Advanced Nonlinear Studies **14** (2014), 1–29

Classification of Global and Blow-up Sign-changing Solutions of a Semilinear Heat Equation in the Subcritical Fujita Range : Second-order Diffusion

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Received (March 2012) in revised form 24 November 2013
Communicated by Shair Ahmad

*This author was supported by the MIUR National Research Project *Quasilinear Elliptic Problems and Related Questions.*

Abstract

It is well known from the seminal paper by Fujita [22] for $1 < p < p_0$, and Hayakawa [36] for the critical case $p = p_0$, that all the solutions $u \geq 0$ of the semilinear heat equation

$$u_t = \Delta u + |u|^{p-1}u \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}_+, \quad \text{in the range } 1 < p \leq p_0 = 1 + \tfrac{2}{N}, \tag{0.1}$$

with arbitrary initial data $u_0(x) \geq 0, \not\equiv 0$, blow-up in finite time, while for $p > p_0$ there exists a class of sufficiently "small" global in time solutions. This fundamental result from the 1960-70s (see also [39] for related contributions), was a cornerstone of further active blow-up research. Nowadays, similar Fujita-type critical exponents p_0, as important characteristics of stability, unstability, and blow-up of solutions, have been calculated for various nonlinear PDEs. The above blow-up conclusion does not include solutions of changing sign, so some of them may remain global even for $p \leq p_0$. Our goal is a thorough description of blow-up and global in time oscillatory solutions in the subcritical range in (0.1) on the basis of various analytic methods including nonlinear capacity, variational, category, fibering, and invariant manifold techniques. Two countable sets of global solutions of changing sign are shown to exist. Most of them *are not* radially symmetric in any dimension $N \geq 2$ (previously, only radial such solutions in $\mathbb{R}^N$ or in the unit ball $B_1 \subset \mathbb{R}^N$ were mostly studied). A countable sequence of critical exponents, at which the whole set of global solutions changes its structure, is detected: $p_l = 1 + \frac{2}{N+l}$, $l = 0, 1, 2, \ldots$. See [47, 48] for earlier interesting contributions on sign changing solutions.

2010 AMS Subject Classification: 35J85, 49J40, 58E05.

Key words. Semilinear heat equations with source, global and blow-up sign-changing solutions, subcritical Fujita range, nonlinear capacity, variational theory, category, bifurcation branches.

1 Introduction

1.1 Semilinear heat equation: classics and new trends

The celebrated results by Fujita [22] for $1 < p < p_0$, and Hayakawa [36] for the critical case $p = p_0$ and $N = 1, 2$, has opened a new paradigm in the study of blow-up theories of different kinds of partial differential equations and systems.

In the last fifty years those results were later extended in several ways to various semilinear and quasilinear PDEs (see a list of monographs to be given below shortly). Fujita and Hayakawa were the first, who established existence of a very special *critical exponent*

$$p = p_0 = 1 + \tfrac{2}{N} \tag{1.1}$$

for the *semilinear heat equation*

$$u_t = \Delta u + |u|^{p-1}u \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}_+, \quad p > 1; \quad u(x,0) = u_0(x) \quad \text{in} \quad \mathbb{R}^N, \tag{1.2}$$

where data $u_0 \in L^1 \cap L^\infty$ are typically, and for simplicity, assumed to decay exponentially fast at infinity. Namely:

(I) *in the subcritical range*

$$1 < p \le p_0 = 1 + \tfrac{2}{N}, \tag{1.3}$$

all the nontrivial **nonnegative** *solutions* (for data $u_0 \ge 0$) *blow-up in finite time*, while

(II) *for $p > p_0$, there exists a class of "small" global nonnegative solutions.*

These two results, (I) and (II), serve as the definition of the first *critical Fujita exponent p_0* for the PDE (1.2). This fundamental result from the 1960-70s initiated a deep research of blow-up solutions and general stability issues for various nonlinear evolution PDEs. Up to now, similar Fujita-type critical exponents p_0, as important characteristics of stability, unstability of the origin $u = 0$, and blow-up of solutions, have been calculated for dozens of various semilinear, quasilinear, and even sometimes fully nonlinear PDEs of the second and higher-order parabolic, hyperbolic, and for Schrödinger-type equations and systems. Such blow-up results have been reflected in detail in a number of monographs devoted to blow-up theory for nonlinear evolution PDEs, embracing a wide range of various nonlinear evolution models. We list a few well-known monographs and contributions from the 1980s and later periods up to 2007, [3, 25, 32, 46, 47, 48, 49, 53, 56, 58], where the history, key references, and further extensions can be found.

The first blow-up conclusion (I) in the parameter range (1.3) does not include *solutions of changing sign*, so some of them may remain global. For sign changing solutions, proving blow-up in the range (1.3) can be rather tricky, as the recent results in [7]–[11] and [2] show. Actually, these papers attracted our attention to the problem of existence of global solutions of (1.2) in the subcritical range (1.3), though we are going to use different approaches to this problem. In particular, we plan to attack the Cauchy problem not in a ball or in a bounded domain as in the most of previous papers.

The Cauchy problem in $\mathbb{R}^N \times \mathbb{R}_+$ leads to different rescaled equations and requires different variational and invariant manifold techniques to detect the corresponding countable families of sign-changing global solutions.

Overall, it is not an exaggeration to say that the recent new results in [7]–[11], [2], (see also [47, 48]) and [52]) on oscillatory solutions revived an extra new interest to blow-up/global phenomena in the classic semilinear models from 1960s after more than forty years of very intensive research in this important nonlinear PDE area. However, nowadays, these require more powerful techniques to describe structures of such blow-up and global solutions of changing sign, which was not done before, and, especially, in the Cauchy problem.

Intuitively, it is clear that, if a sign changing solution of (1.2) contains an essentially "dominant" positive (or negative) part, then such a solution must blow-up in finite time, since the remaining sufficiently small negative (positive) part would play no role as $t \to T^-$. However, a full proof of such a result by standard blow-up approaches is indeed difficult. The main problem of concern is to describe the precise balance between the negative and positive parts of the solutions under consideration that prevents blow-up. This inevitably generates the question of a description of *global solutions* in the range in (1.3), as a natural complement of the study of blow-up solutions.

1.2 Layout for the semilinear heat equation

Our paper is organized as follows. Sections 2–4 are devoted to a classification of global solutions, where we construct two countable families of global self-similar solutions of changing sign. It turns out that most of them *are not* radially symmetric in any dimensions $N \geq 2$. It is worth mentioning that almost all of previous studies of such parabolic blow-up problems were dealt with radial solutions either in $\mathbb{R}^N$ or in a ball $B_1 \subset \mathbb{R}^N$; see [7, 8, 11, 35, 60, 61].

Here the study is done by various analytic methods including variational, category, and fibering approaches. In particular, a countable sequence of critical values is detected:

$$p_l = 1 + \tfrac{2}{N+l}, \quad l = 0, 1, 2, \dots, \tag{1.4}$$

so that Fujita's exponent p_0 is just the first one in this sequence. See the earlier interesting contributions [47, 48] on sign-changing solutions where 1.4 play a crucial role.

These critical exponents occur while using classical Hermitian spectral theory. Indeed, where, locally, the rescaled spatial structure of sign changing solutions well corresponds to classic arbitrary Hermite polynomials in $\mathbb{R}^N$.

In Section 5, we present a linearization technique associated with invariant manifold theory for revealing another countable subset of global patterns, which are not self-similar and are called *linearized* ones. It turns out that, at the critical values (1.4), the overall structure of the whole set of global solutions essentially changes and some global patterns of these two families are interchanged.

1.3 Towards extensions to higher-order reaction-diffusion models

Such extensions to higher-order models are principal in modern nonlinear PDE theory, so we now briefly stress attention on our future work. Namely, in [30], we show that some analogous (but somehow weaker) nonlinear phenomena can be revealed for the $2m$th-order semilinear heat equation for $m \geq 2$,

$$u_t = -(-\Delta)^m u + |u|^p \quad \text{in} \quad \mathbb{R}^N \times \mathbb{R}_+, \quad \text{in the range } \ 1 < p \leq p_0 = 1 + \tfrac{2m}{N}. \tag{1.5}$$

Then blow-up occurs [15] for any solutions with initial data having positive first Fourier coefficient (see [16] for further details and [31] for an alternative proof):

$$\int_{\mathbb{R}^N} u_0(x)\, \mathrm{d}x > 0, \tag{1.6}$$

i.e., again arbitrarily small data lead to blow-up. In [30], construction of countable sets of global sign changing solutions is performed on the basis of bifurcation/branching analysis as well as of a centre-stable manifold one[1]. Here, we apply spectral theory of related non-self-adjoint $2m$th-order operators in [16], which is available for any $m = 2, 3, \dots$. This gives a similar sequence of critical exponents:

$$p_l = 1 + \tfrac{2m}{N+l}, \quad l = 0, 1, 2, \dots. \tag{1.7}$$

[1] For any $m \geq 2$, non-variational problems occur, so that category/fibering theory is useless here.

References and results for analogous global similarity solutions of a different higher-order reaction-diffusion PDEs with a monotone nonlinearity

$$u_t = -(-\Delta)^m u + |u|^{p-1} u \quad (m \geq 2) \tag{1.8}$$

can be found in [27]. It is remarkable (and rather surprising for us) that the bifurcation-branching phenomena therein for (1.8) are entirely different from those for the present equation (1.5). Nevertheless, some approaches developed in [27] will be applied for (1.5) in [30].

