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Cogent Mathematics (2016), 3: 1220670

Natalini & Ricci, *Cogent Mathematics* (2016), 3: 1220670
http://dx.doi.org/10.1080/23311835.2016.1220670

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Remarks on Bell and higher order Bell polynomials and numbers

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Received: 16 June 2016
Accepted: 01 August 2016
Published: 20 August 2016

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Reviewing editor:
Hari M. Srivastava, University of Victoria, Canada

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Abstract: We recover a recurrence relation for representing in an easy form the coefficients $A_{n,k}$ of the Bell polynomials, which are known in literature as the partial Bell polynomials. Several applications in the framework of classical calculus are derived, avoiding the use of operational techniques. Furthermore, we generalize this result to the coefficients $A^{[2]}_{n,k}$ of the second-order Bell polynomials, i.e. of the Bell polynomials relevant to *n*th derivative of a composite function of the type $f(g(h(t)))$. The second-order Bell polynomials $B^{[2]}_n$ and the relevant Bell numbers $b^{[2]}_n$ are introduced. Further extension of the *n*th derivative of *M*-nested functions is also touched on.

Subjects: Advanced Mathematics; Analysis - Mathematics; Mathematics & Statistics; Pure Mathematics; Science; Special Functions

Keywords: Bell polynomials; higher order Bell polynomials and numbers; differentiation of composite functions; combinatorial analysis; partitions; orthogonal polynomials and special functions

AMS subject classifications: 05A10; 26A06; 11P81

1. Introduction

The Bell polynomials are a mathematical tool for representing the *n*th derivative of a composite function.

ABOUT THE AUTHORS

Our research group works by many years in Numerical Analysis, Special functions, Matrix functions, Integral and discrete transforms, Ordinary and partial differential equations, Asymptotic analysis, Eigenvalue problems. Pierpaolo Natalini is a professor and a researcher at the RomaTre University and Paolo Emilio Ricci at the UniNettuno International Telematic University in Rome, being retired from Rome University Sapienza by almost seven years.

Our approach to the Bell polynomials theory is an important part of our research, started more than twenty years ago, and recently included in the Mathematica© project under the name “BellY polynomials”. Many extensions of the classical Bell polynomials were achieved, including the higher order and the multi-variable Bell polynomials. In this article, we show how to compute recursively the higher order Bell polynomials and the relevant Bell numbers, a topic which will permits us to extend our research to Number theory.

PUBLIC INTEREST STATEMENT

The importance and utility of ordinary Bell polynomials in many different frameworks of Mathematics are well known. Being related to partitions, they are used in combinatorial analysis and even in Statistics. Moreover these polynomials have been applied in other contexts, such as the Blissard problem, the representation formulas of Newton sum rules for polynomials’ zeros, the representation of symmetric functions of a countable set of numbers, therefore generalizing the classical algebraic Newton–Girard formulas, and so on. In this article after presenting a short survey of known results, we show applications in connection with several variables Hermite polynomials. In the second part of the article, we extend our results to higher order Bell polynomials, i.e. polynomials associated with differentiation of many nested functions. The relevant higher order Bell numbers are introduced. The arising sequences of integer could be used in cryptography, a subject of wide interest.

Being related to partitions, the Bell polynomials are used in combinatorial analysis (Riordan, 1958), and several applications appeared in different fields, such as: the Blissard problem; the representation of Lucas polynomials of the first and second kind (Bruschi & Ricci, 1980; Di Cave & Ricci, 1980); the representation formulas for Newton sum rules of polynomials zeros (Isoni, Natalini, & Ricci, 2001a, 2001b); the recurrence relations for a class of Freud-type polynomials (Bernardini & Ricci, 2002); the representation formulas for the symmetric functions of a countable set of numbers (generalizing the classical algebraic Newton–Girard formulas). As a consequence of this last application, in Cassisa and Ricci (2000) reduction formulas for the *orthogonal invariants* of a strictly positive compact operator (shortly PCO)—deriving in a simple way the so-called Robert formulas (Robert, 1973)—have been derived.

Some generalized forms of Bell polynomials appeared in literature, see, e.g. Fujiwara (1990), Rai and Singh (1982). Further generalizations, including the multidimensional case, can be found in Bernardini, Natalini, and Ricci (2005), Natalini and Ricci (2003, 2004).

The aim of this article is to give a survey of known results about the classical Bell polynomials; to show some applications in connection with the multi-variable Hermite polynomials (see Srivastava, Özarslan, & Yılmaz, 2014); and lastly to extend the achieved formulas to the higher order Bell polynomials and numbers.

According to this purpose, in the first part of this article, after recalling definitions and the main properties of Bell polynomials, we prove the classical recursion formula useful for computing the polynomial coefficients $A_{n,k}$, also known as the partial Bell polynomials. Consequently, many equations useful in classical calculus can be derived in a quite elementary form. An umbral approach to the same subject, including several extended applications can be also found in a recent paper (Babusci, Dattoli, Górska, & Penson, 2014).

In the second part, we recall the second-order Bell polynomials (see Natalini & Ricci, 2004), and introduce the recursion formula for their polynomial coefficients $A^{[2]}_{n,k}$, we derive the complete $B^{[2]}_n$ and the second-order Bell numbers $b^{[2]}_n$. The extension to the general case of the $(M-1)$-order Bell polynomials is also touched on.

2. Recalling the Bell polynomials

Consider $\Phi(t) := f(g(t))$, i.e. the composition of functions $x = g(t)$ and $y = f(x)$, defined in suitable intervals of the real axis, and suppose that $g(t)$ and $f(x)$ are n times differentiable with respect to the relevant independent variables so that $\Phi(t)$ can be differentiated n times with respect to t, using the differentiation rule of composite functions.

We use the following notations:

$$\Phi_j := D_t^j \Phi(t), \quad f_h := D_x^h f(x)|_{x=g(t)}, \quad g_k := D_t^k g(t).$$

Then, the n-th derivative can be represented by

$$\Phi_n = Y_n(f_1, g_1; f_2, g_2; \dots ; f_n, g_n),$$

where the Y_n are, by definition, the Bell polynomials.

