Double pendulum and θ-divisor

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ABSTRACT. The equations of motion of integrable systems involving hyperelliptic Riemann surfaces of genus 2 and one relevant degree of freedom are integrated in the framework of the Jacobi inversion problem, using a reduction to the θ-divisor on the Jacobi variety, i. e., to the set of zeros of the θ-function. Explicit solutions are given in terms of Kleinian σ-functions and their derivatives. The procedure is applied to the planar double pendulum without gravity, but it is worked out for any Abelian integral of first or second kind.

1. Introduction

Modern investigations in the area of completely integrable mechanical systems with f degrees of freedom have accumulated an impressive number of solutions in cases where the spectral variety is represented by an algebraic curve of genus $g = f$, see [**9**] for a review. Examples are the Jacobi problem of geodesic flows on ellipsoids, the Neumann problem of motion on an n-dimensional sphere in the presence of a quadratic potential, the Kovalevskaya problem of rigid body motion, and the periodic Toda problem. In all these cases, the same hyperelliptic curve carries the motion of all f independent coordinates.

A priori, the requirement $f = g$ seems arbitrary, and indeed, it is easy to conceive of systems where $f < g$. Motion of a particle in a polynomial potential of degree higher than 4 is an obvious example [**23, 11**]. The non-trivial part of the motion of a double pendulum without gravity belongs to this class ($f = 1$, $g = 2$), and so does a symmetric rigid body (Lagrange case) in a Cardan frame of finite moment of inertia ($f = 1$, $g = 3$).

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The integration of such systems leads to the Jacobi inversion problem on the θ-divisor in the Jacobi variety $\mathrm{J}(V)$, i. e., on the set of zeros of the fundamental θ-function. It is classically known [**8**] that the integration of such systems can be executed in terms of appropriate derivatives of θ-functions (see also [**4**], where the corresponding divisors are called *divisors with deficiency*). Traditionally the theory of completely integrable systems [**27**] considers the complement of the θ-divisor as the natural domain for the finite-gap solutions, but the idea that the θ-divisor can also serve as a carrier of integrability attracts now more and more attention, see e.g. [**2, 26, 1, 18, 17**]. Nevertheless, to the best knowledge of the authors there have been no attempts to use the θ-divisor for explicit calculations of the dynamics of systems associated with deficient divisors. We shall do that in this paper.

Namely, we present here the results of an investigation which was motivated by the double pendulum problem but applies to the general case of systems with $f = 1$ where an Abelian integral of first or second kind on a hyperelliptic curve V of genus 2 is to be inverted. The double pendulum has served as a key example of non-integrable, chaotic motion [**24, 25**], but this is not the issue here. Our aim is to understand the integrable limit of high energy, or vanishing gravity. We found it intriguing that a simply defined integrable system seemed to defy attempts to analytically integrate it. We show that solutions can be found by reducing the Jacobi inversion problem to the θ-divisor. We use the fact (contained in Riemann's vanishing theorem) that the Abel map, plus an appropriate constant shift, takes the curve V uniformly to the θ-divisor, and that this relationship can be inverted. We give a detailed recipe how this is done.

Admittedly, the procedure is not very apt for practical purposes. Direct numerical integration of the equations of motion is certainly a faster way to obtain the time course of the motion. But the point of our investigation is (i) to show that an analytic integration of this integrable system is possible, and (ii) to elucidate its nature.

We shall also demonstrate that the Klein-Weierstrass realization of hyperelliptic functions (see [**3, 4**] and also [**5, 6**]) represents a convenient and effective framework to integrate dynamical systems associated with deficient divisors.

The paper is organized as follows. In Section 2 we recall the dynamical equations of the gravity free planar double pendulum and show that time as a function of the relevant coordinate is an Abelian integral of the second kind on a curve of genus 2. This part is elementary. Section 3 reviews the Jacobi inversion problem, the use of θ-functions, and the nature of the θ-divisor. For integrals of the first kind, the solutions of the inversion problem are given explicitly in terms of σ-functions and their derivatives. Most of this is classical material and well known in the community of soliton researchers [**3, 12, 9**], but a restriction of the Jacobi inversion to the theta divisor has not been applied before. Section 4 contains the relevant material for integrals of the second kind, necessary to complete the analysis of the double pendulum. Finally, in Section 5, we comment on generalizations to higher genera g.

Readers who wish to have pictorial representations of the behavior of θ-functions should visit the electronic version of this journal. There we show a color-coded "surface of a unit cell" of the universal covering of the Jacobi variety, and thereby provide an illustration of how the θ-divisor "looks like". We also provide a Maple .mws file where the procedure described in this article is worked out in all computational detail.

2. Double pendulum without gravity

Figure 1 shows a planar double pendulum with a space-fixed axis A_1 and the second axis A_2 fixed in the first body; let their distance be a. It is assumed that the center of gravity of the first body C_1 lies on the line connecting A_1 and A_2, a distance s_1 from A_1 (when C_1 is above A_1, s_1 is taken as negative). The angular position φ_1 of the first pendulum is measured from a fixed direction. Together with the relative angle φ_2 between the lines $\mathrm{A}_1\mathrm{A}_2$ and $\mathrm{A}_2\mathrm{C}_2$, it defines the configuration of the system, $(\varphi_1, \varphi_2) \in T^2$. In the usual exposition of the double pendulum problem [**19**], gravity is assumed to act in the direction $\varphi_1 = 0$. The dynamics is then non-integrable except in two limiting cases: small harmonic oscillations at very low energy, and motion with conserved angular momentum in the absence of gravity. The latter limit applies when the total energy is very large compared to the gravitational potential, or when the motion takes place in a plane perpendicular to the field of gravity. The chaotic motion in the general case of intermediate energies has been analyzed in great detail. It provides a beautiful example for the transition to global chaos via the break-up of a golden KAM torus [**24, 25, 22, 20**]. Mathematical proof for the existence of chaos was given in terms of the application of Melnikov's method [**10**].

Hence, the chaotic nature of the double pendulum is fairly well understood. The same is true for the low energy integrable limit [**15**]. What has been missing so far is the analytic treatment of the integrable limit at high energy, or vanishing gravity.

2.1. The integrable limit of zero gravity.

When gravity is absent, the angle φ_1 is a cyclic variable and the system is trivially

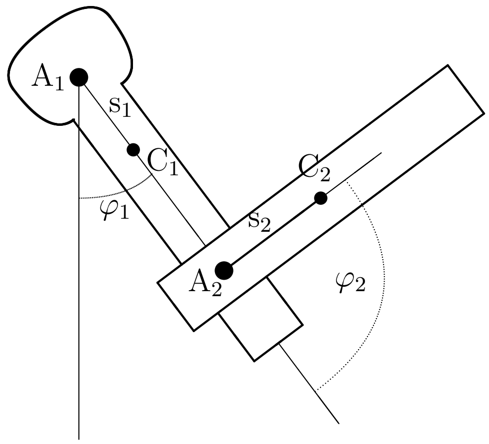

FIGURE 1. Planar double pendulum. The first pendulum swings around a fixed axis A_1 and carries the axis A_2 of the second body. The centers of gravity C_1 and C_2 are distances s_1 and s_2 apart from the axes; the distance between A_1 and A_2 is a. The configuration is determined by the two angles φ_1 and φ_2.

integrable. It is an elementary exercise to show that the Lagrangian is [**25, 20**]

$$L = \tfrac{1}{2}(\Theta_1 + m_2 a^2)\,\dot\varphi_1^2 + \tfrac{1}{2}\Theta_2\,(\dot\varphi_1 + \dot\varphi_2)^2 + m_2 s_2 a\,\dot\varphi_1(\dot\varphi_1 + \dot\varphi_2)\cos\varphi_2, \tag{1}$$

where Θ_1, Θ_2 are the moments of inertia of the two bodies with respect to the respective suspension points, and m_2 is the mass of the second pendulum. Scaling energies with Θ_2, all possible double pendulums are described by the two parameters

$$A := \frac{\Theta_1 + m_2 a^2}{\Theta_2}, \qquad \alpha := \frac{m_2 s_2 a}{\Theta_2}. \tag{2}$$

For the standard textbook case [**19**] with equal point masses at massless rods of equal lengths, the values are $A = 2$ and $\alpha = 1$. In general, the positivity of the moments of inertia implies $A > \alpha^2$.