2 Countable set of p-branches of global self-similar solutions of (1.2): preliminaries and general strategy of research

2.1 Global similarity solutions

This is the first family of *global* solutions of (1.2), which we are looking at. Namely, in the range (1.3), we consider the standard self-similar solutions defined for all $t > 0$:

$$u_S(x,t) = t^{-\frac{1}{p-1}} f(y), \quad y = \tfrac{x}{\sqrt{t}}, \tag{2.9}$$

where f solves a semilinear elliptic equation of the form

$$\begin{cases} \mathbf{A}(f) \equiv \Delta f + \frac{1}{2}\, y \cdot \nabla f + \frac{1}{p-1}\, f + |f|^{p-1} f = 0 \quad \text{in} \quad \mathbb{R}^N, \\ f(y) \quad \text{has exponential decay at infinity.} \end{cases} \tag{2.10}$$

The self-similar solutions of the type (2.9) have been well known from the 1980s, [35]; see [56, Ch. 4] for further history and results. Already in 1985, Weissler [60, 61] proved existence of a countable set of *radially symmetric* similarity profiles $\{f_l\}$ satisfying the corresponding ODE for any $p > 1$.

However, we plan to describe a much wider (and, hopefully, a *whole*) set of *non-radial* global patterns (2.9), so we will need further developments. Let us explain the main ingredients of our bifurcation and variational analysis of non-radial similarity profiles.

2.2 Spectral properties of a self-adjoint operator and bifurcations

We write the elliptic equation in (2.10) as follows:

$$\mathbf{A}(f) \equiv \mathbf{B} f + c_1 f + |f|^{p-1} f = 0, \quad \text{where} \quad c_1 = \tfrac{1}{p-1} - \tfrac{N}{2} = \tfrac{N(p_0 - p)}{2(p-1)} \tag{2.11}$$

and $\mathbf{B}$ is the classic linear *Hermite operator*

$$\mathbf{B} f = \Delta f + \tfrac{1}{2}\, y \cdot \nabla f + \tfrac{N}{2}\, f. \tag{2.12}$$

This can be written in the symmetric form

$$\mathbf{B} f \equiv \tfrac{1}{\rho} \nabla \cdot (\rho \nabla f) + \tfrac{N}{2}\, f, \quad \text{where} \quad \rho(y) = \mathrm{e}^{\frac{1}{4}|y|^2}. \tag{2.13}$$

B is then self-adjoint in the weighted space $L^2_\rho(\mathbb{R}^N)$; see Birman–Solomjak [6, p. 48]. The spectrum of **B** is discrete,

$$\sigma(\mathbf{B}) = \{\lambda_l = -\tfrac{l}{2}, \quad l = |\beta| = 0, 1, 2, ...\}, \quad \text{where} \quad \beta \ \text{ is a multiindex in } \mathbb{R}^N, \tag{2.14}$$

and a complete and closed set of eigenfunctions $\Phi = \{\psi_\beta\}$ is associated with the Hermite polynomials $\{H_\beta\}$ as follows. Denoting by

$$F(y) = (4\pi)^{-\frac{N}{2}} \mathrm{e}^{-\frac{1}{4}|y|^2} \quad (\text{then} \quad \mathbf{B}F = 0) \tag{2.15}$$

the rescaled profile of the fundamental solution of the linear operator $D_t - \Delta$, we have the following well-known *generating formula* of eigenfunctions:

$$\psi_\beta(y) = \tfrac{(-1)^{|\beta|}}{\sqrt{\beta!}} D^\beta F(y) \equiv H_\beta(y)\, \mathrm{e}^{-\frac{1}{4}|y|^2}, \quad \text{with any } \ |\beta| = l = 0, 1, 2, ... , \tag{2.16}$$

where $\beta = (\beta_1, ..., \beta_N)$, $|\beta| = \beta_1 + ... + \beta_N$. Note that Hermite polynomials

$$\psi^*_\beta(y) = b_\beta H_\beta(y) \quad (b_\beta \ \text{ are normalization factors})$$

are then eigenfunctions of the adjoint operator in the usual dual metric of $L^2(\mathbb{R}^N)$:

$$\mathbf{B}^* = \Delta - \tfrac{1}{2}\, y \cdot \nabla;$$

see [18]. Later on, $\langle \cdot, \cdot \rangle$ will denote the L^2-metric, while $\langle \cdot, \cdot \rangle_\rho$ the weighted L^2_ρ-one. Therefore, the orthonormality condition reads: for any σ, γ,

$$\langle \psi_\beta, \psi^*_\sigma \rangle \equiv \langle \psi_\beta, \psi_\sigma \rangle_\rho = \delta_{\beta\sigma}, \quad \text{where} \quad \delta_{\beta\sigma} \ \text{ is Kronecker's delta.} \tag{2.17}$$

2.3 Bifurcations, p-branches, and local structure of solutions

Bifurcation results are next applied to revealing all possible similarity profiles of the problem. In Section 3, we apply classic bifurcation theory [40, 41, 59] to the equation (2.11) looking for small solutions bifurcating from the origin $f = 0$. It follows from (2.11) that the only possible bifurcation points may occur if $\mathbf{B} + c_1 I$ has a non-trivial kernel, i.e., by (2.14), when

$$c_1 = -\lambda_l \quad \Longrightarrow \quad p = p_l = 1 + \tfrac{2}{N+l}, \ \ l = 0, 1, 2, ... , \tag{2.18}$$

so this is how the all the critical exponents (1.4) are revealed. Since **A** is a potential operator, each $p = p_l$ becomes an actual bifurcation point of global similarity profiles f_l, [40, p. 332]. For any $l = 0, 1, 2, ...$, by $\Phi_l = \{f_\sigma\}_{|\sigma|=l}$, with a suitable multiindex σ, we denote the corresponding finite subset of different (up to scalings and rotations or other standard orthogonal changes in $\mathbb{R}^N$) patterns corresponding to the eigenvalue $\lambda_l = -\frac{l}{2}$.

Finally, we rely on the classic fact [14, p. 412] that, for odd higher-order nonlinear perturbations as in (2.11), the number of different bifurcation branches originated at $p = p_l$ from $f = 0$ is not less than the geometric multiplicity of the corresponding eigenvalue $\lambda = \lambda_l$. This defines the overall multiplicity of bifurcation branches. As usual in branching theory (see e.g., a general characterization in [59, p. 329]), the critical points (functions), from which the branches of global similarity profiles are originated at $p = p_l$, are detected from the nonlinear algebraic Lyapunov–Schmidt equation, to be discussed as well on the basis of category theory.

2.4 Variational setting: global p-branches

As a next step, we use the fact that (2.10) admits a variational setting since $\mathbf{A}$ is a Frechét derivative of the following functional:

$$G(f) = -\frac{1}{2}\int \rho|\nabla f|^2 + \frac{1}{2(p-1)}\int \rho f^2 + \frac{1}{p+1}\int \rho|f|^{p+1} \quad \text{in} \quad H^1_\rho(\mathbb{R}^N) \cap L^{p+1}_\rho(\mathbb{R}^N). \tag{2.19}$$

Variational approaches for functionals in weighted spaces of functions in $\mathbb{R}^N$ go back to Kurtz [42]. Later, Weissler [61] applied the variational approach to the elliptic equation (2.10) establishing existence of a countable sequence of similarity patterns (cf. also Escobedo–Kavian [18] for an analogous elliptic problem with absorption). However, the author in [61] made a comment that he did not know whether this variational countable sequence coincided with that obtained simultaneously in [60] by ODE methods for radially symmetric solutions. We show that this is not the case and the variational/fibering family of solutions of (2.10) is uncomparably wider than the ODE (radial) one.

In Section 4, we use the ideas of the fibering method [51] based on Lusternik–Schnirel'man (L–S) category theory of calculus of variations [41] to show that there exists a countable family of global p-bifurcation branches originated at the critical exponents $p = p_l$, (1.4).

Thus, we begin our study of global solutions of (1.2) with bifurcation theory for (2.10).

3 Pitchfork p-bifurcations

3.1 Pitchfork bifurcations at $p = p_l$: local existence of global similarity profiles

We first present a rigorous justification of bifurcation points given in (2.18).

Proposition 1 *For any* $l = 0, 1, 2, 3, ...,$ *the critical exponent in* (2.18) *is a p-bifurcation point for the problem* (2.10).

Proof. This result is standard in elliptic operator theory, where, dealing with the nonlinearity $|f|^{p-1}f$, in order to have a compact embedding of functional spaces involved, one should take into account that (i) p must be always less than the Sobolev critical exponent:

$$1 < p < p_S = \tfrac{N+2}{N-2}, \tag{3.20}$$

and also that (ii) the domain $\mathbb{R}^N$ is unbounded. Since the weight $\rho(y)$ is exponentially growing as $y \to \infty$, the necessary compact embedding holds (see [45, p. 54, 63] and [18]):

$$H^1_\rho(\mathbb{R}^N) \subset L^{p+1}_\rho(\mathbb{R}^N) \quad \text{compactly.} \tag{3.21}$$

Thus, consider in L^2_ρ the equivalent equation

$$\hat{\mathbf{B}} f = -(1 + c_1)f - |f|^{p-1}f, \quad \text{where} \quad \hat{\mathbf{B}} = \mathbf{B} - I. \tag{3.22}$$

The spectrum of $\hat{\mathbf{B}}$ is a translation of that of $\mathbf{B}$, $\sigma(\hat{\mathbf{B}}) = \{-1 - \frac{l}{2}\}$, and consists of strictly negative eigenvalues. The inverse integral operator $\hat{\mathbf{B}}^{-1}$ is known to be compact in L^2_ρ (Theorem 2.1 (iii) in [16]). Therefore, in the corresponding integral equation

$$f = \hat{\mathbf{A}}(f) \equiv -(1+c_1)\hat{\mathbf{B}}^{-1} f - \hat{\mathbf{B}}^{-1}|f|^{p-1}f, \tag{3.23}$$

the right-hand side contains a compact Hammerstein operator in an $L^q_\rho(\mathbb{R}^N)$ space for some $q \geq 1$ [40, p. 38] (see details on the resolvent of $\mathbf{B}$ in [5]). Bifurcations in the truncated problem (3.23) are always guaranteed if the derivative $\hat{\mathbf{A}}'(0) = -(1+c_1)\hat{\mathbf{B}}^{-1}$ has the eigenvalue 1 of an odd multiplicity; see [40, p. 196]. Moreover, for potential operators, any characteristic value, regardless its multiplicity, is a bifurcation point; see [40, p. 332]. As an alternative, we can use Ladyzhenskii's theorem [40, p. 34] establishing compactness in C. For exponentially decaying kernels of $\hat{\mathbf{B}}^{-1}$, the case of unbounded space $\mathbb{R}^N$ can be settled by approximation via a converging in the norm sequence of compact operators in expanding bounded domains. Note that some of the compactness conditions in [40, Ch. 1] are directly applied to arbitrary unbounded domains in the integral operators.

