psi_t^\beta \rangle, \quad \forall a \in \mathcal{L}(\mathcal{H}). \tag{3.2}$$

Under all the assumptions of Section 2, the quantity

$$p_t := \sum_{\beta=1}^{\infty} \|\psi_t^\beta\|^2 \equiv \langle \sigma_t, \mathbb{1} \rangle \tag{3.3}$$

is a probability density and we define a probability measure \hat{P} by Eq. (2.16). Let us stress that we have changed the initial state (a pure state in Section 2 and a mixed state here), but that we are keeping fixed the notations: analogous quantities are denoted by the same symbols.

Similarly to Eq. (2.13), by normalizing σ_t , we define the random states $\hat{\sigma}_t$ by

$$\hat{\sigma}_t := \begin{cases} \sigma_t/p_t, & \text{if } p_t \neq 0, \\ S \text{ (} S \text{ fixed statistical operator)} & \text{if } p_t = 0. \end{cases} \tag{3.4}$$

Finally, we define \bar{Q}_t by

$$\langle \bar{Q}_t, a \rangle := \mathbb{E}_P[\langle \sigma_t, a \rangle] \equiv \mathbb{E}_{\hat{P}}[\langle \hat{\sigma}_t, a \rangle], \quad \forall a \in \mathcal{L}(\mathcal{H}); \tag{3.5}$$

\bar{Q}_t turns out to be a statistical operator. We set also $\bar{Q} := \bar{Q}_0 \equiv \mathbb{E}_P[\varrho]$.

In this section we want to consider the evolution equations for the various families of trace-class operators introduced above. We start by considering \bar{Q}_t .

Let us assume

3.1. $\text{ess sup}_{\omega \in \Omega} \int_0^t \|H_s(\omega)\| ds < +\infty$

(cf. assumptions 2.3).

3.2. Proposition. *Under conditions 2.0, 2.2, 2.3, 3.1, we have, for any $a \in \mathcal{L}(\mathcal{H})$*

$$\langle \bar{Q}_t, a \rangle = \langle \bar{Q}, a \rangle + \int_0^t \mathbb{E}_P[\langle \sigma_{s-}, \mathcal{L}_s[a] \rangle] ds, \tag{3.6}$$

where

$$\begin{aligned} \mathcal{L}_t[a] &:= i[H_t, a] + \frac{1}{2} \sum_{k, k'=1}^r \gamma_{k'k}(t) (L_{k't}^* [a, L_{kt}] + [L_{k't}^*, a] L_{kt}) \\ &\quad + \frac{1}{2} \int_{\mathcal{A}} (L_t(y)^* [a, L_t(y)] + [L_t(y)^*, a] L_t(y)) \nu_t(dy) \\ &\equiv -K_t^* a - a K_t + \sum_{k, k'=1}^r \gamma_{k'k}(t) L_{k't}^* a L_{kt} + \int_{\mathcal{A}} L_t(y)^* a L_t(y) \nu_t(dy). \end{aligned} \tag{3.7}$$

Proof. Under the conditions introduced, \mathcal{L}_t is P -a.s. a weakly*-continuous bounded linear map of $\mathcal{L}(\mathcal{H})$ into itself, such that

$$\text{ess sup}_{\omega \in \Omega} \int_0^t \|\mathcal{L}_s(\omega)\| ds < +\infty, \quad \forall t > 0. \tag{3.8}$$

We first prove Eq. (3.6) for $\varrho = |\psi\rangle\langle\psi|$, when $\langle\sigma_t, a\rangle = \langle\psi_t | a \psi_t\rangle$. We have

$$\mathbb{E}_P[\langle\psi_t | a \psi_t\rangle] = \mathbb{E}_P[\langle\hat{\psi}_t | a \hat{\psi}_t\rangle]. \tag{3.9}$$

Without loss of generality we assume $a \geq 0$. Using Eq. (2.19) and Itô’s formula for $f(\hat{\psi}_t) = \langle\hat{\psi}_t | a \hat{\psi}_t\rangle$, we obtain

$$\begin{aligned} \langle\hat{\psi}_t | a \hat{\psi}_t\rangle &= \langle\psi | a \psi\rangle + \int_{(0,t]} \langle\hat{\psi}_{s-} | \mathcal{L}_s[a] \hat{\psi}_{s-}\rangle ds \\ &\quad + \int_{(0,t]} \sum_{k=1}^r \langle\hat{\psi}_{s-} | (aL_{ks} + L_{ks}^*a - \hat{m}_{ks}a) \hat{\psi}_{s-}\rangle d\hat{M}_{ks} \\ &\quad + \int_{(0,t]} \int_{E_s^+} \left[\frac{\|a^{1/2}(L_s(y) + \mathbb{1})\hat{\psi}_{s-}\|^2}{\hat{I}_s(y)} - \|a^{1/2}\hat{\psi}_{s-}\|^2 \right] \hat{\Pi}(dy, ds). \end{aligned} \tag{3.10}$$

We show that the stochastic integrals in Eq. (3.10) are mean-square convergent. For the integral with respect to \hat{M} this follows easily from assumption 2.3.A and the fact that $\|\hat{\psi}_t\| \equiv 1$. For the square of the integrand of the white random measure we have the inequality

$$\begin{aligned} &\left[\frac{\|a^{1/2}(L_s(y) + \mathbb{1})\hat{\psi}_{s-}\|^2}{\hat{I}_s(y)} - \|a^{1/2}\hat{\psi}_{s-}\|^2 \right]^2 \\ &\leq 2\|a\| \left[\frac{\|a^{1/2}(L_s(y) + \mathbb{1})\hat{\psi}_{s-}\|}{\sqrt{\hat{I}_s(y)}} - \|a^{1/2}\hat{\psi}_{s-}\| \right]^2 \\ &\leq 2\|a\| \left[\hat{I}_s(y)^{-1/2} a^{1/2} L_s(y) \hat{\psi}_{s-} + (\hat{I}_s(y)^{-1/2} - 1) a^{1/2} \hat{\psi}_{s-} \right]^2 \\ &\leq 4\|a\|^2 \left[\hat{I}_s(y)^{-1} \|L_s(y) \hat{\psi}_{s-}\|^2 + (\hat{I}_s(y)^{-1/2} - 1)^2 \right]. \end{aligned} \tag{3.11}$$

This quantity is integrable on $\mathcal{X} \times (0, t]$ with respect to the compensator $\hat{I}_s(y) \nu_s(dy) ds$ of $\hat{\Pi}$, due to assumption 2.3.B and the estimate (2.15). The stochastic integrals are thus square-integrable martingales and have zero expectations. The first integral in Eq. (3.10) converges in the mean due to (3.8). Taking the expectation of Eq. (3.10) we obtain (3.6).

