On codes achieving zero error capacities in limited magnitude error channels

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Abstract*—Shannon in his 1956 seminal paper introduced the concept of the zero error capacity, C_0, of a noisy channel. This is defined as the least upper bound of rates at which it is possible to transmit information with zero probability of error. At present not many codes are known to achieve the zero error capacity. In this paper, some codes which achieve zero error capacities in limited magnitude error channels are described. The code lengths of these zero error capacity achieving codes can be of any finite length $n = 1, 2, \ldots$, in contrast to the long lengths required for the known regular capacity achieving codes such as turbo codes, LDPC codes and polar codes. Both wrap around and non-wrap around limited magnitude error models are considered in this paper. For non-wrap around error model, the exact value of zero error capacities are derived, and optimal non-systematic and systematic codes are designed. The non-systematic codes achieve the zero error capacity with any finite length. The optimal systematic codes achieve the systematic zero error capacity of the channel, which is defined as the zero error capacity with the additional requirements that the communication must be carried out with a systematic code. It is also shown that the rates of the proposed systematic codes are equal to or approximately equal to the zero error capacity of the channel. For the wrap around model bounds are derived for the zero error capacity and in many cases the bounds give the exact value. In addition, optimal wrap around non-systematic and systematic codes are developed which either achieve or are close to achieving the zero error capacity with finite length.

Index terms: Capacity, zero error capacity, asymmetric channel, symmetric channel, limited magnitude errors, positive and negative errors.

I. Introduction

Let the codes be over the alphabet

$$\mathbb{Z}_m = \{0, 1, \ldots, m-1\} \subseteq \mathbb{Z}.$$

In this paper, we are concerned with the limited magnitude channel error models for both wrap around and non-wrap around cases. In particular, for $l_-, l_+ \in \mathbb{N}$ such that $l_- + l_+ \leq m-1$, the wrap around channel error model with negative errors of limited magnitude l_- and positive errors of limited magnitude l_+ (briefly, WA-(l_-, l_+)-channel), is defined by the relations

$$\begin{aligned} &\text{for all } a \in \mathbb{Z}_m \text{ and } i \in \mathbb{Z}, \\ &\quad i \notin [-l_-, l_+] \implies \\ &\qquad P(Y = (a+i) \bmod m | X = a) = 0; \end{aligned} \tag{1}$$

where $X \in \mathbb{Z}_m$ and $Y \in \mathbb{Z}_m$ are the channel input and output symbol random variables, respectively.

When $l_- = 0$ and $l_+ = l$ we get the totally asymmetric wrap around channel with errors of limited magnitude l. On the other hand, when $l_- = l_+ = l$, we get the symmetric wrap around limited magnitude l error model.

For $m = 8$ and $l = 1$, the asymmetric and symmetric error models of limited magnitude 1 with wrap around are shown in Figure 1 and Figure 2, respectively.

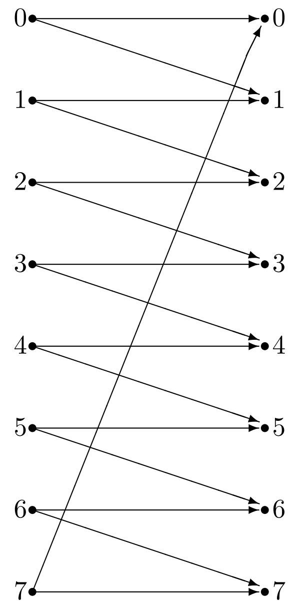

Fig. 1. Limited magnitude, $l = 1$ asymmetric error model with wrap around.

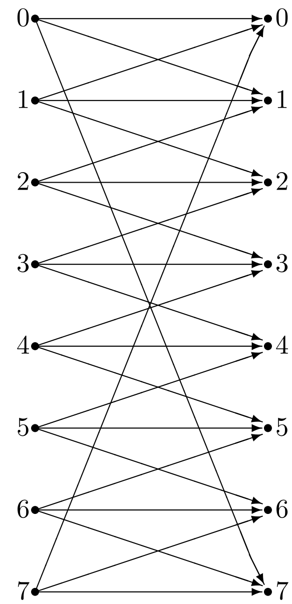

Fig. 2. Limited magnitude, $l = 1$ symmetric error model with wrap around.

In the case of non-wrap around errors, the channel error model with negative errors of limited magnitude l_- and positive errors of limited magnitude l_+ (briefly, NW-(l_-, l_+)-channel), is defined by the relations

$$\begin{aligned} &\text{for all } a \in \mathbb{Z}_m \text{ and } i \in \mathbb{Z}, \\ &\quad i \notin [-l_-, l_+] \text{ or } a + i \in \mathbb{Z} - \mathbb{Z}_m \implies \\ &\qquad P(Y = (a+i) \bmod m | X = a) = 0; \end{aligned} \tag{2}$$

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where, again, $X \in \mathbb{Z}_m$ and $Y \in \mathbb{Z}_m$ are the channel input and output symbol random variables, respectively. From (1) and (2), note that the NW-(l_-, l_+)-channels are particular WA-(l_-, l_+)-channels.

The asymmetric and symmetric limited magnitude error models have applications in flash memories [8], in m-phase shift keying (m-PSK) communication systems [4], [5], [15] and, for $m = +\infty$, in correcting repetition errors [19].

The capacity, C, of a channel is defined as [9]

$$C = \max_{p(x)} I(X,Y) = \max_{p(x)} (H(Y) - H(Y|X)) \qquad (3)$$

where $I(X,Y)$ is the mutual information between X and Y, $H(Y)$ is the entropy of Y, $H(Y|X)$ is the conditional entropy of Y given X and the maximum is taken over all possible input probability distribution $p(x)$. Since $H(Y|X = x)$ is constant with respect to the input symbol $x \in \mathbb{Z}_m$, it is readily seen that the capacity, $C^{(WA)}$, of the general wrap around channel given in (1) is

$$C^{(WA)} = \log m - h\left(p_{-l_-}, \ldots, p_{-1}, p_0, p_1, \ldots, p_{l_+}\right), \qquad (4)$$

where $p_i \stackrel{\text{def}}{=} P(Y = (a+i) \bmod m | X = a)$, for all $a \in \mathbb{Z}_m$ and $i \in \mathbb{Z}$, and

$$h(q_0, q_1, \ldots, q_{D-1}) = -\sum_{h=0}^{D-1} q_h \log q_h$$

is the D variable entropy function. Note that, for all $a \in \mathbb{Z}_m$,

$$P(Y = a | X = a) = p_0 = 1 - \left(\sum_{i \in [-l_-, l_+] - \{0\}} p_i \right).$$

It is well known that $n \in \mathbb{N}$ uses of a discrete memoryless channel does not change the capacity, C, of the channel per single use.

In [18], Shannon introduced the concept of zero error capacity, C_0, of a noisy channel. This is defined as the least upper bound (i. e., the supremum) of rates at which it is possible to transmit information with zero probability of error. In general, the zero error capacity of a channel, C_0, is always less than or equal to the regular capacity of the channel, C; i. e., $C_0 \le C$. We readily note that, unlike the regular capacity, this zero error capacity per single use may depend on the number $n \in \mathbb{N}$ of uses of the discrete memoryless channel, so that

$$C_0 = \sup_{n \in \mathbb{N}} C_0(n);$$

where $C_0(n)$ is the maximum information rate achievable by using the channel n times. This makes the problem of finding the zero error capacity achieving codes a difficult and interesting combinatorial problem [3], [7], [11], [13], [14], [17], [18]. In all these papers except [17] the zero error capacities of graphs are described. Only in [17], some nontrivial limited magnitude one asymmetric error correcting linear codes over $\mathbb{Z}_m$ are described. In [8], the authors consider t (instead of all) limited magnitude l error correcting codes.

***Example** 1.1: For example, the wrap around limited magnitude $l = 1$ asymmetric error channel with $m = 5$ has [18]*

$$C_0(1) = \log 2 < \frac{\log 5}{2} \le C_0(2),$$

and, indeed, it has been proved that $C_0 = (1/2)\log 5$ [11], [13]. In fact, an optimal solution is given by the code [18],

$$\mathcal{C} \stackrel{\text{def}}{=} \{\underline{0}0, \underline{1}2, \underline{2}4, \underline{3}1, \underline{4}3\} \subseteq \mathbb{Z}_5^2;$$

which is a systematic code, where the underlined digit is the information digit.

In a systematic code the information part is separated from the check part and hence, the data processing and the encoding/decoding can be done in parallel. Since here we are also interested in systemaic codes we give the following definition.

***Definition** 1.1 (systematic zero error capacity):* Let the systematic zero error capacity, $C_{0,sys}$, of a noisy channel be defined as the least upper bound (i. e., the supremum) of rates at which it is possible to transmit information with zero probability of error **by using a systematic code**.

Note that, like the Shannon's (non-systematic) zero error capacity, this systematic zero error capacity per single use may depend on the number $n \in \mathbb{N}$ of uses of the discrete memoryless channel, so that

$$C_{0,sys} = \sup_{n \in \mathbb{N}} C_{0,sys}(n);$$

where $C_{0,sys}(n) = k/n$ is the maximum information rate achievable by using the channel n times with a systematic code of length $n \in \mathbb{N}$ conveying $k \in \mathbb{N}$ information digits.

In this paper we describe codes achieving zero error capacities in limited magnitude error channels. Interestingly, the length of these zero error capacity achieving codes can be of any finite length, $n = 1, 2, \ldots$ in contrast to the regular capacity achieving known codes, which require a long code length. Both systematic and non-systematic codes are designed here.

The rest of the paper is organized as follows. In Section II, the combinatorial characterization of error correcting codes for the non-wrap around error model is given based on the max L_1 distance, then the zero error capacity derivation and error correcting code designs for both non-systematic and systematic codes are described. These topics for wrap around limited magnitude error channel model are discussed in Section III. In Section IV, an efficient systematic coding algorithm is presented. Some concluding remarks are given in Section V.

In the following, X, Y, Z, ... indicate vectors or words, $XYZ\ldots$ indicate their concatenation and x_i, y_i, z_i, ..., where $i \in \mathbb{N}$, indicate the i-th component or digit of X, Y, Z, ..., respectively. Also, for $a, b \in \mathbb{N}$, we let

$$\langle a \rangle_b \stackrel{\text{def}}{=} a \bmod b \in \mathbb{Z}_b = \{0, 1, \ldots, b-1\}.$$

Furthermore, given $d, k \in \mathbb{N}$ and $R \in \mathbb{Z}_d^k$, let

$$[R]_d \in \mathbb{N}$$

indicate the natural number whose expression in radix d is R. On the other hand, given $a, b, l \in \mathbb{N}$ with $l \ge \lceil \log_b(a+1) \rceil \in \mathbb{N}$, let

$$(a)_b^{[l]} \in \mathbb{Z}_b^l$$

indicate the length l radix b expression of the natural number a.

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II. Non-wrap around error channel

The case of non-wrap around (l_-, l_+) error channel model is defined by (2).

Here, we design error correcting codes that achieve the zero error capacities for the non-wrap around error model. These codes are capable of correcting **all** limited magnitude of l_- negative errors and l_+ positive errors and are referred to as (l_-, l_+)-AEC ((l_-, l_+)-All Error Correcting) codes. Similarly, codes capable of correcting all asymmetric errors of limited magnitude l and codes capable of correcting all symmetric errors of limited magnitude l are referred to as l-AAEC (l-All Asymmetric Error Correcting) and l-ASEC (l-All Symmetric Error Correcting) codes, respectively. Before describing the code design methods, some preliminaries which are useful for the code designs are given.

A. Combinatorial characterization of error correcting codes for the non-wrap around error model

For $m \in \mathbb{N}$, let $x, y \in \mathbb{Z}_m$. The L_1 distance $D^{(L_1)}(x, y)$ between x and y is defined as the absolute value of the real difference between x and y. That is,

$$D^{(L_1)}(x, y) \stackrel{\text{def}}{=} |x - y|.$$

For example, if $m = 7$, $x = 2$ and $y = 6$, then $D^{(L_1)}(2, 6) = |2 - 6| = 4$. The following distance metric is useful in designing (l_-, l_+)-AEC codes.

***Definition 2.1** (max L_1 distance):* Given $n \in \mathbb{N}$, let $X = (x_{n-1}, x_{n-2}, \ldots, x_0) \in \mathbb{Z}_m^n$ and $Y = (y_{n-1}, y_{n-2}, \ldots, y_0) \in \mathbb{Z}_m^n$. The max L_1 distance, $D_{max}^{(L_1)}(X, Y)$, between X and Y is defined as:

$$D_{max}^{(L_1)}(X, Y) = \max_{i \in [0, n-1]} \left\{ D^{(L_1)}(x_i, y_i) = |x_i - y_i| \right\}. \quad (5)$$

For example, if $m = 7$, $n = 4$, $X = (0, 4, 2, 1)$ and $Y = (4, 2, 3, 1)$ then $D_{max}^{(L_1)}(X, Y) = \max\{4, 2, 1, 0\} = 4$. It is worth noting that $D_{max}^{(L_1)}(X, Y)$ is a metric. Furthermore, $D_{max}^{(L_1)}(X, Y) \in [0, m-1]$ because $D^{(L_1)}(x, y) \in [0, m-1]$ and (5).

The following theorem and the corollaries give the necessary and sufficient conditions on the minimum distance for error correction.