Compactness of the integral operators involved with exponential kernels and weights in $\mathbb{R}^N$ may cause some technical difficulties, especially in the higher-order cases with $m \geq 2$ [30]. Therefore, it is more convenient to use results without compactness assumptions as in [14, p. 412], which we will rely on later.

Thus, since $\sigma(\hat{\mathbf{A}}'(0)) = \{(1+c_1)/(1+\frac{l}{2})\}$, we arrive at the critical values (2.18). By construction, the solutions of (3.23) for $p \approx p_l$ are small in L^2_ρ and, as is seen from the properties of the inverse operator, in H^2_ρ. Since the weight $\rho(y)$ is a monotone exponentially growing function as $|y| \to \infty$, this implies that $f \in H^2_\rho$ is a uniformly bounded, continuous function by standard elliptic regularity and embedding results [45] (as we have mentioned, for compactness in C, this is not necessary). Therefore, for $p \approx p_l$, we have bounded, small solutions only. ■

3.2 Simple eigenvalues

We begin with the case of simple eigenvalues, where the calculus are rather straightforward. Actually, in the elliptic setting, this happens for $l = 0$ only, but, nevertheless, we perform the analysis for any $l \geq 0$ bearing in mind some possible restrictions on the geometry of eigenfunctions (e.g, this happens for any $l = 0, 1, 2, \ldots$ in the radial ODE setting). Since the nonlinear perturbation term in the integral equation (3.23) is an odd sufficiently smooth operator, we arrive at the following result describing the local behaviour of bifurcation branches; see [40] and [41, Ch. 8].

Proposition 2 *Let λ_l be a simple eigenvalue of* $\mathbf{B}$ *with a given eigenfunction ψ_l. Then the problem* (2.10) *has precisely two small solutions for $p \approx p_l^+$, so $p = p_l$ is a supercritical pitchfork bifurcation.*

Observe that the corresponding coefficient of the vector field,

$$\kappa_l = \langle |\psi_l|^{p-1}\psi_l, \psi_l \rangle_\rho \equiv \int \rho |\psi_l|^{p+1} > 0 \quad (p = p_l), \tag{3.24}$$

is strictly positive, so that the bifurcation is always supercritical; see calculus below.

Proof. Thus, we perform the corresponding calculation assuming that λ_l is simple. In order to describe the asymptotics of solutions as $p \to p_l$, we apply the Lyapunov–Schmidt method [41, Ch. 8] to equation (3.23) with the operator $\hat{\mathbf{A}}$ being differentiable at 0. Since, under the assumptions of Proposition 2, the kernel $E_0 = \ker \hat{\mathbf{A}}'(0) = \mathrm{Span}\,\{\psi_l\}$ is one-dimensional, denoting by E_1 the complementary (orthogonal to ψ_l) invariant subspace, we set

$$f = F_0 + F_1, \quad \text{where } F_0 = \varepsilon_l \psi_l \in E_0 \text{ and } F_1 = \textstyle\sum_{(k \neq l)} \varepsilon_k \psi_k \in E_1. \tag{3.25}$$

Let P_0 and P_1, $P_0 + P_1 = I$, be projections onto E_0 and E_1 respectively. Projecting (3.23) onto E_0 yields

$$\gamma_l \varepsilon_l = -\langle \hat{\mathbf{B}}^{-1}(|f|^{p-1} f), \psi_l \rangle_\rho, \quad \text{with} \quad \gamma_l = 1 - \tfrac{1+c_1}{1+\frac{l}{2}} = \tfrac{2(N+l)s}{2(p-1)(2+l)}, \tag{3.26}$$

where $s = p - p_l$. By general bifurcation theory (see e.g. [41, p. 355] and [14, p. 383]; note that operator $\hat{\mathbf{A}}'(0)$ is Fredholm of index zero), the equation for F_1 can be solved and this gives $F_1 = o(\varepsilon_l)$ as $\varepsilon_l \to 0$, so that ε_l is calculated from the Lyapunov bifurcation equation (3.26) as follows:

$$\gamma_l \varepsilon_l = -\varepsilon_l^p \langle \hat{\mathbf{B}}^{-1}(|\psi_l|^{p-1}\psi_l), \psi_l \rangle_\rho + o(\varepsilon_l^p) \implies \boxed{|\varepsilon_l|^{p-1} = \hat{c}_l[(p - p_l) + o(1)],} \tag{3.27}$$

where $\hat{c}_l = \frac{(l+N)^2}{4\kappa_l} > 0$. We have performed these calculations as follows:

$$\langle \hat{\mathbf{B}}^{-1}(|\psi_l|^{p-1}\psi_l), \psi_l \rangle_\rho = \langle |\psi_l|^{p-1}\psi_l, \hat{\mathbf{B}}^{-1}\psi_l \rangle_\rho = -\tfrac{\kappa_l}{1+\frac{l}{2}}.$$

We have shown in (3.24) that $\kappa_l > 0$. Note that, in view of the orthonormality (2.17) of the eigenfunction set $\{\psi_l\}$, for $p = 1$, we have $\kappa_l = 1$. Indeed, the algebraic equation in (3.27) implies a typical and standard pitchfork bifurcation structure of branches at $p = p_l^+$. ■

Thus, we obtain a countable sequence of bifurcation points (2.18) satisfying $p_l \to 1^+$ as $l \to \infty$, with typical supercritical pitchfork bifurcation branches appearing in a right-hand neighbourhood of $p = p_l$, i.e., for $p > p_l$. The behaviour of solutions in H^2_ρ (and uniformly in $\mathbb{R}^N$ by a standard elliptic regularity) takes the form

$$f_l(y) = \pm \left[\hat{c}_l(p - p_l)\right]^{\frac{1}{p-1}} (\psi_l(y) + o(1)) \quad \text{as } \; p \to p_l^+. \tag{3.28}$$

A rather slow rate of such p-bifurcation of the first similarity profile $f_0(y)$ for $N = 1$ as $p \to p_0 = 3^+$ is illustrated by Figure 1, where

$$f_0(y) = \pm \sqrt{\hat{c}_0(p - p_0)}\,(F(y) + o(1)) \quad \text{(with the Gaussian (2.15)).}$$

3.3 Instability of p-branches

A linearized analysis (see some details in [26, 27]) shows that the first p_0-branch is unstable for $p - p_0 > 0$ small in the sense of the rescaled parabolic equation

$$f_\tau = \mathbf{A}(f) \quad \text{for} \quad \tau > 0. \tag{3.29}$$

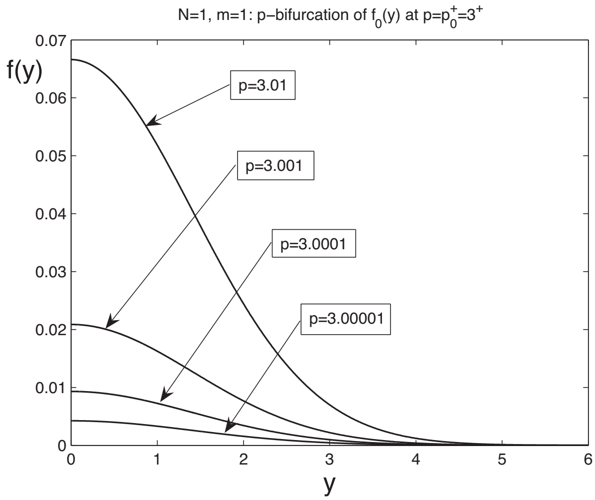


Figure 1: Bifurcation of $f_0(y)$ as $p \to 3^+$ for $N = 1$.

Indeed, by spectral theory, for $p > p_0$, the zero equilibrium $f = 0$ becomes stable, emphasizing existence of global small solutions in this supercritical range. For $p < p_0$, instability of p-branches can be associated with existence of a huge amount of blow-up solutions of (3.29). These results are connected with the spectrum of the linearized operator in (3.29)

$$\mathbf{D}_0 = \mathbf{A}'(f_0) = \mathbf{B}_1 + p|f_0|^{p-1}I, \tag{3.30}$$

which can be sharply estimated for p close to p_0 by the above asymptotic expansion techniques.

In general, studying existence/nonexistence of turning points of such global p-branches is a difficult problem. For a class of variational problems, nonexistence of turning points and hence monotonicity of the branches are known [57], which also holds for ordinary differential higher-order equations with self-adjoint positive operators of special structure of quasi-derivatives [1, 55]. We justify the existence of global continuous branches in the next section. Similarly, all other bifurcation p_l-branches for any $l \geq 1$ are shown to be *unstable* with respect to the rescaled evolution via (3.29).