If ϱ is a general density operator (3.1), by assumptions 2.3 and 3.1 the dominated convergence theorem can be applied and one obtains

$$\begin{aligned} \mathbb{E}_P[\langle \sigma_t, a \rangle] &= \mathbb{E}_P \left[\sum_{\beta=1}^{\infty} \langle \psi_t^\beta | a \psi_t^\beta \rangle \right] \\ &= \sum_{\beta=1}^{\infty} (\mathbb{E}_P[\langle \psi_t^\beta | a \psi_t^\beta \rangle] + \int_0^t \mathbb{E}_P[\langle \psi_{s-}^\beta | \mathcal{L}_s[a] \psi_{s-}^\beta \rangle] ds) \\ &= \langle \bar{\varrho}, a \rangle + \int_0^t \mathbb{E}_P[\langle \sigma_{s-}, \mathcal{L}_s[a] \rangle] ds. \quad \square \end{aligned}$$

As we see from Eq. (3.6), the state $\bar{\varrho}_t$ obeys a closed equation if \mathcal{L}_s is a non-random operator. In the case in which $H_t, L_{kt}, L_t(y), \gamma_{k'k}(t), v_t$ are non-random quantities we have that $\bar{\varrho}_t$ satisfies the *Markovian master equation*

$$\frac{d}{dt} \bar{\varrho}_t = \mathcal{L}_{t*}[\bar{\varrho}_t]; \tag{3.12}$$

the star denotes the pre-adjoint map. In the case of no time-dependence too, $\exp(\mathcal{L}t)$ is a norm-continuous *quantum dynamical semigroup* and one can show that the generic generator of this kind of semigroups can be represented in the form (3.7) (Lindblad, 1976); in other words (3.12) is the generic master equation with bounded *Liouillian* \mathcal{L}_* .

Let us recall that master equations like (3.12) send pure states (one-dimensional projections) into mixed states. So, in the case of non-random \mathcal{L}_t , Eq. (2.3) (or (2.19)) can be seen as a pure-state stochastic representation of Eq. (3.12) (Holevo, 1989, 1991b). Indeed, simple versions of Eqs. (2.3) or (2.19) are used for numerical simulations of master equations (Gisin and Percival, 1992).

3.2. Stochastic evolution equations for states

Let us consider the analogue of Eq. (2.3) in $\mathcal{F}(\mathcal{H})$. We shall need also the following assumption:

3.3. $\sup_{0 \leq s \leq t} \sup_{y \in \mathcal{X}, \omega \in \Omega} \|L_s(y; \omega)\| < \infty$

for all $t > 0$.

3.4. Proposition. *Under the assumptions 2.0, 2.2, 2.3, 3.1, the $\mathcal{F}(\mathcal{H})$ -valued process σ_t satisfies the weak-sense SDE: $\forall a \in \mathcal{L}(\mathcal{H})$*

$$\begin{aligned} d\langle \sigma_t, a \rangle &= \langle \sigma_{t-}, \mathcal{L}_t[a] \rangle dt + \sum_{k=1}^r \langle \sigma_{t-}, L_{kt}^* a + a L_{kt} \rangle dM_{kt} \\ &\quad + \int_{\mathcal{X}} \langle \sigma_{t-}, L_t(y)^* a L_t(y) + a L_t(y) + L_t(y)^* a \rangle \tilde{\Pi}(dy, dt). \end{aligned} \tag{3.13}$$

Moreover, p_t , given by (3.3), satisfies Eqs. (2.10) and (2.11) with

$$\hat{m}_{kt} := 2 \operatorname{Re} \langle \hat{\sigma}_{t-}, L_{kt} \rangle, \quad \hat{I}_t(y) := \langle \hat{\sigma}_{t-}, (L_t(y)^* + 1)(L_t(y) + 1) \rangle. \tag{3.14}$$

When also assumption 3.3. holds, σ_t given by Eq. (3.2) is the unique solution of Eq. (3.13) with initial condition (3.1).

Proof. If $\varrho = |\psi\rangle\langle\psi|$, then Eq. (3.13) follows from (2.3) through application of Itô’s formula to the function $f(\psi_t) = \langle \psi_t | a \psi_t \rangle$. By linearity it follows that (3.13) holds in the case of initial condition $\varrho = \varrho^N \equiv \sum_{\beta=1}^N |\psi^\beta\rangle\langle\psi^\beta|$, $N < \infty$. For general ϱ we shall obtain (3.13) by an approximation with respect to N .

We first show that p_t satisfies Eqs. (2.10), (2.11), (3.14). By setting

$$p_t^N := \sum_{\beta=1}^N \|\psi_t^\beta\|^2,$$

we have an equation for p_t^N similar to (2.10), (2.11), (3.14), were \hat{m}_{kt} and $\hat{I}_t(y)$ are replaced by

$$\hat{m}_{kt}^N := \frac{2}{p_{t-}^N} \operatorname{Re} \sum_{\beta=1}^N \langle \psi_{t-}^\beta | L_{kt} \psi_{t-}^\beta \rangle, \quad \hat{I}_t^N(y) := \frac{1}{p_{t-}^N} \sum_{\beta=1}^N \|(L_t(y) + 1)\psi_{t-}^\beta\|^2,$$

with obvious modifications on the set $\{\omega \in \Omega: p_{t-}^N(\omega) = 0\}$.

By setting $q_t^N := \sqrt{p_t^N}$, we have

$$\begin{aligned} q_t^N &= q_0^N + \frac{1}{2} \int_{(0,t]} q_{s-}^N \sum_{k=1}^r \hat{m}_{ks}^N dM_{ks} - \frac{1}{8} \int_{(0,t]} q_{s-}^N \sum_{k,k'=1}^r \hat{m}_{k's}^N \gamma_{k'k}(s) \hat{m}_{ks}^N ds \\ &\quad + \int_{(0,t]} q_{s-}^N \int_{\mathcal{A}} (\sqrt{\hat{I}_s^N(y)} - 1) \tilde{\Pi}(dy, ds) \\ &\quad - \frac{1}{2} \int_{(0,t]} q_{s-}^N \int_{\mathcal{A}} (\sqrt{\hat{I}_s^N(y)} - 1)^2 \nu_s(dy) ds. \end{aligned} \tag{3.15}$$