With the scaled Lagrangian

$$L = \tfrac{1}{2}A\dot\varphi_1^2 + \tfrac{1}{2}(\dot\varphi_1 + \dot\varphi_2)^2 + \alpha\dot\varphi_1(\dot\varphi_1 + \dot\varphi_2)\cos\varphi_2, \tag{3}$$

the angular momenta are

$$
\begin{aligned}
p_1 &= (A+1+2\alpha\cos\varphi_2)\,\dot\varphi_1 + (1+\alpha\cos\varphi_2)\,\dot\varphi_2,\\
p_2 &= (1+\alpha\cos\varphi_2)\,\dot\varphi_1 + \dot\varphi_2,
\end{aligned} \tag{4}
$$

and the Hamiltonian becomes

$$
H = \frac{1}{2}\,\frac{p_1^2 - 2Q(\cos\varphi_2)\,p_1p_2 + P_1(\cos\varphi_2)\,p_2^2}{P_2(\cos\varphi_2)}, \tag{5}
$$

where

$$
Q(x) = 1+\alpha x, \quad P_1(x) = A+1+2\alpha x, \quad P_2(x) = A-\alpha^2x^2. \tag{6}
$$

Energy and total angular momentum are first integrals, $H =: h$, $p_1 =: l$, hence for fixed h, l the motion is restricted to a Liouville torus $\mathrm{T}_{h,l}$.

The equations of motion are easily derived with the canonical formalism. Their integration conveniently starts with $\dot\varphi_2 = \dot\varphi_2(\varphi_2, p_2; l)$ where p_2 is then replaced with the solution of (5) for $p_2(\varphi_2, l, h)$. The result is

$$
t = \int_0^{\varphi_2} \frac{\mathrm{d}\varphi_2}{\dot\varphi_2} = \int_0^{\varphi_2} P_2(\cos\varphi_2)\frac{\mathrm{d}\varphi_2}{w}, \tag{7}
$$

where

$$
w^2 = P_2(\cos\varphi_2)[2hP_1(\cos\varphi_2) - l^2]. \tag{8}
$$

This gives time t as a function of φ_2. The complete integral over a cycle of φ_2 gives the period T_2.

The angle φ_1 can also be obtained as a function of φ_2, after dividing the two equations for $\dot\varphi_1$ and $\dot\varphi_2$:

$$
\varphi_1 = -\int_0^{\varphi_2} \frac{Q(\cos\varphi_2)}{P_1(\cos\varphi_2)}\mathrm{d}\varphi_2 + l\int_0^{\varphi_2} \frac{P_2(\cos\varphi_2)}{P_1(\cos\varphi_2)}\frac{\mathrm{d}\varphi_2}{w}. \tag{9}
$$

The complete integral over a cycle of φ_2 gives $\Delta\varphi_1 =: 2\pi W$, where W is the winding number of the orbit on $\mathrm{T}_{h,l}$.

The above integrals can also be derived with action angle variables $(\phi_1, \phi_2, I_1, I_2)$. The actions are defined as

$$
I_i = \frac{1}{2\pi}\oint_{\gamma_i} (p_1\mathrm{d}\varphi_1 + p_2\mathrm{d}\varphi_2) \qquad (i = 1, 2), \tag{10}
$$

where γ_i are two fundamental paths on $\mathrm{T}_{h,l}$. We choose $\gamma_1 : \mathrm{d}\varphi_2 = 0$ and $\gamma_2 : \mathrm{d}\varphi_1 = 0$. The first action I_1 is then simply $p_1 = l$, the second is the complete integral

$$
2\pi I_2 = l\oint_{\gamma_2} \frac{Q(\cos\varphi_2)}{P_1(\cos\varphi_2)}\mathrm{d}\varphi_2 + \oint_{\gamma_2} \frac{w}{P_1(\cos\varphi_2)}\mathrm{d}\varphi_2. \tag{11}
$$

With $l = I_1$ this is an implicit representation of the new Hamiltonian $h = h(I_1, I_2)$. We then require that the transformation $(\varphi_1, \varphi_2, p_1, p_2) \to (\phi_1, \phi_2, I_1, I_2)$ be canonical; this fixes the angle variables associated with the new momenta I_1, I_2. The corresponding generating function that achieves this is

$$F(\varphi_1, \varphi_2, I_1, I_2) = I_1\varphi_1 + \int_0^{\varphi_2} p_2(\varphi_2, I_1, I_2)\mathrm{d}\varphi_2. \tag{12}$$

From there, the new angles are obtained as

$$\phi_1 = \frac{\partial F}{\partial I_1} = \varphi_1 + \int_0^{\varphi_2} \frac{\partial p_2}{\partial I_1}\mathrm{d}\varphi_2 = \Omega_1 t, \tag{13}$$

$$\phi_2 = \frac{\partial F}{\partial I_2} = \int_0^{\varphi_2} \frac{\partial p_2}{\partial I_2}\mathrm{d}\varphi_2 = \Omega_2 t. \tag{14}$$

The constant frequencies Ω_i are given as $\Omega_i = \partial h/\partial I_i$. For $\Omega_2 \equiv 2\pi/T_2$ the identity

$$T_2 = 2\pi \left.\frac{\partial I_2}{\partial h}\right|_{I_1} \tag{15}$$

gives the complete form of the integral (7). To get the first period Ω_1, we use the relation

$$W = \frac{\Delta\varphi_1}{2\pi} = \frac{\Omega_1}{\Omega_2} = -\left.\frac{\partial I_2}{\partial I_1}\right|_h \tag{16}$$

and obtain for $\Delta\varphi_1$ the complete form of the integral (9).

2.2. The hyperelliptic nature of the problem. Let us now introduce the more convenient coordinate $x := \cos\varphi_2$, $\mathrm{d}x = -\sqrt{1-x^2}\,\mathrm{d}\varphi_2$, and the polynomial of degree 5

$$P_5(x) = 4\big(1-x^2\big)\Big(\frac{A}{\alpha^2} - x^2\Big)\Big(x - \frac{l^2}{4\alpha h} + \frac{A+1}{2\alpha}\Big) =: 4\prod_{i=1}^{5}(x - e_i), \tag{17}$$

which defines the hyperelliptic curve $V(z)$ of genus 2

$$V := \{z = (x, y) \in \mathbb{C}^2 : y^2 = P_5(x)\}. \tag{18}$$

Its branch points $(e_k, 0)$ all lie on the real x-axis; we call them e_k for short and arrange them in the order $e_1 \le e_2 \le \ldots \le e_5 < e_6 := \infty$. Two of them are ± 1; they are the boundaries of the physically accessible range $x^2 \equiv \cos^2\varphi_2 \le 1$. Two other roots of $P_5(x)$ are $\pm\sqrt{A}/\alpha$; they only depend on the parameters of the double pendulum and lie outside the physical range. The root