***Theorem 2.1** (characterization of (l_-, l_+)-AEC for non-wrap around errors):* Let $m, n \in \mathbb{N}$ and $l_-, l_+ \in \mathbb{N}$. A code $\mathcal{C} \subseteq \mathbb{Z}_m^n$ is capable of correcting all negative errors of magnitude up to l_- and all positive errors of magnitude up to l_+ if, and only if,

$$\text{for all } X, Y \in \mathcal{C}, \quad X \neq Y \implies D_{max}^{(L_1)}(X, Y) > l_- + l_+. \quad (6)$$

Proof: Given (2), for all $n = 1, 2, \ldots$, let

$$S_{m,n,l_-,l_+}^{(NW)}(X) \stackrel{\text{def}}{=} \left\{ Z \in \mathbb{Z}_m^n \;\middle|\; \begin{array}{c} \text{for all } i \in [0, n-1], \\ z_i = x_i + e_i \text{ and} \\ -l_- \leq e_i \leq l_+ \end{array} \right\} = \quad (7)$$
$$\{Z \in \mathbb{Z}_m^n : \text{ for all } i \in [0, n-1],\ z_i \in [x_i - l_-, x_i + l_+]\}$$

be the set of all m-ary vectors obtained from $X \in \mathbb{Z}_m^n$ due to any number of negative errors of magnitude up to l_- and any number of positive errors of magnitude up to l_+.

First note that, $\mathcal{C}$ is (l_-, l_+)-AEC if, and only if,

$$\text{for all } X, Y \in \mathcal{C}, \quad X \neq Y \implies S_{m,n,l_-,l_+}^{(NW)}(X) \cap S_{m,n,l_-,l_+}^{(NW)}(Y) = \emptyset. \quad (8)$$

So, the equivalence between (6) and (8) must be shown.

Suppose (6) holds. From (5), this implies that for all $X, Y \in \mathcal{C}$ with $X \neq Y$, there exists an index $i \in [0, n-1]$ such that $|x_i - y_i| > l_- + l_+$. Let $x \stackrel{\text{def}}{=} \max\{x_i, y_i\} > \min\{x_i, y_i\} \stackrel{\text{def}}{=} y$. So,

$$\begin{aligned}
&x - y = |x_i - y_i| > l_- + l_+ \implies \\
&y + l_+ < x - l_- \implies \\
&[y - l_-, y + l_+] \cap [x - l_-, x + l_+] = \emptyset \implies \\
&S_{m,n,l_-,l_+}^{(NW)}(x) \cap S_{m,n,l_-,l_+}^{(NW)}(y) = \\
&\quad [x - l_-, x + l_+] \cap [y - l_-, y + l_+] \cap \mathbb{Z}_m = \emptyset \implies \\
&\qquad S_{m,n,l_-,l_+}^{(NW)}(X) \cap S_{m,n,l_-,l_+}^{(NW)}(Y) = \emptyset;
\end{aligned}$$

that is, (8) holds.

Conversely, if (6) does not hold then there exists $X, Y \in \mathcal{C}$ such that $D_{max}^{(L_1)}(X, Y) \leq l_- + l_+$. Hence, from $l_-, l_+ \in \mathbb{N}$, $X, Y \in \mathbb{Z}_m^n$ and (5), for all integer $i \in [0, n-1]$,

$$\begin{aligned}
&\max\{x_i, y_i\} - \min\{x_i, y_i\} = |x_i - y_i| \leq l_- + l_+ \implies \\
&\max\{x_i, y_i\} - l_- \leq \min\{x_i, y_i\} + l_+ \implies \\
&\qquad [\max\{x_i, y_i\} - l_-, \min\{x_i, y_i\} + l_+] \cap \mathbb{Z}_m \neq \emptyset.
\end{aligned}$$

So, for all integer $i \in [0, n-1]$, choose $z_i \in [\max\{x_i, y_i\} - l_-, \min\{x_i, y_i\} + l_+] \cap \mathbb{Z}_m$ and let the well defined vector, $Z \in \mathbb{Z}_m^n$, be $Z \stackrel{\text{def}}{=} (z_{n-1}, z_{n-2}, \ldots, z_0)$. The word $Z \in S_{m,n,l_-,l_+}^{(NW)}(X) \cap S_{m,n,l_-,l_+}^{(NW)}(Y) \neq \emptyset$; that is, (8) does not hold. ■

***Corollary 2.2** (characterization of l-AAEC for non-wrap around errors):* Let $m, n, l \in \mathbb{N}$. A code $\mathcal{C} \subseteq \mathbb{Z}_m^n$ is an l-AAEC code if, and only if,

$$\text{for all } X, Y \in \mathcal{C}, \quad X \neq Y \implies D_{max}^{(L_1)}(X, Y) > l. \quad (9)$$

Proof: Let $l_- = 0$ and $l_+ = l$ in the above Theorem 2.1. ■

***Corollary 2.3** (characterization of l-ASEC for non-wrap around errors):* Let $m, n, l \in \mathbb{N}$. A code $\mathcal{C} \subseteq \mathbb{Z}_m^n$ is an l-ASEC code if, and only if,

$$\text{for all } X, Y \in \mathcal{C}, \quad X \neq Y \implies D_{max}^{(L_1)}(X, Y) > 2l. \quad (10)$$

Proof: Let $l_- = l_+ = l$ in the above Theorem 2.1. ■

Suppose $X, Y \in \mathbb{Z}_m^n$. If $m - 1 \leq l_- + l_+$ then $D_{max}^{(L_1)}(X, Y) \leq m - 1 \leq l_- + l_+$ and so, the (l_-, l_+)-AEC code can have at most only one codeword, which is not interesting. So, in the rest of the paper, for the non-wrap around error model, it is assumed that $l_- + l_+ < m - 1$.

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B. Zero error capacity derivations

Given $m, n, l_-, l_+ \in \mathbb{N}$, let

$$D \stackrel{\text{def}}{=} l_- + l_+ + 1 \in \mathbb{N},$$

$A_{NW}(m, n, l_-, l_+) \in \mathbb{N}$ be the maximum cardinality for a non-wrap around (l_-, l_+)-AEC code of length n, and

$$C_0^{(NW)}(m, n, l_-, l_+) = \frac{\log A_{NW}(m, n, l_-, l_+)}{n} \in \mathbb{R}$$

be the maximum information rate achievable by using the channel n times, where $\log$ indicates a base m logarithm.

The zero error capacity of the m-ary NW-(l_-, l_+)-channel model is defined as

$$C_0^{(NW)}(m, l_-, l_+) = \sup_{n \in \mathbb{N}} C_0^{(NW)}(m, n, l_-, l_+)$$

From Theorem 2.5 below, it is readily seen that the non-wrap around zero error capacity behaves like the normal capacity and does not depend on the number of independent uses of the channel. Indeed, it is possible to determine $A_{NW}(m, n, l_-, l_+)$ exactly as below. First note that, from Theorem 2.1, we have

$$A_{NW}(m, n, l_-, l_+) = A_{NW}(m, n, 0, l_- + l_+) \stackrel{\text{def}}{=} A_{NW}(m, n, D-1). \tag{11}$$

Now, the following theorem can be found in [1] and the proof is given here for completeness.

***Theorem** 2.4 (the maximum cardinality of non-wrap around L-AAEC codes [1]):* If $m, n, D \in \mathbb{N}$ then

$$A_{NW}(m, n, D-1) = \left\lceil \frac{m}{D} \right\rceil^n.$$

Proof: If $\mathcal{C} \subseteq \mathbb{Z}_m^n$ is an $(D-1)$-AAEC code then

$$|\mathcal{C}| \leq \left\lceil \frac{m}{D} \right\rceil^n. \tag{12}$$

In fact, let

$$\varphi : \mathbb{Z}_m \to \mathbb{Z}_{\lfloor (m-1)/D \rfloor + 1} = \left\{ 0, 1, \ldots, \left\lfloor \frac{m-1}{D} \right\rfloor \right\}$$

be the function defined as

$$\varphi(x) = \left\lfloor \frac{x}{D} \right\rfloor, \quad \text{for all } x \in \mathbb{Z}_m = \{0, 1, \ldots, m-1\}. \tag{13}$$

Also, let $\varphi : \mathbb{Z}_m^n \to \mathbb{Z}_{\lfloor (m-1)/D \rfloor + 1}^n$ be defined as

$$\varphi(X) = (\varphi(x_{n-1}), \varphi(x_{n-2}), \ldots, \varphi(x_0)), \quad \text{for all } X = (x_{n-1}, x_{n-2}, \ldots, x_0) \in \mathbb{Z}_m^n;$$

and

$$\varphi_{\mathcal{C}} : \mathcal{C} \to \mathbb{Z}_{\lfloor (m-1)/D \rfloor + 1}^n$$

be the restriction of the above φ to $\mathcal{C} \subseteq \mathbb{Z}_m^n$ defined by $\varphi_{\mathcal{C}}(X) = \varphi(X)$, for all $X \in \mathcal{C}$. Now, the function $\varphi_{\mathcal{C}}$ is injective. In fact, if $X, Y \in \mathcal{C}$ and $\varphi_{\mathcal{C}}(X) = \varphi_{\mathcal{C}}(Y)$ then, for all $i = 0, 1, \ldots, n-1$,

$$\left\lfloor \frac{x_i}{D} \right\rfloor = \varphi(x_i) = \varphi_{\mathcal{C}}(x_i) = \varphi_{\mathcal{C}}(y_i) = \varphi(y_i) = \left\lfloor \frac{y_i}{D} \right\rfloor \implies$$
$$x_i - \langle x_i \rangle_D = \left\lfloor \frac{x_i}{D} \right\rfloor D = \left\lfloor \frac{y_i}{D} \right\rfloor D = y_i - \langle y_i \rangle_D \implies$$
$$x_i - y_i = \langle x_i \rangle_D - \langle y_i \rangle_D \in [-(D-1), (D-1)] \implies$$
$$D^{(L_1)}(x_i, y_i) \leq |x_i - y_i| \leq D - 1.$$

Since $D^{(L_1)}(x_i, y_i) \leq D - 1$, for all integer $i \in [0, n-1]$, it follows that $D^{(L_1)}_{max}((X, Y) \leq D - 1$. From Corollary 2.2, this implies $X = Y$ because $X, Y \in \mathcal{C}$ and $\mathcal{C}$ is an $(D-1)$-AAEC code. Since $\varphi_{\mathcal{C}}$ is injective it follows that

$$|\mathcal{C}| = |\varphi_{\mathcal{C}}(\mathcal{C})| \leq \left| \mathbb{Z}_{\lfloor (m-1)/D \rfloor + 1}^n \right| = \left(\left\lfloor \frac{m-1}{D} \right\rfloor + 1 \right)^n = \left\lceil \frac{m}{D} \right\rceil^n;$$

and (12) is proved.

Recall that $\langle a \rangle_b \stackrel{\text{def}}{=} a \bmod b \in \mathbb{Z}_b$, for $a, b \in \mathbb{N}$, and consider the m-ary codes of length $n \in \mathbb{N}$ defined as follows.

$$\text{For all } V \stackrel{\text{def}}{=} (v_{n-1}, v_{n-2}, \ldots, v_0) \in \mathbb{Z}_D^n \subseteq \mathbb{Z}_m^n, \tag{14}$$
$$\mathcal{C} \stackrel{\text{def}}{=} \mathcal{C}_n(V) \stackrel{\text{def}}{=} \prod_{i=0}^{n-1} \{x \in \mathbb{Z}_m : \langle x \rangle_D = v_i\} = \prod_{i=0}^{n-1} \mathcal{C}_1(v_i),$$

where the product indicates a cartesian product. Note that the minimum max L_1 distance of the code $\mathcal{C}$ is

$$D^{(L_1)}_{max}(\mathcal{C}) \stackrel{\text{def}}{=} \max_{X, Y \in \mathcal{C}:\, X \neq Y} D^{(L_1)}_{max}(X, Y) \geq D$$

because, up to a constant, every codeword component is a multiple of D. When $V = (0, 0, \ldots, 0) = \mathbf{0} \in \mathbb{Z}_D^n$, the code

$$\mathcal{C}_n(\mathbf{0}) \stackrel{\text{def}}{=} \{(x_{n-1}, x_{n-2}, \ldots, x_0) \in \mathbb{Z}_m^n : \langle x_i \rangle_D = 0, \text{ for all integer } i \in [0, n-1]\} = \{x \in \mathbb{Z}_m : \langle x \rangle_D = 0\}^n = \mathcal{C}_1(0)^n \tag{15}$$

has a cardinality which is exactly equal to $|\mathcal{C}_1(0)|^n = \lceil m/D \rceil^n$. Thus, from Corollary 2.2, this code is a $(D-1)$-AAEC code with cardinality $\lceil m/D \rceil^n$ and the equality in the statement is proved. ■

We note that the above function (13) is an adjacent reducing mapping [18] for the non-wrap around asymmetric channel. So, the theorem may follow from Theorem 3 in [18].

***Theorem** 2.5 (zero error capacity for the m-ary non-wrap around (l_-, l_+) error model):* If $m, n, l_-, l_+, D = l_- + l_+ + 1 \in \mathbb{N}$ then

$$C_0^{(NW)}(m, n, l_-, l_+) = \frac{\log A_{NW}(m, n, l_-, l_+)}{n} = \log \left\lceil \frac{m}{D} \right\rceil.$$

Hence, the zero error capacity is independent of n and

$$C_0^{(NW)}(m, l_-, l_+) = \sup_{n \in \mathbb{N}} C_0^{(NW)}(m, n, l_-, l_+) = C_0^{(NW)}(m, 1, l_-, l_+) = \log \left\lceil \frac{m}{D} \right\rceil, \tag{16}$$

for all $m, l_-, l_+, D = l_- + l_+ + 1 \in \mathbb{N}$.