The proof of this equation is similar to that of (2.26). As $N \rightarrow \infty$, we have $p_t^N \uparrow p_t$, $q_t^N \uparrow q_t = \sqrt{p_t}$, $\hat{m}_{kt}^N \rightarrow \hat{m}_{kt}$, $\hat{I}_t^N(y) \rightarrow \hat{I}_t(y)$. Moreover, the stochastic integrals in (3.15) converge in the mean square, while the integral with respect to ds converges in the mean. We shall prove this assertion only for the case of the stochastic integral with respect to the white random measure. Inequality (2.15) implies

$$(q_{t-}^N)^2 (\sqrt{\hat{I}_t^N(y)} - 1)^2 \leq \sum_{\beta=1}^N \|L_t(y)\psi_{t-}^\beta\|^2,$$

whence, by condition 2.0.B, the fact that $\mathbb{E}_P[p_t^N] \leq \mathbb{E}_P[p_0^N] \leq 1$ and the Lebesgue dominated convergence theorem, the assertion follows. Thus, q_t satisfies Eq. (3.15) with \hat{m}_{kt}^N and $\hat{I}_t^N(y)$ replaced by \hat{m}_{kt} and $\hat{I}_t(y)$. Then, applying Itô’s formula to $f(q_t) = q_t^2$, we get (3.14). From the general theory of Doléans equations (Liptser and Shirayev, 1986) it follows that p_t is a regular process and therefore

$$\sup_{s \leq t} p_s < \infty \quad \text{a.s.} \tag{3.16}$$

Passing to approximation in Eq. (3.13), we can assume $a \geq 0$. Then, we have

$$0 \leq \langle \sigma_t - \sigma_t^N, a \rangle = \sum_{\beta=N+1}^{\infty} \|\sqrt{a}\psi_t^\beta\|^2 \leq \|a\| \sum_{\beta=N+1}^{\infty} \|\psi_t^\beta\|^2.$$

Therefore, $\langle \sigma_t^N, a \rangle \rightarrow \langle \sigma_t, a \rangle$ a.s. Using similar estimates and conditions 2.0 we can show that

$$\int_0^t \langle \sigma_{s-}^N, \mathcal{L}_s[a] \rangle ds \rightarrow \int_0^t \langle \sigma_{s-}, \mathcal{L}_s[a] \rangle ds.$$

For the stochastic integral with respect to M we have to show that

$$\begin{aligned} & \int_0^t \sum_{k,k'=1}^r \left[\operatorname{Re} \sum_{\beta=N+1}^{\infty} \langle a\psi_{s-}^\beta - |L_{k's}\psi_{s-}^\beta \rangle \right] \gamma_{k'k}(s) \\ & \times \left[\operatorname{Re} \sum_{\beta=N+1}^{\infty} \langle a\psi_{s-}^\beta - |L_{ks}\psi_{s-}^\beta \rangle \right] ds \rightarrow 0 \quad \text{a.s.} \end{aligned}$$

By the Cauchy–Schwarz inequality, this is estimated as

$$\begin{aligned} & \int_0^t \sum_{k,k'=1}^r \sum_{\beta=N+1}^{\infty} \|a\psi_{s-}^\beta\|^2 \sum_{\alpha=N+1}^{\infty} \|L_{k's}\psi_{s-}^\alpha\| \gamma_{k'k}(s) \|L_{ks}\psi_{s-}^\alpha\| ds \\ & \leq \|a\|^2 \sup_{r < t} p_r \int_0^t \|S_s\| \sum_{\beta=N+1}^{\infty} \|\psi_{s-}^\beta\|^2 ds; \end{aligned}$$

the coefficient is a.s. bounded by (3.16), while the integral tends to zero as $N \rightarrow \infty$ by assumption 2.0.A and the Lebesgue dominated convergence theorem.

For the stochastic integral with respect to the white random measure we have to show that

$$\int_0^t \int_{\mathbb{R}} \frac{|F_s^N(y)|^2}{1 + |F_s^N(y)|} \nu_s(dy) ds \rightarrow 0 \quad \text{a.s.,} \tag{3.17}$$

where

$$F_s^N(y) := \sum_{\beta=N+1}^{\infty} [\|\sqrt{a}(L_s(y) + 1)\psi_{s-}^\beta\|^2 - \|\sqrt{a}\psi_{s-}^\beta\|^2].$$

By estimates of the type (2.15) and the Cauchy–Schwarz inequality we have

$$\begin{aligned} |F_s^N(y)| & \leq \sum_{\beta=N+1}^{\infty} \|\sqrt{a}L_s(y)\psi_{s-}^\beta\| [\|\sqrt{a}(L_s(y) + 1)\psi_{s-}^\beta\| + \|\sqrt{a}\psi_{s-}^\beta\|] \\ & \leq \|a\| \left\{ \sum_{\alpha=N+1}^{\infty} \|L_s(y)\psi_{s-}^\alpha\|^2 \sum_{\beta=N+1}^{\infty} [\|(L_s(y) + 1)\psi_{s-}^\beta\| + \|\psi_{s-}^\beta\|]^2 \right\}^{1/2}. \end{aligned} \tag{3.18}$$

We take a constant $\kappa > 1$ and separate the integral (3.17) into two domains \mathcal{D} and $\bar{\mathcal{D}}$, where

$$\mathcal{D} := \left\{ (y, s): \sum_{\beta=N+1}^{\infty} \|(L_s(y) + \mathbb{1})\psi_{s-}^{\beta}\|^2 \leq \kappa^2 \sum_{\beta=N+1}^{\infty} \|\psi_{s-}^{\beta}\|^2 \right\}.$$

On \mathcal{D} we have by (3.18)

$$\frac{|F_s^N(y)|^2}{1 + |F_s^N(y)|} \leq |F_s^N(y)|^2 \leq \|a\|^2 \sum_{\alpha=1}^{\infty} \langle \psi_{s-}^{\alpha} | L_s(y) * L_s(y) \psi_{s-}^{\alpha} \rangle (1 + \kappa)^2 \sum_{\beta=N+1}^{\infty} \|\psi_{s-}^{\beta}\|^2;$$

therefore

$$\begin{aligned} & \int_0^t \int_{\mathcal{D}} \frac{|F_s^N(y)|^2}{1 + |F_s^N(y)|} \nu_s(dy) ds \\ & \leq \|a\|^2 (1 + \kappa)^2 \sup_{r < t} p_r \int_0^t \|R_s\| \sum_{\beta=N+1}^{\infty} \|\psi_{s-}^{\beta}\|^2 ds. \end{aligned}$$

By the Lebesgue dominated convergence theorem and assumption 2.0.B this tends to zero a.s.