$$r := \frac{l^2}{4\alpha h} - \frac{A+1}{2\alpha} \tag{19}$$

depends on the angular momentum; its collisions with the fixed roots indicate bifurcations in the set of Liouville tori. At the maximum possible value of l^2, $l^2_{\max} := 2h(A+1+2\alpha)$, we have $r = 1$. With decreasing l^2, the root r moves towards the point -1 and reaches it at $l^2_{\mathrm{sep}} := 2h(A+1-2\alpha)$. This marks the bifurcation from oscillatory to rotational behavior of the angle φ_2. There is another collision of roots when $r = -\sqrt{A}/\alpha$; this happens at $l^2 = l^2_{\mathrm{res}} := 2h\big(\sqrt{A}-1\big)^2$ inside the rotational regime, but it does not involve a critical Liouville torus. Instead, it marks the resonance $W = 0$ between the two angular motions ($\Omega_1 = 0$). Finally, at $l^2 = 0$ we have $r = -(A+1)/2\alpha$. Summing up, there are three regimes with physical motion:

$$
\begin{aligned}
&e_1 = -\frac{\sqrt{A}}{\alpha}, && e_2 = -1, && e_3 = \ \ r, && (l^2_{\mathrm{sep}} < l^2 < l^2_{\max});\\
&e_1 = -\frac{\sqrt{A}}{\alpha}, && e_2 = \ \ r, && e_3 = -1, && (l^2_{\mathrm{res}} < l^2 < l^2_{\mathrm{sep}});\\
&e_1 = \ \ r, && e_2 = -\frac{\sqrt{A}}{\alpha}, && e_3 = -1, && (0 < l^2 < l^2_{\mathrm{res}}).
\end{aligned}
\tag{20}
$$

The roots $e_4 = 1$ and $e_5 = \sqrt{A}/\alpha$ are always the same. Writing the polynomial $P_5(x)$ as $\sum_{k=0}^{5} \lambda_k x^k$, we have the coefficients

$$
\begin{aligned}
\lambda_5 &= \ \ 4 & \lambda_4 &= \ \frac{2}{\alpha}\Big(A+1-\frac{l^2}{2h}\Big),\\
\lambda_1 &= \ \ \frac{4A}{\alpha^2}, & \lambda_0 &= \ \frac{\lambda_4\, A}{\alpha^2},\\
\lambda_3 &= -(\lambda_1+\lambda_5), & \lambda_2 &= -(\lambda_0+\lambda_4).
\end{aligned}
\tag{21}
$$

The Riemann surface of the curve V shall now be equipped with the homology basis $(\mathfrak{a}_1, \mathfrak{a}_2; \mathfrak{b}_1, \mathfrak{b}_2) \in H_1(V, \mathbb{Z})$ shown in Figure 2. The physical motion takes place in the range $e_3 \le x \le e_4$ which means that $\mathfrak{a}_2$ is the cycle of interest. As $e_4 = 1$ corresponds to the angle $\varphi_2 = 0$, we choose this point as the starting point of integration. Let us collect the relevant integrals:

$$t = \int_{e_4}^{x} (A - \alpha^2 x^2) \frac{\mathrm{d}x}{y} \qquad \text{(time)} \tag{22}$$

$$T_2 = \oint_{\mathfrak{a}_2} (A - \alpha^2 x^2) \frac{\mathrm{d}x}{y} \qquad \text{(period)} \tag{23}$$

$$2\pi W = -c + l \oint_{\mathfrak{a}_2} \frac{A - \alpha^2 x^2}{A + 1 + 2\alpha x} \frac{\mathrm{d}x}{y} \qquad \text{(winding number)} \tag{24}$$

$$2\pi I_2 = -2\pi W l \pm 2hT_2 \qquad \text{(action)} \tag{25}$$

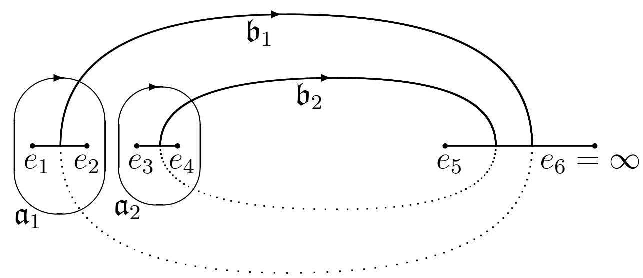

FIGURE 2. Homology basis on the Riemann surface of the curve $V(z)$ with real branch points $e_1 < e_2 < \ldots < e_6 = \infty$ (upper sheet). The cuts are drawn from e_{2i-1} to e_{2i}, $i = 1, 2, 3$. The $\mathfrak{b}$–cycles are completed on the lower sheet (dotted lines).

The constant c is determined with the theorem of residues:

$$c = 2\int_{-1}^{1} \frac{1+\alpha x}{A+1+2\alpha x}\frac{\mathrm{d}x}{\sqrt{1-x^2}} = \pi\Big(1 - \frac{A-1}{\sqrt{(A+1)^2-4\alpha^2}}\Big). \tag{26}$$

The difficult part of the problem is the inversion of the Abelian integral (22). In integrable systems where it has been solved (see, e. g., [**9**]), the number f of degrees of freedom coincides with the genus g of the Riemann surface which is shared by all $f = g$ coordinates. These coordinates $x_1, \ldots, x_g$ are confined to g mutually different branch cuts, and together, as a set, they are determined by the Jacobi inversion of the Abel mapping. In our case, like in many others that occur in physics, the genus of the hyperelliptic curve, $g = 2$, is larger than the number of the effective degrees of freedom, $f = 1$: the motion of φ_2 takes place along the branch cut $\mathfrak{a}_2$, and φ_1 is passively coupled to it. There is no dynamic role to the other real branch cuts. But then, how do we solve the inversion problem?

The answer involves the θ-divisor of the Jacobi variety.

3. Jacobi's inversion problem and the θ-divisor

Let $V(x, y) \in \mathbb{C}^2$ be a hyperelliptic Riemann surface of genus $g \geq 2$, and $z \equiv (x, y)$ a point on it. An Abelian integral

$$u = u(z) = \int_{z_0}^{z} R(x, y)\mathrm{d}x =: \int_{z_0}^{z} \mathrm{d}u, \tag{27}$$

where z_0 is any fixed reference point and $R(x,y)$ a rational function in x and y, cannot be considered a one-to-one map $V(z) \to \mathbb{C}(u)$ because its inverse would have to be a $2g$-periodic function on $\mathbb{C}$, and such functions do not exist (in contrast to the case $g=1$ where the doubly periodic elliptic functions are the inverse of Abel maps). Jacobi realized that the problem ought to be formulated in terms of a g-dimensional complex variety, namely, the *Jacobi variety* $\mathrm{J}(V)$ of the curve V.