Proof: It follows from (11) and Theorem 2.4. ■

So, from Theorem 2.5, for the non-wrap around asymmetric limited magnitude error channel (where $l_- = 0$ and $l_+ = l$), the zero error capacity is given by

$$C_0^{(NW)}(m, 0, l) = \log \left\lceil \frac{m}{(l+1)} \right\rceil. \tag{17}$$

Similarly, for the non-wrap around symmetric limited magnitude error channel (where $l_- = l_+ = l$), the zero error capacity is given by

$$C_0^{(NW)}(m, l, l) = \log \left\lceil \frac{m}{(2l+1)} \right\rceil. \tag{18}$$

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C. Code designs

First, we describe the non-systematic (l_-, l_+)-AEC code design, where $l_-, l_+ \in \mathbb{N}$ and $D-1 = l_- + l_+ < m-1$. In Subsection II-B, we have seen that some zero error capacity achieving codes are the ones given in (15) and can be obtained by taking the code symbols input to the channel as bD, where $b = 0, 1, 2\ldots, \lceil m/D \rceil - 1$. For these codes, a received symbol $y \in \mathbb{Z}_m$ is decoded into the code symbol $x \in \mathbb{Z}_m$ by rounding it to a multiple of D according to the following rule.

$$x = \begin{cases} \left\lfloor \dfrac{y}{D} \right\rfloor D = y - \langle y \rangle_D \\ \qquad\qquad\qquad \text{if } \langle y \rangle_D \in [0, l_+], \\ \left(\left\lfloor \dfrac{y}{D} \right\rfloor + 1 \right) D = y + D - \langle y \rangle_D \\ \qquad\qquad\qquad \text{if } \langle y \rangle_D \in [D - l_-, D-1]. \end{cases} \tag{19}$$

***Example** 2.1:* *Let $m = 16$ and consider the code $\mathcal{C} = \{0, 5, 10, 15\} \subseteq \mathbb{Z}_{16}$. For any $n \in \mathbb{N}$, the code $\mathcal{C}^n$ has a minimum max L_1 distance of $D = 5$ and so it is capable of correcting all symmetric errors of magnitude 2 (that is, $\mathcal{C}^n$ is 2-ASEC), or, all asymmetric errors of magnitude 4 (that is, $\mathcal{C}^n$ is 4-AAEC), or, in general, all negative errors of magnitude up to $l_- = l$ and all positive errors of magnitude up to $l_+ = 4 - l$, for all $l \in \{0, 1, 2, 3, 4\}$. Also, the code $\mathcal{C}^n$ is optimal because its rate is $R = (1/n) \log 4^n = \log 4 = \log \lceil 16/5 \rceil = C_0^{(NW)}(m, l_-, l_+)$.*

Now, we turn our attention to the (l_-, l_+)-AEC systematic code design. Before that, we derive the following bound which is useful in proving the optimality of the proposed systematic codes. The following theorem generalizes Theorem 4 and 10 given in [16] and shows a lower bound on the number of check digits to design a systematic (l_-, l_+)-AEC code with k information digits.

***Theorem** 2.6:* Let $\mathcal{C}$ be a systematic m-ary non-wrap around (l_-, l_+)-AEC code such that the number of information digits in a codeword is k. Then, the number of check digits, r, satisfies the following condition.

$$r \geq \left\lceil \frac{k \log D}{\log \lceil m/D \rceil} \right\rceil; \tag{20}$$

where $D = l_- + l_+ + 1$.

Proof: Because of Theorem 2.1, $\mathcal{C}$ is a systematic m-ary $(D-1)$-AAEC code. So, consider the subset of information vectors,

$$V = \{(v_{k-1}, v_{k-2}, \ldots, v_0) :\ 0 \leq v_i < D \text{ and } i \in [0, k-1]\}.$$

Vectors of V can be viewed as the set of all vectors of length k over $\mathbb{Z}_D$. Hence, for all $X, Y \in V$, $D_{max}^{(L_1)}(X, Y) < D$, and $|V| = D^k$. Therefore, from Theorem 2.1, the $D_{max}^{(L_1)}$ of every two checks assigned to vectors in V must be at least D. Theorem 2.4 gives a bound on the number of vectors satisfying such criterion and we get

$$|V| = D^k \leq A_{NW}(m, r, D-1) = \left\lceil \frac{m}{D} \right\rceil^r.$$

Taking the $\log$ on both sides of the above inequality we get the desired property. ■

By using the same systematic code design given in [16] for $(D-1)$-AAEC codes, we get optimal systematic (l_-, l_+)-AEC codes with the number of check digits reaching the lower bound in (20), i. e., the codes require exactly

$$r = \left\lceil \frac{k \log D}{\log \lceil m/D \rceil} \right\rceil \tag{21}$$

check digits to encode any number $k \in \mathbb{N}$ of information digits, with $D = l_- + l_+ + 1 \in \mathbb{N}$. We briefly describe this code design.

***Algorithm** 2.1 (Encoding Algorithm):*
Input: The information word:

$$X = (x_{k-1}, x_{k-2}, \ldots, x_0) \in \mathbb{Z}_m^k.$$

Output: The codeword:

$$X\,A = (x_{k-1}, x_{k-2}, \ldots, x_0, a_{r-1}, a_{r-2}, \ldots, a_0) \in \mathbb{Z}_m^{k+r}.$$

Perform the following steps.

1) Perform the following steps.
   1.a) Compute the component-wise $\bmod D$ operation of the vector X:
   $$Y = (y_{k-1}, y_{k-2}, \ldots, y_0) = (\langle x_{k-1} \rangle_D, \langle x_{k-2} \rangle_D, \ldots, \langle x_0 \rangle_D) = \langle X \rangle_D \in \mathbb{Z}_D^k.$$
   1.b) Compute the natural number whose expression in radix D is Y:
   $$z = [Y]_D = \sum_{i=0}^{k-1} y_i D^i \in \mathbb{Z}_{D^k} \subseteq \mathbb{N}.$$
2) Represent the number z in base $b = \lceil m/D \rceil$ as a vector:
   $$Z = (z_{r-1}, z_{r-2}, \ldots, z_0) = (z)_b^{[r]} \in \mathbb{Z}_b^r.$$
3) Compute the check symbol as
   $$A = (a_{r-1}, a_{r-2}, \ldots, a_0) = (Dz_{r-1}, Dz_{r-2}, \ldots, Dz_0) \in \mathbb{Z}_m^r.$$
4) Output the codeword $X\,A$ and **exit**.

***Example** 2.2:* *Let $m = 10$, $D = 3$ and $k = 4$. We encode the word $X = (6, 2, 8, 1)$ over $\mathbb{Z}_{10}$, assuming a maximum error level of $l_- = l_+ = 1$. The number of check digits needed is $r = \lceil 4 \log(2+1) / \log \lceil 10/(2+1) \rceil \rceil = 4$. With notation as above,*

$$z = 0 \times 3^3 + 2 \times 3^2 + 2 \times 3^1 + 1 \times 3^0 = 25$$

and thus $Z = (0, 1, 2, 1)$ is the representation of z in base $b = \lceil m/D \rceil = \lceil 10/3 \rceil = 4$. Therefore, the check is $A = (0, 3, 6, 3)$ and the encoded codeword is $X\,A = (6, 2, 8, 1, 0, 3, 6, 3)$.

The (l_-, l_+)-AEC decoding for these $(D-1)$-AAEC codes is done as follows.

***Algorithm** 2.2 (Decoding Algorithm):*
Input: The channel output:

$$X'\,A' = (x'_{k-1}, x'_{k-2}, \ldots, x'_0, a'_{r-1}, a'_{r-2}, \ldots, a'_0) \in \mathbb{Z}_m^{k+r}.$$

Output: The recovered codeword:

$$X\,A = (x_{k-1}, x_{k-2}, \dots, x_0, a_{r-1}, a_{r-2}, \dots, a_0) \in \mathbb{Z}_m^{k+r}.$$

Perform the following steps.

1) Recover the check word, $A = (a_{r-1}, a_{r-2}, \dots, a_0)$, with (19); that is, either subtract at most l_+ or add at most l_- from each a'_i so that each a_i is a multiple of D.
2) Perform the following steps.
   2.a) Compute the vector
   $$Z = (z_{r-1}, z_{r-2}, \dots, z_0) = \left(\frac{a_{r-1}}{D}, \frac{a_{r-2}}{D}, \dots, \frac{a_0}{D}\right) = \frac{A}{D} \in \mathbb{Z}_b^r.$$
   2.b) Compute the natural number whose expression in radix $b = \lceil m/D \rceil$ is Z:
   $$z = [Z]_b = \sum_{i=0}^{k-1} z_i b^i \in \mathbb{Z}_{b^k} \subseteq \mathbb{N}.$$
3) Represent the number z in base D as a vector:
   $$Y = (y_{k-1}, y_{k-2}, \dots, y_0) = (z)_D^{[k]} \in \mathbb{Z}_D^k.$$
4) Perform the following steps.
   4.a) Compute the error vector
   $$E = (e_{k-1}, e_{k-2}, \dots, e_0) \in [-l_-, l_+]^k$$
   such that:
   $$e_i \equiv \langle y_i - x'_i \rangle_D \bmod D \text{ and } e_i \in [-l_-, l_+] \subseteq \mathbb{Z}.$$
   4.b) Recover the information word as $X = X' + E$.
5) Output the codeword $X\,A$ and **exit**.

***Example** 2.3: Let the encoded word be as in Example 2.2. Let the channel output be $X'\,A' = (5, 3, 7, 2, 1, 2, 5, 4)$. Rounding the check symbols to the nearest multiple of 3, we get $A = (0, 3, 6, 3)$. As in Steps 2) and 3) of the algorithm, we compute $z = [0121]_4 = 25$, and $Y = (25)_3^4 = (0, 2, 2, 1)$. As in Step 4.a), we compute $E \in [-l_-, l_+]^4$ such that $E \equiv \langle Y - X' \rangle_3 \bmod 3 \equiv \langle(-5, -1, -5, -1)\rangle_3 \bmod 3 \equiv (1, 2, 1, 2) \bmod 3$. Thus, $E = (1, -1, 1, -1)$. As in Step 4.b), we compute $X = X' + E = (5, 3, 7, 2) + (1, -1, 1, -1) = (6, 2, 8, 1)$. Therefore, the correct codeword is $X\,A = (6, 2, 8, 1, 0, 3, 6, 3)$.*

A more general decoding algorithm for the codes is explained in Section IV of this paper.

From (21), the information rate of these optimal non-systematic non wrap-around (l_-, l_+)-AEC codes of length $n(k) = k + r \in \mathbb{N}$ is

$$R_{sys}^{(NW)}(k) \stackrel{\text{def}}{=} R_{sys}^{(NW)}(m, D; n(k)) \stackrel{\text{def}}{=} \frac{k}{k+r} = \frac{k}{k + \lceil k \log D / \log \lceil m/D \rceil \rceil}; \tag{22}$$

where $k \in \mathbb{N}$ is the number of information digits. Note that, from $x \le \lceil x \rceil \le x + 1$,

$$\frac{1}{1 + \log D / \log \lceil m/D \rceil + 1/k} \le R_{sys}^{(NW)}(k) \le \frac{1}{1 + \log D / \log \lceil m/D \rceil}. \tag{23}$$

Hence, from the code optimality shown in Theorem 2.6, the systematic zero error capacity

$$C_{0,sys}^{(NW)}(m, l_-, l_+) = \lim_{k\to\infty} R_{sys}^{(NW)}(k) = \frac{1}{1 + \log D / \log \lceil m/D \rceil} = \frac{\log \lceil m/D \rceil}{\log (D \cdot \lceil m/D \rceil)}. \tag{24}$$

In order to estimate how fast the sequence

$$\{R_{sys}^{(NW)}(k) : \ k \in \mathbb{N}\}$$

converges to the above quantity, consider the real function $f(x) \stackrel{\text{def}}{=} (1+x)^{-1}$ and let $\bar{x} \stackrel{\text{def}}{=} \log D / \log \lceil m/D \rceil \in \mathbb{R}^+$. Note that, from the mean value theorem,

$$0 \le f(\bar{x}) - f\left(\bar{x} + \frac{1}{k}\right) = |f'(\xi)| \cdot \left|\frac{1}{k}\right| \le \frac{1}{(1+\bar{x})^2 k}, \quad \text{with } \xi \in (\bar{x}, \bar{x} + 1/k).$$

So, from this, (23) and (24) it follows,

$$C_{0,sys}^{(NW)} - R_{sys}^{(NW)}(k) \le \frac{1}{1 + \log D / \log \lceil m/D \rceil} - \frac{1}{1 + \log D / \log \lceil m/D \rceil + 1/k} = f(\bar{x}) - f\left(\bar{x} + \frac{1}{k}\right) \le \frac{1}{(1+\bar{x})^2 k} \le \frac{1}{k}.$$

Hence, the sequence $\{R_{sys}^{(NW)}(k) : \ k \in \mathbb{N}\}$ reaches $C_{0,sys}^{(NW)}$ in such a way that $R_{sys}^{(NW)}(k)$ is very close to $C_{0,sys}^{(NW)}$ even for small values of k. In particular, if

$$\bar{k} \stackrel{\text{def}}{=} \frac{1}{\bar{x}} = \frac{\log \lceil m/D \rceil}{\log D} \in \mathbb{N} \tag{25}$$

then, from (22) and (24),

$$R_{sys}^{(NW)}(m, D; n(\bar{k})) = \frac{\bar{k}}{\bar{k}+1} = C_{0,sys}^{(NW)}(m, l_-, l_+) \le C_0^{(NW)}(m, l_-, l_+); \tag{26}$$

and so the systematic zero error capacity is reached with finite length $\bar{n} \stackrel{\text{def}}{=} \bar{k} + 1$. For example, if $m \in [253, 256]$ and $D = 4$ then $\bar{n} = 4$ and $\bar{k} = 3$. Furthermore, if (25) holds and $D|m$ then

$$R_{sys}^{(NW)}(m, D; n(\bar{k})) = C_{0,sys}^{(NW)}(m, l_-, l_+) = C_0^{(NW)}(m, l_-, l_+) = \log_m\left(\frac{m}{D}\right)$$

because of (16), (24) and (26). Hence, in this case, the proposed optimal systematic (l_-, l_+)-AEC codes also reach the zero error capacity with finite length $\bar{n}$. For example, $R(256, 4; 4) = C_{0,sys}^{(NW)}(256, l_-, l_+) = C_0^{(NW)}(256, l_-, l_+) = 0.75$. In general, if $\log \lceil m/D \rceil \simeq \log (m/D)$ then

$$R_{sys}^{(NW)}(m, D; n(\bar{k})) \simeq C_{0,sys}^{(NW)}(m, l_-, l_+) \simeq \frac{\log \lceil m/D \rceil}{\log m} = \log_m \left\lceil \frac{m}{D} \right\rceil = C_0^{(NW)}(m, l_-, l_+)$$

because of (16).