For example, one has:

$$\begin{aligned} &Y_1(f_1, g_1) = f_1 g_1 \\ &Y_2(f_1, g_1; f_2, g_2) = f_1 g_2 + f_2 g_1^2 \\ &Y_3(f_1, g_1; f_2, g_2; f_3, g_3) = f_1 g_3 + f_2(3 g_2 g_1) + f_3 g_1^3. \end{aligned} \tag{2.1}$$

Further examples can be found in Riordan (1958, p. 49).

The general form is as follows:

$$Y_n(f_1, g_1; f_2, g_2; \dots; f_n, g_n) = \sum_{k=1}^{n} A_{n,k}(g_1, g_2, \dots, g_n) f_k, \tag{2.2}$$

where the coefficient $A_{n,k}$, for all $k = 1, \dots, n$, is a polynomial in $g_1, g_2, \dots, g_n$, homogeneous of degree k and *isobaric* of weight n (i.e. it is a linear combination of monomials $g_1^{k_1} g_2^{k_2} \cdots g_n^{k_n}$ whose weight is constantly given by $k_1 + 2k_2 + \dots + nk_n = n$).

Since the coefficients $A_{n,k}$ are independent of f, their construction can be performed by choosing

$$f = e^{ax},$$

where a is an arbitrary constant.

Then, for example, we find:

- for $n = 1$: $e^{-ag} D_t e^{ag} = ag_1$, so that: $A_{1,1} = g_1$.
- for $n = 2$: $e^{-ag} D_t^2 e^{ag} = ag_2 + a^2 g_1^2$, so that: $A_{2,1} = g_2,\ A_{2,2} = g_1^2$, and so on.

It is easy to prove the following known result:

Proposition 2.1 The Bell polynomials satisfy the recurrence relation:

$$\begin{cases} Y_0 := f_1; \\ Y_{n+1}(f_1, g_1; \dots; f_n, g_n; f_{n+1}, g_{n+1}) = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} Y_{n-k}(f_2, g_1; f_3, g_2; \dots; f_{n-k+1}, g_{n-k}) g_{k+1}. \end{cases} \tag{2.3}$$

An explicit expression for the Bell polynomials is given by the Faà di Bruno formula:

$$\Phi_n = Y_n(f_1, g_1; f_2, g_2; \dots; f_n, g_n) = \sum_{\pi(n)} \frac{n!}{r_1! r_2! \dots r_n!} f_r \left(\frac{g_1}{1!}\right)^{r_1} \left(\frac{g_2}{2!}\right)^{r_2} \cdots \left(\frac{g_n}{n!}\right)^{r_n}, \tag{2.4}$$

where the sum runs over all partitions $\pi(n)$ of the integer n, r_i denotes the number of parts of size i, and $r = r_1 + r_2 + \cdots + r_n$ denotes the number of parts of the considered partition.

A proof of the Faà di Bruno formula can be found in Riordan (1958). In Roman (1980), the proof is based on the *umbral calculus* (see Roman & Rota, 1978 and references therein).

It is worth noting that the formula (2.4) was previously stated by Arbogast in (1800). See also the historical article by Johnson (2002), where the formula is ascribed to Tiburce Abadie (1850), but the priority of L.F.A. Arbogast is indubitable.

The polynomial coefficients $A_{n,k}$ coincide with the partial Bell polynomials $B_{n,k}$, however, we use in this article the same notation of our preceding papers (see e.g. Bernardini et al., 2005; Natalini & Ricci, 2004, 2006; Noschese & Ricci, 2003), since it is borrowed from the already mentioned classical book (Riordan, 1958).

We recall, in the next section a classical recurrence relation for the $A_{n,k}$ coefficients, i.e. for the partial Bell polynomials, and derive, in Sections 4 and 5, some applications of the complete Bell polynomials.

3. A recursion for the $A_{n,k}$ coefficients

In this section, we recover a recurrence relation for the partial Bell polynomials, by proving the following theorem:

THEOREM 3.1 *We have,* $\forall\, n$*:*

$$A_{n+1,1} = g_{n+1}, \quad A_{n+1,n+1} = g_1^{n+1}. \tag{3.1}$$

Furthermore, $\forall\, k = 1, 2, \dots, n-1$, the $A_{n,k}$ coefficients appearing in the Bell formula (2.2) can be computed by the recurrence relation

$$A_{n+1,k+1}(g_1, g_2, \dots, g_{n+1}) = \sum_{h=0}^{n-k} \binom{n}{h} A_{n-h,k}(g_1, g_2, \dots, g_{n-h})\, g_{h+1}. \tag{3.2}$$

Proof Equation (3.1) is a direct consequence of the Definition (2.2). In order to prove Equation (3.2), note that, taking into account the first relation in (3.1), we can write Equation (2.2) in the form:

$$\begin{aligned} Y_{n+1}(f_1, g_1; f_2, g_2; \dots; f_{n+1}, g_{n+1}) &= \sum_{k=0}^{n} A_{n+1,k+1}(g_1, g_2, \dots, g_{n+1})\, f_{k+1} \\ &= g_{n+1}\, f_1 + \sum_{k=1}^{n} A_{n+1,k+1}(g_1, g_2, \dots, g_{n+1})\, f_{k+1}, \end{aligned}$$

and, recalling Equation $(2.3)_1$, the second-hand side of $(2.3)_2$ becomes:

$$\sum_{h=0}^{n} \binom{n}{h} Y_{n-h}(f_2, g_1; f_3, g_2; \dots; f_{n-h+1}, g_{n-h})\, g_{h+1} = f_1\, g_{n+1} + \sum_{h=0}^{n-1} Y_{n-h}(f_2, g_1; f_3, g_2; \dots; f_{n-h+1}, g_{n-h})\, g_{h+1},$$

so that, neglecting the first term in both the above sums, we find:

$$\sum_{k=1}^{n} A_{n+1,k+1}(g_1, g_2, \dots, g_{n+1})\, f_{k+1} = \sum_{h=0}^{n-1} \binom{n}{h} \left(\sum_{\ell=1}^{n-h} A_{n-h,\ell}\,(g_1, g_2, \dots, g_{n-h}) f_{\ell+1} \right) g_{h+1},$$

and inverting summations by the Dirichlet formula:

$$\sum_{k=1}^{n} A_{n+1,k+1}(g_1, g_2, \dots, g_{n+1})\, f_{k+1} = \sum_{\ell=1}^{n} \left(\sum_{h=0}^{n-\ell} \binom{n}{h} A_{n-h,\ell}\,(g_1, g_2, \dots, g_{n-h}) g_{h+1} \right) f_{\ell+1}.$$

Therefore, changing ℓ into k in the last formula, and equating the coefficients of f_{k+1}, our result follows.