3.1. The case of genus 2: preliminaries. As the double pendulum involves a curve of genus 2, we treat this case in detail. To start, we choose a basis of canonical holomorphic differentials $\mathrm{d}\boldsymbol{u}^t = (\mathrm{d}u_1, \mathrm{d}u_2)$ and associated meromorphic differentials of the second kind, $\mathrm{d}\boldsymbol{r}^t = (\mathrm{d}r_1, \mathrm{d}r_2)$, in such a way that their periods

$$2\omega_{ik} = \oint_{\mathfrak{a}_k} \mathrm{d}u_i, \qquad 2\omega'_{ik} = \oint_{\mathfrak{b}_k} \mathrm{d}u_i, \tag{28}$$

$$2\eta_{ik} = -\oint_{\mathfrak{a}_k} \mathrm{d}r_i, \qquad 2\eta'_{ik} = -\oint_{\mathfrak{b}_k} \mathrm{d}r_i, \tag{29}$$

satisfy the generalized Legendre relation

$$\begin{pmatrix} \omega & \omega' \\ \eta & \eta' \end{pmatrix} \begin{pmatrix} 0 & -1_2 \\ 1_2 & 0 \end{pmatrix} \begin{pmatrix} \omega & \omega' \\ \eta & \eta' \end{pmatrix}^t = -\frac{\mathrm{i}\pi}{2} \begin{pmatrix} 0 & -1_2 \\ 1_2 & 0 \end{pmatrix}, \tag{30}$$

where 1_2 is the 2×2 unit matrix. Such a set of differentials can be realized with (see [**3**])

$$\mathrm{d}u_1 = \frac{\mathrm{d}x}{y}, \quad \mathrm{d}u_2 = \frac{x\,\mathrm{d}x}{y}, \tag{31}$$

and

$$\mathrm{d}r_1 = \frac{\lambda_3 x + 2\lambda_4 x^2 + 12x^3}{4y}\mathrm{d}x, \quad \mathrm{d}r_2 = \frac{x^2}{y}\mathrm{d}x. \tag{32}$$

The periods $2\omega_{ik}$, $2\eta_{ik}$ are real, the periods $2\omega'_{ik}$, $2\eta'_{ik}$ imaginary. We shall also need the normalized holomorphic differentials

$$\mathrm{d}\boldsymbol{v} = (2\omega)^{-1}\mathrm{d}\boldsymbol{u}, \tag{33}$$

as well as the symmetric matrices of periods

$$\tau := \omega^{-1}\omega', \qquad \kappa := \eta(2\omega)^{-1}. \tag{34}$$

It is an important fact that τ is a *Riemann matrix*, i. e., $\tau\mathrm{i}$ is negative definite.

In the example of the double pendulum's time differential, we have from (22)

$$\mathrm{d}t = A\,\mathrm{d}u_1 - \alpha^2\,\mathrm{d}r_2, \tag{35}$$

which involves only two of the four basic differentials. But let us develop the solution procedure for a general differential of the first or second kind,

$$\mathrm{d}t = a\,\mathrm{d}u_1 + b\,\mathrm{d}u_2 + c\,\mathrm{d}r_1 + d\,\mathrm{d}r_2. \tag{36}$$

The Jacobi variety $\mathrm{J}(V)$ is a two-dimensional complex torus $\mathbb{C}^2/\Gamma$, where Γ is the lattice generated by the periods of the canonical holomorphic differentials; denote as $\widetilde{\mathrm{J}}(V)$ the complex torus $\mathbb{C}^2/\widetilde{\Gamma}$, where $\widetilde{\Gamma}$ is the lattice generated by the periods of the normalized holomorphic differentials. The *Abel maps* $\boldsymbol{u}: V \to \mathrm{J}(V)$ or $\boldsymbol{v}: V \to \widetilde{\mathrm{J}}(V)$, defined by

$$u_i(z) = \int_{z_0}^{z} \mathrm{d}u_i, \qquad v_i(z) = \int_{z_0}^{z} \mathrm{d}v_i, \qquad (i = 1, 2) \tag{37}$$

respectively, generate one-dimensional images of the Riemann surface V in the two-dimensional Jacobi variety. Obviously, with these maps, it would not make sense to look for preimages of *every* point in $\mathrm{J}(V)$ or $\widetilde{\mathrm{J}}(V)$. But if we consider the *Abel-Jacobi map* $\mathfrak{A}: S^2V \to \mathrm{J}(V)$ from the set of *pairs of points* $\{z_1, z_2\}$ to the Jacobi variety, defined by

$$\boldsymbol{u}(\{z_1, z_2\}) = \int_{z_0}^{z_1} \mathrm{d}\boldsymbol{u} + \int_{z_0}^{z_2} \mathrm{d}\boldsymbol{u} = \boldsymbol{u}(z_1) + \boldsymbol{u}(z_2) \tag{38}$$

(and similarly for the normalized version), then almost everywhere this map establishes a one-to-one correspondence between points $\{z_1, z_2\} \in \mathrm{S}^2V$ and $\boldsymbol{u} \in \mathrm{J}(V)$. This is Jacobi's setting for the inversion problem.

3.2. Theta functions. The key to its solution are *theta functions* [**13, 12**]. They come in two forms, and both are needed. First, the canonical θ-function $\theta(\boldsymbol{v}|\tau)$ is a map $\widetilde{\mathrm{J}}(V) \to \mathbb{C}$:

$$\theta(\boldsymbol{v}|\tau) = \sum_{\boldsymbol{m}\in\mathbb{Z}^2} \exp \mathrm{i}\pi\big\{\boldsymbol{m}^t\tau\boldsymbol{m} + 2\boldsymbol{v}^t\boldsymbol{m}\big\}. \tag{39}$$

As function on the universal covering $\mathbb{C}^2$ of $\widetilde{\mathrm{J}}(V)$, it is even, periodic in the real, or ω-directions, and "quasi-periodic" in the imaginary, or ω'-directions:

$$\theta(-\boldsymbol{v}|\tau) = \theta(\boldsymbol{v}|\tau), \tag{40}$$

$$\theta(\boldsymbol{v} + \boldsymbol{n}|\tau) = \theta(\boldsymbol{v}|\tau), \tag{41}$$

$$\theta(\boldsymbol{v} + \tau\boldsymbol{n}|\tau) = \mathrm{e}^{-\mathrm{i}\pi\boldsymbol{n}^t\tau\boldsymbol{n} - 2\mathrm{i}\pi\boldsymbol{v}^t\boldsymbol{n}}\theta(\boldsymbol{v}|\tau), \tag{42}$$

where $\boldsymbol{n}$ is any vector from $\mathbb{Z}^2$. The exponential multiplier in (42) makes $\theta(-\boldsymbol{v}|\tau)$ a multivalued function on $\widetilde{\mathrm{J}}(V)$.

Second, *Riemann's* $\theta_{\boldsymbol{e}}$-function $\theta_{\boldsymbol{e}}(\,\cdot\,|\tau):\ V \to \mathbb{C}$, is the composition of Abel's map, normalization, translation, and the canonical θ-function:

$$\theta_{\boldsymbol{e}}(z|\tau) = \theta\big((2\omega)^{-1}\boldsymbol{u}(z) - \boldsymbol{e}|\tau\big) = \theta\big(\boldsymbol{v}(z) - \boldsymbol{e}|\tau\big), \tag{43}$$

where $\boldsymbol{e}^t = (e_1, e_2) \in \widetilde{\mathrm{J}}(V)$ is an arbitrary fixed vector.

The zeros of these two θ-functions are of particular importance. Notice first that the set of zeros has four-fold periodicity in $\mathbb{C}^2$ because the multiplier in (42) is always non-zero. Clearly, the solutions of $\theta(\boldsymbol{v}|\tau) = 0$ form a set Θ_{zero} of complex co-dimension 1 in $\widetilde{\mathrm{J}}(V)$, called the θ-*divisor*. As to the zeros of the $\theta_{\boldsymbol{e}}$-function, they are precisely those $z \in V$ which the Abel map, followed by the shift $\boldsymbol{e}$, carries to Θ_{zero}. Riemann's *vanishing theorem (Nullstellensatz)* relates them to the solution of the inversion problem [**13, 12**]:

Theorem 3.1 (Riemann's vanishing theorem in the case of genus 2). *The function $\theta_{\boldsymbol{e}}(z|\tau)$ either vanishes identically on V or else has precisely $g = 2$ zeros. In the latter case, the zeros z_1, z_2 fulfill the identity*

$$\boldsymbol{v}(\{z_1, z_2\}) = \int_{z_0}^{z_1} \mathrm{d}\boldsymbol{v} + \int_{z_0}^{z_2} \mathrm{d}\boldsymbol{v} = \boldsymbol{e} + \boldsymbol{K}_{z_0}, \tag{44}$$

where $\boldsymbol{K}_{z_0} = (K_1, K_2)^t$ is the Riemann vector *associated with the base point z_0:*

$$\begin{aligned} K_1 &= \frac{1+\tau_{11}}{2} - \oint_{\mathfrak{a}_2} \mathrm{d}v_2(z) \int_{z_0}^{z} \mathrm{d}v_1, \\ K_2 &= \frac{1+\tau_{22}}{2} - \oint_{\mathfrak{a}_1} \mathrm{d}v_1(z) \int_{z_0}^{z} \mathrm{d}v_2. \end{aligned} \tag{45}$$

The Riemann vectors for two different base points z, z_0 are related by

$$\boldsymbol{K}_z = \boldsymbol{K}_{z_0} + \int_{z_0}^{z} \mathrm{d}\boldsymbol{v}. \tag{46}$$

For all $\boldsymbol{v} = \boldsymbol{K}_z$, the canonical θ-function vanishes, $\theta(\boldsymbol{K}_z|\tau) = 0$. Hence, the Riemann θ-function vanishes identically when $\boldsymbol{e}$ is chosen as $-\boldsymbol{K}_{z_0}$ because then

$$\theta_{-\boldsymbol{K}_{z_0}}\Big(\int_{z_0}^{z} \mathrm{d}\boldsymbol{v}|\tau\Big) = \theta\Big(\int_{z_0}^{z} \mathrm{d}\boldsymbol{v} + \boldsymbol{K}_{z_0}|\tau\Big) = \theta(\boldsymbol{K}_z|\tau) = 0. \tag{47}$$

In the usual applications of the theorem [**9**], the two preimages z_1, z_2 of a point $\boldsymbol{v} \in \widetilde{\mathrm{J}}(V)$ are obtained with $\boldsymbol{e} = \boldsymbol{v} - \boldsymbol{K}_{z_0}$, assuming that $\theta_{\boldsymbol{e}}$ does not vanish identically on V. Explicit solutions of the inversion problem will be given later, see (63). We shall be interested in the opposite case: the restriction of $\boldsymbol{v}$ to Θ_{zero}.

Before we address this problem, let us get familiar with the Riemann vectors. The formal definition (45) looks cumbersome, but Riemann developed a surprisingly simple characterization in explicit terms [**13**].

At this point it is convenient to introduce the *half-integer characteristics* $[\epsilon]$,

$$[\epsilon] = \begin{bmatrix} \boldsymbol{\epsilon}'^t \\ \boldsymbol{\epsilon}^t \end{bmatrix} = \begin{bmatrix} \epsilon_1' & \epsilon_2' \\ \epsilon_1 & \epsilon_2 \end{bmatrix}, \tag{48}$$

where ϵ_1, ϵ_2 and ϵ_1', ϵ_2' are taken from the set $\{0, \frac{1}{2}\}$. *Theta-functions with characteristics* are defined as

$$\theta[\epsilon](\boldsymbol{v}|\tau) = \sum_{\boldsymbol{m}\in\mathbb{Z}^2} \exp \mathrm{i}\pi\big\{(\boldsymbol{m}+\boldsymbol{\epsilon}')^t\tau(\boldsymbol{m}+\boldsymbol{\epsilon}')+2(\boldsymbol{v}+\boldsymbol{\epsilon})^t(\boldsymbol{m}+\boldsymbol{\epsilon}')\big\}. \tag{49}$$

Their importance lies in the fact that they are a convenient manner to describe θ-functions with shifted arguments. A simple relation holds between $\theta[\epsilon](\boldsymbol{v}|\tau)$ and the fundamental θ-function $\theta(\boldsymbol{v}|\tau) = \theta\left[\begin{smallmatrix}00\\00\end{smallmatrix}\right](\boldsymbol{v}|\tau)$:

$$\theta[\epsilon](\boldsymbol{v}|\tau) = \exp\big\{2\pi\mathrm{i}\boldsymbol{\epsilon}'^t(\boldsymbol{v}+\boldsymbol{\epsilon}+\tfrac{1}{2}\tau\boldsymbol{\epsilon}')\big\}\theta(\boldsymbol{v}+\boldsymbol{\epsilon}+\tau\boldsymbol{\epsilon}'|\tau). \tag{50}$$

It follows that a shift by a vector from the half-lattice,

$$\boldsymbol{v} \to \boldsymbol{v}+\boldsymbol{\epsilon}+\tau\boldsymbol{\epsilon}', \tag{51}$$

transforms the θ-functions in a simple way. In particular, if $\theta[\epsilon](\mathbf{0}|\tau)$ vanishes, then $\theta(\boldsymbol{\epsilon}+\tau\boldsymbol{\epsilon}') = 0$. It can be checked that under inversion $\boldsymbol{v} \to -\boldsymbol{v}$, all θ-functions with half-integer characteristics are either even or odd:

$$\theta[\epsilon](-\boldsymbol{v}|\tau) = \mathrm{e}^{-4\pi\mathrm{i}\boldsymbol{\epsilon}^t\boldsymbol{\epsilon}'}\theta[\epsilon](\boldsymbol{v}|\tau). \tag{52}$$

Among the 16 possible half-integer characteristics $[\epsilon]$, there are 6 for which $4\boldsymbol{\epsilon}^t\boldsymbol{\epsilon}' = 1$; these are the *odd characteristics*. For them it follows that $\theta[\epsilon](\mathbf{0}|\tau) = 0 = \theta(\boldsymbol{\epsilon}+\tau\boldsymbol{\epsilon}')$. The other 10 half-characteristics are called *even*; there $\theta[\epsilon](\mathbf{0}|\tau)$ and hence $\theta(\boldsymbol{\epsilon}+\tau\boldsymbol{\epsilon}')$ does not vanish.

Let us list the six points of the half-lattice in $\widetilde{\mathrm{J}}(V)$ where $\theta(\boldsymbol{v}|\tau)$ vanishes. They are

$$\begin{aligned} \boldsymbol{v}_1 &= \tfrac{1}{2}(0,1)^t + \tfrac{1}{2}\tau(0,1)^t, &\qquad \boldsymbol{v}_2 &= \tfrac{1}{2}(1,1)^t + \tfrac{1}{2}\tau(0,1)^t, \\ \boldsymbol{v}_3 &= \tfrac{1}{2}(1,1)^t + \tfrac{1}{2}\tau(1,0)^t, &\qquad \boldsymbol{v}_4 &= \tfrac{1}{2}(1,0)^t + \tfrac{1}{2}\tau(1,0)^t, \\ \boldsymbol{v}_5 &= \tfrac{1}{2}(1,0)^t + \tfrac{1}{2}\tau(1,1)^t, &\qquad \boldsymbol{v}_6 &= \tfrac{1}{2}(0,1)^t + \tfrac{1}{2}\tau(1,1)^t. \end{aligned} \tag{53}$$

By definition, they are part of the θ-divisor. A look at the homology basis of Fig. 2 and the definition of periods (28) shows that modulo lattice vectors, $\boldsymbol{v}_k - \boldsymbol{v}_j = \int_{e_j}^{e_k} \mathrm{d}\boldsymbol{v}$. This is the specialization of (46) to the branching points of our Riemann curve,

$$\boldsymbol{K}_{e_i} = \boldsymbol{v}_i. \tag{54}$$

More generally we may say that the θ-divisor is the set of all possible Riemann vectors.