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III. WRAP AROUND ERROR CHANNEL

The wrap around (l_-, l_+) error channel model is defined by (1). In this section, the probelm of designing systematic and non-systematic efficient m-ary wrap around (l_-, l_+)-AEC, $m, l_-, l_+ \in \mathbb{N}$, codes is addressed for this channel model.

A. Combinatorial characterization of error correcting codes for the wrap around error model

In order to design codes for the wrap around error model, a slightly different distance metric, which is defined below, is useful.

For $m \in \mathbb{N}$, let $x, y \in \mathbb{Z}_m$. The m-ary Lee distance

$$D^{(Lee)}(m; x, y) \stackrel{\text{def}}{=} D^{(Lee)}(x, y)$$

between x and y is defined as

$$\begin{aligned} D^{(Lee)}(x, y) &\stackrel{\text{def}}{=} \min\{\langle x-y\rangle_m, \langle y-x\rangle_m\} \qquad (27) \\ &= \min\{\langle x-y\rangle_m, m - \langle x-y\rangle_m\} \\ &= \min\left\{D^{(L_1)}(x, y), m - D^{(L_1)}(x, y)\right\} \\ &= \min\{|x-y|, m-|x-y|\} \\ &= D^{(Lee)}(0, |x-y|) \in \left[0, \left\lfloor \frac{m}{2} \right\rfloor\right]. \end{aligned}$$

For example, when $m = 7$, $x = 2$ and $y = 6$, $D^{(Lee)}(2, 6) = \min\{3, 4\} = 3$. The following distance metric is useful in designing (l_-, l_+)-AEC codes.

***Definition** 3.1 (m-ary max Lee distance):* Given $n \in \mathbb{N}$, let $X = (x_{n-1}, x_{n-2}, \dots, x_0) \in \mathbb{Z}_m^n$ and $Y = (y_{n-1}, y_{n-2}, \dots, y_0) \in \mathbb{Z}_m^n$. The m-ary max Lee distance,

$$D_{max}^{(Lee)}(m; X, Y) \stackrel{\text{def}}{=} D_{max}^{(Lee)}(X, Y),$$

between X and Y is defined as:

$$D_{max}^{(Lee)}(X, Y) = \max_{i \in [0, n-1]} \left\{ D^{(Lee)}(x_i, y_i) = \min\{|x_i - y_i|, m - |x_i - y_i|\} \right\}. \qquad (28)$$

For example, if $m = 7$, $n = 4$, $X = (0, 4, 2, 1)$ and $Y = (4, 2, 3, 1)$ then $D_{max}^{(Lee)}(X, Y) = \max\{3, 2, 1, 0\} = 3$. It is worth noting that, as $D_{max}^{(L_1)}(X, Y)$, $D_{max}^{(Lee)}(X, Y)$ defines a metric. Furthermore, for all $m, n \in \mathbb{N}$,

$$\text{for all } X, Y \in \mathbb{Z}_m^n,\ D_{max}^{(Lee)}(m; X, Y) \le D_{max}^{(L_1)}(X, Y); \qquad (29)$$

because of (27), (28) and (5). Also, $D_{max}^{(Lee)}(X, Y) \in [0, \lfloor m/2 \rfloor]$ because of (27).

The following theorem gives the necessary and sufficient conditions on the minimum distance for error correction.

***Theorem** 3.1 (characterization of (l_-, l_+)-AEC for wrap around errors):* Let $m, n \in \mathbb{N}$ and $l_-, l_+ \in \mathbb{N}$. An m-ary code $\mathcal{C} \subseteq \mathbb{Z}_m^n$ is capable of correcting all negative wrap around errors of magnitude up to l_- and all positive wrap around errors of magnitude up to l_+ if, and only if,

$$\begin{aligned} &\text{for all } X, Y \in \mathcal{C}, \qquad (30) \\ &X \ne Y \implies D_{max}^{(Lee)}(m; X, Y) > l_- + l_+. \end{aligned}$$

Proof: Given (1), for all $n = 1, 2, \dots$, let

$$S_{m,n,l_-,l_+}^{(WA)}(X) \stackrel{\text{def}}{=} \left\{ Z \in \mathbb{Z}_m^n \;\middle|\; \begin{array}{l} \text{for all } i \in [0, n-1], \\ z_i = \langle x_i + e_i \rangle_m \text{ and} \\ e_i \in [-l_-, l_+] \cap \mathbb{Z} \end{array} \right\} \qquad (31)$$

be the set of m-ary vectors obtained from $X \in \mathbb{Z}_m^n$ due to any number of negative errors of magnitude up to l_- and any number of positive errors of magnitude up to l_+, where the errors may wrap around. Given (31), the proof follows similarly to the proof of Theorem 2.1. ■

From Theorem 3.1, the following corollaries follow analogously to the non-wrap around case.

***Corollary** 3.2 (characterization of l-AAEC for wrap around errors):* Let $m, n, l \in \mathbb{N}$. A code $\mathcal{C} \subseteq \mathbb{Z}_m^n$ is an l-AAEC code if, and only if,

$$\text{for all } X, Y \in \mathcal{C}, \quad X \ne Y \implies D_{max}^{(L_1)}(X, Y) > l.$$

***Corollary** 3.3 (characterization of l-ASEC for wrap around errors):* Let $m, n, l \in \mathbb{N}$. A code $\mathcal{C} \subseteq \mathbb{Z}_m^n$ is an l-ASEC code if, and only if,

$$\text{for all } X, Y \in \mathcal{C}, \quad X \ne Y \implies D_{max}^{(L_1)}(X, Y) > 2l.$$

Suppose $X, Y \in \mathbb{Z}_m^n$. If $\lfloor m/2 \rfloor \le l_- + l_+$ then

$$D_{max}^{(Lee)}(X, Y) \le \left\lfloor \frac{m}{2} \right\rfloor \le l_- + l_+$$

and so, the (l_-, l_+)-AEC code can have only one code word, which is not interesting. So, in the rest of the paper, for the wrap around error model, it is assumed that $l_- + l_+ < \lfloor m/2 \rfloor$; that is, $2(l_- + l_+) \le m - 1$. Also, note that any wrap around (l_-, l_+)-AEC code is a non-wrap around (l_-, l_+)-AEC code because of Theorem 2.1, Theorem 3.1 and (29).

B. Zero error capacity derivations and bounds

As in Subsection II-B, given $m, n, l_-, l_+ \in \mathbb{N}$, let

$$D \stackrel{\text{def}}{=} l_- + l_+ + 1 \in \mathbb{N},$$

$A_{WA}(m, n, l_-, l_+) \in \mathbb{N}$ be the maximum cardinality for a wrap around (l_-, l_+)-AEC code of length n,

$$C_0^{(WA)}(m, n, l_-, l_+) = \frac{\log A_{WA}(m, n, l_-, l_+)}{n}$$

be the maximum information rate achievable by using the channel n times and

$$C_0^{(WA)}(m, l_-, l_+) = \sup_{n \in \mathbb{N}} C_0^{(WA)}(m, n, l_-, l_+)$$

be the the zero error capacity of the m-ary WA-(l_-, l_+)-channel model. As in the non-wrap around case, we have

$$\begin{aligned} A_{WA}(m, n, l_-, l_+) = A_{WA}(m, n, 0, l_- + l_+) &\stackrel{\text{def}}{=} \qquad (32) \\ A_{WA}(m, n, D-1). \end{aligned}$$

This comes from Theorem 3.1. However, as the 5-ary WA-$(0, 1)$ channel example mentioned in Example 1.1 shows, the wrap around zero error capacity may depend on the number, n, of independent uses of the channel and determining the exact value of $A_{WA}(m, n, l_-, l_+)$ may be challenging, unless

D divides m, in which case, we get an exact value as derived below.

***Theorem** 3.4:* If $m, n, l_-, l_+, D = l_- + l_+ + 1 \in \mathbb{N}$ then

$$\left(\left\lfloor \frac{m}{D} \right\rfloor\right)^n \le A_{WA}(m,n,l_-,l_+) \le \left\lfloor \left(\frac{m}{D}\right)^n \right\rfloor \le \left(\frac{m}{D}\right)^n. \quad (33)$$

Hence, if $D|m$ then

$$A_{WA}(m,n,l_-,l_+) = \left(\frac{m}{D}\right)^n,$$

and so, the zero error capacity is

$$C_0^{(WA)}(m,l_-,l_+) = \sup_{\nu \in \mathbb{N}} C_0^{(WA)}(m,\nu,l_-,l_+) = C_0^{(WA)}(m,n,l_-,l_+) = \frac{\log A_{WA}(m,n,l_-,l_+)}{n} = \log\left(\frac{m}{D}\right).$$

Thus, if $D|m$ then the zero error capacity is independent of n.

Proof: Unlike the non-wrap around case, a sphere packing argument can be used to derive a good upper-bound because the error spheres in (31) have the same cardinality given by

$$|S^{(WA)}_{m,n,l_-,l_+}(X)| = D^n, \qquad \text{for all } X \in \mathbb{Z}_m^n.$$

Thus, if $\mathcal{C}$ is any (l_-, l_+)-AEC code of length n then

$$|\mathcal{C}|D^n = \sum_{X \in \mathcal{C}} |S^{(WA)}_{m,n,l_-,l_+}(X)| \le |\mathbb{Z}_m^n| = m^n;$$

and so, the following upper-bound holds.

$$|\mathcal{C}| \le A_{WA}(m,n,l_-,l_+) \le \left(\frac{m}{D}\right)^n. \quad (34)$$

With respect to a lower-bound, consider the m-ary codes of length $n \in \mathbb{N}$ defined as follows.

$$\text{For all } V \stackrel{\text{def}}{=} (v_{n-1}, v_{n-2}, \ldots, v_0) \in \mathbb{Z}_m^n, \quad (35)$$
$$\mathcal{C} \stackrel{\text{def}}{=} \mathcal{C}_n(V) \stackrel{\text{def}}{=} \prod_{i=0}^{n-1} \left\{x \in \mathbb{Z}_m : x = \langle v_i + bD \rangle_m \text{ and } b \in \mathbb{Z}_{\lfloor m/D \rfloor}\right\} = \prod_{i=0}^{n-1} \mathcal{C}_1(v_i),$$

where, as in (14), the product indicates a cartesian product. Note that the minimum max Lee distance of these codes is D because, up to a constant, every codeword component is a multiple of D under the $\bmod m$ operation and because $b \le \lfloor m/D \rfloor - 1$. Note also that the cardinality of the above codes is

$$|\mathcal{C}_n(V)| = |\mathcal{C}_1(0)|^n = \left|\mathbb{Z}_{\lfloor m/D \rfloor}\right|^n = \left\lfloor \frac{m}{D} \right\rfloor^n \le A_{WA}(m,n,l_-,l_+). \quad (36)$$

The rest of the theorem follows from (34), (36) and the zero error capacity definition. ■

When D does not divide m, determining the exact value of the zero error capacity is still an open problem [11]. However, the bounds in (33) are still valid, and so the following bounds hold for the wrap around zero error capacity and the maximum achievable information rate.

$$\log\left\lfloor \frac{m}{D} \right\rfloor = C_0^{(WA)}(m, n=1, l_-, l_+) \le C_0^{(WA)}(m, n=2, l_-, l_+) \le C_0^{(WA)}(m, l_-, l_+) \le \log\left(\frac{m}{D}\right). \quad (37)$$

We note that, from the general Theorem 1 in [18], the above upper-bound $C_0^{(WA)}(m,l_-,l_+) \le \log(m/D)$ can also be derived by taking the minimum of the normal capacity $C^{(WA)} = C^{(WA)}(p_{-l_-}, p_{-l_-+1}, \ldots, p_{+l_+})$ in (4) over the probability distribution defined by the p_i's for $i \in [-l_-, l_+]$.

From Theorem 3.4, note that when $D = l_- + l_+ + 1$ divides m, the zero error capacity for the wrap around asymmetric limited magnitude error channel (where $l_- = 0$ and $l_+ = l$) is given by

$$C_0^{(WA)}(m,0,l) = \log\left(\frac{m}{l+1}\right). \quad (38)$$

Similarly, when $D|m$, the zero error capacity for the wrap around symmetric limited magnitude error channel (where $l_- = l_+ = l$) is given by

$$C_0^{(WA)}(m,l,l) = \log\left(\frac{m}{2l+1}\right). \quad (39)$$

In general, for both cases, the bounds in (37) hold for $C_0^{(WA)}(m,0,l)$ and $C_0^{(WA)}(m,l,l)$.

C. Code designs

As before, first, we describe the non-systematic (l_-, l_+)-AEC code design, where $l_-, l_+ \in \mathbb{N}$ and $D - 1 = l_- + l_+ \le \lfloor m/2 \rfloor$. In Subsection III-B, we have seen that the codes in (35) are zero error capacity achieving (i. e., optimal) if D divides m. The codes are obtained by taking the code symbols input to the channel as

$$\langle v + bD \rangle_m = (v + bD) \bmod m,$$

where $b = 0, 1 \ldots, \lfloor m/D \rfloor - 1$ and $v \in \mathbb{Z}_m$ is a fixed element. For example, by letting $v = 0$ for every component, the following codes are obtained.

$$\mathcal{C}_n(\mathbf{0}) \stackrel{\text{def}}{=} \{(x_{n-1}, x_{n-2}, \ldots, x_0) \in \mathbb{Z}_m^n : x_i/D \in \mathbb{Z}_{\lfloor m/D \rfloor} \text{ for all } i \in [0, n-1]\} = \left\{x \in \mathbb{Z}_m : x = bD \text{ with } b \in \mathbb{Z}_{\lfloor m/D \rfloor}\right\}^n = \mathcal{C}_1(0)^n \quad (40)$$

For these last codes, a received symbol $y \in \mathbb{Z}_m$ is decoded into the code symbol $x \in \mathbb{Z}_m$ by rounding it to the closest integer multiple of D according to the rule in (19), taking into account the $\bmod m$ operation (i. e., the wrap around).

