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Using some results about the Lie evolution of differential operators to obtain the Fokker-Planck equation for non-Hamiltonian dynamical systems of interest

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Finding the generalized Fokker-Planck Equation (FPE) for the reduced probability density function of a subpart of a given complex system is a classical issue of statistical mechanics. Zwanzig projection perturbation approach to this issue leads to the trouble of resumming a series of commutators of differential operators that we show to correspond to solving the Lie evolution of first order differential operators along the unperturbed Liouvillian of the dynamical system of interest. In this paper, we develop in a systematic way the procedure to formally solve this problem. In particular, here we show which the basic assumptions are, concerning the dynamical system of interest, necessary for the Lie evolution to be a group on the space of first order differential operators, and we obtain the coefficients of the so-evolved operators. It is thus demonstrated that if the Liouvillian of the system of interest is not a first order differential operator, in general, the FPE structure breaks down and the master equation contains all the power of the partial derivatives, up to infinity. Therefore, this work shed some light on the trouble of the ubiquitous emergence of both thermodynamics from microscopic systems and regular regression laws at macroscopic scales. However these results are very general and can be applied also in other contexts that are non-Hamiltonian as, for example, geophysical fluid dynamics, where important events, like El Niño, can be considered as large time scale phenomena emerging from the observation of few ocean degrees of freedom of a more complex system, including the interaction with the atmosphere. https://doi.org/10.1063/1.5037656

I. INTRODUCTION

As it is well known and it is also shown in Sec. III, in perturbation approaches in classical statistical mechanics, and in particular when we exploit the powerful Zwanzig projection formalism,[1–5] we face the problem of solving the interaction representation of the perturbation Liouville operator, namely, the following compound of operators: $\exp(\mathcal{L}_0 t)\mathcal{L}_I \exp(-\mathcal{L}_0 u)$, where $\mathcal{L}$ is the Liouvillian and the subscripts "0" and "I" stand for "unperturbed" and "interaction" (or "perturbation"), respectively. Using the Hadamard lemma we can write the above expression as a power series of commutators,

$$e^{\mathcal{L}_0 u}\mathcal{L}_I e^{-\mathcal{L}_0 u} = \mathcal{L}_I + [\mathcal{L}_0, \mathcal{L}_I]u + [\mathcal{L}_0, [\mathcal{L}_0, \mathcal{L}_I]]u^2/2! + [\mathcal{L}_0[\mathcal{L}_0, [\mathcal{L}_0, \mathcal{L}_I]]]u^3/3! \dots .$$
$$\equiv e^{\mathcal{L}_0^{\times} u}[\mathcal{L}_I], \tag{1}$$

where $\mathcal{L}_0^{\times}[\mathcal{L}_I] \equiv [\mathcal{L}_0, \mathcal{L}_I] \equiv \mathcal{L}_0\mathcal{L}_I - \mathcal{L}_I\mathcal{L}_0$. For "solving" the above expression, we mean to find a corresponding explicit, exact or approximate, partial differential operator, possibly of finite maximum order in the power of derivatives. Notice that, in principle, this series leads to a partial differential

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operator with derivatives of all orders, up to infinity, a result that would make useless the corresponding master equation for the Density Function (DF) of the system of interest. As we show in Sec. III, to obtain just first and second order partial derivatives in the master equation from the Zwanzig projection procedure, we need that the series of Eq. (1) give rise to a first order partial differential operator.

Using the Lie formalism, solving Eq. (1) corresponds to find the "evolution" of the interaction Liouvillian $\mathcal{L}_I$, along the unperturbed Liouvillian $\mathcal{L}_0$ in some appropriate phase space. Thus looking for first order partial differential operator solutions of the same equation means studying the condition for which the Lie evolution along a given Liouvillian is a group in the space of the first order partial differential operators: we can say that Eq. (1) leads us to work with the Lie algebra. Actually, introducing a Lie algebra is not necessary and it is not done in standard papers exploiting the Zwanzig projection formalism. This is because most of them concern the foundation of statistical mechanics issue, where the "fundamental" approach requires staying in a Hamiltonian framework.[6–10] The Hamiltonian nature of the system and the fact that in this field canonical equilibrium DF is an "*a priori*" hypothesis impose both a Fokker-Planck-like structure for the effective reduced Liouvillian of the part of interest and a fundamental relationship between the diffusion (second order partial derivatives) and the dissipation (first order partial derivatives) terms. This relationship, often cited as the fluctuation dissipation theorem, allow us to deal only with the deterministic parts of the Fokker-Planck Equation (FPE), i.e., the first order partial derivatives, and to obtain the diffusion coefficients just by using the fluctuation dissipation relationship (see Ref. 10 for a clear example of such an approach). From the physicist point of view, in fact, in the Hamiltonian cases, fluctuation and dissipation processes come out from the perturbation and the corresponding reaction forces, respectively, of the fast microscopic degrees of freedom (often called thermal bath) to the slow system of interest. They are the way the exchange of energy between the two systems is balanced at equilibrium, from a statistical point of view. In a nutshell, from a formal point of view, the fact that in the FPE for the reduced DF of the system of interest the coefficients of the second order partial derivatives are strictly related to those of the first order ones allows us to skip the problem of solving Eq. (1), focusing all the attention to the easier task of finding the drift terms.[8,10–12]