Notice that modulo integer lattice vectors, we have

$$\boldsymbol{v}_1 + \boldsymbol{v}_3 + \boldsymbol{v}_5 = \boldsymbol{v}_2 + \boldsymbol{v}_4 + \boldsymbol{v}_6 = \mathbf{0}. \tag{55}$$

This reflects the fact that with the basis in Fig. 2, the cycle around the branch cut from e_5 to e_6 is homologous to $-(\mathfrak{a}_1 + \mathfrak{a}_2)$, while the cycle around e_2 and e_3, going back on the lower sheet, is homologous to $\mathfrak{b}_1 - \mathfrak{b}_2$. It follows, for example, that

$$\boldsymbol{K}_{e_6} = \boldsymbol{v}_6 = -\boldsymbol{v}_2 - \boldsymbol{v}_4 = -\int_{e_6}^{e_2} \mathrm{d}\boldsymbol{v} - \int_{e_6}^{e_4} \mathrm{d}\boldsymbol{v}, \tag{56}$$

but the signs do not matter because $2\boldsymbol{K}_{e_6}$ is a lattice vector.

3.3. Explicit solutions. How does this help solving the Jacobi inversion problem? Consider first the case where $\boldsymbol{v} = (2\omega)^{-1}\boldsymbol{u}$ does *not* lie on Θ_{zero}, and take, for example, e_6 as the base point for integration in (38). Then Riemann's theorem states that z_1, z_2 are the two zeros of

$$\theta_{\boldsymbol{e}}(z) = \theta\Big(\int_{e_6}^{z} \mathrm{d}\boldsymbol{v} - \boldsymbol{e}|\tau\Big) = \theta\Big(\int_{e_6}^{z} \mathrm{d}\boldsymbol{v} - \int_{e_2}^{z_1} \mathrm{d}\boldsymbol{v} - \int_{e_4}^{z_2} \mathrm{d}\boldsymbol{v}|\tau\Big) \tag{57}$$

because $\boldsymbol{e} = \int_{e_6}^{z_1} \mathrm{d}\boldsymbol{v} + \int_{e_6}^{z_2} \mathrm{d}\boldsymbol{v} - \boldsymbol{K}_{e_6} = \int_{e_2}^{z_1} \mathrm{d}\boldsymbol{v} + \int_{e_4}^{z_2} \mathrm{d}\boldsymbol{v}$. Let us check that $\theta_{\boldsymbol{e}}(z_1) = 0$:

$$\begin{aligned} \theta\Big(\int_{e_6}^{z_1} \mathrm{d}\boldsymbol{v} - \boldsymbol{e}|\tau\Big) &= \theta\Big(\int_{e_6}^{e_2} \mathrm{d}\boldsymbol{v} - \int_{e_4}^{z_2} \mathrm{d}\boldsymbol{v}|\tau\Big) \\ &= \theta\Big(-\boldsymbol{K}_{e_4} - \int_{e_4}^{z_2} \mathrm{d}\boldsymbol{v}|\tau\Big) = \theta\big(-\boldsymbol{K}_{z_2}|\tau\big) = 0. \end{aligned} \tag{58}$$

In a similar way, we find $\theta_{\boldsymbol{e}}(z_2) = 0$.

Let us absorb the Riemann vector in the definition of a *shifted Abel-Jacobi map*. Instead of (38) consider

$$\boldsymbol{u}(\{z_1, z_2\}) = \int_{e_6}^{z_1} \mathrm{d}\boldsymbol{u} + \int_{e_6}^{z_2} \mathrm{d}\boldsymbol{u} - 2\omega\boldsymbol{K}_{e_6} = \int_{e_2}^{z_1} \mathrm{d}\boldsymbol{u} + \int_{e_4}^{z_2} \mathrm{d}\boldsymbol{u}. \tag{59}$$

Then $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$ are the zeros of $\theta_{\boldsymbol{e}}(z)$ with $\boldsymbol{e} = (2\omega)^{-1}\boldsymbol{u}$. The theory of Abelian functions now tells us [**4, 6**] that this inversion problem is equivalent to finding the roots of the quadratic equation

$$x^2 - \wp_{22}(\boldsymbol{u})x - \wp_{12}(\boldsymbol{u}) = 0, \tag{60}$$

where $\wp_{ij}$ are second logarithmic derivatives of the fundamental σ-function,

$$\wp_{ij}(\boldsymbol{u}) = -\frac{\partial^2 \ln \sigma(\boldsymbol{u})}{\partial u_i \partial u_j} = \frac{\sigma_i \sigma_j - \sigma \sigma_{ij}}{\sigma^2}. \tag{61}$$

The function $\sigma(\boldsymbol{u})$ is closely related to the θ-function:

$$\sigma(\boldsymbol{u}) = C \exp\{\boldsymbol{u}^t \kappa \boldsymbol{u}\} \theta\big((2\omega)^{-1}\boldsymbol{u}|\tau\big), \tag{62}$$

and the indices i, j at σ mean corresponding derivatives with respect to u_i, u_j. The modulus $\kappa = \eta(2\omega)^{-1}$ was defined in (34); it contains the periods of the differentials of the second kind. The constant C can be given explicitly but does not matter here.

Solving equation (60) we find for the symmetric combinations of the two roots

$$\begin{aligned} x_1 + x_2 &= \wp_{22}(\boldsymbol{u}), \\ -x_1 x_2 &= \wp_{12}(\boldsymbol{u}). \end{aligned} \tag{63}$$

Furthermore, the corresponding y_k can be expressed as

$$y_k = \wp_{222} x_k + \wp_{122}, \qquad k = 1, 2. \tag{64}$$

This solves the inversion problem in explicit terms, as far as integrals of the first kind are concerned. (In the next section, integrals of the second kind will also be considered.)

In a typical physical situation where this analysis applies [**9**], the two points (x_1, y_1) and (x_2, y_2) on the Riemann surface lie in the regions $e_1 \le x_1 \le e_2$ and $e_3 \le x_2 \le e_4$, respectively, cf. figure 2. x_1 and x_2 represent two degrees of freedom of the system. Their time development is given by (63) because $\boldsymbol{u}$ varies linearly with t, as (36) shows (ignore the second order differentials for a moment).

But what if only one of the ranges (e_1, e_2) or (e_3, e_4) has physical meaning, as in the case of the double pendulum? Then we need a setting where there is a one-to-one correspondence between points on the Riemann surface and points in the Jacobi variety. This can be established by the requirement that $\boldsymbol{v} = (2\omega)^{-1}\boldsymbol{u}$ lie on the θ-divisor. The curve V and Θ_{zero} are isomorphic manifolds of complex dimension 1, so it ought to be possible to recover a *single* point z that maps to $\boldsymbol{u} \in \Theta_{\text{zero}}$ under the Abel map. Let us see how this can be done.

Notice first that Θ_{zero} is also the set of zeros of the σ-function. Therefore a naive application of (63) to points of Θ_{zero} would produce infinities for $x_1 + x_2$ and $x_1 x_2$. We must proceed with some care and consider their ratio. Take the shifted Abel-Jacobi map (59) and let

$x_2 \to e_6 = \infty$. Then (46) tells us that

$$\boldsymbol{u}(\{z,\infty\}) = \int_{e_2}^{z} \mathrm{d}\boldsymbol{u} + \int_{e_4}^{\infty} \mathrm{d}\boldsymbol{u} = \int_{e_2}^{z} \mathrm{d}\boldsymbol{u} + \boldsymbol{K}_{e_2} = \boldsymbol{K}_z \in \Theta_{\mathrm{zero}} \tag{65}$$

for any $z \in V$, i. e., the map $\boldsymbol{u}_0 : V \to \mathrm{J}(V)$ defined by

$$z \mapsto \boldsymbol{u}_0(z) = \boldsymbol{u}(\{z,\infty\}) = \boldsymbol{K}_z \in \Theta_{\mathrm{zero}} \subset \mathrm{J}(V), \tag{66}$$

takes a point z to its "home position" on Θ_{zero}. If we now approach the point $\boldsymbol{u}(z,\infty)$ as $\lim_{x_2\to\infty} \boldsymbol{u}(\{z,x_2\})$, we find with (63) that x may be consistently determined from

$$x = \lim_{x_2\to\infty} \frac{x x_2}{x + x_2} = \lim_{\sigma\to 0} \frac{\sigma\sigma_{12} - \sigma_1\sigma_2}{\sigma_2^2 - \sigma\sigma_{22}} = -\frac{\sigma_1}{\sigma_2}(\boldsymbol{u}_0). \tag{67}$$

This is the explicit inversion of the map (66). We remark that the equality (67) was obtained in [**14**].