***Example** 3.1: Let $m = 16$ and consider the code $\mathcal{C} = \{0, 4, 8, 12\} \subseteq \mathbb{Z}_{16}$. For any $n \in \mathbb{N}$, the code $\mathcal{C}^n$ has a minimum max Lee distance of $D = 4$ and so it is capable of correcting all negative errors of magnitude up to $l_- = l$ and all positive errors of magnitude up to $l_+ = 3 - l$, for all $l \in \{0, 1, 2, 3\}$. Also, the code $\mathcal{C}^n$ is optimal because its rate is $R = (1/n)\log 4^n = \log 4 = \log\lfloor 16/4 \rfloor = \log(16/4) = C_0^{(WA)}(m, l_-, l_+)$.*

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When D does not divide m, the codes in (35), which are obtained by concatenating optimal non-systematic codes of length $n = 1$, are close to optimal non-systematic (l_-, l_+)-AEC codes for the wrap around error model. However, they may not be optimal in general, as the following Example 3.2 shows.

***Example** 3.2: Let $m = 16$ and consider the code $\mathcal{C} = \{0, 5, 10\} \subseteq \mathbb{Z}_{16}$ (note that, unlike the code in Example 2.1 for the non-wrap around model, the symbol 15 cannot be in $\mathcal{C}$). For any $n \in \mathbb{N}$, the code $\mathcal{C}^n$ has a minimum max Lee distance of $D = 5$ and so it has exactly the same error correcting abilities of the code in Example 2.1, but for the wrap around error model. Also, the code rate is $R = \log(3)$, wheras*

$$C_0^{(WA)}(m = 16, l_-, l_+) \in \left[\log\left\lfloor\frac{16}{5}\right\rfloor, \log\left(\frac{16}{5}\right)\right] = [\log(3), \log(3.2)].$$

By considering the concatenation of codes of length $n = 2$, some improvement can be obtained, as shown in the following two examples.

***Example** 3.3: Let $m = 8$, $n = 2$ and $D = 3$. Consider the code*

$$\mathcal{C} = \{00, 13, 36, 41, 64\} \subseteq \mathbb{Z}_8^2.$$

The code $\mathcal{C}$ has length 2, cardinality 5 (> 4), a minimum max Lee distance of $D = 3$ and, hence, an information rate of

$$R = \frac{1}{2}\log(5) = \log\left(\sqrt{5}\right) = \log(2.236).$$

On the other hand,

$$C_0^{(WA)}(m = 8, l_-, l_+) \in \left[\log\left\lfloor\frac{8}{3}\right\rfloor, \log\left(\frac{8}{3}\right)\right] = [\log(2), \log(2.666)].$$

***Example** 3.4: Let $m = 16$, $n = 2$ and $D = 3$. Consider the code*

$$\begin{aligned}\mathcal{C} = \{&\underline{0}\,\underline{0}, \underline{0}\,\underline{4}, \underline{1}\,\underline{7}, \underline{1}\,\underline{10}, \underline{2}\,\underline{13},\\ &\underline{3}\,\underline{0}, \underline{3}\,\underline{3}, \underline{4}\,\underline{6}, \underline{4}\,\underline{9}, \underline{5}\,\underline{12},\\ &\underline{6}\,\underline{2}, \underline{6}\,\underline{15}, \underline{7}\,\underline{5}, \underline{7}\,\underline{8}, \underline{8}\,\underline{11},\\ &\underline{9}\,\underline{1}, \underline{9}\,\underline{14}, \underline{10}\,\underline{4}, \underline{10}\,\underline{7}, \underline{11}\,\underline{10},\\ &\underline{12}\,\underline{13}, \underline{13}\,\underline{0}, \underline{13}\,\underline{3}, \underline{13}\,\underline{6}, \underline{14}\,\underline{9},\\ &\underline{15}\,\underline{13}\} \subseteq \mathbb{Z}_{16}^2.\end{aligned}$$

The code $\mathcal{C}$ has length 2, cardinality 26 (> 25), a minimum max Lee distance of $D = 3$ and, hence, an information rate of

$$R = \frac{1}{2}\log(26) = \log\left(\sqrt{26}\right) = \log(5.099).$$

On the other hand,

$$C_0^{(WA)}(m = 16, l_-, l_+) \in \left[\log\left\lfloor\frac{16}{3}\right\rfloor, \log\left(\frac{16}{3}\right)\right] = [\log(5), \log(5.333)].$$

We suspect all the above examples to be indeed optimal non-systematic minimum max Lee distance D codes. Also, a general method of how to construct these codes is still an open problem.

Now we focus our attention on systematic code design. Before describing the code design method, first we investigate the minimum number of check digits needed to encode information vectors of a certain length in any systematic (l_-, l_+)-AEC code.

***Theorem** 3.5:* Let $\mathcal{C}$ be any systematic m-ary wrap around (l_-, l_+)-AEC code of length $n \in \mathbb{N}$ such that the number of information digits in a codeword is $k \in \mathbb{N}$. Then, the number of check digits, $r = n - k \in \mathbb{N}$, satisfies the relation,

$$r \geq \left\lceil \frac{k \log D}{\log(m/D)} \right\rceil; \tag{41}$$

where $D = l_- + l_+ + 1$.

Proof: The proof follows exactly as that of Theorem 2.6. ∎

Let $d \in \mathbb{N}$ be defined as

$$d \stackrel{\text{def}}{=} \left\lceil \frac{m}{\lfloor m/D \rfloor} \right\rceil \geq \frac{m}{\lfloor m/D \rfloor} \geq D \tag{42}$$

and note that if $D|m$ then $d = D$. Furthermore,

$$\left\lfloor \frac{m}{D} \right\rfloor \geq \left\lceil \frac{m}{d} \right\rceil \geq \frac{m}{d}. \tag{43}$$

In the rest of the paper, given $m, l_-, l_+, L \in \mathbb{N}$ with $D = l_- + l_+ + 1$, optimal or close to optimal m-ary (l_-, l_+)-AEC systematic codes which require

$$r = \left\lceil \frac{k \log d}{\log \lfloor m/D \rfloor} \right\rceil \leq \left\lceil \frac{k \log d}{\log(m/d)} \right\rceil \in \mathbb{N} \tag{44}$$

check digits are presented, where $k \in \mathbb{N}$ is the number of information digits. From (42), (43) and (44), when k is large, the information rate of these codes of length $n(k) = k + r \in \mathbb{N}$ is

$$\begin{aligned} R_{sys}^{(WA)}(m, D; n(k)) &\stackrel{\text{def}}{=} \frac{k}{k+r} = \\ &\frac{k}{k + \lceil k \log d / \log \lfloor m/D \rfloor \rceil} = \\ &\frac{k}{k + \lceil k \log \lceil m/\lfloor m/D \rfloor \rceil / \log \lfloor m/D \rfloor \rceil} \simeq \\ &\frac{1}{1 + \log \lceil m/\lfloor m/D \rfloor \rceil / \log \lfloor m/D \rfloor} = \\ &\frac{\log \lfloor m/D \rfloor}{\log \lfloor m/D \rfloor + \log \lceil m/\lfloor m/D \rfloor \rceil} = \\ &\frac{\log \lfloor m/D \rfloor}{\log(\lfloor m/D \rfloor \cdot \lceil m/\lfloor m/D \rfloor \rceil)}; \end{aligned} \tag{45}$$

and, if $\bar{k} \stackrel{\text{def}}{=} \log \lfloor m/D \rfloor / \log \lceil m/\lfloor m/D \rfloor \rceil \in \mathbb{N}$ then with finite length $\bar{n} = \bar{k} + 1$,

$$R_{sys}^{(WA)}(m, D; \bar{n}) = \frac{\bar{k}}{\bar{k}+1} = \frac{\log \lfloor m/D \rfloor}{\log(\lfloor m/D \rfloor \cdot \lceil m/\lfloor m/D \rfloor \rceil)}.$$

Furthermore, from (41) and (37), if $D|m$ then $d = D$ and the proposed systematic (l_-, l_+)-AEC codes are optimal systematic and non-systematic (l_-, l_+)-AEC codes (i. e., they achieve the zero error capacities). If instead D does not divide m and d is close to D then the codes are close to optimal; however, in general, there is some room for improvement, especially when m/D is low (see Example 4.9 and Example 4.10).

IV. General systematic code design

For both wrap around and non-wrap around error model, a very general systematic coding scheme can be designed. In fact, this systematic code construction is so general that it can be applied to do error correction for any channel model. The idea is very simple and can be described as follows. For example, assume wrap around model. Given $m, s, d \in \mathbb{N}$, let

$$\mathcal{P} = \{\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_{d-1}\} \tag{46}$$

be a partition (or, more generally, a covering set with d to be as small as possible) of $\mathbb{Z}_m^s$ into (l_-, l_+)-AEC codes where, as before, $l_-, l_+, D = l_- + l_+ + 1 \in \mathbb{N}$. Let

$$\rho : \mathbb{Z}_m^s \to \mathbb{Z}_d$$

be the function which associates every $X \in \mathbb{Z}_m^s$ with the index of the code in the partition which contains X; formally,

$$\rho(X) = i \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad X \in \mathcal{C}_i \in \mathcal{P}. \tag{47}$$

Example *4.1: If $m = 17$, $D = 3$ and $s = 1$ then a value of $d = 4$ can be obtained with the following partition of $\mathbb{Z}_m^s = \mathbb{Z}_{17}$ into minimum max Lee distance $D = 3$ codes.*

$$\mathcal{P} = \left\{ \begin{array}{l} \mathcal{C}_0 \stackrel{\text{def}}{=} \{0, 4, 8, 11, 14\}, \\ \mathcal{C}_1 \stackrel{\text{def}}{=} \{1, 5, 9, 12, 15\}, \\ \mathcal{C}_2 \stackrel{\text{def}}{=} \{2, 6, 10, 13, 16\}, \\ \mathcal{C}_3 \stackrel{\text{def}}{=} \{3, 7\} \end{array} \right\}. \tag{48}$$

In this way,

$$\begin{array}{ll} \rho(0) = \rho(4) = \rho(8) & = \rho(11) = \rho(14) = 0, \\ \rho(1) = \rho(5) = \rho(9) & = \rho(12) = \rho(15) = 1, \\ \rho(2) = \rho(6) = \rho(10) & = \rho(13) = \rho(16) = 2, \\ \rho(3) = \rho(7) & \qquad\qquad\qquad\quad = 3. \end{array}$$

For simplicity, let the number of information digits be $K \in \mathbb{N}$ where $K/s \stackrel{\text{def}}{=} k \in \mathbb{N}$ so that K is a multiple of s. In this way, the function ρ can be easily componentwise extended to a vector function

$$\rho : (\mathbb{Z}_m^s)^k \equiv \mathbb{Z}_m^K \to \mathbb{Z}_d^k \tag{49}$$

defined as

$$\rho(X) = (\rho(X_{k-1}), \rho(X_{k-2}), \ldots, \rho(X_0)),$$
$$\text{for all } X = X_{k-1}X_{k-1} \ldots X_0 \in \mathbb{Z}_m^K.$$

Example *4.2 (Example 4.1 continued): Continuing the previous example with $m = 17$, $D = 3$, $s = 1$ and $d = 4$, for $K = k = 8$ we have*

$$\begin{array}{rl} X = & \underline{1}\ \underline{0}\ \underline{16}\ \underline{7}\ \underline{1}\ \underline{12}\ \underline{13}\ \underline{2} \quad \Longrightarrow \\ \rho(X) = & 1\ 0\ 2\ 3\ 1\ 1\ 2\ 2. \end{array} \tag{50}$$

Now, let us assume $X \in \mathbb{Z}_m^K$ is the sent information part of a codeword and $Y \in \mathbb{Z}_m^K$ is its received version. Note that if the receiver knows $\rho(X)$ then it is capable of performing (l_-, l_+)-All Error Correction because, for every received part $Y_i \in \mathbb{Z}_m^s$, $i \in [0, k-1]$, the receiver knows that the corresponding $X_i \in \mathcal{C}_{\rho(X_i)}$, and so, it recover X_i by applying the (l_-, l_+)-AEC procedure of $\mathcal{C}_{\rho(X_i)}$ with input Y_i.

Example *4.3 (Example 4.1 continued): Continuing the example with $m = 17$, $K = k = 8$, $D = 3$. Assume we want to perform $(l_- = 1, l_+ = 1)$-AEC and $X = (1, 0, 16, 7, 1, 12, 13, 2)$ in (50) is the sent information part. Suppose the received information vector is $Y = (1, 16, 0, 8, 2, 12, 12, 2)$. Since it is assumed that $\rho(X)$ is known to the receiver, it knows*

$$\begin{array}{rl} Y = & \underline{1}\ \underline{16}\ \underline{0}\ \underline{8}\ \underline{2}\ \underline{12}\ \underline{12}\ \underline{2}, \\ \rho(X) = & 1\ 0\ 2\ 3\ 1\ 1\ 2\ 2. \end{array}$$

Hence, since $\mathcal{C}_0$, $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ are $(1, 1)$-AEC codes, the receiver,

- *knows that $X_7 \in \mathcal{C}_1 = \{1, 5, 9, 12, 15\}$ and $Y_7 = 1$. So it can apply the $(1, 1)$-AEC of $\mathcal{C}_1$ on input $Y_7 = 1$ and recover $X_7 = 1$,*
- *knows that $X_6 \in \mathcal{C}_0 = \{0, 4, 8, 11, 14\}$ and $Y_6 = 16$. So it can apply the $(1, 1)$-AEC of $\mathcal{C}_0$ on input $Y_6 = 16$ and recover $X_6 = 0$,*
- *knows that $X_5 \in \mathcal{C}_2 = \{2, 6, 10, 13, 16\}$ and $Y_5 = 0$. So it can apply the $(1, 1)$-AEC of $\mathcal{C}_2$ on input $Y_5 = 0$ and recover $X_5 = 16$,*
- *knows that $X_4 \in \mathcal{C}_3 = \{3, 7\}$ and $Y_4 = 8$. So it can apply the $(1, 1)$-AEC of $\mathcal{C}_3$ on input $Y_4 = 8$ and recover $X_4 = 7$,*
- *and so on.*

At this point the code design idea for an m-ary systematic (l_-, l_+)-AEC code $\mathcal{C}$ is very simple and is captured by the following algorithm.