However, the Zwanzig formalism can be applied to obtain statistical information from dynamical systems in a great variety of cases, well beyond the classical problem of foundation of thermodynamics and statistical mechanics. For example, it can be applied to climate dynamics problems[13] or to some geophysical fluid dynamics phenomena like El Niño/La Niña episodes[14] where the system of interest of the corresponding models is usually intrinsically dissipative and also the interaction with the atmosphere is not of Hamiltonian type.[14–18] In non-Hamiltonian cases, fluctuation and dissipation could not balance each other, and they could be of different "strength" and origin. Moreover, the equilibrium DF (if any) is not known "*a priori*" and often is not possible to obtain an analytical expression for it. This means that to obtain the Liouville equation for the reduced DF of the part of interest, we are forced to solve Eq. (1) and go deeper into the Lie algebra formalism.

Of course, Lie algebra concepts are not new for physicists, who know and use them since the first half of the previous century. For example, they are largely used in theoretical, classical, and quantum physics as the generators of "infinitesimal" (Lie) group transformations. In classical mechanics, the dynamics of Hamiltonian systems is usually described as symplectic flows where the Lie algebra is associated with the Poisson Bracket (PB) operation between *functions* of the phase space of the system.

Moreover, generalized PB between functions defines also co-symplectic fluxes, as in the case of the equation of motion for the rigid rotor and fluid dynamics theories in Eulerian variables,[19–23] where the associated semi-simple Lie algebra generates the SO(3) group. Lie integrators have been introduced to exactly preserve conserved quantities in the numerical simulation of Hamilton-Jacobi equations.[24,25] It is also well known that the classical to quantum mechanics correspondence for a given dynamical system is obtained by transforming the Lie-Poisson brackets among functions to commutators (i.e., Lie algebras) among the operators associated with observables.

Here, as we have stated above, we are interested in finding the coefficients of the differential operator corresponding to Eq. (1), and thus we shall deal with Lie algebras from a different point of view with respect to the above cited standard cases: we shall focus on the Lie evolution of differential operators along some Liouvillian. However, to achieve this goal, we have to put the problem in a more general and formal context that leads us to some results that can be useful also beyond the specific problem of statistical mechanics here addressed. For example, we see that for the Lie evolution of functions, along a generic Liouvillian, we get an effective antisymmetric property of the Liouvillian operator also for *non*-Hamiltonian (symplectic or co-symplectic) fluxes. Moreover, the same, but under the limit of validity of Assumption C (see Sec. IV), holds also for the Lie evolution of differential operators. This fact could be used, for example, for an eigenvalue/eigenvector approach to the Lie evolution of operators, something that we do not address in the present paper.

For those interested in further generalizations, we think worthwhile observe that most of the results we shall obtain hold under the sole assumption of working with differential operators acting on "enough" smooth functions; thus, in principle, we could not require to have some physical system to which we refer. Namely, in principle, we could develop the present results staying in the formal framework of the pure differential geometry field.

Having said that, we want to stress again that the present work is mainly devoted to contribute to solve specific physics problems in statistical mechanics. In fact, the main result of this paper, from a physics point of view, shall be the generalized FPE stemming from a so large class of dynamical systems that it can contribute to shed some light on the ubiquitous emergence, in nature, of canonical/Gaussian DF and regular and linear Onsager-like regression laws and, at the same time, to formally justify the possible departure from these standard statistical behaviors.

For those interested in a more formal and mathematical approach, we shall insert sometime, sparsely, without following a true rigorous criterion, some hints on how to generalize in a more abstract way the approach developed here.

This paper is organized as follows. In Sec. II, we give a general formal definition of the dynamical system we are interested in. This part is devoted to those interested in a fundamental approach, from a mathematical point of view, of the physical problem we focus on in the present work. In Sec. III, we introduce, with a more physical point of view, the specific problem we want to contribute to solve: finding the time evolution of the reduced DF for a dynamical system of interest weakly interacting with other dynamical system. Here we show that the perturbation approach of this problem, and in particular the Zwanzig projection procedure, leads us to face with the trouble of the Lie evolution of differential operators along the Liouvillian of the system of interest. Thus in Secs. IV and V we give a context adapted introduction of the Lie derivative and of the Lie evolution of differential operators, and we find a representation of this evolution operation, in terms of the formal analytic expression of the coefficients of the basis of the vector field. In Sec. VI we apply the results of Sec. V to get the FPE for the reduced DF for the system of interest and we give a couple of simple specific physical examples where these results are used. Finally, in Sec. VII, we consider the special cases where the system of interest is Hamiltonian apart from some linear (dissipative or explosive) terms and we show how it is possible to find, in an alternative way with respect to that of Sec. VI, the explicit coefficients of the FPE. Section VIII is devoted to the conclusions of the present work.