The last piece in the puzzle is to find an expression for the time $t = \int \mathrm{d}t$ according to (36), with $\boldsymbol{u} \in \Theta_{\mathrm{zero}}$.

4. Solution for integrals of first and second kind

If $\mathrm{d}t$ were a holomorphic differential, $t = a\,u_1 + b\,u_2$, the solution of our inversion problem could proceed along the following lines. Start from the definition $\sigma(\boldsymbol{u}_0) = 0$ of the θ-divisor, and use it to express u_2 as a function of u_1. This gives $t = t(u_1)$, and $x = x(u_1)$ by Eq. (67). Invert the first relation to obtain $u_1 = u_1(t)$, hence $x = x(u_1(t))$. The bulk of the technical computations resides in the necessary manipulations of the σ-function and its derivatives.

With meromorphic differentials there is more work to do. We need to find σ-function expressions for the integrals $\int^z \mathrm{d}r_1$ and $\int^z \mathrm{d}r_2$. To this end, we employ a few more relations from the theory of hyperelliptic functions [**5, 6**]. In traditional notation, the first logarithmic derivative of σ is called a ζ-function,

$$\zeta_i(\boldsymbol{u}) = \frac{\partial \log \sigma(\boldsymbol{u})}{\partial u_i} = \frac{\sigma_i}{\sigma}(\boldsymbol{u}). \tag{68}$$

Then the relations

$$\int_{e_2}^{z_1} \mathrm{d}r_1 + \int_{e_4}^{z_2} \mathrm{d}r_1 = -\zeta_1(\boldsymbol{u}) + \frac{1}{2}\wp_{222}(\boldsymbol{u}), \tag{69}$$

$$\int_{e_2}^{z_1} \mathrm{d}r_2 + \int_{e_4}^{z_2} \mathrm{d}r_2 = -\zeta_2(\boldsymbol{u}) \tag{70}$$

hold [**5, 6**] and are used in the solution of the Jacobi inversion problem. The difficulty here is that we are interested in the limit $z_2 \to \infty$ and

$\sigma \to 0$, i. e., we must investigate the behavior of the σ-function and its derivatives as $\boldsymbol{u} \equiv \boldsymbol{u}(\{z_1, z_2\}) \to \boldsymbol{u}(\{z_1, \infty\}) \equiv \boldsymbol{u}_0$. We have

$$\boldsymbol{u} - \boldsymbol{u}_0 = \int_{e_6}^{z_2} \mathrm{d}\boldsymbol{u} \tag{71}$$

and consider the following Taylor expansion:

$$\begin{aligned} \zeta_i(\boldsymbol{u}) &= \frac{\sigma_i}{\sigma}\big(\boldsymbol{u}_0 + (\boldsymbol{u} - \boldsymbol{u}_0)\big) \\ &= \frac{\sigma_i(\boldsymbol{u}_0) + \sigma_{i1}(\boldsymbol{u}_0)(u_1 - u_{0,1}) + \sigma_{i2}(\boldsymbol{u}_0)(u_2 - u_{0,2}) + \dots}{\sigma(\boldsymbol{u}_0) + \sigma_1(\boldsymbol{u}_0)(u_1 - u_{0,1}) + \sigma_2(\boldsymbol{u}_0)(u_2 - u_{0,2}) + \dots}. \end{aligned} \tag{72}$$

We need expressions for $u_i - u_{0,i} = \int_{e_6}^{z_2} \mathrm{d}u_i$. Since $\infty = e_6$ is a branch point of our hyperelliptic curve, the appropriate local coordinate is ξ: $x = 1/\xi^2$. Inserting this into the definition of $\mathrm{d}u_i$, see (31), we find (using $e_1 + e_2 + e_3 + e_4 + e_5 = -\lambda_4/4$)

$$\int_{e_6}^{z_2} \mathrm{d}u_1 = -\frac{1}{3}\xi^3 + \frac{\lambda_4}{40}\xi^5 + O(\xi^7), \tag{73}$$

$$\int_{e_6}^{z_2} \mathrm{d}u_2 = -\xi + \frac{\lambda_4}{24}\xi^3 + O(\xi^5). \tag{74}$$

This is inserted into (72) and gives

$$\begin{aligned} \zeta_i(\boldsymbol{u}) &= \frac{\sigma_i(\boldsymbol{u}_0) - \xi\sigma_{i2}(\boldsymbol{u}_0) + O(\xi^3)}{-\xi\sigma_2(\boldsymbol{u}_0) + \frac{1}{2}\xi^2\sigma_{22}(\boldsymbol{u}_0) + O(\xi^3)} \\ &= -\frac{1}{\xi}\frac{\sigma_i}{\sigma_2} + \frac{\sigma_{i2}}{\sigma_2} - \frac{\sigma_i\sigma_{22}}{2\sigma_2^2} + O(\xi). \end{aligned} \tag{75}$$

In order to use this in the expressions (69) and (70), we must also determine the asymptotic behavior of the second kind integrals:

$$\int_{e_6}^{z_2} \mathrm{d}r_1 = \frac{1}{\xi^3} + \frac{\lambda_4}{8\xi} + O(\xi), \tag{76}$$

$$\int_{e_6}^{z_2} \mathrm{d}r_2 = \frac{1}{\xi} + \frac{\lambda_4}{8}\xi + O(\xi^3). \tag{77}$$

For $\int \mathrm{d}r_2$ we see immediately that the leading singularities $1/\xi$ cancel on the two sides of (70). Hence we have the result that

$$\int_{e_4}^{z_1} \mathrm{d}r_2 = -\frac{\sigma_{22}}{2\sigma_2}(\boldsymbol{u}_0) + c_2, \tag{78}$$

where

$$c_2 = \frac{\sigma_{22}}{2\sigma_2}(\boldsymbol{u}_{e_4}) \tag{79}$$

and $\boldsymbol{u}_{e_4} = (u_{e_4,1}, u_{e_4,2})^t = \boldsymbol{u}(\{e_4,\infty\}) = \int_{e_4}^{e_4} \mathrm{d}\boldsymbol{u} + \int_{e_2}^{e_6} \mathrm{d}\boldsymbol{u} = \int_{e_2}^{e_6} \mathrm{d}\boldsymbol{u}$.