Algorithm *4.1 (*General Encoding Algorithm):
Input: the information word $X \in \mathbb{Z}_m^K$, $K = sk$ with $s, k \in \mathbb{N}$.
Output: the codeword

$$\mathcal{E}(X) = X\,A \in \mathcal{C} \subseteq \mathbb{Z}_m^n$$

where $n = K + r$ and $A \in \mathbb{Z}_m^r$ is the check word.
Perform steps S1, S2, S3 and S4.
S1: Compute $\rho(X) \in \mathbb{Z}_d^k$ according to (47) and (49).
S2: Encode $\rho(X)$ into an m-ary (l_-, l_+)-AEC code $\tilde{\mathcal{C}}$ with the smallest possible length $r \in \mathbb{N}$. Let $\tilde{\mathcal{E}}(\rho(X)) \in \tilde{\mathcal{C}}$ be this encoding of $\rho(X)$, where

$$\tilde{\mathcal{E}} : \mathbb{Z}_d^k \to \tilde{\mathcal{C}} \subseteq \mathbb{Z}_m^r. \tag{51}$$

S3: Append the check word $A \stackrel{\text{def}}{=} \tilde{\mathcal{E}}(\rho(X)) \in \mathbb{Z}_m^r$ to X to get the systematic encoding of X as

$$\mathcal{E}(X) = X\,\tilde{\mathcal{E}}(\rho(X)) = X\,A \in \mathbb{Z}_m^n.$$

S4: Output $\mathcal{E}(X) \in \mathcal{C}$ and **exit**.

Example *4.4: If we need to design a systematic wrap around m-ary (l_-, l_+)-AEC code, $\mathcal{C}$, with k information digits with the following parameters: $m = 12$, $K = k = 5$ (i. e., $s = 1$), $l_- = l_+ = 1$ then Algorithm 4.1 can be implemented as follows. Since $D = 3$ divides $m = 12$, the partition in (46) can be defined with the codes of length $n = 1$ in (35) by letting,*

$$\begin{array}{l} \mathcal{C}_v \stackrel{\text{def}}{=} \mathcal{C}_1(v) = \\ \qquad \{x \in \mathbb{Z}_{12} :\ x = v + b \cdot 3 \ \textit{ with } b \in \{0, 1, 2, 3\}\} = \\ \qquad (3\mathbb{Z} + v) \cap \mathbb{N}, \quad \textit{for all } v \in \mathbb{Z}_3. \end{array}$$

In this way, $d = D = 3$ *and*

$$\mathcal{P} = \left\{ \begin{array}{l} \mathcal{C}_0 \stackrel{\text{def}}{=} \{0,3,6,9\}, \\ \mathcal{C}_1 \stackrel{\text{def}}{=} \{1,4,7,10\}, \\ \mathcal{C}_2 \stackrel{\text{def}}{=} \{2,5,8,11\} \end{array} \right\}.$$

Suppose $X = \underline{6}\,\underline{2}\,\underline{8}\,\underline{1}\,\underline{11} \in \mathbb{Z}_{12}$ *needs to be encoded. Then, in Step S1 of Algorithm 4.1, the function* ρ *in (49) can be defined as the remainder* $\bmod 3$ *vector*

$$\rho(X) = X \bmod 3 = \underline{0}\,\underline{2}\,\underline{2}\,\underline{1}\,\underline{2} \in \mathbb{Z}_3^5.$$

In (51), we can let $\tilde{\mathcal{C}} \stackrel{\text{def}}{=} \mathcal{C}_0^r = \{0,3,6,9\}^r$ *where the length* r *of* $\tilde{\mathcal{C}}$ *is the smallest possible natural such that there exists an encoding function* $\mathcal{E} : \mathbb{Z}_3^5 \to \tilde{\mathcal{C}} \subseteq \mathbb{Z}_{12}^r$*; that is, such that* $3^5 = |\mathbb{Z}_3^5| \le |\tilde{\mathcal{C}}| = |\mathcal{C}_0|^r = 4^r$*. Then, in Step S2 of Algorithm 4.1, this least redundant encoding function* $\tilde{\mathcal{E}}$ *can be defined as follows.*

1) *compute the natural number whose radix* $d = 3$ *expression is* $\rho(X) \in \mathbb{Z}_3^5$ *as*

$$[\rho(X)]_d = \underline{0}\times 3^4 + \underline{2}\times 3^3 + \underline{2}\times 3^2 + \underline{1}\times 3^1 + \underline{2}\times 3^0 = 77;$$

2) *convert this number in radix* $b = |\tilde{\mathcal{C}}| = \lceil m/d \rceil = 12/3 = 4$ *form with smallest possible length*

$$r = \lceil \log_b |\mathbb{Z}_d^k| \rceil = \lceil \log_4 3^5 \rceil = \lceil 3.962 \rceil = 4.$$

That is,

$$([\rho(X)]_d)_b^{[r]} = (77)_4^{[4]} = \underline{1}\,\underline{0}\,\underline{3}\,\underline{1} \in \mathbb{Z}_4^4;$$

3) *multiply each component of* $([\rho(X)]_d)_b^{[r]}$ *by* $D = 3$ *to obtain the vector*

$$A = 3 \cdot ([\rho(X)]_d)_b^{[r]} = \underline{3}\,\underline{0}\,\underline{9}\,\underline{3} \in \tilde{\mathcal{C}}^4 \subseteq \mathbb{Z}_{12}^4.$$

Therefore, in Step S3 of Algorithm 4.1, the encoded word is

$$X\,A = \underline{6}\,\underline{2}\,\underline{8}\,\underline{1}\,\underline{11}\,\underline{3}\,\underline{0}\,\underline{9}\,\underline{3} \in \mathcal{C}. \tag{52}$$

Given the encoding Algorithm 4.1, the decoding of the above m-ary systematic (l_-, l_+)-AEC code $\mathcal{C}$ can be performed as follows.

***Algorithm** 4.2* (General Decoding Algorithm):
Input: the received word $Y\,A' \in \mathbb{Z}_m^n$ where $n = K + r$, $K = sk$ with $s, k \in \mathbb{N}$, $Y \in \mathbb{Z}_m^K$ and $A' \in \mathbb{Z}_m^r$.
Output: the information word $X \in \mathbb{Z}_m^K$.
Perform steps S1, S2, S3 and S4.
S1: Apply the (l_-, l_+)-AEC procedure of $\tilde{\mathcal{C}}$ with input A' and correct all the errors in the check part. In this way, the output from this procedure is exactly $A = \tilde{\mathcal{E}}(\rho(X)) \in \tilde{\mathcal{C}} \subseteq \mathbb{Z}_m^r$.
S2: From A, compute $\rho(X) = \tilde{\mathcal{E}}^{-1}(A) \in \mathbb{Z}_d^k$ according to the definition of $\tilde{\mathcal{E}}$ in (47) and (49). At this point the receiver knows the received information part Y and $\rho(X)$.
S3: From Y and $\rho(X)$, correct all the errors in the information part Y to get X. Note that the error values must be in $[-l_-, l_+] \cap \mathbb{Z}$.
S4: Output $X \in \mathbb{Z}_m^K$ and **exit**.

***Example** 4.5 (Example 4.4 continued): Let the encoded word be the one in (52) and let the channel output, given as input to Algorithm 4.2, be*

$$Y\,A' = \underline{5}\,\underline{3}\,\underline{7}\,\underline{2}\,\underline{0}\;\; \underline{2}\,\underline{11}\,\underline{9}\,\underline{2} \in \mathbb{Z}_{12}^9.$$

In Step S1 of Algorithm 4.2, by rounding the check symbols to the nearest multiple of 3 *and then taking the componentwise* $\bmod m = \bmod 12$ *operation we correct the errors in the check part and get*

$$A = \underline{3}\,\underline{0}\,\underline{9}\,\underline{3} \in \mathbb{Z}_{12}^4.$$

In Step S2 of Algorithm 4.2, we compute $\rho(X) = \tilde{\mathcal{E}}^{-1}(A)$ *as follows.*

1) *Divide each component of* A *by* $D = 3$ *to get the vector*

$$\frac{A}{3} = \underline{1}\,\underline{0}\,\underline{3}\,\underline{1} = ([\rho(X)]_d)_b^{[r]} \in \mathbb{Z}_4^4;$$

2) *compute the number whose expression in radix* $b = 4$ *is* $A/3$*, i. e.,*

$$\left[\frac{A}{3}\right]_4 = [\underline{1}\,\underline{0}\,\underline{3}\,\underline{1}]_4 = \underline{1}\times 4^3 + \underline{0}\times 4^2 + \underline{3}\times 4^1 + \underline{1}\times 4^0 = 77 = [\rho(X)]_d;$$

3) *convert* $[A/3]_4$ *in radix* $d = 3$ *form of length* $k = 5$*, as*

$$\left(\left[\frac{A}{3}\right]_4\right)_3^{[5]} = (77)_3^{[5]} = \underline{0}\,\underline{2}\,\underline{2}\,\underline{1}\,\underline{2} = \rho(X) = X \bmod 3.$$

In Step S3 of Algorithm 4.2, componentwise compute the integer error vector, $E \in [-1,+1]^5$*, as*

$$\begin{aligned} E = {} & (Y - X) \bmod 3 = \\ & \langle 5-0\rangle_3\,\langle 3-2\rangle_3\,\langle 7-2\rangle_3\,\langle 2-1\rangle_3\,\langle 0-2\rangle_3 = \\ & \langle 2\rangle_3\,\langle 1\rangle_3\,\langle 2\rangle_3\,\langle 1\rangle_3\,\langle 1\rangle_3 \equiv \\ & \underline{-1}\,\underline{+1}\,\underline{-1}\,\underline{+1}\,\underline{+1} \quad \bmod 3 \end{aligned}$$

and subtract it under $\bmod 12$ *to the received information part* Y *to obtain the correct sent information word* X *as*

$$\begin{aligned} X = {} & (Y - E) \bmod 12 = \\ & \langle 5-(-1)\rangle_{12}\,\langle 3-(+1)\rangle_{12}\,\langle 7-(-1)\rangle_{12} \\ & \qquad \langle 2-(+1)\rangle_{12}\,\langle 0-(+1)\rangle_{12} = \\ & \langle 6\rangle_{12}\,\langle 2\rangle_{12}\,\langle 8\rangle_{12}\,\langle 1\rangle_{12}\,\langle -1\rangle_{12} = \\ & \underline{6}\,\underline{2}\,\underline{8}\,\underline{1}\,\underline{11}. \end{aligned}$$

Now, let us focus on the number of redundant digits required. In the proposed coding schemes, $s = 1$. Note that the encoding is done as follows. The function

$$\tilde{\mathcal{E}} : \mathbb{Z}_d^k \to \mathbb{Z}_m^r$$

to encode $\rho(X) \in \mathbb{Z}_d^k$ given in (51) is defined as follows. First, express the number $[\rho(X)]_d$ in base

$$b \stackrel{\text{def}}{=} \max_{v \in \mathbb{Z}_d} |\mathcal{C}_1(v)| \stackrel{\text{def}}{=} |\mathcal{C}_1(0)|,$$

and then encode each b-ary digit of $([\rho(X)]_d)_b^{[r]} \in \mathbb{Z}_m^r$ into the code $\mathcal{C}_1(0)$ as defined in (15) or (40) for the non-wrap around case or the wrap around case, respectively. In both cases, the encoding is defined as

$$\tilde{\mathcal{E}}(\rho(X)) = D \cdot ([\rho(X)]_d)_b^{[r]} \in \mathbb{Z}_m^r;$$

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where "·" indicates a scalar times a vector product. Since $\rho(X)\in\mathbb{Z}_d^k$ can be any element, the redundancy of the coding scheme can be

$$r=\left\lceil \log_b\left(\max_{\rho\in\mathbb{Z}_d^k}[\rho]_d+1\right)\right\rceil=\left\lceil \log_b d^k\right\rceil= \qquad (53)$$
$$\lceil k\log_b(d)\rceil=\left\lceil \frac{k\log(d)}{\log(b)}\right\rceil.$$

In general $s\geq 1$ and if

$$R(\tilde{\mathcal{C}})=\frac{\log|\tilde{\mathcal{C}}|}{r}\simeq\frac{k\log d}{r}$$

is the information rate of the (l_-,l_+)-AEC code $\tilde{\mathcal{C}}$ then the information rate of the code $\mathcal{C}$ is given by

$$R(\mathcal{C})=\frac{K}{n}=\frac{K}{K+r}\simeq\frac{s}{s+\log(d)/R(\tilde{\mathcal{C}})}. \qquad (54)$$

Note that, if a systematic code is needed with $m=9$ in the above Example 4.4 for the non-wrap around error model then Algorithm 4.1 and Algorithm 4.2 can be implemented in exactily the same way and they are exactly equal to the optimal algorithms given in [16]. In fact, for the non-wrap around error model, this holds true in general for the code design obtained with the partitioning method given in Subsection IV-A.

A. Optimal partitioning method for $s=1$

From (54), it is clear that to reduce the redundancy the cardinality, d, of the partition in (46) should be as small as possible. Given $m,D,l_-,l_+\in\mathbb{N}$, with $D=l_-+l_++1$, in this subsection, we describe an optimal partitioning of $\mathbb{Z}_m$ into minimum max L_1/Lee distance D codes.