Remark 3.2 A recursion like this was already known, but derived using generating functions, and it is very convenient by the computational point of view, owing the high complexity of the Faà di Bruno formula, making use of partitions (see also Cvijović, 2011 for more recent results).

Note that the BellY polynomials, according to the recent Mathematica© notation, see https://reference.wolfram.com/language/ref/BellY.html, were considered in past literature only in our preceding works Bernardini et al. (2005), Natalini and Ricci (2004, 2006, 2015).

Remark 3.3 Note that, by letting $f_k = 1$, $\forall k$ in Equation (2.2), i.e. considering the nth derivative *at the origin* of the composite function $\Phi_n(t) := e^{g(t)}$, the Bell polynomials usually introduced in literature Andrews (1998) appear:

$$B_n(g_1, g_2, \dots, g_n) := Y_n(1, g_1; 1, g_2; \dots; 1, g_n) = \sum_{k=1}^{n} A_{n,k}(g_1, g_2, \dots, g_n),$$

so that the recursion (3.2) can be used to compute the $B_n(g_1, g_2, \dots, g_n)$.

Furthermore, since the nth Bell number b_n is given by

$$b_n = B_n(1, 1, \dots, 1) := Y_n(1, 1; 1, 1; \dots; 1, 1) = \sum_{k=1}^{n} A_{n,k}(1, 1, \dots, 1),$$

the same recursion (3.2) can be used to compute the b_n too. A table of the b_n numbers up to the index 1,000 can be found at the home page: http://www.dnull.com/bells/bell1000.html.

It is worth to recall that the number b_n is the number of partitions of a set of size n, (a partition of a set S is a set of nonempty, pairwise disjoint subsets of S whose union is S).

3.1. The $A_{10,k}$ coefficients

As an example, we write down the values $A_{10,k}$, $(k = 1, 2, \dots, 10)$, obtained recursively, using a Mathematica© program:

$$\begin{aligned}
A_{10,1} &= g_{10} \\
A_{10,2} &= 126g_5^2 + 210g_4g_6 + 120g_3g_7 + 45g_2g_8 + 10g_1g_9 \\
A_{10,3} &= 2100g_3^2g_4 + 1575g_2g_4^2 + 2520g_2g_3g_5 + 1260g_1g_4g_5 \\
&\quad + 630g_2^2g_6 + 840g_1g_3g_6 + 360g_1g_2g_7 + 45g_1^2g_8 \\
A_{10,4} &= 6300g_2^2g_3^2 + 2800g_1g_3^3 + 3150g_2^3g_4 + 12600g_1g_2g_3g_4 + 1575g_1^2g_4^2 \\
&\quad + 3780g_1g_2^2g_5 + 2520g_1^2g_3g_5 + 1260g_1^2g_2g_6 + 120g_1^3g_7 \\
A_{10,5} &= 945g_2^5 + 12600g_1g_2^3g_3 + 12600g_1^2g_2g_3^2 + 9450g_1^2g_2^2g_4 + 4200g_1^3g_3g_4 \\
&\quad + 2520g_1^3g_2g_5 + 210g_1^4g_6 \\
A_{10,6} &= 4725g_1^2g_2^4 + 12600g_1^3g_2^2g_3 + 2100g_1^4g_3^2 + 3150g_1^4g_2g_4 + 252g_1^5g_5 \\
A_{10,7} &= 3150g_1^4g_2^3 + 2520g_1^5g_2g_3 + 210g_1^6g_4 \\
A_{10,8} &= 630g_1^6g_2^2 + 120g_1^7g_3 \\
A_{10,9} &= 45g_1^8g_2 \\
A_{10,10} &= g_1^{10}
\end{aligned}$$

Remark 3.4 By adding a few instructions to the above-mentioned program, it is possible to compute recursively the Bell polynomials

$$B_n(g_1, g_2, \dots, g_n) = \sum_{k=1}^{n} A_{n,k}(g_1, g_2, \dots, g_n),$$

and the Bell numbers

$$b_n = B_n(1, 1, \dots, 1) = \sum_{k=1}^{n} A_{n,k}(1, 1, \dots, 1),$$

both up to the desired index, recovering the results already known in literature.

For example, by summing up all the numerical coefficients of the above $A_{n,k}$, $(k = 1, 2, \dots, n)$, we write down explicitly the Bell numbers b_n, $(n = 1, 2, \dots, 10)$,

1, 2, 5, 15, 52, 203, 877, 4140, 21147, 115975.

Of course, this can be done for every choice of the integer n, and we find the results under the # A000110 in the Encyclopedia of Integer Sequences (Sloane & Plouffe, 1995).

Theorem 3.1 allows to obtain in a friendly form the nth differentiation formula for the composite function $f(x^\alpha)$, ($\alpha \in \boldsymbol{R}$), since assuming:

$$g_1 = \alpha x^{\alpha-1},\ g_2 = \alpha(\alpha-1)x^{\alpha-2},\ \ldots; g_h = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-h+1)}\,x^{\alpha-h}; \ldots$$

We write Equation (2.2) in the form:

$$D_x^n(f(x^\alpha)) = \sum_{k=1}^{n} A_{n,k}\left(\alpha x^{\alpha-1}, \alpha(\alpha-1)x^{\alpha-2}, \ldots, \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n+1)}\,x^{\alpha-n}\right)f_k, \tag{3.3}$$

where the coefficients $A_{n,k}$ can be computed recursively using Theorem 3.1.

Equation (3.3) includes obviously the particular cases

$$D_x^n f(x^m),\ (m \in \mathbf{N});\quad D_x^n f(\sqrt{x});\quad D_x^n f(1/x);\quad \text{etc.}$$

4. The particular case of integer powers

In the following, we examine the particular case when $\alpha = m$, an integer number, recovering the classical formulas derived in Babusci et al. (2014), using operational techniques. Here, we use only the above-mentioned properties of Bell polynomials.