3.4 Lyapunov–Schmidt branching equation in the general multiple case: non-radial patterns

Let now $\lambda_l = -\frac{l}{2}$ have multiplicity $m = m(l) > 1$ given by the binomial coefficient

$$m(l) = \dim W^c(\mathbf{B} - \lambda_l I) = C^l_{N+l-1} = \tfrac{(N+l-1)!}{l!(N-1)!}, \quad \text{so that} \tag{3.31}$$

$$E_0 = \ker(\mathbf{B} - \lambda_l I) = \text{Span}\{\psi_{l1}, ..., \psi_{lm}\}. \tag{3.32}$$

Then, similar to calculations associated with (3.25) and (3.26), looking for a solution

$$f = f_0 + f_1, \quad \text{with} \quad f_0 = a_1\psi_{l1} + ... + a_m\psi_{lm}, \quad \text{where} \quad f_1 \bot E_0, \tag{3.33}$$

and substituting into the equation (3.23), multiplying by ψ_{li}, and denoting, as usual, $s = p - p_l \approx 0^+$, we obtain the following *generating* system of m algebraic equations:

$$a_i = \tfrac{4}{s(N+l)^2} \int \rho |a_1\psi_{l1} + ... + a_m\psi_{lm}|^{p-1}(a_1\psi_{l1} + ... + a_m\psi_{lm})\psi_{li} \equiv D_i \tag{3.34}$$

for $i = 1, 2, ..., m$. Here $p = p_l$. Denoting $x = (a_1, ..., a_m)^T \in \mathbb{R}^m$, the system (3.34) is written as a fixed point problem for the given nonlinear operator $\mathbf{D} = (D_1(x), ..., D_m(x))^T$,

$$x = \mathbf{D}(x) \quad \text{in} \quad \mathbb{R}^m. \tag{3.35}$$

Above, we have studied the scalar case $m = 1$, which always gave a single (up to "±") non-trivial solution. Obviously, there are several other *one-dimensional* settings that lead to a scalar equation for a single coefficient a in (3.34). However, since (3.35) admits an obvious variational setting for the corresponding functional

$$H(x) = \tfrac{1}{p+1}\, \tfrac{4}{s(N+l)^2} \int \rho |a_1\psi_{l1} + ... + a_m\psi_{lm}|^{p+1} - \frac{1}{2}\,(a_1^2 + ... + a_m^2), \tag{3.36}$$

so that $H'(x) = \mathbf{D}(x) - x$.

We have the following:

Proposition 3 *The problem* (3.35) *has at least m distinct nontrivial solutions.*

Proof. This follows by applying to the functional (3.36) the fibering method and using the fact that the unit sphere in $\mathbb{R}^m$ has the category m, so (3.36) has at least m distinct critical values and points; see Section 4 for details on L–S category/genus theory. ■

Of course, this result well corresponds to classic variational bifurcation theory [14, p. 412], so that, at $p = p_l$, there occur at least $m = m(l)$ distinct pairs (with different signs, ±) of branches, where m is the geometric multiplicity (the maximal number of linear independent eigenvectors in $\ker(\mathbf{B} - \lambda_l I)$) of the eigenvalue $\lambda_l = -\frac{l}{2}$. Note that some of these patterns may be identical up to scaling and orthogonal (rotational) invariance.

3.5 Toward a classification of patterns and their nodal sets in $\mathbb{R}^2$

It follows from (3.33) that, locally, for $p \approx p_l$, the similarity profiles $f(y)$ are structurally close to the corresponding eigenfunctions of the linear operator (2.13). To verify possible shapes of such $f(y)$, for the case $N = 2$, so that the multiindex is $\sigma = (\sigma_1, \sigma_2)$, we present the Hermite polynomials and hence a possible classification of geometric shapes of global similarity patterns close to bifurcation points ("∼" means equality up to a non-zero multiplier):

$$\begin{aligned}
&\underline{l=0:} \quad f_0 \sim \psi_0 = F \sim \mathrm{e}^{-\frac{1}{4}(y_1^2+y_2^2)} \quad \text{(generic pattern);}\\
&\underline{l=1:} \quad f_{1,0} \sim \psi_{1,0} = D_{y_1}F \sim y_1\, \mathrm{e}^{-\frac{1}{4}(y_1^2+y_2^2)} \quad \text{(1-dipole pattern);}\\
&\underline{l=2:} \quad f_{2,0} \sim \psi_{2,0} = D_{y_1^2}F \sim (-\tfrac{1}{2}\, y_1^2 + 1)\, \mathrm{e}^{-\frac{1}{4}(y_1^2+y_2^2)},\\
&\qquad f_{1,1} \sim \psi_{1,1} = D_{y_1y_2}F \sim y_1y_2\, \mathrm{e}^{-\frac{1}{4}(y_1^2+y_2^2)} \quad \text{(2-dipole);}
\end{aligned} \tag{3.37}$$

$$\begin{aligned}
&\underline{l=3:}\quad f_{3,0} \sim \psi_{3,0} = D_{y_1^3}F \sim (\tfrac{1}{4}\,y_1^3 - \tfrac{3}{2}\,y_1)\mathrm{e}^{-\frac{1}{4}(y_1^2+y_2^2)} \quad \text{(3-dipole)};\\
&f_{2,1} \sim \psi_{2,1} = D_{y_1^2 y_2}F \sim (1-\tfrac{1}{2}\,y_1^2)y_2\mathrm{e}^{-\frac{1}{4}(y_1^2+y_2^2)};\\
&\underline{l=4:}\quad \psi_{4,0} \sim D_{y_1^4}F \sim (-\tfrac{1}{8}\,y_1^4 + \tfrac{3}{2}\,y_1^2 - \tfrac{3}{2})\,\mathrm{e}^{-\frac{1}{4}(y_1^2+y_2^2)},\\
&\psi_{3,1} \sim D_{y_1^3 y_2}F \sim (\tfrac{1}{4}\,y_1^3 - \tfrac{3}{2}y_1)y_2\,\mathrm{e}^{-\frac{1}{4}(y_1^2+y_2^2)} \quad \text{((3,1)-dipole)}\\
&\psi_{2,2} \sim D_{y_1^2 y_2^2}F \sim (-\tfrac{1}{2}y_1^2+1)(-\tfrac{1}{2}y_2^2+1)\,\mathrm{e}^{-\frac{1}{4}(y_1^2+y_2^2)};\ \text{etc.}
\end{aligned} \tag{3.38}$$

Among this abundance of patterns, there are three radial ones for even $l = 0, 2, 4$:

(i) the first radial one is $f_0 \sim F$ ($l = 0$);

(ii) the second one is generated by a linear combination of two eigenfunctions for $l = 2$:

$$f_2 \sim \psi_{2,0} + \psi_{0,2} \sim (1 - \tfrac{1}{2}\,(y_1^2 + y_2^2))\,\mathrm{e}^{-\frac{1}{4}(y_1^2+y_2^2)} \equiv (1 - \tfrac{1}{2}\,r^2)\,\mathrm{e}^{-\frac{1}{4}\,r^2};$$

(iii) for $l = 4$, the third radial pattern has the form

$$f_4 = \psi_{4,0} + \tfrac{1}{2}\,\psi_{2,2} + \psi_{0,4} = (-\tfrac{1}{8}\,r^4 + \tfrac{5}{4}\,r^2 - \tfrac{5}{2})\,\mathrm{e}^{-\frac{1}{4}\,r^2}.$$

All other bifurcation linearized patterns *are not radially symmetric* and generate non-radial similarity profiles at $p = p_l$.

4 Calculating global p-bifurcation branches: variational fibering

4.1 Application of variational and category theory

We first note that global extension of bifurcation branches can be performed by classic theory [40, 41], so that, for the present potential operator (2.19), from all the bifurcation points, we obtain bifurcation branches for $p > p_l$ for any $l \geq 0$, which are globally continued in $p > 1$. Nevertheless, according to global theory (see general results in [14, p. 401]), such branches are allowed to end up at some further bifurcation points, say, at some $p = p_s^-$. This may lead to closed bifurcation branches (loops) that actually occur in some problems including those for the semilinear heat equations with absorption with a different choice of bifurcation parameters; see [33, § 6.4]. However, (3.28) shows that such a subcritical bifurcation at $p = p_s^-$ is not allowed, so the only way for a p-branch to be globally non-extensible is to have a turning point or to blow-up. The latter can happen as $p \to p_{\mathrm{S}}^-$, i.e., for $N \geq 3$ only.

We are now going to fully use the variational structure of the problem (2.10) and apply the fibering method [50, 51] as a convenient generalization of previous versions [13, 54] of Lusternik–Schnirel'man (further denoted by L–S) classic category theory [43]. Namely, by L–S theory, the number of critical points of the functional (2.19) depends on the *category* (or *genus*) of the functional subset on which fibering is taking place; see precise definitions and results in Berger [4, p. 378].

Namely [50, 51], critical points of G are obtained via the *radial fibering*

$$f = r(v)v, \tag{4.39}$$

where $r(v) \ge 0$ is a scalar functional, and v belongs to the subset $\Omega_p \subset H^1_\rho(\mathbb{R}^N) \cap L^2_\rho(\mathbb{R}^N)$ given as follows:

$$\Omega_p = \{v \in H^1_\rho \cap L^2_\rho : \quad \int \rho |\nabla v|^2 - \frac{1}{p-1} \int \rho v^2 = 1\}. \tag{4.40}$$

The new functional

$$H(r, v) = -\tfrac{1}{2} r^2 + \tfrac{|r|^{p+1}}{p+1} \int \rho |v|^{p+1} \tag{4.41}$$

has the minimum point at

$$\bar r(v) = (\int \rho |v|^{p+1})^{-\frac{1}{p-1}}, \quad \text{at which} \quad H(\bar r(v), v) = -\frac{p-1}{2(p+1)} \, \bar r^2(v).$$

Therefore, introducing the new simpler functional

$$\tilde H(v) = [-H(\bar r(v), v)]^{-\frac{p-1}{2}} \equiv [\tfrac{2(p+1)}{p-1}]^{\frac{p-1}{2}} \int \rho |v|^{p+1}, \tag{4.42}$$

we arrive at a homogeneous non-negative convex and uniformly differentiable functional, to which classic L-S theory applies, [41]; see also [14, p. 353]. Existence of critical points is guaranteed by the compact embedding (3.21) under the hypothesis (3.20). As in [4, § 6.7], the number of the critical points of $\tilde H$ on Ω_p is associated with the category of Ω_p. It follows from (4.40) that the category of Ω_p is equal to the total multiplicity of all the eigenvalues $\lambda_l = -\frac{l}{2}$ of the operator (2.12) that satisfy

$$-\lambda_l = \tfrac{l}{2} > c_1 = \tfrac{1}{p-1} - \tfrac{N}{2}. \tag{4.43}$$

Notice that the equality in (4.43) leads to the critical exponents as in (2.18). Obviously, (4.43) implies that

$$\boxed{\operatorname{cat} \Omega_p = +\infty \quad \text{for any} \quad p > 1.} \tag{4.44}$$

In addition, the inequality (4.43) correctly explains that, for creating a set S_k of a given category k, only eigenfunctions ψ_β with sufficiently large $|\beta| > c_1$ can be used. This somehow reflects the actual geometry of nodal sets of nonlinear eigenfunctions $f_l(y)$ and confirms the local structure of p_l-bifurcation branches from Section 2.3. Finally, the fibering method [51] (cf. another version in [4, p. 376]) guarantees the following:

Proposition 4 *For any* $1 < p < p_S$, *in view of* (4.44), *the functional* (4.42), *and hence the original functional* (2.19), *has infinitely many distinct critical points in* Ω_p.