Let us set $\tilde{\kappa} := (\kappa + 1)/(\kappa - 1)$; then, on $\bar{\mathcal{D}}$, we have

$$\begin{aligned} & \sum_{\beta=N+1}^{\infty} [\|(L_s(y) + \mathbb{1})\psi_{s-}^{\beta}\| + \|\psi_{s-}^{\beta}\|]^2 \\ & \leq \tilde{\kappa}^2 \sum_{\beta=N+1}^{\infty} [\|(L_s(y) + \mathbb{1})\psi_{s-}^{\beta}\| - \|\psi_{s-}^{\beta}\|]^2 \leq \tilde{\kappa}^2 \sum_{\beta=N+1}^{\infty} \|L_s(y)\psi_{s-}^{\beta}\|^2; \end{aligned}$$

therefore, by (3.18),

$$\frac{|F_s^N(y)|^2}{1 + |F_s^N(y)|} \leq |F_s^N(y)| \leq \tilde{\kappa} \|a\| \sum_{\beta=N+1}^{\infty} \langle \psi_{s-}^{\beta} | L_s(y) * L_s(y) \psi_{s-}^{\beta} \rangle$$

and

$$\int_0^t \int_{\bar{\mathcal{D}}} \frac{|F_s^N(y)|^2}{1 + |F_s^N(y)|} \nu_s(dy) ds \leq \tilde{\kappa} \|a\| \int_0^t \|R_s\| \sum_{\beta=N+1}^{\infty} \|\psi_{s-}^{\beta}\|^2 ds,$$

which tends to zero a.s.

About uniqueness, let us regard Eq. (3.13) as an equation for the Hilbert–Schmidt operators. By using the same theorems used in the proof of Proposition 2.1 for existence and uniqueness of the solutions of SDE’s in Hilbert spaces, one sees that condition 3.3 is sufficient to assure such a uniqueness. \square

3.5. Remark. As in Section 2 we have that, under the law \hat{P} , the process \hat{M} , defined by (2.17), is a continuous martingale with the same matrix of quadratic variation as before and that $\hat{\Gamma}$, defined by (2.18), is a white random measure; now \hat{m}_{kt} and $\hat{I}_t(y)$ are given by Eq. (3.14).

3.6. Remark. Essentially with the same reasoning used for proving Eq. (2.19), one can prove now that the random states (3.4) satisfy the non-linear SDE

$$\begin{aligned}
 d\langle \hat{\sigma}_t, a \rangle &= \langle \hat{\sigma}_{t-}, \mathcal{L}_t[a] \rangle dt + \sum_{k=1}^r \langle \hat{\sigma}_{t-}, L_{kt}^* a + a L_{kt} - \hat{m}_{kt} a \rangle d\hat{M}_{kt} \\
 &+ \int_{E_t^c} [\hat{I}_t(y)^{-1} \langle \hat{\sigma}_{t-}, (L_t(y)^* + \mathbb{1}) a (L_t(y) + \mathbb{1}) \rangle - \langle \hat{\sigma}_{t-}, a \rangle] \hat{\Pi}(dy, dt);
 \end{aligned}
 \tag{3.19}$$

E_t^c is a set defined as in Theorem 2.7.

4. Stochastic representation of the continuous measurement process

4.1. The output process

Starting from the SDE’s and the probability measures introduced in Sections 2 and 3, it is possible to arrive at describing measurements continuous in time in quantum mechanics. After a technical assumption, the first step is to describe the results of the measurements: only some *classical* process X_t is observed, which we call the *output process*.

4.1. We assume the point process Π to be such that the relation

$$\mu_t(dx) := \sum_{j=1}^{\infty} \nu_t(dx, \{j\})
 \tag{4.1}$$

defines a predictable random measure on $\mathbb{R}^d \setminus \{0\}$ such that

$$\int_{\mathbb{R}^d \setminus \{0\}} \varphi(x) \mu_t(dx) < \infty, \quad \varphi(x) := \frac{|x|^2}{1 + |x|^2}.
 \tag{4.2}$$

Then,

$$N(dx, dt) := \sum_{j=1}^{\infty} \Pi(dx, \{j\}, dt)
 \tag{4.3}$$

is a point process of order φ (Métivier, 1982, p. 218) on $(\mathbb{R}^d \setminus \{0\}) \times \mathbb{R}^+$ with the compensator $\mu_t(dx)dt$ (under the law P). Moreover, we set

$$\tilde{N}(dx, dt) := N(dx, dt) - \mu_t(dx)dt.
 \tag{4.4}$$

Let us consider the \mathbb{R}^d -valued RRC process X_t defined, for $i = 1, \dots, d$, by

$$\begin{aligned}
 X_{it} &= X_{i0} + \int_{(0,t]} \left\{ b_i(s) ds + \sum_{k=1}^r a_{ik}(s) dM_{ks} \right. \\
 &\left. + \int_{0 < |x| \leq 1} x_i \tilde{N}(dx, ds) + \int_{|x| > 1} x_i N(dx, ds) \right\},
 \end{aligned}
 \tag{4.5}$$

where X_{i0} is \mathcal{F}_0 -measurable, $a_{ik}(\cdot)$ and $b_i(\cdot)$ are RLC processes.

With respect to the law P and the filtration \mathcal{F}_t , X_t is a *semimartingale* with triplet of predictable characteristics (Liptser and Shirayayev, 1986, pp. 188–191): $B_{ii} := \int_0^t b_i(s) ds$ (predictable process of bounded variation), $\int_0^t b_{ij}(s) ds$ (matrix of the quadratic variation of the continuous part), $\mu_t(dx) dt$ (compensator of the jump measure), where

$$b_{ij}(t) := \sum_{k, k'=1}^r a_{ik'}(t) \gamma_{k'k}(t) a_{jk}(t). \tag{4.6}$$

Let us introduce now the proper filtration of the semimartingale X_t . By using the notation $\sigma\{\dots\}$ for the σ -algebra generated by \dots , we set

$$\mathcal{E}_t := \sigma\{X_s, 0 \leq s \leq t\} \vee \mathcal{N}, \quad \mathcal{E} := \bigvee_{t \geq 0} \mathcal{E}_t, \tag{4.7}$$

where \mathcal{N} is the collection of sets in \mathcal{F} with P -measure zero. In what follows we assume that this filtration is right continuous.

4.2. Remark. Let us denote by $\sigma_t(\varrho)$ the solution of Eq. (3.13) with a generic non-random trace-class operator ϱ as initial condition. Then, the equation

$$\langle \varrho, \mathcal{I}_t(E)[a] \rangle = \mathbb{E}_P[1_E \langle \sigma_t(\varrho), a \rangle], \quad \forall E \in \mathcal{E}_t, \forall \varrho \in \mathcal{T}(\mathcal{H}), \forall a \in \mathcal{L}(\mathcal{H}), \tag{4.8}$$

defines a map-valued measure \mathcal{I}_t with the properties: (i) $\forall E \in \mathcal{E}_t, \mathcal{I}_t(E)$ is a bounded linear operator from $\mathcal{L}(\mathcal{H})$ into itself and it is completely positive and normal (Lindblad, 1976); (ii) $\forall \varrho \in \mathcal{L}(\mathcal{H}), \forall a \in \mathcal{L}(\mathcal{H}), a \geq 0, \langle \varrho, \mathcal{I}_t(\cdot)[a] \rangle$ is σ -additive; (iii) $\mathcal{I}_t(\Omega)[\mathbb{1}] = \mathbb{1}$. Such a map-valued measure is called an *instrument* with *value space* (Ω, \mathcal{E}_t) (Davies, 1976).