4.1. The case $\alpha = 2$

Putting $\Phi(t) := f(g(t)) = f(t^2)$, i.e. $g(t) = t^2$, we have:

$$g_1 = 2t,\quad g_2 = 2,\quad \text{and}\quad g_k = 0,\ \forall k \geq 3.$$

Using the Faà di Bruno formula (2.4), we find

$$D_t^n(f(t^2)) = \sum_{\pi(n)} \frac{n!}{r_1!r_2!}\,f_r\left(\frac{g_1}{1!}\right)^{r_1}\left(\frac{g_2}{2!}\right)^{r_2} = n!\sum_{\pi(n)}\frac{(2t)^{r_1}}{r_1!r_2!}\,f_r, \tag{4.1}$$

and letting $r_2 = k$, $r_1 = n - 2k \geq 0$, (so that $0 \leq k \leq [n/2]$), $r = r_1 + r_2 = n - k$, (here and in the following the square brackets denote the *integer part*),

$$D_t^n(f(t^2)) = n!\sum_{k=0}^{[n/2]}\frac{(2t)^{n-2k}}{(n-2k)!\,k!}\,D_{t^2}^{n-k}f(t^2). \tag{4.2}$$

In particular, if $f(t^2) = e^{at^2}$,

$$D_t^n\left(e^{at^2}\right) = n!\sum_{k=0}^{[n/2]}\frac{(2at)^{n-2k}a^k}{(n-2k)!\,k!}\,e^{at^2} = H_n^{(2)}(2at, a)\,e^{at^2}, \tag{4.3}$$

where $H_n^{(2)}(x, y)$ is the nth Hermite–Kampé de Fériet polynomial in two variables (see Srivastava et al., 2014).

As a consequence of Equation (4.2), putting $f(t^2) = g(t^2)h(t^2)$, and recalling the ordinary Leibniz rule:

$$D_{t^2}^{n-k}\left(g(t^2)h(t^2)\right) = \sum_{j=0}^{n-k}\binom{n-k}{j}D_{t^2}^{n-k-j}g(t^2)D_{t^2}^{j}\,h(t^2),$$

the generalized Leibniz rule introduced in Babusci et al. (2014) follows:

$$D_t^n\left(g(t^2)h(t^2)\right) = n! \sum_{k=0}^{[n/2]} \frac{(2t)^{n-2k}}{(n-2k)!\,k!} \sum_{j=0}^{n-k} \binom{n-k}{j} D_{t^2}^{n-k-j} g(t^2) D_{t^2}^{j}\, h(t^2).$$

4.2. The case $\alpha = 3$

Putting $\Phi(t) := f(g(t)) = f(t^3)$, i.e. $g(t) = t^3$, we have:

$$g_1 = 3t^2, \quad g_2 = 6t, \quad g_3 = 6, \quad \text{and} \quad g_k = 0,\ \forall k \geq 4.$$

Using the Faà di Bruno formula (2.4), we find

$$D_t^n(f(t^3)) = n! \sum_{\pi(n)} \frac{(3t^2)^{r_1}(3t)^{r_2}}{r_1!r_2!r_3!} f_r, \tag{4.4}$$

and letting $r_3 = k$, $r_2 = h$, (so that $n - 3k \geq 0$ and therefore $k \leq [n/3]$), $r_1 = n - 3k - 2h \geq 0$, (so that $0 \leq k \leq [(n-3k)/2]$), $r = r_1 + r_2 + r_3 = n - 2k - h$,

$$D_t^n(f(t^3)) = n! \sum_{k=0}^{[n/3]} \sum_{h=0}^{[(n-3k)/2]} \frac{(3t^2)^{n-3k-2h}(3t)^h}{(n-3k-2h)!h!k!} D_{t^3}^{n-2k-h} f(t^3). \tag{4.5}$$

In particular, if $f(t^3) = e^{at^3}$,

$$\begin{aligned} D_t^n\left(e^{at^3}\right) &= n! \sum_{k=0}^{[n/3]} \sum_{h=0}^{[(n-3k)/2]} \frac{(3at^2)^{n-3k-2h}(3at)^h a^k}{(n-3k-2h)!h!k!} e^{at^3} \\ &= H_n^{(3)}(3at^2, 3at, a)\, e^{at^3}, \end{aligned} \tag{4.6}$$

where $H_n^{(3)}(x, y, z)$ is the nth Hermite-Kampé de Fériet polynomial in three variables (see Srivastava et al., 2014).

As a consequence of Equation (4.2), putting $f(t^3) = g(t^2)h(t^3)$, and recalling the ordinary Leibniz rule:

$$D_{t^3}^{n-k}\left(g(t^3)h(t^3)\right) = \sum_{j=0}^{n-k} \binom{n-k}{j} D_{t^3}^{n-k-j} g(t^3) D_{t^3}^{j}\, h(t^3),$$

the generalized Leibniz rule introduced in Babusci et al. (2014) follows:

$$D_t^n\left(g(t^3)h(t^3)\right) = n! \sum_{k=0}^{[n/3]} \sum_{h=0}^{[(n-3k)/2]} \frac{(3t^2)^{n-3k-2h}(3t)^h}{(n-3k-2h)!h!k!} \sum_{j=0}^{n-k} \binom{n-k}{j} D_{t^3}^{n-k-j} g(t^3) D_{t^3}^{j}\, h(t^3).$$

4.3. The case $\alpha = m$ ($m \in N$)

The results of the preceding subsections can be extended to the general case, even if the relevant formulas are quite cumbersome.