As usual, not all the critical points lead to essentially different solutions, which can coincide by orthogonal transformations and other symmetries. In particular, Proposition 4 establishes a kind of a one-to-one correspondence between infinitely many bifurcation points at the critical values (2.18) and p_l-bifurcation branches that appear at these. In other words, this once more confirms that all p_l-bifurcation branches are global in $p > p_l$.

4.2 Nodal set classification of similarity profiles

Thus, for a fixed $p \in (0, p_S)$, we denote by S_p the set of all critical points of (2.19):

$$S_p = \{f_\beta\}_{|\beta| \geq 0}, \quad \text{with critical values} \quad c_\beta = G(f_\beta) < 0; \tag{4.45}$$

for $G(v_0) \geq 0$ there occurs blow-up in the rescaled equation, see below.

Note that $\{c_\beta\}$ can be represented as a monotone increasing sequence of $|\beta|$ and

$$c_\beta \to 0^- \quad \text{as} \quad |\beta| \to +\infty. \tag{4.46}$$

We recommend [4, § 6.7C] for further details and examples of typical bifurcation diagrams with unbounded and ordered branches; see also [57] for more recent results.

Finally, let us underline and specify some important properties of the constructed self-similar stationary solutions:

(i) For $N = 1$, the set $S_p = \{f_k\}$ satisfies the Sturmian property (a corollary of the Maximum Principle): *each profile $f_k(y)$ has precisely k zeros (sign changes) in $\mathbb{R}$.* Moreover, the even ones $\{f_{2k}(y)\}$ are even functions, while the odd ones $\{f_{2k+1}(y)\}$ are odd. The first profile $f_0 = f_0(|y|) > 0$ is symmetric and is an *unstable* stationary solution of the rescaled parabolic flow (see (5.50) below).

(ii) For $N \geq 2$, the L–S category construction of critical values and points makes the nodal set of each $f_\beta(y)$ more complicated as $|\beta|$ increases. A simple logical characterization of such nodal sets for general non-radial self-similar profiles $f_\beta(y)$ for $|\beta| \gg 1$ is not easy.

For small β this can be done: e.g., $f_0(y)$ is always a positive radial solution, with an empty nodal set (cf. (3.37)). The second dipole profile f_{β_1}, with, say, $\beta_1 = \{1, 0, ..., 0\}$, has, as the nodal set, the only hyperplane $\{y_1 = 0\}$ (cf. (3.37)) and can be constructed as the unique positive critical point of the potential (2.19) in the half-space $\{y_1 > 0\}$ with the zero Dirichlet condition on this boundary. Then the full resulting dipole profile is obtained by the negative reflection:

$$f_{\beta_1}(-y_1, ...) = -f_{\beta_1}(y_1, ...). \tag{4.47}$$

For $|\beta| = 2$ in $\mathbb{R}^2$, the first pattern is radial, and has a unique nodal sphere; cf. the first one in (3.37). The second one, as in (3.37), has nodal sets consisting in two hyperplanes:

$$\{y_1 = 0\} \quad \text{and} \quad \{y_2 = 0\}, \tag{4.48}$$

so that the variational construction is performed in the corresponding “corner”, with zeros on the boundary, giving a positive pattern therein, with further suitable negative (odd) reflections about the hyperplanes to get a pattern in $\mathbb{R}^N$.

It follows from (3.37), (3.38), etc., that a similar radial and hyperplane reflection construction can be done for some higher-order profiles f_β with $|\beta| \geq 3$, but not for all of them. It is seen from (3.38) that there appear stationary profiles with more complicated nodal sets, e.g., combining radial and hyperplane structures on certain subspaces.

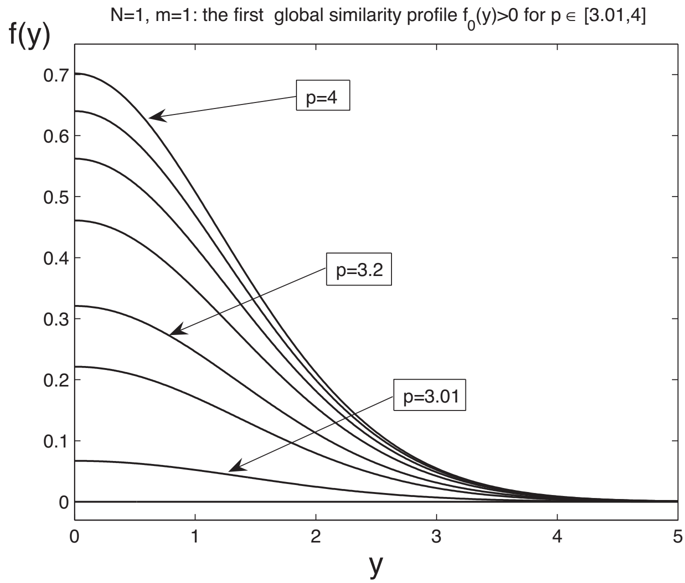


Figure 2: The first profile $f_0(y)$ of (2.10) for $N = 1$ and $p \in [3.01, 4]$.

However, we claim that, though such a "Sturmian" classification of nodal sets of various $f_\beta(y)$ gets rather complicated in $\mathbb{R}^N$, it is available and a complete nodal classification of S_p actually exists (though, probably, not being that useful).

Note that the complicated structure of nodal sets of $f_\beta(y)$ for large $|\beta|$ also reflects the increasing co-dimension of their *stable set* according to the rescaled parabolic evolution via (5.50). Actually, this co-dimension can be characterized by the Morse index (the number of positive eigenvalues) of the linearized self-adjoint operator $\mathbf{A}'(f_\beta)$, which increases with $|\beta|$.

4.3 Numerical illustrations: similarity profiles and p-branches

Though the above rigorous results explain existence of an infinite number of p-branches, it is convenient to describe numerically the actual properties of profiles $\{f_l(y)\}$ and the corresponding p-bifurcation diagrams.

In Figure 2, we show the first positive profile $f_0(y)$ for $N = 1$ in the supercritical range $p > p_0 = 3$ (for $p \le p_0$, f_0 is nonexistent). The N-dependence of the first radially symmetric profile $f_0(y)$ for $p = 4$ is explained in Figure 3.

We next explain in Figure 4 typical features of higher-order profiles $f_l(y)$ for $N = 1$, $p = 4$, belonging to different global p_l-branches that appear at supercritical pitchfork bifurcation points $p = p_l^+$. By symmetry of the ODE for $N = 1$, the profiles $f_{2k}(y)$ are even and $f_{2k+1}(y)$ are odd. By classic theory, the Sturmian zero property holds: each solution $f_l(y)$ has l sign changes (zeros) for $y \in \mathbb{R}$.

Since $p = 4 > p_l$ for any $l \ge 0$, all the profiles $\{f_l\}_{l \ge 0}$ are available in this case. Let us present the

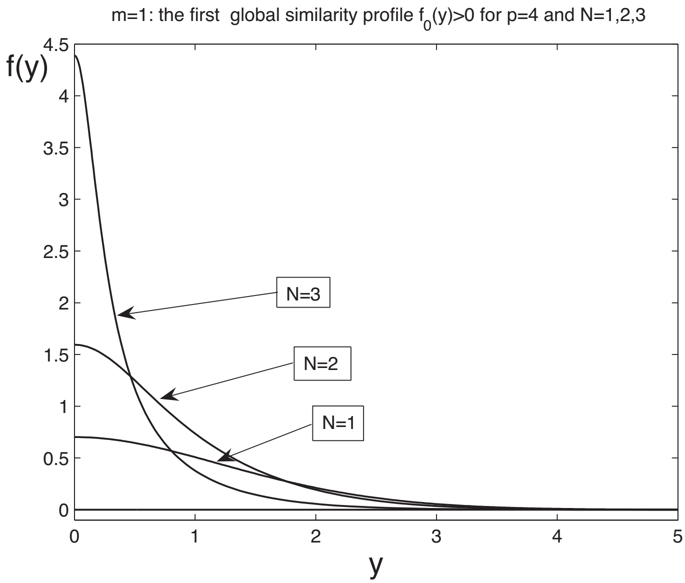


Figure 3: The first profile $f_0(y)$ of (2.10) for $p = 4$ in dimensions $N = 1, 2, 3$.

necessary parameters obtained numerically: for even solutions in (a),

$$f_0(0) = 0.7018..., \ f_2(0) = 1.741..., \ f_4(0) = 2.445..., \ f_6(0) = 3.047..., \ f_8(0) = 3.5903... .$$

For odd solutions in (b):

$$f_1'(0) = 1.367..., \quad f_3'(0) = 4.1949..., \quad f_5'(0) = 8.05828... .$$

For $N = 1$ and $p = 2$, first six profiles $f_l(y)$ are shown in Figure 5. Note that, since

$$p = 2 < p_0 = 3, \quad p = 2 \le p_1 = 1 + \tfrac{2}{1+1} = 2, \quad \text{and} \quad p_2 = \tfrac{5}{3} < 2,$$