Let us note that, by using a non-random state ϱ as initial condition in Eq. (3.13), we have

$$\langle \varrho, \mathcal{I}_t(E)[\mathbb{1}] \rangle = \hat{P}(E) \tag{4.9}$$

for $E \in \mathcal{E}_t$. By the fact that $\{\mathcal{E}_t, t \geq 0\}$ is the proper filtration of the process X , we can say that \mathcal{I}_t describes the measuring of the quantities $X_s, 0 \leq s \leq t$, and this measurement gives (in the state ϱ) the probability $\hat{P}|_{\mathcal{E}_t}$ for this “output” process.

4.3. Remark. By Remark 3.5, we have immediately that, with respect to the law \hat{P} defined by Eqs. (2.16) and (3.3), X_t is again a semimartingale with triplet of predictable characteristics (Liptser and Shirayayev, 1986, Ch. 4, Section 5): $\hat{B}_{ii} := \int_0^t \hat{b}_i(s) ds, \int_0^t b_{ij}(s) ds, \sum_{j=1}^{\infty} \hat{I}_t(x, j) \nu_j(dx, \{j\}) dt$, where $b_{ij}(t)$ is given by Eq. (4.6),

$$\begin{aligned} \hat{b}_i(s) := & b_i(s) + \sum_{k, k'=1}^r a_{ik'}(s) \gamma_{k'k}(s) \hat{m}_{ks} \\ & + \sum_{j=1}^{\infty} \int_{0 < |x| \leq 1} x_i (\hat{I}_s(x, j) - 1) \nu_s(dx, \{j\}), \end{aligned} \tag{4.10}$$

and \hat{m}_{kt} and \hat{I}_t are defined by Eq. (3.14).

We introduce now the random trace-class operators ϱ_t by

$$\langle \varrho_t, a \rangle := \mathbb{E}_P[\langle \sigma_t, a \rangle | \mathcal{E}_t], \quad \forall a \in \mathcal{L}(\mathcal{H}), \tag{4.11}$$

and the random states $\hat{\varrho}_t$ by

$$\hat{\varrho}_t := \begin{cases} \varrho_t / \langle \varrho_t, \mathbb{1} \rangle, & \text{if } \langle \varrho_t, \mathbb{1} \rangle \neq 0, \\ S \text{ (S fixed statistical operator)} & \text{if } \langle \varrho_t, \mathbb{1} \rangle = 0. \end{cases} \tag{4.12}$$

Note that $\langle \varrho_t, \mathbb{1} \rangle = \mathbb{E}_P[p_t | \mathcal{E}_t]$; therefore, $\langle \varrho_t, \mathbb{1} \rangle$ is the density of $\hat{P}|_{\mathcal{E}_t}$ with respect to $P|_{\mathcal{E}_t}$. Moreover, by Eqs. (3.3),(3.4),(4.11) and (4.12), for every bounded and \mathcal{E}_t -measurable random variable X we have

$$\begin{aligned} \mathbb{E}_P[X \langle \hat{\varrho}_t, a \rangle] &= \mathbb{E}_P[\mathbb{E}_P[p_t | \mathcal{E}_t] X \langle \hat{\varrho}_t, a \rangle] = \mathbb{E}_P[X \langle \varrho_t, a \rangle] \\ &= \mathbb{E}_{\hat{P}}[X \langle \sigma_t, a \rangle] = \mathbb{E}_P[p_t X \langle \hat{\sigma}_t, a \rangle] = \mathbb{E}_{\hat{P}}[X \langle \hat{\sigma}_t, a \rangle]. \end{aligned}$$

This means that

$$\langle \hat{\varrho}_t, a \rangle = \mathbb{E}_{\hat{P}}[\langle \hat{\sigma}_t, a \rangle | \mathcal{E}_t], \quad \forall a \in \mathcal{L}(\mathcal{H}). \tag{4.13}$$

4.4. Remark. We take a non-random initial condition $\varrho \in \mathcal{S}(\mathcal{H})$, so that $\varrho = \sigma_0 = \varrho_0 = \hat{\varrho}_0$. Then, $\hat{\varrho}_t$ is the a posteriori state for the instrument \mathcal{I}_t and the premeasurement state ϱ , in the sense that (Ozawa, 1985)

$$\langle \varrho, \mathcal{I}_t(E)[a] \rangle = \int_E \langle \hat{\varrho}_t(\omega), a \rangle \langle \varrho, \mathcal{I}_t(d\omega)[\mathbb{1}] \rangle, \quad \forall E \in \mathcal{E}_t, \quad \forall \varrho \in \mathcal{S}(\mathcal{H}), \quad \forall a \in \mathcal{L}(\mathcal{H}) \tag{4.14}$$

Indeed, by Eqs. (4.8) and (4.9), we have

$$\begin{aligned} \langle \varrho, \mathcal{I}_t(E)[a] \rangle &= \mathbb{E}_P[1_E \langle \sigma_t, a \rangle] = \mathbb{E}_P[1_E \mathbb{E}_P[\langle \sigma_t, a \rangle | \mathcal{E}_t]] = \mathbb{E}_P[1_E \langle \varrho_t, a \rangle] \\ &= \mathbb{E}_P[1_E \langle \hat{\varrho}_t, a \rangle \mathbb{E}_P[p_t | \mathcal{E}_t]] = \mathbb{E}_P[\mathbb{E}_P[1_E \langle \hat{\varrho}_t, a \rangle p_t | \mathcal{E}_t]] \\ &= \mathbb{E}_{\hat{P}}[1_E \langle \hat{\varrho}_t, a \rangle] = \int_E \langle \hat{\varrho}_t(\omega), a \rangle \langle \varrho, \mathcal{I}_t(d\omega)[\mathbb{1}] \rangle. \end{aligned}$$

The statistical operator $\hat{\varrho}_t(\omega)$ is the state one attributes to the quantum system at time t if the trajectory $X_s(\omega)$, $0 \leq s \leq t$, of the output process has been observed. Moreover, if we consider the state $\bar{\varrho}_t$ defined by Eq. (3.5) and we take Eqs. (4.8) and (4.14) for $E = \Omega$, we obtain, $\forall a \in \mathcal{L}(\mathcal{H})$,

$$\langle \bar{\varrho}_t, a \rangle = \int_{\Omega} \langle \hat{\varrho}_t(\omega), a \rangle \langle \varrho, \mathcal{I}_t(d\omega)[\mathbb{1}] \rangle; \tag{4.15}$$

we can read this equation by saying that $\bar{\varrho}_t$ is a mixture of the a posteriori states $\hat{\varrho}_t(\omega)$ with respect to the probability measure $\langle \varrho, \mathcal{I}_t(d\omega)[\mathbb{1}] \rangle$: $\bar{\varrho}_t$ is called the a priori state at time t because it is the state we attribute to the system at time t before knowing the results of the measurement.