Let $\Phi(t) := f(g(t)) = f(t^m)$, i.e. $g(t) = t^m$, we have:

$$g_1 = mt^{m-1},\ g_2 = m(m-1)t^{m-2}, \ldots,\ g_{m-1} = m!t,\ g_m = m!, \ \text{and}\ g_k = 0,\ \forall k \geq m+1.$$

The Faà di Bruno formula gives

$$D_t^n(f(t^m)) = n! \sum_{\pi(n)} \frac{(mt^{m-1}/1!)^{r_1}(m(m-1)t^{m-2}/2!)^{r_2} \cdots (m!t/(m-1)!)^{r_{m-1}}}{r_1!r_2!\cdots r_m!} f_r.$$

Putting:

$$r_m = k_m, \quad \text{so that} \quad n - mk_m \geq 0 \quad \Rightarrow \quad 0 \leq k_m \leq [n/m],$$
$$r_{m-1} = k_{m-1} \quad \text{so that} \quad n - (m-1)k_{m-1} - mk_m \geq 0 \quad \Rightarrow \quad 0 \leq k_{m-1} \leq [(n - mk_m)/(m-1)],$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$r_2 = k_2, \quad \text{so that} \quad n - 2k_2 - 3k_3 - \cdots - mk_m \geq 0 \quad \Rightarrow \quad 0 \leq k_2 \leq [(n - 3k_3 - \cdots - mk_m)/2)],$$
$$n = r_1 + 2k_2 + \cdots + mk_m \quad \Rightarrow \quad r_1 = n - mk_m - \cdots - 2k_2,$$

and for shortness:

$$\ell_2 = \frac{1}{2}(n - 3k_3 - \cdots - mk_m), \dots, \ell_{m-1} = \frac{1}{m-1}(n - mk_m)$$

we find the equation:

$$D_t^n(f(t^m)) = n! \sum_{k_m=0}^{[n/m]} \sum_{k_{m-1}=0}^{[\ell_{m-1}]} \cdots \sum_{k_2=0}^{[\ell_2]} \times \frac{(mt^{m-1})^{n-mk_m-\cdots-2k_2}(m(m-1)t^{m-2}/2!)^{k_2} \cdots (mt)^{k_{m-1}}}{(n - mk_m - \cdots - 2k_2)!k_2! \dots k_{m-1}!k_m!} D_{t^m}^r f(t^m),$$

where $r = n - (m-1)k_m - \cdots - 2k_3 - k_2$.

In particular, if $f(t^m) = e^{at^m}$,

$$D_t^n\left(e^{at^m}\right) = H_n^{(m)}\left(\binom{m}{1}at^{m-1}, \binom{m}{2}at^{m-2}, \dots, \binom{m}{m-1}at, a\right) e^{at^m}, \tag{4.7}$$

where $H_n^{(m)}(x_1, x_2, \dots, x_m)$ is the nth Hermite-Kampé de Fériet polynomial in m variables.

Further results and interesting applications to several properties of special functions can be found in the already mentioned article (Babusci et al., 2014).

5. Miscellaneous results

Some particular cases of the above equations are derived in this section.

Considering the generalized Leibniz rules in Section 4, and assuming $g(t^2) \equiv h(t^2)$, find:

$$D_t^n\left(h^2(t^2)\right) = n! \sum_{k=0}^{[n/2]} \frac{(2t)^{n-2k}}{(n-2k)!k!} \sum_{j=0}^{n-k} \binom{n-k}{j} D_{t^2}^{n-k} h(t^2),$$

In particular:

$$D_t^n\left(e^{2t^2}\right) = n! \sum_{k=0}^{[n/2]} \frac{(2t)^{n-2k}}{(n-2k)!k!} \sum_{j=0}^{n-k} \binom{n-k}{j} e^{2t^2}.$$

$$D_t^n\left(h^2(t^3)\right) = n! \sum_{k=0}^{[n/3]} \sum_{h=0}^{[(n-3k)/2]} \frac{(3t^2)^{n-3k-2h}(3t)^h}{(n-3k-2h)!h!k!} \sum_{j=0}^{n-k} \binom{n-k}{j} D_{t^3}^{n-k} h(t^3),$$

In particular:

$$D_t^n\left(e^{2t^3}\right) = n! \sum_{k=0}^{[n/3]} \sum_{h=0}^{[(n-3k)/2]} \frac{(3t^2)^{n-3k-2h}(3t)^h}{(n-3k-2h)!h!k!} \sum_{j=0}^{n-k} \binom{n-k}{j} e^{2t^3}.$$

Introducing the polynomial coefficient, by putting

$$\binom{m}{j_1, j_2, \ldots, j_n} := \frac{m!}{j_1! j_2! \ldots j_n!},$$

where $0 \le j_\ell \le m, (\ell = 1, 2, \ldots, n)$ and $j_1 + j_2 + \cdots + j_n = m$, we find the following extensions of the above formulas:

$$D_t^n\left(h^r(t^2)\right) = n! \sum_{k=0}^{[n/2]} \frac{(2t)^{n-2k}}{(n-2k)!k!} \sum_{j_1, j_2, \ldots, j_r} \binom{n-k}{j_1, j_2, \ldots, j_r} D_{t^2}^{n-k} h(t^2),$$

In particular:

$$D_t^n\left(e^{rt^2}\right) = n! \sum_{k=0}^{[n/2]} \frac{(2t)^{n-2k}}{(n-2k)!k!} \sum_{j_1, j_2, \ldots, j_r} \binom{n-k}{j_1, j_2, \ldots, j_r} e^{rt^2}.$$

It is easy to generalize the last equations to the general case when t is raised to the power $\alpha = m$, but the relevant equations are still more complicate, so that we leave them to the reader.

6. The case of second-order Bell polynomials

We consider now the case of the second-order Bell polynomials, introduced in Natalini and Ricci (2004): $Y_n^{[2]}(f_1, g_1, h_1; f_2, g_2, h_2; \ldots; f_n, g_n, h_n)$, generated by the nth derivative of the composite function $\Phi(t) := f(g(h(t)))$, whose definition is as follows.

Consider the functions $x = h(t)$, $z = g(x)$, and $y = f(z)$, defined in suitable intervals of the real axis, and suppose that $h(t)$, $g(x)$, and $f(z)$ are n times differentiable with respect to the relevant independent variables, so that the composite function $\Phi(t) := f(g(h(t)))$ can be differentiated n times with respect to t, using the differentiation rule of composite functions.

We use, as before, the following notations:

$$\Phi_j := D_t^j \Phi(t), \quad f_h := D_y^h f(y)|_{y=g(x)}, \quad g_k := D_x^k g(x)|_{x=h(t)}, \quad h_r := D_t^r h(t).$$

Then the n-th derivative can be represented by

$$\Phi_n = Y_n^{[2]}(f_1, g_1, h_1; f_2, g_2, h_2; \ldots; f_n, g_n, h_n),$$

where the $Y_n^{[2]}$ are, by definition, the second-order Bell polynomials.