In the non-wrap around case, $b=|\mathcal{C}_1(0)|=\lceil m/D\rceil$ and the partition (46) can be defined as

$$\mathcal{C}_v=\mathcal{C}_1(v)=[D\cdot\mathbb{Z}+v]\cap\mathbb{Z}_m,\quad \text{for all } v\in\mathbb{Z}_D.$$

***Example** 4.6: For example, when $m=14$ and $D=3$, the partition is*

$$\mathcal{P}=\left\{\begin{array}{l}\mathcal{C}_0\overset{\text{def}}{=}\{0,3,6,9,12\},\\ \mathcal{C}_1\overset{\text{def}}{=}\{1,4,7,10,13\},\\ \mathcal{C}_2\overset{\text{def}}{=}\{2,5,8,11\}\end{array}\right\}.$$

Hence, $d=|\mathcal{P}|=D$. So, for the non-wrap around case, the redundancy is the one given in (21) and the codes are optimal systematic (l_-,l_+)-AEC codes because of Theorem 2.6.

On the other hand, in the wrap around case, $b=|\mathcal{C}_1(0)|=\lfloor m/D\rfloor$ is the maximum possible because of Theorem 3.4 with $n=1$; and so, since $\mathcal{P}$ is a covering of $\mathbb{Z}_m$,

$$d\geq\left\lceil\frac{|\mathbb{Z}_m|}{|\mathcal{C}_1(0)|}\right\rceil=\left\lceil\frac{m}{\lfloor m/D\rfloor}\right\rceil.$$

Indeed, as Example 4.1 shows, it is not difficult to design a partition (46) whose cardinality d reaches the above lower-bound. It can be done as follows. Let $b\overset{\text{def}}{=}\lfloor m/D\rfloor$, recall (42) and let

$$d\overset{\text{def}}{=}\left\lceil\frac{m}{\lfloor m/D\rfloor}\right\rceil=\left\lceil\frac{m}{b}\right\rceil\geq D. \qquad (55)$$

Without restriction, $D\in\mathbb{N}$ can be assumed to be the greatest natural such that $b=\lfloor m/D\rfloor$; otherwise, the partition can be designed for a natural $D'>D$ (for example, assume $m=89$ and $D=15$. Then, $b=\lfloor 89/15\rfloor=\lfloor 89/16\rfloor=\lfloor 89/17\rfloor=5$ and $\lfloor 89/18\rfloor=4$. Thus, we can design a code with a minimum distance up to $D'=17>15=D$). In this way, we can assume $b=\lfloor m/D\rfloor>\lfloor m/(D+1)\rfloor$. Hence, since

$$m=\left\lfloor\frac{m}{D}\right\rfloor D+\langle m\rangle_D=bD+\langle m\rangle_D, \qquad (56)$$

it follows

$$b=\left\lfloor\frac{m}{D}\right\rfloor>\left\lfloor\frac{m}{D+1}\right\rfloor\implies b>\frac{m}{D+1}\implies$$
$$\frac{m}{b}<D+1\implies bD+\langle m\rangle_D=m<bD+b;$$

and so,

$$\langle m\rangle_D<b. \qquad (57)$$

Also, from (55), (56) and (57),

$$d=\left\lceil\frac{m}{b}\right\rceil=\begin{cases}D & \text{if } \langle m\rangle_D=0,\\ D+1 & \text{if } \langle m\rangle_D\neq 0.\end{cases} \qquad (58)$$

Now, with regard to the partition, consider the following codes,

$$\mathcal{A}_0\overset{\text{def}}{=}\{0,(D+1),2(D+1),\ldots,(\langle m\rangle_D-1)(D+1)\}=(D+1)\cdot\mathbb{Z}_{\langle m\rangle_D},$$
$$\mathcal{B}_0\overset{\text{def}}{=}\{\langle m\rangle_D(D+1),\langle m\rangle_D(D+1)+D,\langle m\rangle_D(D+1)+2D,\ldots,\langle m\rangle_D(D+1)+(b-\langle m\rangle_D-1)D\}=(D+1)\langle m\rangle_D+D\cdot\mathbb{Z}_{b-\langle m\rangle_D},$$
$$\mathcal{C}_0\overset{\text{def}}{=}\mathcal{A}_0\cup\mathcal{B}_0.$$

Note that the above codes are well defined because of (57).

***Example** 4.7: For example, let $m=29$ and $D=5$. Then $b=\lfloor 29/5\rfloor=5$, $d=\lceil 29/5\rceil=6$, $\langle 29\rangle_5=4$ and*

$$\mathcal{A}_0\overset{\text{def}}{=}\{0,6,12,18\},$$
$$\mathcal{B}_0\overset{\text{def}}{=}\{24\},$$
$$\mathcal{C}_0\overset{\text{def}}{=}\{0,6,12,18,24\}.$$

Note that $|\mathcal{C}_0|=b=\lfloor m/D\rfloor$. Furthermore, $D_{max}^{Lee}(\mathcal{C}_0)\geq D$ because $D_{max}^{Lee}(\mathcal{A}_0)\geq D$, $D_{max}^{Lee}(\mathcal{B}_0)\geq D$ and the Lee distance between the last word in $\mathcal{B}_0\subseteq\mathcal{C}_0$ and $0\in\mathcal{A}_0\subseteq\mathcal{C}_0$ is

$$m-[\langle m\rangle_D(D+1)+(b-\langle m\rangle_D-1)D]=D.$$

This last relation follows from (56). At this point, by an integer translation, we let

$$\mathcal{C}_v\overset{\text{def}}{=}\mathcal{C}_0+v=(\mathcal{A}_0+v)\cup(\mathcal{B}_0+v),\ \text{for all integer } v\in\mathbb{Z}_D;$$

and

$$\mathcal{C}_D\overset{\text{def}}{=}\mathbb{Z}_m-\bigcup_{v\in\mathbb{Z}_D}\mathcal{C}_v=\mathcal{A}_D=\mathcal{A}_0+D=\{D,D+(D+1),D+2(D+1),\ldots,D+(\langle m\rangle_D-1)(D+1)\},$$

where $\langle m\rangle_D=0$ if, and only if, $\mathcal{C}_D=\emptyset$. From the construction, the above codes are well defined and, as above for $\mathcal{C}_0$, from (56) and (57), it follows that $D^{Lee}_{max}(\mathcal{C}_v) \geq D$, for all $v \in \mathbb{Z}_{D+1}$.

Example *4.8 (Example 4.7 continued): For the given example with $m = 29$ and $D = 5$ the partition elements are*

$$\begin{aligned}
\mathcal{C}_0 &\overset{\text{def}}{=} \{0, 6, 12, 18, 24\}\,, \\
\mathcal{C}_1 &\overset{\text{def}}{=} \{1, 7, 13, 19, 25\}\,, \\
\mathcal{C}_2 &\overset{\text{def}}{=} \{2, 8, 14, 20, 26\}\,, \\
\mathcal{C}_3 &\overset{\text{def}}{=} \{3, 9, 15, 21, 27\}\,, \\
\mathcal{C}_4 &\overset{\text{def}}{=} \{4, 10, 16, 22, 28\}\,, \\
\mathcal{C}_5 &\overset{\text{def}}{=} \{5, 11, 17, 23\}\,.
\end{aligned}$$

In this way, the sought partition (46) of cardinality d into minimum max Lee distance D codes is well defined. The partition given in (48) is also obtained with the method just described.

Hence, in the wrap around case, the redundancy is the one given in (44).

B. Improved codes based on partitioning $\mathbb{Z}^s_m$ for $s > 1$

As mentioned earlier, in general, there may be some room for improvement in the case of systematic code design for the wrap around error model, especially when m/D is low. In fact, a little higher information rate may be achieved for $s = 2$ (or, even more for $s > 2$) as the following examples show.

Example *4.9: If $m = 8$, $D = 3$ and $s = 2$ then a value of $d = 15$ can be obtained with the following partition $\mathcal{P} = \{\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_{14}\}$ of $\mathbb{Z}^s_m = \mathbb{Z}^2_8$ into minimum max Lee distance $D = l_- + l_+ + 1 = 3$ (i. e., (l_-, l_+)-AEC) codes.*

$$\begin{aligned}
\mathcal{C}_0 &\overset{\text{def}}{=} \{00, 13, 36, 41, 64\} \text{ (this is the code in Example 3.3)}, \\
\mathcal{C}_1 &\overset{\text{def}}{=} \mathcal{C}_0 + 01, \\
\mathcal{C}_2 &\overset{\text{def}}{=} \mathcal{C}_0 + 02, \\
\mathcal{C}_3 &\overset{\text{def}}{=} \mathcal{C}_0 + 10, \\
\mathcal{C}_4 &\overset{\text{def}}{=} \mathcal{C}_0 + 11, \\
\mathcal{C}_5 &\overset{\text{def}}{=} \mathcal{C}_0 + 12, \\
\mathcal{C}_6 &\overset{\text{def}}{=} \mathcal{C}_0 + 20, \\
\mathcal{C}_7 &\overset{\text{def}}{=} \mathcal{C}_0 + 21, \\
\mathcal{C}_8 &\overset{\text{def}}{=} \mathcal{C}_0 + 22, \\
\mathcal{C}_9 &\overset{\text{def}}{=} \mathcal{C}_0 + 70, \\
\mathcal{C}_{10} &\overset{\text{def}}{=} \{17, 32, 45, 60, 73\}, \\
\mathcal{C}_{11} &\overset{\text{def}}{=} \{16, 44, 67, 72\}, \\
\mathcal{C}_{12} &\overset{\text{def}}{=} \{27, 77\}, \\
\mathcal{C}_{13} &\overset{\text{def}}{=} \{07, 55\}, \\
\mathcal{C}_{14} &\overset{\text{def}}{=} \{71\}.
\end{aligned}$$

Assume that the code $\tilde{\mathcal{C}}$ in (51) is obtained by concatenation of the above code $\mathcal{C}_0$. Then, from (54), $s = 2$, $d = 15$ and $R(\tilde{\mathcal{C}}) = (1/2)\log(5)$, the information rate of this systematic code is

$$R(\mathcal{C}) = \frac{s}{s + \log(d)/R(\tilde{\mathcal{C}})} = \frac{1}{1 + \log(15)/\log(5)} = \frac{1}{1 + \log_5(15)} \simeq 0.3728. \tag{59}$$

On the other hand, if $s = 1$ then $d = 4$ and $R(\tilde{\mathcal{C}}) = \log(2)$, and so, the code design gives an information rate of

$$R(\mathcal{C}) = \frac{s}{s + \log(d)/R(\tilde{\mathcal{C}})} = \frac{1}{1 + \log(4)/\log(2)} = \frac{1}{1 + \log_2(4)} = \frac{1}{3} \simeq 0.3334 < 0.3728.$$

In any case, from (37), the maximum possible information rate of a systematic 8-ary code with minimum max distance of 3 is

$$R_{opt} \leq \frac{\log(m/D)}{\log(m)} = \log_8(8/3) \simeq 0.4717.$$

The systematic information rate gap may be filled by considering a smaller partition for $s = 2$ (if there exists one) or other small partitions for $s > 2$.

Example *4.10: If $m = 16$, $D = 3$ and $s = 2$ then a value of $d = 12$ can be obtained with the following partition $\mathcal{P} = \{\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_{13}\}$ of $\mathbb{Z}^s_m = \mathbb{Z}^2_{16}$ into minimum max Lee distance $D = l_- + l_+ + 1 = 3$ (i. e., (l_-, l_+)-AEC) codes. Let*

$$\begin{aligned}
\mathcal{C} = \{&\underline{0}\,\underline{0}, \underline{0}\,\underline{4}, \underline{1}\,\underline{7}, \underline{1}\,\underline{10}, \underline{2}\,\underline{13}, \\
&\underline{3}\,\underline{0}, \underline{3}\,\underline{3}, \underline{4}\,\underline{6}, \underline{4}\,\underline{9}, \underline{5}\,\underline{12}, \\
&\underline{6}\,\underline{2}, \underline{6}\,\underline{15}, \underline{7}\,\underline{5}, \underline{7}\,\underline{8}, \underline{8}\,\underline{11}, \\
&\underline{9}\,\underline{1}, \underline{9}\,\underline{14}, \underline{10}\,\underline{4}, \underline{10}\,\underline{7}, \underline{11}\,\underline{10}, \\
&\underline{12}\,\underline{13}, \underline{13}\,\underline{0}, \underline{13}\,\underline{3}, \underline{13}\,\underline{6}, \underline{14}\,\underline{9}, \\
&\underline{15}\,\underline{13}\} \subseteq \mathbb{Z}^2_{16}
\end{aligned}$$

be the code of length 2, cardinality 26 and minimum max Lee distance of $D = 3$ considered in Example 3.4. Then, the elements of the partition $\mathcal{P}$ are

$$\begin{aligned}
\mathcal{C}_0 &\overset{\text{def}}{=} \mathcal{C} \qquad \text{(this is the code in Example 3.4)}, \\
\mathcal{C}_1 &\overset{\text{def}}{=} \mathcal{C}_0 + 01, \\
\mathcal{C}_2 &\overset{\text{def}}{=} \mathcal{C}_0 + 02, \\
\mathcal{C}_3 &\overset{\text{def}}{=} \mathcal{C}_0 + 10, \\
\mathcal{C}_4 &\overset{\text{def}}{=} \mathcal{C}_0 + 11, \\
\mathcal{C}_5 &\overset{\text{def}}{=} \mathcal{C}_0 + 12, \\
\mathcal{C}_6 &\overset{\text{def}}{=} \mathcal{C}_0 + 20, \\
\mathcal{C}_7 &\overset{\text{def}}{=} \mathcal{C}_0 + 21, \\
\mathcal{C}_8 &\overset{\text{def}}{=} \mathcal{C}_0 + 22, \\
\mathcal{C}_9 &\overset{\text{def}}{=} \{\underline{0}\,\underline{3}, \underline{0}\,\underline{7}, \underline{3}\,\underline{6}, \underline{4}\,\underline{12}, \underline{5}\,\underline{15}, \\
&\qquad \underline{6}\,\underline{5}, \underline{7}\,\underline{11}, \underline{8}\,\underline{14}, \underline{9}\,\underline{4}, \underline{10}\,\underline{10}, \\
&\qquad \underline{11}\,\underline{13}, \underline{12}\,\underline{0}, \underline{12}\,\underline{3}, \underline{13}\,\underline{9}, \underline{14}\,\underline{12}\}, \\
\mathcal{C}_{10} &\overset{\text{def}}{=} \{\underline{0}\,\underline{8}, \underline{0}\,\underline{12}, \underline{1}\,\underline{3}, \underline{12}\,\underline{1}\},
\end{aligned}$$

$$\mathcal{C}_{11} \stackrel{\text{def}}{=} \{\underline{2}\,\underline{3}, \underline{12}\,\underline{2}, \underline{15}\,\underline{12}\}.$$