The case of $\int \mathrm{d}r_1$ is more involved. As (69) shows, we must consider the expansion of $\wp_{222}$ as $\boldsymbol{u} \to \boldsymbol{u}_0$. The formal expansion first gives

$$\wp_{222} = \frac{2}{\xi^3} - \frac{2}{\xi}\Big(\frac{\sigma_1}{\sigma_2} - \frac{\lambda_4}{8}\Big) - \frac{\sigma_{2222}\sigma_2^2 - 2\sigma_{222}\sigma_{22}\sigma_2 + \sigma_{22}^3}{4\sigma_2^3} + O(\xi). \tag{80}$$

But the ξ-independent term can be shown to be zero! To do so, we employ one of the KdV-type equations which the $\wp_{ij}$ functions fulfill, namely [**5, 6**],

$$\wp_{2222} = 6\wp_{22}^2 + \frac{1}{2}\lambda_3 + \lambda_4\wp_{22} + 4\wp_{12}. \tag{81}$$

Considering its expansion in ξ, and comparing the leading terms of order ξ^{-2} and ξ^{-1} on both sides, we find

$$\begin{aligned} \sigma_{222} &= \frac{3}{4}\frac{\sigma_{22}^2}{\sigma_2} + \frac{\lambda_4}{4}\sigma_2 + \sigma_1, \\ \sigma_{2222} &= \frac{\sigma_{22}}{\sigma_2}\Big(\frac{\sigma_{22}^2}{2\sigma_2} + \frac{\lambda_4}{2}\sigma_2 + 2\sigma_1\Big) \end{aligned} \tag{82}$$

on the θ-divisor. We remark that formulae of the form (82) were derived and implemented in [**14, 21**]. Inserting them into (80) makes the constant term vanish.

Collecting all the terms involved in (69), we see that

$$\begin{aligned} \int_{e_4}^{z_1} \mathrm{d}r_1 &= -\frac{\sigma_{12}(\boldsymbol{u}_0)}{\sigma_2(\boldsymbol{u}_0)} + \frac{\sigma_1(\boldsymbol{u}_0)\sigma_{22}(\boldsymbol{u}_0)}{2\sigma_2(\boldsymbol{u}_0)^2} + c_1 \\ &= -\frac{\sigma_{12}(\boldsymbol{u}_0)}{\sigma_2(\boldsymbol{u}_0)} - x_1\frac{\sigma_{22}(\boldsymbol{u}_0)}{2\sigma_2(\boldsymbol{u}_0)} + c_1, \end{aligned} \tag{83}$$

where we used (67) for the last equality and

$$c_1 = \frac{\sigma_{12}(\boldsymbol{u}_{e_4})}{\sigma_2(\boldsymbol{u}_{e_4})} - \frac{\sigma_1(\boldsymbol{u}_{e_4})\sigma_{22}(\boldsymbol{u}_{e_4})}{2\sigma_2(\boldsymbol{u}_{e_4})^2}. \tag{84}$$

We can now return to the differential $\mathrm{d}t$ in (36) and outline a possible inversion procedure for its integral $t(x) = \int_{e_4}^{x} \mathrm{d}t$. We require that $\boldsymbol{u}_0 = (u_1, u_2)^t + \boldsymbol{u}_{e_4}$ lie on Θ_{zero}, i. e., we use its definition $\sigma(\boldsymbol{u}_0) = 0$ to express u_2 as a function of u_1. Then Eq. (67) allows us to express x as a function of u_1,

$$x = -\frac{\sigma_1}{\sigma_2}\big(\boldsymbol{u}_0\big) = -\frac{\sigma_1}{\sigma_2}\big(u_1 + u_{e_4,1}, u_2(u_1) + u_{e_4,2}\big) = x(u_1). \tag{85}$$

Next we express the time integral as a function of $\boldsymbol{u}_0$ on the θ-divisor,

$$
\begin{aligned}
t(x) &= \int_{e_4}^{z} \mathrm{d}t \\
&= a\,u_1 + b\,u_2 + c\left(-\frac{\sigma_{12}}{\sigma_2} - x_1\frac{\sigma_{22}}{2\sigma_2} + c_1\right) + d\left(-\frac{\sigma_{22}}{2\sigma_2} + c_2\right),
\end{aligned} \tag{86}
$$

where the constants c_1 and c_2 are given in (84) and (79). Using again $\sigma = 0$ to eliminate u_2, we get $t = t(u_1)$. The final step is to invert this relation and insert it into (85) to obtain $x = x(u_1(t))$.

5. Comments and outlook

We found that the θ-divisor in the Jacobi variety of the system's hyperelliptic curve can be used to solve the inversion problem, and we presented a procedure which we checked in detail by numerical computations. As far as the double pendulum at zero gravity is concerned, the analytic nature of its integration has thereby been clarified.

The analysis ought to be extended to encompass even more involved situations. For example, the time differential $\mathrm{d}t$ may have contributions of the third kind; this situation has not yet been considered. Another interesting generalization refers to higher genera. The work of Jorgensen [**16**] and Ônishi [**21**] contains hints as to how one would proceed. Consider a Riemann surface with genus $g = 3$ as an example. The number f of degrees of freedom may be 1, 2, or 3. The case $f = g$ is the standard situation where $\boldsymbol{u}$ moves linearly with time t (ignoring complications from differentials of second and third kind). The case $f = g - 1 = 2$ involves a restriction to a codimension 1 manifold. This may again be the theta divisor. With $\mathrm{d}\boldsymbol{u} = (\mathrm{d}u_1, \mathrm{d}u_2, \mathrm{d}u_3)$ and $\mathrm{d}u_k = x^{k-1}\mathrm{d}x/y$, the theta divisor Θ_{zero} is the two-dimensional manifold (z_0 considered as a fixed base point on V)

$$
\Theta^2 := \{(z_1, z_2) : \int_{z_0}^{z_1} \mathrm{d}\boldsymbol{u} + \int_{z_0}^{z_2} \mathrm{d}\boldsymbol{u} + \boldsymbol{K}_{z_0}\}. \tag{87}
$$

It is again characterized by $\sigma = 0$, and the inversion problem from $\boldsymbol{u} \in \Theta^2$ to $z_1, z_2 \in V$ is solved with $x_1 + x_2 = \sigma_1/\sigma_3$ and $x_1x_2 = -\sigma_2/\sigma_3$.

The case $f = g - 2 = 1$ requires a further restriction to the set

$$
\Theta^1 := \{z : \int_{z_0}^{z} \mathrm{d}\boldsymbol{u} + \boldsymbol{K}_{z_0}\} \tag{88}
$$

which is characterized by $\sigma = 0$ and $\sigma_3 = 0$. The inversion from $\boldsymbol{u} \in \Theta^1$ to $z \in V$ is solved with $x = -\sigma_1/\sigma_2$ as in (67).

The generalization to arbitrary genus is obvious, at least in principle. The sets $\Theta^0 := \{\boldsymbol{K}_{z_0}\}$ and

$$\Theta^f := \{(z_1, \ldots, z_f) : \int_{z_0}^{z_1} \mathrm{d}\boldsymbol{u} + \ldots + \int_{z_0}^{z_f} \mathrm{d}\boldsymbol{u} + \boldsymbol{K}_{z_0}\} \tag{89}$$

$(f = 1, \ldots, g)$ define a stratification

$$\Theta^0 \subset \Theta^1 \subset \ldots \Theta^{g-1} = \Theta_{\text{zero}} \subset \Theta^g = \mathrm{J}(V), \tag{90}$$

and problems with f degrees of freedom should be treated on the stratum Θ^f.

We are not aware of any previous attempt to put the θ-divisor to this use, let alone its generalizations Θ^f. Let us conclude with a remark on the computational aspects of this procedure. To identify the Jacobi variety, we need the periods ω_{ij} and ω'_{ij}, and if differentials of the second kind play a role, the periods η_{ij} and η'_{ij} as well. These periods can only be obtained by numerical integration. But given these periods, we can discuss the time development in terms of θ- and σ-functions for which there exist efficient computational procedures in the spirit of Weierstrass recursions [**7**] . Unfortunately, the picture is not as simple as in the usual case $f = g$ where the dynamics in the Jacobi variety is linear. The θ-divisor is a non-linear object, not simpler than the Riemann surface itself, hence its usefulness for practical purposes is limited.

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