4.2. *Non-linear stochastic equation for the a posteriori states*

We want to introduce now new assumptions in order to reach two goals. The first one is to obtain closed SDE's for ϱ_t and $\hat{\varrho}_t$. The second one is to introduce (random)

instruments related to arbitrary time intervals and to obtain some rule for composing instruments related to consecutive intervals, in such a way that this composition gives the instrument related to the union of the time intervals. The simplest way of obtaining these goals is to have that $\{\mathcal{E}_t, t \geq 0\}$ coincide with the filtration generated by N and some of the components of M , that all the coefficients appearing in the various equations be adapted to such a filtration and that Π and M reduces to Poisson and Wiener processes, respectively.

4.5. We take $\gamma_{k'k}(t) = \delta_{k'k}$. Then, $M_k, k = 1, \dots, r$, are independent standard Wiener processes and we denote them by W_k .

4.6. We take $\nu_t(dy)$ to be a family of non-random measures. In other words, Π is a Poisson point process with, eventually, a time-dependent intensity and the same holds for N .

4.7. We assume the rank d' of the matrix $b_{ij}(t)$ to be non-random and time-independent (obviously we have $0 \leq d' \leq d, d' \leq r$); then, we take $a_{ik}(t) \equiv 0$ for $k > d'$. Moreover, we fix $X_0 \equiv 0$.

4.8. We take $H_t, L_{kt}, L_t(y), b_i(t), a_{ik}(t)$ to be adapted to the filtration generated by the set of processes $W_k, k = 1, \dots, d'$, and N .

Under these further assumptions we have also, $\forall a \in \mathcal{L}(\mathcal{H})$,

$$\langle \varrho_t, a \rangle = \mathbb{E}_P[\langle \sigma_t, a \rangle | \mathcal{E}], \quad \langle \hat{\varrho}_t, a \rangle = \mathbb{E}_{\hat{P}}[\langle \hat{\sigma}_t, a \rangle | \mathcal{E}]. \tag{4.16}$$

By the previous assumptions the \mathcal{E} -conditional expectation of Eq. (3.13) gives

$$\begin{aligned} d\langle \varrho_t, a \rangle &= \langle \varrho_{t-}, \mathcal{L}_t[a] \rangle dt + \sum_{k=1}^{d'} \langle \varrho_{t-}, a L_{kt} + L_{kt}^* a \rangle dW_{kt} \\ &\quad + \int_{\mathbb{R}^d \setminus \{0\}} \langle \varrho_{t-}, J_t(x)[a] - a \rangle \tilde{N}(dx, dt), \end{aligned} \tag{4.17}$$

where \mathcal{L}_t is given by Eq. (3.7) and

$$J_t(x)[a] := \sum_{j=1}^{\infty} \frac{\nu_t(dx, \{j\})}{\mu_t(dx)} (L_t(x, j)^* + \mathbb{1}) a (L_t(x, j) + \mathbb{1}). \tag{4.18}$$

In the derivation of this result it is useful to recall the rules:

$$\begin{aligned} \mathbb{E}_P[\Pi(\{(0, j)\}, dt) | \mathcal{E}] &= \nu_t(\{(0, j)\}) dt, \\ \mathbb{E}_P[\Pi(dx, \{j\}, dt) | \mathcal{E}] &= \frac{\nu_t(dx, \{j\})}{\mu_t(dx)} N(dx, dt) \end{aligned} \tag{4.19}$$

for $dx \subset \mathbb{R}^d \setminus \{0\}$.

Similarly to Proposition 2.5 and Remarks 2.6 and 3.5, we have that in the probability space $(\Omega, \mathcal{E}, \hat{P})$ the processes

$$\check{W}_{kt} := W_{kt} - \int_{(0,t)} m_{ks} ds, \quad k = 1, \dots, d', \tag{4.20}$$

are independent standard Wiener processes and N is a point process, whose associated white measure is

$$\check{N}(dx, dt) := N(dx, dt) - I_t(x)\mu_t(dx)dt; \tag{4.21}$$

we have defined

$$m_{kt} := 2 \operatorname{Re} \langle \hat{Q}_{t-}, L_{kt} \rangle, \quad I_t(x) := \langle \hat{Q}_{t-}, J_t(x)[\mathbb{1}] \rangle. \tag{4.22}$$

Finally, as in Theorem 2.7 and in Remark 3.6, we obtain the non-linear SDE for the a posteriori states:

$$\begin{aligned} d\langle \hat{Q}_t, a \rangle &= \langle \hat{Q}_{t-}, \mathcal{L}_t[a] \rangle dt + \sum_{k=1}^{d'} \langle \hat{Q}_{t-}, L_{kt}^* a + a L_{kt} - m_{kt} a \rangle d\check{W}_{kt} \\ &+ \int_{\check{E}_t^c} [I_t(x)^{-1} \langle \hat{Q}_{t-}, J_t(x)[a] \rangle - \langle \hat{Q}_{t-}, a \rangle] \check{N}(dx, dt), \end{aligned} \tag{4.23}$$

where $\check{E}_t := \{x \in \mathbb{R}^d \setminus \{0\} : I_t(x) = 0\}$ and \check{E}_t^c is its complement.

4.3. “Chain rule” for the instruments

Let us consider Eq. (4.17) as defining a (random) linear map from the trace-class (the initial conditions) into itself. Let us take $0 \leq s \leq t$ and denote by $A_t^s[\varrho]$ the solution of Eq. (4.17) with initial condition $A_s^s[\varrho] = \varrho$; note that $A_t^s = A_t^u \circ A_u^s$ for $s \leq u \leq t$. Moreover we denote by \mathcal{E}_t^s the σ -algebra generated by the increments of the process X , i.e.

$$\mathcal{E}_t^s := \sigma\{X_u - X_r, s \leq r \leq u \leq t\} \vee \mathcal{N}; \tag{4.24}$$

note that $\mathcal{E}_t^0 \equiv \mathcal{E}_t$.