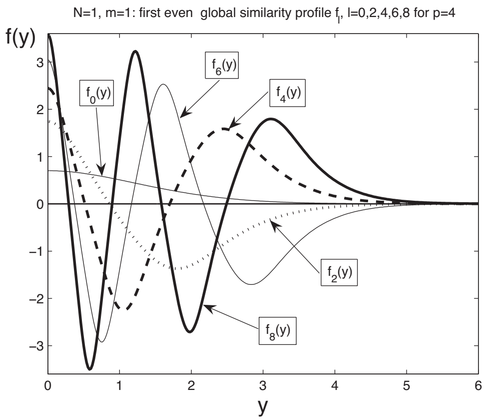
the profiles of the type $f_0(y)$ and $f_1(y)$ are nonexistent for $p = 2$, while all others are available, according to the structure of their p-diagrams to be shown later on. Parameters: for even solutions in (a),

$$f_2(0) = 1.215..., \quad f_4(0) = 4.88..., \quad f_6(0) = 10.13... ,$$

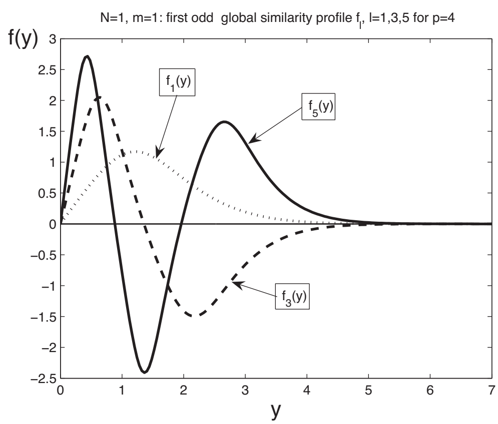
while for odd ones in (b),

$$f_3'(0) = 4.675..., \quad f_5'(0) = 17.415..., \quad f_7'(0) = 41.53... .$$

Finally, in Figure 6, we show the actual structure of the global p-branch of the first similarity profile $f_0(y)$, while Figure 7 explains the corresponding p-deformation of f_0 in this parameter range. Other p_l-branches with pitchfork bifurcations at $p = p_l^+$ for all $l \ge 1$ look similar.

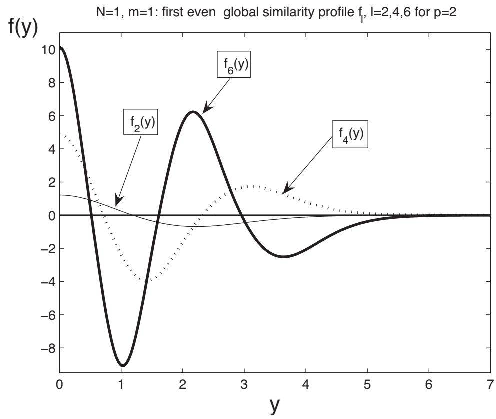


(a) even profiles $f_{0,2,4,6,8}$

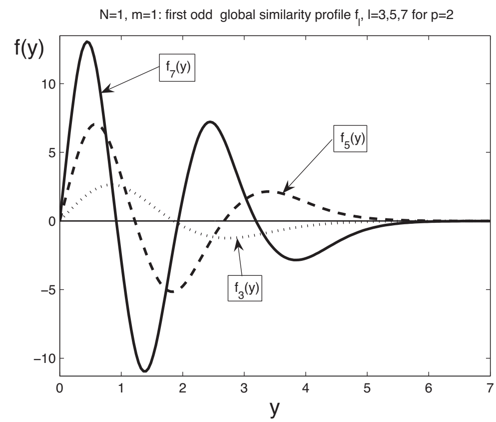


(b) odd profiles $f_{1,3,5}$

Figure 4: Eight profiles $f_l(y)$ for $N = 1$, $p = 4$: even profiles (a) and the odd ones (b).



(a) even profiles $f_{2,4,6}$



(b) odd profiles $f_{1,3,5}$

Figure 5: Six profiles $f_l(y)$ for $N = 1$, $p = 2$: even profiles (a) and the odd ones (b).

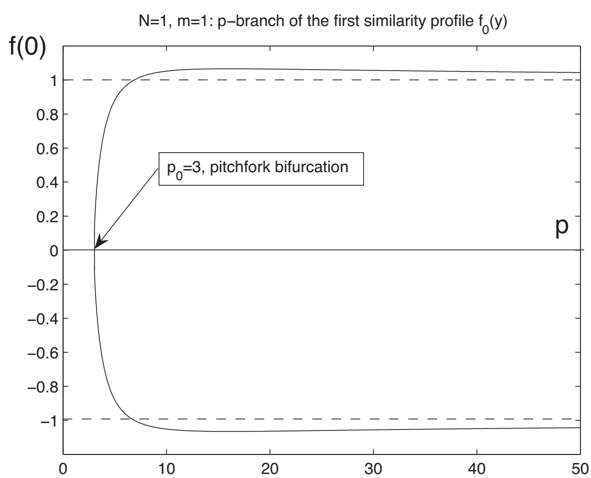


Figure 6: The p-branch of the first profile $f_0(y)$ of (2.10) for $N = 1$ and $p \in [3, 50]$.

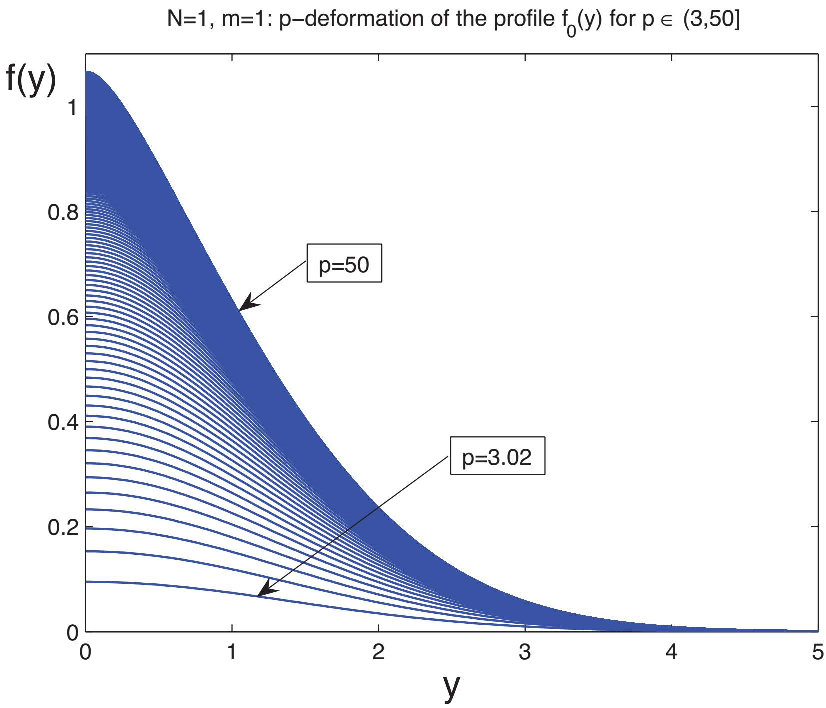


Figure 7: Deformation of the first profile $f_0(y)$ from Figure 6, $N = 1$, $p \in [3, 50]$.

5 Countable family of global linearized patterns

5.1 Stable manifold patterns

This construction is easier and is based on standard and well established stable manifold techniques; see Lunardi [44]. Namely, we perform the same scaling (2.9) of a global solution $u(x, t)$ of (1.2) for $t \gg 1$,

$$u(x,t) = t^{-\frac{1}{p-1}} v(y,\tau), \quad y = \tfrac{x}{\sqrt{t}}, \quad \tau = \ln t, \tag{5.49}$$

to get the rescaled equation (the same as in (3.29)) with the operator $\mathbf{A}$ in (2.10):

$$v_\tau = \mathbf{A}(v) \equiv \Delta v + \tfrac{1}{2}\, y \cdot \nabla v + \tfrac{1}{p-1}\, v + |v|^{p-1} v. \tag{5.50}$$

It follows that

$$\mathbf{A}'(0) = \mathbf{B} + c_1 I, \quad \text{where} \quad c_1 = \tfrac{1}{p-1} - \tfrac{N}{2} = \tfrac{N(p_0-p)}{2(p-1)} > 0 \text{ for } p < p_0. \tag{5.51}$$

Therefore, the derivative $\mathbf{A}'(0)$ has an infinite-dimensional stable subspace,

$$E^s = \text{Span}\{\psi_\beta : \ \lambda_\beta + c_1 \equiv -\tfrac{|\beta|}{2} + c_1 < 0, \ \text{i.e., } |\beta| > 2c_1\}. \tag{5.52}$$

Thus, using good spectral properties of the self-adjoint operator $\mathbf{B}$ [6], by invariant manifold theory for parabolic equations [44, Ch. 9], we arrive at the following (see [23] as a sample for an absorption-diffusion equation):

Proposition 5 *For any multiindex $|\beta| = l > 2c_1$, equation* (5.50) *admits global solutions with the asymptotic behaviour as $\tau \to +\infty$*

$$v_\beta(y,\tau) = \mathrm{e}^{(\lambda_\beta + c_1)\tau} \varphi_\beta(y)(1+o(1)), \quad \textit{where} \quad \varphi_\beta \in \mathrm{Span}\{\psi_\beta : \ |\beta| = l\}, \ \ \varphi_\beta \ne 0. \tag{5.53}$$

In the original variables (5.49), the global patterns (5.53) take the form:

$$u_\beta(x,t) = t^{-\frac{N+|\beta|}{2}} \varphi_\beta\big(\tfrac{x}{\sqrt{t}}\big)(1+o(1)) \quad \text{as} \quad t \to +\infty. \tag{5.54}$$

5.2 Nonexistence of center manifold patterns

Such patterns may occur if

$$\lambda_\beta + c_1 = 0 \quad \Longrightarrow \quad l = |\beta| = 2c_1 > 0, \quad \text{or} \quad p = p_l. \tag{5.55}$$

Studying the centre manifold behaviour of the simplest 1D type

$$v(\tau) = a_l(\tau)\psi_l + w^\perp \quad \text{as} \quad \tau \to +\infty, \tag{5.56}$$

we obtain from (5.50) the following equations for the expansion coefficient:

$$\dot a_l = \kappa_l \, |a_l|^{p-1} a_l (1+o(1)), \quad \text{where} \quad \kappa_l = \langle |\psi_l|^{p-1}\psi_l, \psi_l^* \rangle \quad (p = p_l). \tag{5.57}$$

However, by (3.24), $\kappa_l = \displaystyle\int \rho |\psi_l|^{p-1}\psi_l\psi_l = \int \rho|\psi_l|^{p+1} > 0$, so that the asymptotic ODE (5.57) describes the unstable (actually, blow-up-like) behaviour and no global orbits approach the centre subspace. Overall, it seems that finding a nontrivial centre subspace global behaviour for (5.50) is hopeless. But this can exist for higher-order equations with absorption [23, 30], where we present new examples of such logarithmically perturbed asymptotic patterns.