For example, one has:

$$\begin{aligned}
&Y_1^{[2]}(f_1, g_1, h_1) = f_1 g_1 h_1;\\
&Y_2^{[2]}(f_1, g_1, h_1; f_2, g_2, h_2) = f_1 g_1 h_2 + f_1 g_2 h_1^2 + f_2 g_1^2 h_1^2;\\
&Y_3^{[2]}(f_1, g_1, h_1; f_2, g_2, h_2; f_3, g_3, h_3) = f_1 g_1 h_3 + f_1 g_3 h_1^3\\
&\qquad + 3 f_1 g_2 h_1 h_2 + 3 f_2 g_1^2 h_1 h_2 + 3 f_2 g_1 g_2 h_1^3 + f_3 g_1^3 h_1^3.
\end{aligned}$$

As in the case of the ordinary Bell, we can write, in general:

$$Y_n^{[2]}(f_1, g_1, h_1; \ldots; f_n, g_n, h_n) = \sum_{k=1}^{n} A_{n,k}^{[2]}(g_1, h_1; \ldots; g_n, h_n) f_k \tag{6.1}$$

where the partial second-order Bell polynomials $A_{n,k}^{[2]}$ are introduced.

The connections with the ordinary Bell polynomials are expressed below.

THEOREM 6.1 For every integer n, the polynomials $Y_n^{[2]}$ are expressed in terms of the ordinary Bell ones, by means of the following equation

$$Y_n^{[2]}(f_1, g_1, h_1; \dots; f_n, g_n, h_n)$$
$$= Y_n\big(f_1, Y_1(g_1, h_1); f_2, Y_2(g_1, h_1; g_2, h_2); \dots; f_n, Y_n(g_1, h_1; g_2, h_2; \dots; g_n, h_n)\big). \tag{6.2}$$

Proof Proceeding by induction we have that Equation (6.2) is true for $n = 1$, since

$$Y_1^{[2]}(f_1, g_1, h_1) = f_1\ g_1\ h_1 = f_1\ Y_1(g_1, h_1) = Y_1\big(f_1, Y_1(g_1, h_1)\big).$$

We assume now Equation (6.2) is true. Then,

$$Y_{n+1}^{[2]}(f_1, g_1, h_1; \dots; f_{n+1}, g_{n+1}, h_{n+1}) = D_t\ Y_n^{[2]}(f_1, g_1, h_1; \dots; f_n, g_n, h_n)$$
$$= D_t\ Y_n\big(f_1, Y_1(g_1, h_1); \dots; f_n, Y_n(g_1, h_1; g_2, h_2; \dots; g_n, h_n)\big)$$
$$= Y_{n+1}\big(f_1, Y_1(g_1, h_1); \dots; f_{n+1}, Y_{n+1}(g_1, h_1; g_2, h_2; \dots; g_{n+1}, h_{n+1})\big)$$

See Natalini and Ricci (2004) for the proof of next results.

THEOREM 6.2 The second-order Bell polynomials satisfy the recursion:
$Y_0^{[2]} = f_1;$

$$Y_{n+1}^{[2]}(f_1, g_1, h_1; \dots; f_{n+1}, g_{n+1}, h_{n+1}) = \sum_{k=0}^{n} \binom{n}{k}$$
$$\times Y_{n-k}^{[2]}(f_2, g_1, h_1; f_3, g_2, h_2; \dots; f_{n-k+1}, g_{n-k}, h_{n-k}) Y_{k+1}(g_1, h_1; \dots; g_{k+1}, h_{k+1}). \tag{6.3}$$

THEOREM 6.3 The generalized Faà di Bruno formula holds:

$$Y_n^{[2]}(f_1, g_1, h_1; \dots; f_n, g_n, h_n)$$
$$= \sum_{\pi(n)} \frac{n!}{r_1! r_2! \dots r_n!} f_r \left[\frac{Y_1(g_1, h_1)}{1!}\right]^{r_1} \left[\frac{Y_2(g_1, h_1; g_2, h_2)}{2!}\right]^{r_2} \cdots \left[\frac{Y_n(g_1, h_1; \dots; g_n, h_n)}{n!}\right]^{r_n}. \tag{6.4}$$

According to the above results, we find:

THEOREM 6.4 We have, $\forall\, n$
$A_{n+1,1}^{[2]} = Y_{n+1}(g_1, h_1; \dots; g_{n+1}, h_{n+1}), \quad A_{n+1,n+1}^{[2]} = Y_1^{n+1}(g_1, h_1) = g_1^{n+1} h_1^{n+1}.$

Furthermore, $\forall\, k = 1, 2, \dots, n-1$, the second-order partial Bell polynomials $A_{n,k}^{[2]}$ satisfy the recursion:

$$A_{n+1,k+1}^{[2]}(g_1, h_1; \dots; g_{n+1}, h_{n+1})$$
$$= \sum_{j=0}^{n-k} \binom{n}{j} A_{n-j,k}^{[2]}(g_1, h_1; \dots; g_{n-j}, h_{n-j}) \cdot Y_{j+1}(g_1, h_1; \dots; g_{j+1}, h_{j+1}) \tag{6.5}$$

Proof According to Equation (6.1), and using Theorem 6.1, we can write

$$Y_n^{[2]}(f_1, g_1, h_1; \dots; f_n, g_n, h_n) = \sum_{k=1}^{n} A_{n,k}^{[2]}(g_1, h_1; \dots; g_n, h_n) f_k$$
$$= Y_n\big(f_1, Y_1(g_1, h_1); \dots; f_n, Y_n(g_1, h_1; \dots; g_n, h_n)\big)$$
$$= \sum_{k=1}^{n} A_{n,k}\big(Y_1(g_1, h_1); \dots; Y_n(g_1, h_1; \dots; g_n, h_n)\big) f_k,$$

so that

$$A^{[2]}_{n,k}(g_1,h_1;\dots;g_n,h_n) = A_{n,k}\big(Y_1(g_1,h_1);\dots;Y_n(g_1,h_1;\dots;g_n,h_n)\big).$$

Using the recursion (3.2) of Theorem 3.1, we find:

$$\begin{aligned} &A_{n+1,k+1}\big(Y_1(g_1,h_1),\dots,Y_{n+1}(g_1,h_1;\dots;g_{n+1},h_{n+1})\big) \\ &= \sum_{j=0}^{n-k} A_{n-j,k}\Big(Y_1(g_1,h_1);\dots;Y_{n-j}(g_1,h_1;\dots;g_{n-j},h_{n-j})\Big)\cdot Y_{j+1}(g_1,h_1;\dots;g_{j+1},h_{j+1}),\end{aligned}$$

so that we have proved our result.