Now we can define a family of (random) instruments \mathcal{I}_t^s on $(\Omega, \mathcal{E}_t^s)$ by

$$\langle \varrho, \mathcal{I}_t^s(E)[a] \rangle = \mathbb{E}_P[1_E \langle A_t^s[\varrho], a \rangle | \mathcal{E}_s], \tag{4.25}$$

$\varrho \in \mathcal{T}(\mathcal{H})$, $a \in \mathcal{L}(\mathcal{H})$, $E \in \mathcal{E}_t^s$, $0 \leq s \leq t$. Let us note that \mathcal{I}_t^0 coincides with \mathcal{I}_t , the non-random instrument previously defined.

Let us take now $0 \leq s < u < t$, $E \in \mathcal{E}_u^s$, $F \in \mathcal{E}_t^u$. By using the properties of conditional expectations we have

$$\begin{aligned} \mathbb{E}_P[1_{E \cap F} \langle A_t^s[\varrho], a \rangle | \mathcal{E}_s] &= \mathbb{E}_P[1_E \mathbb{E}_P[1_F \langle A_t^u \circ A_u^s[\varrho], a \rangle | \mathcal{E}_u] | \mathcal{E}_s] \\ &= \mathbb{E}_P[1_E \langle A_u^s[\varrho], \mathcal{I}_t^u(F)[a] \rangle | \mathcal{E}_s] \end{aligned}$$

and this equation can be written as a “chain rule” for our family of instruments:

$$\langle \varrho, \mathcal{I}_t^s(E \cap F)[a] \rangle = \int_E \langle \varrho, \mathcal{I}_u^s(d\omega) \circ \mathcal{I}_t^u(F; \omega)[a] \rangle. \tag{4.26}$$

The dynamics of the system is described by the operators

$$\mathcal{U}(s, t) := \mathcal{I}_t^s(\Omega), \quad \mathcal{U}(t) \equiv \mathcal{U}(0, t) = \mathcal{I}_t(\Omega). \tag{4.27}$$

By Eq. (4.26), we have

$$\langle \varrho, \mathcal{U}(s, t)[a] \rangle = \int_{\Omega} \langle \varrho, \mathcal{I}_u^s(d\omega) \circ \mathcal{U}(u, t; \omega)[a] \rangle \tag{4.28}$$

and, by Eq. (3.6),

$$\frac{d}{dt} \langle \varrho, \mathcal{U}(t)[a] \rangle = \mathbb{E}_P[\langle \mathcal{A}_t^0[\varrho], \mathcal{L}_t[a] \rangle] = \int_{\Omega} \langle \varrho, \mathcal{I}_t^0(d\omega) \circ \mathcal{L}_t(\omega)[a] \rangle. \tag{4.29}$$

In the particular case of non-random coefficients in Eqs. (4.5) and (4.17), we obtain that $\mathcal{I}_t^s, \mathcal{U}(s, t), \mathcal{L}_t$ are non-random maps and that Eqs. (4.26), (4.28) and (4.29) become

$$\mathcal{I}_t^s(E \cap F) = \mathcal{I}_u^s(E) \circ \mathcal{I}_t^u(F), \quad \mathcal{U}(s, t) = \mathcal{U}(s, u) \circ \mathcal{U}(u, t), \tag{4.30}$$

$$\frac{d}{dt} \langle \varrho, \mathcal{U}(t)[a] \rangle = \langle \varrho, \mathcal{U}(t) \circ \mathcal{L}_t[a] \rangle; \tag{4.31}$$

Eq. (4.31) is nothing but the master equation (3.12).

4.4. The Markov case

Up to now applications to concrete physical models concerned the case of no randomness in the various coefficients appearing in our stochastic equations. A conceptually interesting and promising case, which is intermediate in between the full generality of this paper and the case of non-random coefficients, is the Markov case: all the coefficients appearing in assumption 4.8 depend on ω only through X_t . Let us give a sketch of what can be done.

First let us take the observable process $X_{it}, i = 1, \dots, d$, introduced in Eq. (4.5), to be solution of the SDE

$$\begin{aligned} dX_{it} &= b_i(X_{t-})dt + \sum_{k=1}^{d'} a_{ik}(X_{t-})dW_{kt} \\ &+ \int_{0 < |y| \leq 1} y_i \tilde{N}(dy, dt) + \int_{|y| > 1} y_i N(dy, dt), \end{aligned} \tag{4.32}$$

where $b_i(x), a_{ik}(x), \mu(dy)$ (which we take time-independent) satisfy the “global Lipschitz” conditions for existence, uniqueness and non-explosion of Métivier (1982), already used in Section 2. Then, X is a Markov process. An equivalent way to define X is to say that for any function $f \in C_2(\mathbb{R}^d)$ (two times continuously differentiable

functions) we have

$$df(X_t) = K[f](X_t)dt + \sum_{i=1}^d \sum_{k=1}^{d'} \frac{\partial f(X_t)}{\partial x_i} a_{ik}(X_t) dW_{kt} + \int_{\mathbb{R}^d \setminus \{0\}} [f(y + X_t) - f(X_t)] \tilde{N}(dy, dt), \tag{4.33}$$

$$K[f](x) := \sum_{i=1}^d \tilde{b}_i(x) \frac{\partial f(x)}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^d b_{ij}(x) \frac{\partial^2 f(x)}{\partial x_i \partial x_j} + \int_{\mathbb{R}^d \setminus \{0\}} \left[f(y + x) - f(x) - \sum_{i=1}^d \frac{\partial f(x)}{\partial x_i} \frac{y_i}{1 + |y|^2} \right] \mu(dy), \tag{4.34}$$

$$\tilde{b}_i(x) := b_i(x) - \int_{0 < |y| \leq 1} y_i \varphi(y) \mu(dy) + \int_{|y| > 1} \frac{y_i}{1 + |y|^2} \mu(dy). \tag{4.35}$$

Secondly, let us take $L_t(y; \omega) = L(y; X_t(\omega))$, $L_{k_t}(\omega) = L_k(X_t(\omega))$, $H_t(\omega) = H(X_t(\omega))$, where the functions $L(y; x)$, $L_k(x)$, $H(x)$ satisfy conditions of types 2.3, 3.1 and 3.3 uniformly in x ; we take $\nu(dy)$ to be time-independent.