Finally, let us mention that both countable families of global solutions of (1.5) given by (2.9) and (5.54) exhibit some clear similarity-like scaling invariance, which is crucially necessary for solutions to remain in a given subclass for any $t > 0$ and hence to be global in time. This explains why any "linear" λ-scaling applied to initial data u_0 in [7, 8] (or any other that is not coherent with the above "self-similar" ones) will lead to blow-up of the corresponding solutions.

6 Some structural properties of the set of global solutions via critical points: blow-up, transversality, and connecting orbits

6.1 Blow-up and global solutions via a Lyapunov function

Let us write down (5.50) in the divergence form:

$$v_\tau = \mathbf{A}(v) \equiv \tfrac{1}{\rho} \nabla \cdot (\rho \nabla v) + \tfrac{1}{p-1}\, v + |v|^{p-1} v \quad \text{for} \quad \tau > 0, \quad v(0) = v_0. \tag{6.58}$$

We naturally introduce two sets of initial data: (i) the **global set**,

$$\mathcal{G} = \{v_0 \in H^2_\rho(\mathbb{R}^N) \cap C(\mathbb{R}^N) : \quad (6.58) \;\; \text{has a global solution } v(\tau)\}, \tag{6.59}$$

and the **blow-up set**:

$$\mathcal{B} = \{v_0 \in H^2_\rho(\mathbb{R}^N) \cap C(\mathbb{R}^N) : \quad \text{the solution of } (6.58) \;\; \text{blows up in a finite time.}\}, \tag{6.60}$$

It follows that there exists a direct decomposition:

$$H^2_\rho(\mathbb{R}^N) \cap C(\mathbb{R}^N) = \mathcal{G} \oplus \mathcal{B}. \tag{6.61}$$

Next, as is well known, (6.58) is a smooth gradient system with the monotone in time Lyapunov function denoted by $L(v)(\tau) \equiv -G(v)(\tau)$ (see (2.19)), i.e.,

$$\tfrac{\mathrm{d}}{\mathrm{d}\tau} L(v)(\tau) \equiv \tfrac{\mathrm{d}}{\mathrm{d}\tau} \big(\tfrac{1}{2} \int \rho |\nabla v|^2 - \frac{1}{2(p-1)} \int \rho v^2 - \frac{1}{p+1} \int \rho |v|^{p+1}\big) = - \int \rho (v_\tau)^2 \le 0. \tag{6.62}$$

Some of our "structural" conclusions follow from the next simple and well known blow-up result.

Proposition 6 *The following holds,*

$$L(v_0) < 0 \quad \Longrightarrow \quad \textit{finite-time blow-up in } (6.58). \tag{6.63}$$

Proof. Multiplying (6.58) in L^2_ρ by v and v_τ, after simple standard manipulations, which are well known since 1970s, one obtains that

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}\tau} \int \rho v^2 \ge -2L_0 + \frac{p-1}{p+1} \int \rho |v|^{p+1}, \quad \text{where} \quad L_0 = L(v_0). \tag{6.64}$$

Thus the fact that the solution blows up in finite τ, follows by applying the Hölder inequality leading to a simple ordinary differential inequality for the L^2_ρ-norm of $v(\tau)$:

$$\tfrac{1}{2} \tfrac{\mathrm{d}}{\mathrm{d}\tau} \int \rho v^2 \ge \frac{p-1}{p+1} \int \rho |v|^{p+1} \ge \frac{p-1}{p+1} \|\rho\|_{L^1}^{-\frac{1-p}{2}} \big(\int \rho v^2 \big)^{\frac{p+1}{2}}. \tag{6.65}$$

■

Corollary 1 *The following hold for the Cauchy problem* (6.58)*:*

$$\begin{aligned} L(v_0) < 0 \quad &\Longrightarrow \quad v_0 \in \mathcal{B}, \\ v_0 \in \mathcal{G} \quad &\Longrightarrow \quad L(v_0) \ge 0. \end{aligned} \tag{6.66}$$

6.2 Transversality of intersections

First of all, these issues have been recently solved *only* for a scalar reaction-diffusion equation on a circle

$$u_t = \mathbf{A}(u) \equiv u_{xx} + g(x, u, u_x), \quad x \in S^1 = \mathbb{R}/2\pi\mathbb{Z}, \tag{6.67}$$

where the nonlinearity $g(\cdot)$ satisfies necessary conditions for existence of global classical bounded solutions for arbitrary bounded smooth initial data. Namely, it is known that if f is a hyperbolic *equilibrium* of $\mathbf{A}$, $\mathbf{A}(f) = 0$, known to be generic (or a *rotating wave*), then the global stable and unstable subspaces of $\mathbf{A}'(f)$ span the whole functional space $X^\alpha = H^{2\alpha}(S^1)$, $\alpha \in (\frac{3}{2}, 1)$, where the global semiflow is naturally defined, i.e.,

$$W^s(\mathbf{A}'(f)) \oplus W^u(\mathbf{A}'(f)) = X^\alpha, \tag{6.68}$$

so that these subspaces *intersect transversely*. It is crucial that such a complete analysis can be performed in 1D only, since it is based on Sturmian zero set arguments (see [25] for main references and various extensions of these fundamental ideas), so, in principle, cannot be extended to equation in $\mathbb{R}^N$. We refer to most recent papers [12, 20, 37], where earlier key references and most advances results on the transversality and connecting orbits can be found.

Of course, even knowing (assume, for a moment) the whole set of equilibria $\{f_\beta\}$ for the problem (6.58) in $\mathbb{R}^N$, obtained in Section 4, we do not have any chance to establish a suitable and general transversality result for a generic hyperbolic equilibrium f_β. However, curiously, we can do something like that close to the bifurcation points $p \approx p_l$ in (1.4) by using bifurcation theory from Section 3:

Proposition 7 (i) *Fix, for a given $p \approx p_l$, $p \ne p_l$, a hyperbolic equilibrium f_β, with a $|\beta| = l$, of the operator* $\mathbf{A}$ *in* (6.58). *Then the transversality conclusion holds:*

$$W^s(\mathbf{A}'(f_\beta)) \oplus W^u(\mathbf{A}'(f_\beta)) = H^2_\rho(\mathbb{R}^N). \tag{6.69}$$

(ii) *Particularly, for $l = 0$, for any $p \approx p_0$, $p \ne p_0$, the first equilibrium $f_0(y)$ is hyperbolic, at least, for any $N \le 10$.*

Proof. (i) It follows from (6.58) and the expansion (3.28) that, for $p = p_l + \varepsilon$, with $\varepsilon \ll 1$,

$$\begin{aligned} \mathbf{A}'(f_\beta) &= \tfrac{1}{\rho}\, \nabla \cdot (\rho \nabla) + \tfrac{1}{p-1}\, I + p|f_\beta|^{p-1} I \\ &= (\mathbf{B} - \lambda_l I) + \tfrac{\varepsilon (N+l)^2}{4} \big[\tfrac{1}{\kappa_l}\, \tfrac{N+l+2}{N+l} |\psi_\beta|^{\frac{2}{N+l}} - 1\big] I + O(\varepsilon^2). \end{aligned} \tag{6.70}$$

Therefore, for $p = p_l$, the following analogy of (6.69) is valid:

$$\mathbf{A}'(f_\beta) = \mathbf{B} - \lambda_l I \quad \Longrightarrow \quad W^s(\mathbf{B} - \lambda_l I) \oplus W^u(\mathbf{B} - \lambda_l I) \oplus W^c(\mathbf{B} - \lambda_l I) = H^2_\rho(\mathbb{R}^N), \tag{6.71}$$

and $\dim W^c(\mathbf{B} - \lambda_l I)$ is equal to the algebraic multiplicity (3.31) of $\lambda_l = -\frac{l}{2}$: By the assumption of the hyperbolicity of f_β, and in view of small perturbations (see, e.g., [6, 38]; this is a standard result in spectral theory of self-adjoint operators) of all the eigenfunctions of $\mathbf{A}'(f_\beta)$ for any $|\varepsilon| \ll 1$, $\varepsilon \neq 0$, which remain complete and closed as for $p = p_l$, we arrive at (6.69). Recall that, since by (6.70), $\mathbf{A}'(f_\beta)$, with eigenfunction $\{\hat{\psi}_\beta\}$, is a small perturbation of $\mathbf{B} - \lambda_\beta I$ (eigenfunctions $\{\psi_\beta\}$) and, in addition, the perturbation is exponentially small as $y \to \infty$, the "perturbed" eigenfunctions remain a small perturbation of the Hermite polynomials in any bounded ball, and sharply approximate those as $y \to +\infty$. Therefore, close to $p = p_l$, there is no doubt that the well-known condition of completeness/closure of $\{\hat{\psi}_\beta\}$ (the so-called *stability of the basis*) is, indeed, valid:

$$\textstyle\sum_{(\beta)} \|\psi_\beta\|_\rho \, \|\hat{\psi}_\beta - \psi_\beta\|_\rho < 1.$$

(ii) For $l = 0$, we can easily estimate κ_0 in (3.24) and substitute into (6.70). Then, the $O(\varepsilon)$-term therein takes the form

$$... + \tfrac{\varepsilon N^2}{4} \left[\tfrac{N+2}{N^2} 4^{N+3} \pi^{\frac{N+4}{2}} \left(\tfrac{N+2}{4N}\right)^{\frac{N}{2}} |F(y)|^{\frac{2}{N}} - 1 \right], \tag{6.72}$$

where F is the rescaled Gaussian (2.15). Eventually, at the origin $y = 0$ (and hence in a sufficient neighbourhood around, in our weighted metric), we estimate this square bracket as follows:

$$[\,\cdot\,](0) = \tfrac{N+2}{N^2} 2^6 \pi^2 \left(\tfrac{N+2}{N}\right)^{\frac{N}{2}} - 1, \tag{6.73}$$

and the positivity for small $\varepsilon > 0$ (and a strict negativity for $\varepsilon < 0$) is clearly seen for $N \leq 10$ as the following table shows:

Table 1: Values of $[\cdot](0)$ in (6.73) for $N = 1, 2, ..., 1000$

N	$[\cdot](0)$
1	3281.2
2	1262.3
3	754.06
4	531.96
5	409.16
6	331.72
7	278.60
8	239.96
10	187.61
100	16.3415
1000	0.7187

In view of such large positive values at $y = 0$ for any $N \leq 10$, the Gaussian $(F(y))^{\frac{2}{N}}$ in (6.72) spreads out such a positivity in a sufficient neighbourhood of the origin, which, in the metric of L^2_ρ will guarantee that no centre subspace is available for any $|\varepsilon| > 0$ small enough.