Assume that the code $\tilde{\mathcal{C}}$ in (51) is obtained by concatenation of the above code $\mathcal{C}_0$. So, from (54), $s = 2$, $d = 12$ and $R(\tilde{\mathcal{C}}) = (1/2)\log(26)$, the information rate of this systematic code is

$$R(\mathcal{C}) = \frac{s}{s + \log(d)/R(\tilde{\mathcal{C}})} = \frac{1}{1+\log(12)/\log(26)} = \frac{1}{1+\log_{26}(12)} \simeq 0.5673. \tag{60}$$

On the other hand, if $s = 1$ then $d = 4$ and $R(\tilde{\mathcal{C}}) = \log(5)$, and so, the code design gives an information rate of

$$R(\mathcal{C}) = \frac{s}{s + \log(d)/R(\tilde{\mathcal{C}})} = \frac{1}{1+\log(4)/\log(5)} = \frac{1}{1+\log_5(4)} \simeq 0.5372 < 0.5673.$$

In any case, from (37), the maximum possible information rate of a systematic 16-ary code with minimum max distance of 3 is

$$R_{opt} \leq \frac{\log(m/D)}{\log(m)} = \log_{16}(16/3) \simeq 0.6038.$$

This systematic information rate gap may be filled by considering a smaller partition for $s = 2$ (if there exists one) or other small partitions for $s > 2$.

V. Concluding remarks

In this paper, we have described some non-systematic and systematic (l_-, l_+)-AEC, l-ASEC and l-AAEC codes which all achieve or are close to achieving Shannon's zero error capacities with any finite length. Codes for both wrap around and non-wrap around limited magnitude m-ary (l_-, l_+)-error channels are described and it is shown that the error correcting code parameters depend on l_- and l_+ only through $D = l_- + l_+ + 1$. In particular, given $m, l_-, l_+, D = l_- + l_+ + 1, n \in \mathbb{N}$, let

$$C_0 \stackrel{\text{def}}{=} C_0(m, D) = \sup_{n\in\mathbb{N}} C_0(m, D; n)$$

be the zero error capacity where $C_0(n) \stackrel{\text{def}}{=} C_0(m, D; n)$ is the maximum information rate achievable for zero error transmission with n independent uses of the m-ary (l_-, l_+)-channel. For the non wrap-around error model $C_0(n)$ does not depend on n and

$$C_0^{(NW)} = \log\left\lceil\frac{m}{D}\right\rceil \geq C_{0,sys}^{(NW)} = \frac{\log\lceil m/D\rceil}{\log\left(D\cdot\lceil m/D\rceil\right)} \geq \log\left(\frac{m}{D}\right). \tag{61}$$

Optimal non-systematic non wrap-around (l_-, l_+)-AEC codes are given in (15) with information rate

$$R_{nsy}^{(NW)}(n) \stackrel{\text{def}}{=} R_{nsy}^{(NW)}(m, D; n) = C_0^{(NW)} = \log\left\lceil\frac{m}{D}\right\rceil$$

of any length $n \in \mathbb{N}$. Also, optimal systematic zero error capacity achieving non wrap-around (l_-, l_+)-AEC codes of length $n(k) \in \mathbb{N}$ are given with information rate as in (22),

$$R_{sys}^{(NW)}(k) \stackrel{\text{def}}{=} R_{sys}^{(NW)}(m, D; n(k)) = \frac{k}{k + \lceil k\log D/\log\lceil m/D\rceil\rceil};$$

where $k \in \mathbb{N}$ is the number of information digits. These codes are systematic zero error capacity achieving codes; that is,

$$R_{sys}^{(NW)}(\infty) \stackrel{\text{def}}{=} \lim_{k\to\infty} R_{sys}^{(NW)}(k) = C_{0,sys}^{(NW)}.$$

It is also shown that even for small values of k their information rate $R_{sys}^{(NW)}(k)$ is very close to $C_{0,sys}^{(NW)}$. Also, for the values of $m, D \in \mathbb{N}$ such that $\bar{k} = \log_D\lceil m/D\rceil \in \mathbb{N}$ holds, it is shown that $R_{sys}^{(NW)}(k) = C_{0,sys}^{(NW)}$; that is, the codes achieve the systematic zero error capacity with finite length. If $D|m$ then all the quantities in (61) are equal to $\log(m/D)$, and so, the codes achieve the (non-systematic) zero error capacity and, if $\bar{k} = \log_D(m/D) \in \mathbb{N}$ (that is, m is a power of D) then they achieve the zero error capacity with finite length $n(\bar{k}) = n(\log_D(m/D))$.

For the wrap-around error model $C_0(n)$ may depend on n, unless $D|m$. Also, for the non-systematic case,

$$\log\left(\frac{m}{D}\right) \geq C_0^{(WA)} \geq C_0^{(WA)}(2) \geq C_0^{(WA)}(1) = \log\left\lfloor\frac{m}{D}\right\rfloor. \tag{62}$$

Whereas, for the systematic case,

$$\log\left(\frac{m}{D}\right) \geq C_0^{(WA)} \geq C_{0,sys}^{(WA)} \geq \frac{\log\lfloor m/D\rfloor}{\log\left[\lfloor m/D\rfloor\cdot\lceil m/\lfloor m/D\rfloor\rceil\right]}. \tag{63}$$

Suboptimal non-systematic wrap-around (l_-, l_+)-AEC codes are given in (40) with information rate

$$R_{nsy}^{(WA)}(n) \stackrel{\text{def}}{=} R_{nsy}^{(WA)}(m, D; n) = C_0^{(WA)}(1) = \log\lfloor m/D\rfloor$$

of any length $n \in \mathbb{N}$, and two new examples (Example 3.3 and Example 3.4) where $R_{nsy}^{(WA)}(n) > C_0^{(WA)}(1)$ are given. Close to optimal systematic wrap-around (l_-, l_+)-AEC codes of length $n(k) \in \mathbb{N}$ are derived with information rate as in (45),

$$R_{sys}^{(WA)}(k) \stackrel{\text{def}}{=} R_{sys}^{(WA)}(m, D; n(k)) = \frac{k}{k + \lceil k\log\lceil m/\lfloor m/D\rfloor\rceil/\log\lfloor m/D\rfloor\rceil};$$

where $k \in \mathbb{N}$ is the number of information digits. For these codes,

$$R_{sys}^{(WA)}(\infty) \stackrel{\text{def}}{=} \lim_{k\to\infty} R_{sys}^{(WA)}(k) = \frac{\log\lfloor m/D\rfloor}{\log\left[\lfloor m/D\rfloor\cdot\lceil m/\lfloor m/D\rfloor\rceil\right]}. \tag{64}$$

If $D|m$ then all the quantities in (61), (62) and (63) are equal to $\log(m/D)$, and so, either

$$\lim_{k\to\infty} R_{sys}^{(WA)}(k) = C_{0,sys}^{(WA)} = C_0^{(WA)} = C_{0,sys}^{(NW)} = C_0^{(NW)}$$

or

$$R_{sys}^{(WA)}(\bar{k}) = C_{0,sys}^{(WA)} = C_0^{(WA)} = C_{0,sys}^{(NW)} = C_0^{(NW)}$$

for some finite $\bar{k} \stackrel{\text{def}}{=} \log_D(m/D) \in \mathbb{N}$. Also, only if $D|m$, the optimal systematic and non-systematic code designs and the decoding algorithms for the non-wrap around error model are exactly the same as the algorithms for the wrap around error model. Improved codes are discussed by giving two non-trivial examples (Example 4.9 and Example 4.10).

For some values of m and D, Table I gives the information rates of the optimal systematic and non-systematic codes and these rates are respectively equal to or asymptotically equal to the systematic and non-systematic zero error capacities of the m-ary NW-(l_-, l_+)-channel. Analogously, Table II compares the information rates of the optimal/close to optimal non-systematic and systematic code designs with the bounds given in (62) and (63) on the zero error capacities of the m-ary WA-(l_-, l_+)-channel.

The regular capacity achieving codes such as turbo codes [6], [12], LDPC codes [10] and polar codes [2] require long lengths to achieve the capacity of the error channels, whereas the proposed zero error capacity achieving codes can be designed for any finite length $n = 1, 2, \ldots$. These codes have simple encoding and decoding algorithms and can be systematic. The implication of these zero error capacity achieving codes is that, when the data words are sent through these limited magnitude error channels, using the proposed (l_-, l_+)-AEC, l-AAEC and l-ASEC codes, it is possible to recover the original data words with 100% error free.

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TABLE I
COMPARISONS OF CODE PARAMETERS FOR THE NON WRAP AROUND CHANNEL MODEL.

m	D	$R_{nsy}^{(NW)}(n{=}1) = C_0^{(NW)} = \log \lceil \frac{m}{D} \rceil$	$R_{sys}^{(NW)}(k{=}\infty) = C_{0,sys}^{(NW)}$
4	2	0.5	0.5
	3	0.5	0.3869
8	2	0.6667	0.6667
	3	0.5283	0.5
	4	0.3333	0.3333
	5	0.3333	0.301
	6	0.3333	0.2789
	7	0.3333	0.2626
16	2	0.75	0.75
	3	0.6462	0.6199
	4	0.5	0.5
	5	0.5	0.4628
	6	0.3962	0.3801
	7	0.3962	0.3608
	8	0.25	0.25
	9	0.25	0.2398
32	2	0.8	0.8
	3	0.6919	0.6858
	4	0.6	0.6
	5	0.5615	0.5473
	6	0.517	0.5
	7	0.4644	0.4527
	8	0.4	0.4
	9	0.4	0.3869
	17	0.2	0.1966
64	2	0.8333	0.8333
	3	0.7432	0.7378
	4	0.6667	0.6667
	5	0.6167	0.6144
	6	0.5766	0.5723
	7	0.5537	0.542
	8	0.5	0.5
	9	0.5	0.4862
	17	0.3333	0.3285
	33	0.1667	0.1655
256	2	0.875	0.875
	3	0.8033	0.8022
	4	0.75	0.75
	5	0.7126	0.7106
	6	0.6783	0.6773
	7	0.6512	0.6498
	8	0.625	0.625
	9	0.6072	0.6051
	17	0.5	0.4946
	33	0.375	0.3729
	65	0.25	0.2493
	129	0.125	0.1248

The values in the third and fourth columns give the information rates of the proposed non-systematic and systematic codes, respectively, and are computed with (61).

TABLE II
COMPARISONS OF CODE PARAMETERS FOR THE WRAP AROUND CHANNEL MODEL.

m	D	$\log\left(\frac{m}{D}\right)$	$R_{nsy}^{(WA)}(n{=}1) = \log\lfloor \frac{m}{D} \rfloor$	$R_{sys}^{(WA)}(k{=}\infty)$ for $s=1$ (left), $R_{sys}^{(WA)}(k{=}\infty)$ for $s=2$ (right)	
4	2	0.5	0.5	0.5	
8	2	0.6667	0.6667	0.6667	
	3	0.4717	0.3333	0.3333	0.3728
	4	0.3333	0.3333	0.3333	
16	2	0.75	0.75	0.75	
	3	0.6038	0.5805	0.5372	0.5673
	4	0.5	0.5	0.5	
	5	0.4195	0.3962	0.3801	
	6	0.3538	0.25	0.25	
	7	0.2982	0.25	0.25	
	8	0.25	0.25	0.25	
32	2	0.8	0.8	0.8	
	3	0.683	0.6644	0.6242	
	4	0.6	0.6	0.6	
	5	0.5356	0.517	0.5	
	6	0.483	0.4644	0.4527	
	7	0.4385	0.4	0.4	
	8	0.4	0.4	0.4	
	9	0.366	0.317	0.3142	
	10	0.3356	0.317	0.3142	
	11	0.3081	0.2	0.2	
64	2	0.8333	0.8333	0.8333	
	3	0.7358	0.7321	0.6871	
	4	0.6667	0.6667	0.6667	
	5	0.613	0.5975	0.581	
	6	0.5692	0.5537	0.542	
	7	0.5321	0.5283	0.5138	
	8	0.5	0.5	0.5	
	9	0.4717	0.4679	0.458	
	10	0.4463	0.4308	0.4277	
	11	0.4234	0.387	0.3856	
	17	0.3188	0.2642	0.2622	
256	2	0.875	0.875	0.875	
	3	0.8019	0.8012	0.7622	
	4	0.75	0.75	0.75	
	5	0.7098	0.7091	0.687	
	6	0.6769	0.674	0.6576	
	7	0.6491	0.6462	0.6328	
	8	0.625	0.625	0.625	
	9	0.6038	0.6009	0.5914	
	10	0.5848	0.5805	0.5731	
	11	0.5676	0.5654	0.5579	
	17	0.4891	0.4884	0.4837	
	33	0.3695	0.3509	0.3502	
	65	0.2472	0.1981	0.1978	

The value in the third and fourth column give respectively the upper and lower bounds of the zero error capacity derived in (62). The value in the fourth column is also equal to the information rate of the proposed non-systematic codes. The value in the left side of the fifth column is the information rate computed with (64) of the proposed systematic code designs with $s = 1$. The right side of the fifth column gives the information rates, (59) and (60), obtained for the given examples with $s = 2$.