Now we can define “instrumental” transition probabilities on the Borel sets in \mathbb{R}^d by an analogue of Eq. (4.25):

$$\langle \varrho, \mathcal{P}_t(B|x)[a] \rangle := \mathbb{E}_P[1_{\{X_{s+t} - X_s \in B - x\}} \langle A_{s+t}^s[\varrho], a \rangle | X_s = x]; \tag{4.36}$$

no dependence on s appears because we are treating the time homogeneous case. Then, analogously to Eq. (4.26), we get

$$\langle \varrho, \mathcal{P}_{t+s}(B|x)[a] \rangle = \int_{\mathbb{R}^d} \langle \varrho, \mathcal{P}_s(dy|x) \circ \mathcal{P}_t(B|y)[a] \rangle. \tag{4.37}$$

This composition law for transition probabilities corresponds to the existence of the analogue of the Markov semigroup. Let $A \in C_2(\mathbb{R}^d; \mathcal{L}(\mathcal{H}))$ (two times continuously differentiable functions from \mathbb{R}^d into $\mathcal{L}(\mathcal{H})$); then the relation

$$\langle \varrho, T_t[A](x) \rangle = \mathbb{E}_P[\langle A_{s+t}^s[\varrho], A(X_{s+t}) \rangle | X_s = x] \tag{4.38}$$

defines a semigroup on $C_2(\mathbb{R}^d; \mathcal{L}(\mathcal{H}))$.

By differentiating Eq. (4.38) with respect to time, we obtain the analogue of Kolmogorov equation. It is sufficient to give the generator \mathcal{K} of T_t in the case $A(x) = f(x)a$, $a \in \mathcal{L}(\mathcal{H})$, $f \in C_2(\mathbb{R}^d)$. By using Itô’s formula in computing $d(\langle \varrho_t, a \rangle f(X_t))$ from Eqs. (4.17) and (4.33) and taking the conditional expectation, we obtain

$$\begin{aligned} \mathcal{K}[a \otimes f](x) &= f(x) \mathcal{L}(x)[a] + \sum_{j=1}^d \tilde{b}_j(x) \frac{\partial f(x)}{\partial x_j} a \\ &+ \sum_{j=1}^d \sum_{k=1}^{d'} \frac{\partial f(x)}{\partial x_j} a_{jk}(x) (L_k(x) * a + a L_k(x)) + \frac{1}{2} a \sum_{j,i=1}^d b_{ji}(x) \frac{\partial^2 f(x)}{\partial x_i \partial x_j} \\ &+ \int_{\mathbb{R}^d \setminus \{0\}} \left\{ [f(x + y) - f(x)] J_x(y)[a] - \sum_{j=1}^d \frac{\partial f(x)}{\partial x_j} \frac{y_j}{1 + |y|^2} a \right\} \mu(dy), \end{aligned} \tag{4.39}$$

where $\mathcal{L}(x)$ and $J_x(y)$ are given by obvious modifications of Eqs. (3.7) and (4.18). Such a kind of generators and the related semigroups have been studied rigorously by Barchielli et al. (1993), in the case of x -independent coefficients.

Another possibility, inside the Markov case, is to let N to be a point process with “space” dependent intensity $\mu(X_t; dy)dt$. In any case X is a Markov process under the “reference” probability measure P , not under the final “physical” law \hat{P} .

4.5. The classical case

The situation we have described in this paper (a linear SDE, a related non-linear one and a change of measure) is very reminiscent of what happens in classical filtering theory (think to Zakai’s equation for unnormalized a posteriori densities in the case of diffusive processes and to its generalizations for jump processes – see Pardoux, 1991; Kliemann et al., 1990, and references therein). In this respect it is instructive to particularize some of our equations to a simple commutative case.

For simplicity let us start by taking $H_t \equiv 0$, $L_t(y) \equiv 0$. Then, let us assume that all the quantities $L_{kt}(\omega)$ are commuting operators. This means that we can take $\mathcal{H} = L^2(A, d\lambda)$ and represent $L_{kt}(\omega)$ by essentially bounded functions $L_{kt}(\lambda, \omega)$, $\lambda \in A$. Now in Eq. (4.17) we consider only operators a in the algebra of multiplication operators: this means that only the “diagonal” part of ϱ_t matters, or, better, that it is enough to represent $\varrho_t(\omega)$ by an $L^1(A, d\lambda)$ -function $\varrho_t(\lambda, \omega)$.

In this case Eq. (3.13) becomes a Doléans equation for a (local) probability density in $A \times \Omega$:

$$d\varrho_t(\lambda, \omega) = \varrho_t(\lambda, \omega)l_{kt}(\lambda, \omega)dW_{kt}(\omega), \tag{4.40}$$

where

$$l_{kt}(\lambda, \omega) = 2 \operatorname{Re} L_{kt}(\lambda, \omega). \tag{4.41}$$

Given a non-random initial condition

$$\varrho_0(\lambda, \omega) = \varrho(\lambda) \geq 0, \quad \int_A \varrho(\lambda) d\lambda = 1, \tag{4.42}$$

the solution of Eq. (4.40) is

$$\varrho_t(\lambda, \omega) = \varrho(\lambda) \exp \left[\int_0^t \sum_{k=1}^{d'} l_{ks}(\lambda, \omega) dW_{ks}(\omega) - \frac{1}{2} \int_0^t \sum_{k=1}^{d'} l_{ks}(\lambda, \omega)^2 ds \right]. \tag{4.43}$$

Then, the probability measure $\hat{\mu}_t(d\lambda, d\omega) := \varrho_t(\lambda, \omega) d\lambda P(d\omega)$ is the same as the joint distribution of $(\lambda, Y_{ks}, s \leq t)$, where Y_{kt} is given by

$$dY_{kt} = l_{kt} dt + dW_{kt} \tag{4.44}$$

and the initial distribution $d\lambda P(d\omega)$ is considered. Thus, the meaning of Eq. (3.13) is to describe “signal + noise” in terms of quantum states.

The a posteriori state

$$\hat{Q}_t(\lambda | \omega) = Q_t(\lambda, \omega) \Big/ \int_{\mathcal{A}} Q_t(\lambda, \omega) d\lambda \quad (4.45)$$

is the conditional probability density of λ with respect to the observation of W_{k_s} up to time t (under the law $\hat{\mu}_t$). Eq. (4.22) now gives the a posteriori mean of l_{kt} , that is

$$m_{kt}(\omega) = \int_{\mathcal{A}} l_{kt}(\lambda, \omega) \hat{Q}_t(\lambda | \omega) d\lambda, \quad (4.46)$$

and Eq. (4.23) for a posteriori states becomes

$$d\hat{Q}_t(\lambda | \omega) = \hat{Q}_t(\lambda | \omega) \sum_{k=1}^{d'} [l_{kt}(\lambda, \omega) - m_{kt}(\omega)] d\check{W}_{kt}(\omega), \quad (4.47)$$

where the \check{W}_{kt} are given by Eq. (4.20) and are Wiener processes under the new measure $\hat{P}_t(d\omega) = \int_{\mathcal{A}} \hat{\mu}_t(d\lambda, d\omega)$ in the trajectory space Ω . We can think to λ as to a random parameter subject to Bayes estimation.

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