Note also that, as Table 1 shows, even for $N = 100$, we also observe a rather strong positive (for $\varepsilon > 0$) dominance of the perturbation. Moreover, only for $N \sim 1000$ this becomes not that clear, i.e., the negative part of the perturbation associated with "-1" in (6.72) may reduce an eigenvalue of $\mathbf{A}'(f_0)$ to zero creating its centre subspace. ■

Corollary 2 *Under the conditions of Proposition* 7*, there exists an explicit expression of the Morse index* $M(\mathbf{A}'(f_\beta))$*, with* $|\beta| = l \geq 0$ (*the number of positive eigenvalues*) *of the self-adjoint operator* $\mathbf{A}'(f_\beta)$*:*

$$M(\mathbf{A}'(f_\beta)) = \textstyle\sum_{(k \leq l)} C^k_{N+k-1}. \tag{6.74}$$

Obviously, the Morse index is given by a direct summation of all the multiplicities (as in (3.31)) of positive eigenvalues $\hat{\lambda}_l \approx \lambda_l$.

Thus, close to any bifurcation point $p = p_l$, we precisely know not only the dimensions of the unstable manifold of $\mathbf{A}'(f_\beta)$ of any hyperbolic equilibrium f_β (and, sometimes, we can prove the latter), we also approximately know the corresponding eigenfunctions $\{\hat{\psi}_\beta\}$:

$$\text{by continuity, for all } \; p \approx p_l: \quad \hat{\lambda}_l \approx -\lambda_l = \tfrac{l}{2} \quad \text{and} \quad \hat{\psi}_\beta \approx \psi_\beta, \tag{6.75}$$

where convergence of eigenfunctions as $p \to p_l^-$ is guaranteed in L^2_ρ and uniformly in $\mathbb{R}^N$.

Furthermore, moving along the given bifurcation p-branch, the transversality persists until a saddle-node bifurcation occurs, when a centre subspace for $\mathbf{A}'(f_\beta)$ occurs, and hence (6.69) does not apply. If such a "turning" point of the p-branch this does not appear, the transversality persists globally in p. In other words, this question is directly related to the *strict monotonicity* of p-branches, a problem, which was studies for a number of semilinear elliptic equations; see [4, § 6.7C] and [57] for typical examples and results.

6.3 Connecting orbits for $p < p_0$: a first step

The above analysis makes it possible to claim some first general connecting orbit principles in the rescaled problem (6.58). First of all, since blow-up in the subcritical Fujita range (1.3) is a generic property for even arbitrarily small initial data, we then expect the following:

(I) Any $v_0 \in W^u(\mathbf{A}'(f_\beta))$ leads to finite time blow-up;

(II) For $p \approx p_l$, the stable subspace $W^s(\mathbf{A}'(f_\beta))$ connects f_β with some "higher-degree" (category) equilibrium f_γ, where $|\gamma| \geq |\beta|$;

(III) Moreover, in view of a simple differentiating nature of eigenfunctions (2.16) of $\mathbf{B}$ and the approximation result (6.75), connections within manifolds of the same degree (category) are not possible, i.e., always $|\gamma| > |\beta|$ in **(II)**.

(IV) Looking again at the generating formula (2.16) and to (2.14), (2.16), etc., one can observe that connecting orbits

$$f_\gamma \to f_\beta, \quad \text{with} \quad |\gamma| > |\beta|,$$

can be achieved provided that a simplification of symmetries of equilibrium occurs on the connection. For instance, if $f_\gamma(y)$ is anti-symmetric relative a hyperplane, then $f_\beta(y)$ can be symmetric (even) relative to it, but not *vice versa*. Evidently, this is explained by different stabilities of these two configurations in a natural linearization sense (the even one is always "more" stable than the odd one).

Of course, these first conclusions, which are not easy to prove at all, require further study and extensions.

7 On evolution completeness

This issue is principal: do the two countable families of global solutions (2.9) (with their stable manifolds) and (5.54) describe **all possible global solutions** of the problem (1.2), (1.3)? In other words, are these families (plus stable manifolds if any) *evolutionary complete*[2] in the whole set of global solutions? If the answer is "YES", this leads to a simple blow-up conclusion: any other solution with data not on the corresponding manifolds blows up in finite time[3].

Proving evolution completeness in the case, where the whole set of patterns consists of, on one hand, nonlinear ones (2.9) being self-similar solutions of (1.2), and, on the other hand, of non-similarity linearized patterns (5.54) (assuming centre manifold patterns nonexistent, which also needs a proper proof) is a difficult problem. However, for the present variational and gradient case, there is a definite hope that such a goal can be achieved. Here, we briefly discuss such an opportunity by developing a general view to such a problem, and do not pretend for a full analysis.

Thus, consider the rescaled equation (5.50) for an arbitrary proper global orbit $\{v(\cdot, \tau)\}$. There are two cases:

7.1 Case 1: uniformly bounded orbit, $|v(\tau)| \le C$

This is an easy case: as we already know, (5.50) is a smooth gradient system with the monotone in time Lyapunov function $L(v)(\tau) \equiv -G(v)(\tau)$ in (2.19) given by (6.62). Therefore, the ω-limit set $\omega(v_0)$ consists of stationary points [34], so that, those are either nontrivial similarity profiles as in (2.9), or the orbit approaches zero, so gets into the framework of the linearized construction leading to patterns (5.53) (*modulo* centre manifold ones). In the last case, the completeness-closure of the eigenfunctions $\{\psi_\beta\}$ guarantees that no other asymptotic patterns can appear when the orbit approaches the trivial equilibrium.

7.2 Case 2: blow-up at infinity

This means that the global orbit is unbounded as $\tau \to +\infty$, so that there exists sequences $\{x_k\} \subset \mathbb{R}^N$ and $\{\tau_k\} \to +\infty$ such that

$$\sup_{x \in \mathbb{R}^N} |v(x, \tau_k)| \equiv |v(x_k, \tau_k)| = C_k \to +\infty \quad \text{as} \quad k \to \infty \tag{7.76}$$

monotonically. This case is more difficult, since it is known that blow-up at infinity is available, but in the critical Sobolev case $p = p_S$; see [28] for history, references, and typical results. However, this does not concern us, since, by (3.20), we have to be in the subcritical range.

Let us show how the critical Sobolev exponent actually appears in the study of global unbounded solutions (GUS). The analysis is based on a scaling argument; see [29] for earlier references and

[2] See [24] for first examples of evolutionary complete sets of "nonlinear eigenfunctions" in quasilinear parabolic problems.

[3] We use here the fact that in the Sobolev subcritical range, blow-up in infinite time is not available (see comments below); for $p = p_S$, this is already not the case,[28].

results. Namely, we perform the scaling:

$$v = C_k w, \quad y = y_k + a_k z, \quad \tau = \tau_k + b_k s, \quad \text{where} \quad a_k = C_k^{-\frac{p-1}{2}}, \; b_k = a_k^2, \tag{7.77}$$

so that the sequence of functions $\{w_k(z,s)\}$ solves the following perturbed equation:

$$w_s = \Delta w + |w|^{p-1} w + \delta_k \big(\tfrac{1}{2}\, y \cdot \nabla w + \tfrac{1}{p-1}\, w \big), \quad \text{where} \quad \delta_k = a_k^2 = C_k^{1-p} \to 0. \tag{7.78}$$

By construction, each $w_k(z,s)$ is uniformly bounded for all admissible $s < 0$ and

$$|w_k(z,s)| \le 1 \quad \text{for} \quad s \le 0 \quad \text{and} \quad \sup_z |w_k(z,0)| = 1. \tag{7.79}$$

By classic parabolic regularity theory [21, 17], it follows by passing to the limit $k \to \infty$ in (7.78) that

$$w_k \to \hat{w}, \quad \text{where} \quad \hat{w}_s = \Delta \hat{w} + |\hat{w}|^{p-1} \hat{w}, \tag{7.80}$$

and $\hat{w}(z,s)$ satisfies the same estimates (7.76) for all $s \le 0$. In other words, $\hat{w}$ is an *ancient* solution of (7.80) (in R.S. Hamilton's terminology). Indeed, (7.80) is a simpler gradient system in the L^2-metric, so that ω-limit sets of uniformly bounded orbits must consist of equilibria. The set of equilibria for the equation (7.80) changes their structure precisely at the *critical Sobolev exponent* given in (3.20). Recall that existence of a GUS in [28] was established for $p = p_S$ only.

In other words, for $p < p_S$, the set of stationary solutions of (7.80) is too "poor" to support existence of a suitable ancient solution satisfying (7.79) *under the assumption that the orbit* $\{\hat{w}(s)\}$ *is global in time*. This explains how p_S occurs in this analysis, but not proves the evolution completeness in the general case.

Acknowledgement. The authors would like to thank the anonymous referee for making valuable suggestion that have enhanced the overall quality of our paper.

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