Therefore, the complete second-order Bell polynomials $B^{[2]}_n$ are defined by the equation:

$$B^{[2]}_n(g_1,h_1;\dots;g_n,h_n) = Y^{[2]}_n(1,g_1,h_1;\dots;1,g_n,h_n) = \sum_{k=1}^{n} A^{[2]}_{n,k}(g_1,h_1;\dots;g_n,h_n),$$

and the second-order Bell numbers by

$$b^{[2]}_n = Y^{[2]}_n(1,1,1;\dots;1,1,1) = \sum_{k=1}^{n} A^{[2]}_{n,k}(1,1;\dots;1,1).$$

6.1. The $B^{[2]}_n$ polynomials and $b^{[2]}_n$ numbers

The second-order Bell polynomials $B^{[2]}_n(g_1,h_1;\dots;g_n,h_n)$ for $(n=1,2,\dots,5)$, computed by a Mathematica© program, are given by:

$$\begin{aligned} B^{[2]}_1 &= g_1h_1 \\ B^{[2]}_2 &= g_1^2h_1^2 + g_2h_1^2 + g_1h_2 \\ B^{[2]}_3 &= g_1^3h_1^3 + 3g_1g_2h_1^3 + g_3h_1^3 + 3g_1^2h_1h_2 + 3g_2h_1h_2 + g_1h_3 \\ B^{[2]}_4 &= g_1^4h_1^4 + 6g_1^2g_2h_1^4 + 3g_2^2h_1^4 + 4g_1g_3h_1^4 + g_4h_1^4 + 6g_1^3h_1^2h_2 + 18g_1g_2h_1^2h_2 \\ &\quad + 6g_3h_1^2h_2 + 3g_1^2h_2^2 + 3g_2h_2^2 + 4g_1^2h_1h_3 + 4g_2h_1h_3 + g_1h_4 \\ B^{[2]}_5 &= g_1^5h_1^5 + 10g_1^3g_2h_1^5 + 15g_1g_2^2h_1^5 + 10g_1^2g_3h_1^5 + 10g_2g_3h_1^5 + 5g_1g_4h_1^5 + g_5h_1^5 \\ &\quad + 10g_1^4h_1^3h_2 + 60g_1^2g_2h_1^3h_2 + 30g_2^2h_1^3h_2 + 40g_1g_3h_1^3h_2 + 10g_4h_1^3h_2 + 15g_1^3h_1h_2^2 \\ &\quad + 45g_1g_2h_1h_2^2 + 15g_3h_1h_2^2 + 10g_1^3h_1^2h_3 + 30g_1g_2h_1^2h_3 + 10g_3h_1^2h_3 + 10g_1^2h_2h_3 \\ &\quad + 10g_2h_2h_3 + 5g_1^2h_1h_4 + 5g_2h_1h_4 + g_1h_5\,. \end{aligned}$$

Furthermore, we write down explicitly the $b^{[2]}_n$, $(n=1,2,\dots,10)$,

$$1, 3, 12, 60, 358, 2471, 19302, 167894, 1606137, 16733779$$

and we remark that the sequence $b^{[2]}_1, b^{[2]}_2, b^{[2]}_3, \dots$ appears in the Encyclopedia of Integer Sequences (Sloane & Plouffe, 1995) with # A000258, arising from a different problem of combinatorial analysis.

7. The general case of $M-1$-order Bell polynomials

Using the same methods, the above results can be generalized to the case of the Bell polynomials of higher order.

Consider $\Phi(t) := f_{(1)}(f_{(2)}(\cdots(f_{(M)}(t))))$, i.e. the composition of functions $x_{M-1} = f_{(M)}(t), \dots, x_1 = f_{(2)}(x_2)$, $y = f_{(1)}(x_1)$, defined in suitable intervals of the real axis, and suppose that the functions $f_{(M)}, \dots, f_{(2)}, f_{(1)}$ are n times differentiable with respect to the relevant independent variables so that $\Phi(t)$ can be differentiated n times with respect to t, using the differentiation rule of composite functions. By definition we put $x_M := t$, so that $y = \Phi(t)$.

We use the following notations:

$$\Phi_h: = D_t^h \Phi(t),$$
$$f_{(1),h}: = D_{x_1}^h f_{(1)}|_{x_1 = f_{(2)}(\cdots(f_{(M)}(t)))},$$
$$f_{(2),k}: = D_{x_2}^k f_{(2)}|_{x_2 = f_{(3)}(\cdots(f_{(M)}(t)))},$$
$$\dots\dots\dots\dots\dots$$
$$f_{(M),j}: = D_{x_M}^j f_{(M)}|_{x_M = t}.$$

Then, the nth derivative can be represented by

$$\Phi_n = Y_n^{[M-1]}(f_{(1),1}, \dots, f_{(M),1}; f_{(1),2}, \dots, f_{(M),2}; \dots; f_{(1),n}, \dots, f_{(M),n}),$$

where the $Y_n^{[M-1]}$ are, by definition, the Bell polynomials of order $M-1$.

The theorems of Section 6 can be generalized as follows:

THEOREM 7.1 For every integer n, the polynomials $Y_n^{[M-1]}$ are expressed in terms of the Bell polynomials of lower order, by means of the following equation:

$$\begin{aligned} &Y_n^{[M-1]}(f_{(1),1}, \dots, f_{(M),1}; \dots; f_{(1),n}, \dots, f_{(M),n}) \\ &= Y_n\Big(f_{(1),1}, Y_1^{[M-2]}(f_{(2),1}, \dots, f_{(M),1}); f_{(1),2}, Y_2^{[M-2]}(f_{(2),1}, \dots, f_{(M),1}; f_{(2),2}, \dots, f_{(M),2}); \\ &\quad \dots; f_{(1),n}, Y_n^{[M-2]}(f_{(2),1}, \dots, f_{(M),1}; \dots; f_{(2),n}, \dots, f_{(M),n})\Big). \end{aligned} \tag{7.1}$$

The proof can be achieved by induction, as it was done in the previous Theorem 6.1.

THEOREM 7.2 The following recurrence relation for the Bell polynomials $Y_n^{[M-1]}$ holds true:

$$\begin{aligned} &Y_n^{[M-1]}(f_{(1),1}, \dots, f_{(M),1}; \dots; f_{(1),n}, \dots, f_{(M),n}) \\ &= \sum_{\pi(n)} \frac{n!}{r_1! r_2! \dots r_n!} f_{(1),r} \left[\frac{Y_1^{[M-2]}(f_{(2),1}, \dots, f_{(M),1})}{1!}\right]^{r_1} \\ &\times \left[\frac{Y_2^{[M-2]}(f_{(2),1}, \dots, f_{(M),1}; f_{(2),2}, \dots, f_{(M),2})}{2!}\right]^{r_2} \\ &\cdots \times \left[\frac{Y_n^{[M-2]}(f_{(2),1}, \dots, f_{(M),1}; \dots; f_{(2),n}, \dots, f_{(M),n})}{n!}\right]^{r_n}. \end{aligned} \tag{7.2}$$

THEOREM 7.3 The generalized Faà di Bruno formula holds true:

$$\begin{aligned} &Y_0^{[M-1]} = f_{(1),1}; \\ &Y_{n+1}^{[M-1]}(f_{(1),1}, \dots, f_{(M),1}; \dots; f_{(1),n+1}, \dots, f_{(M),n+1}) \\ &= \sum_{k=0}^{n} \binom{n}{k} Y_{n-k}^{[M-1]}(f_{(1),2}, f_{(2),1}, \dots, f_{(M),1}; f_{(1),3}, f_{(2),2}, \dots, f_{(M),2}; \\ &\quad \dots; f_{(1),n-k+1}, f_{(2),n-k}, \dots, f_{(M),n-k}) \\ &\times Y_{k+1}^{[M-2]}(f_{(2),1}, \dots, f_{(M),1}; \dots; f_{(2),k+1}, \dots, f_{(M),k+1}). \end{aligned} \tag{7.3}$$

Using the same technique of Theorem 6.4, we find

THEOREM 7.4 We have, $\forall\, n$

$$A_{n+1,1}^{[M-1]} = Y_{n+1}^{[M-2]}(f_{(2),1}, \dots, f_{(M),1}; \dots; f_{(2),n+1}, \dots, f_{(M),n+1}),$$
$$A_{n+1,n+1}^{[M-1]} = \left(Y_1^{[M-2]}(f_{(2),1}, \dots, f_{(M),1})\right)^{n+1}.$$

Furthermore, $\forall k = 1, 2, \dots, n-1$, the $(M-1)$thorder partial Bell polynomials $A_{n,k}^{[M-1]}$ satisfy the recursion:

$$\begin{aligned} & A_{n+1,k+1}^{[M-1]}\left(f_{(2),1}, \dots, f_{(M),1}; \dots; f_{(2),n+1}, \dots, f_{(M),n+1}\right) \\ & = \sum_{j=0}^{n-k} \binom{n}{j} A_{n-j,k}^{[M-1]}\left(f_{(2),1}, \dots, f_{(M),1}; \dots; f_{(2),n-j}, \dots, f_{(M),n-j}\right) \\ & \times Y_{j+1}^{[M-2]}\left(f_{(2),1}, \dots, f_{(M),1}; \dots; f_{(2),j+1}, \dots, f_{(M),j+1}\right). \end{aligned} \tag{7.4}$$

Proof As a consequence of the equations

$$\begin{aligned} & Y_n^{[M-1]}(f_{(1),1}, \dots, f_{(M),1}; \dots; f_{(1),n}, \dots, f_{(M),n}) \\ & = \sum_{k=1}^{n} A_{n,k}^{[M-1]}(f_{(2),1}, \dots, f_{(M),1}; \dots; f_{(2),n}, \dots, f_{(M),n}) f_{(1),k} \\ & = Y_n\Big(f_{(1),1}, Y_1^{[M-2]}(f_{(2),1}, \dots, f_{(M),1}); \dots \\ & \quad \dots; f_{(1),n}, Y_n^{[M-2]}(f_{(2),1}, \dots, f_{(M),1}; \dots; f_{(2),n}, \dots, f_{(M),n})\Big) \\ & = \sum_{k=1}^{n} A_{n,k}\Big(Y_1^{[M-2]}(f_{(2),1}, \dots, f_{(M),1}); \dots \\ & \quad \dots; Y_n^{[M-2]}(f_{(2),1}, \dots, f_{(M),1}; \dots; f_{(2),n}, \dots, f_{(M),n})\Big) f_{(1),k}, \end{aligned}$$

we find the relations between the polynomial coefficients:

$$\begin{aligned} & A_{n,k}^{[M-1]}(f_{(2),1}, \dots, f_{(M),1}; \dots; f_{(2),n}, \dots, f_{(M),n}) \\ & = A_{n,k}\Big(Y_1^{[M-2]}(f_{(2),1}, \dots, f_{(M),1}); \dots; Y_n^{[M-2]}(f_{(2),1}, \dots, f_{(M),1}; \dots; f_{(2),n}, \dots, f_{(M),n})\Big) \end{aligned}$$

so that, recalling Equation (3.2) in Theorem 3.1, our result follows.

The results contained in this section can be used in order to introduce the higher order Bell polynomials and the relevant higher order Bell numbers. Tables of these mathematical objects will be included in a separate article.

Dedication
Dedicated to Prof. Dr Hari M. Srivastava.

Funding
The authors received no direct funding for this research.

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Citation information
Cite this article as: Remarks on Bell and higher order Bell polynomials and numbers, Pierpaolo Natalini & Paolo Emilio Ricci, *Cogent Mathematics* (2016), 3: 1220670.

Cover image
Source: Author.

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