II. DEFINITION OF THE DYNAMICAL SYSTEM

The general, but in some way, well-defined, dynamical system we are interested in is given by

$$\frac{d}{dt}\boldsymbol{x}(t) = \boldsymbol{V}(\boldsymbol{x};\xi), \tag{2}$$

where bold case is for vectors, here $\boldsymbol{x}, \boldsymbol{V} \in \mathbb{R}^N$ (or a subset of $\mathbb{R}^N$), i.e., $\boldsymbol{x} \equiv (x_1, \ldots, x_N)$, $\boldsymbol{V}(\boldsymbol{x};\xi) \equiv (V_1(\boldsymbol{x};\xi), \ldots, V_N(\boldsymbol{x};\xi)) \in \mathbb{R}^N$, and $V_i(\boldsymbol{x};\xi) \in C^k(\mathbb{R}^N) : \mathbb{R}^N \to \mathbb{R}$, $i \in \mathbb{N}$, $1 \le i \le N$, and k is an appropriate integer. ξ is, for now, just a parameter. As we shall see later, our approach is useful for cases where the velocity vector field $\boldsymbol{V}(\boldsymbol{x};\xi)$ can be separated in two parts,

$$\boldsymbol{V}(\boldsymbol{x};\xi) = -\boldsymbol{C}(\boldsymbol{x}) - \epsilon\, \boldsymbol{I}(\boldsymbol{x})\xi, \tag{3}$$

where the "forcing" $-\epsilon\, \boldsymbol{I}(\boldsymbol{x})\xi$ represents a perturbation. We shall say that the system is unperturbed when the coupling parameter ϵ is vanishing. The starting point of the present work is that the unperturbed time evolution $\boldsymbol{x}_0(t+u)$ [u negative or positive, $\boldsymbol{x}(t) \equiv \boldsymbol{x}$] of the system is known, namely, that *we are able to integrate analytically, or numerically*, Eqs. (2)–(3) for $\epsilon = 0$.

For any function $f(\boldsymbol{x}) \in C^\infty(\mathbb{R}^N)$, the time derivative along the flux generated by Eq. (2) can be written as (repeated indices imply summation from 1 to N)

$$\frac{d}{dt} f(\boldsymbol{x}(t)) = \left\{ V_i(\boldsymbol{x};\xi) \frac{\partial}{\partial x_i} f(\boldsymbol{x}) \right\}_{\boldsymbol{x}=\boldsymbol{x}(t)}. \tag{4}$$

Thus $V_i(x;\xi)$ are the coordinates of the vector field of the flux $\Phi^t_{\boldsymbol{V}} : \mathbb{R}^N \to \mathbb{R}^N$ for $t \in \mathbb{R}$, generated by Eq. (2), on the basis $(\partial/\partial x_1, \ldots, \partial/\partial x_N)$.

More in general, it can happen that the phase space of the system is mapped only locally by $\mathbb{R}^N$, and thus we should consider a given compact smooth Riemannian manifold $\boldsymbol{M}$ and a tangent vector field $\boldsymbol{V}$, which can be thought as a smooth assignment of a tangent vector $\boldsymbol{V}\,(p;\,\xi)$ to each point $p \in \boldsymbol{M}$. The vector field defines a one-parameter family of maps, $\Phi^t_{\boldsymbol{V}} : \boldsymbol{M} \to \boldsymbol{M}$ for $t \in \mathbb{R}$, called the flow of $\boldsymbol{V}$. The flow is formally defined as the unique solution of the differential equation,

$$\frac{d}{dt}\Phi^t_{\boldsymbol{V}}(p) = \boldsymbol{V}(p;\xi), \quad \Phi^0_{\boldsymbol{V}}(p) = p. \tag{5}$$

Then, for a function $f(p) \in C^\infty(\boldsymbol{M})$, the covariant derivative $D_{\boldsymbol{V}}\,(f)$ of f with respect to $\boldsymbol{V}$ is given by the inner product, in the tangent space of p, of the tangent vector $\boldsymbol{V}$ and ∇_p (the gradient of f on p),

$$D_{\boldsymbol{V}}(f)(p) = \boldsymbol{V}(p;\xi) \cdot \boldsymbol{\nabla}_p f(p). \tag{6}$$

Notice that, as we have already stressed, apart from the present section, including Subsection II A, devoted to define the general formal framework we are considering, for the sake of simplicity, we will focus on the case where the manifold $\boldsymbol{M}$ is $\mathbb{R}^N$ (or a subset of $\mathbb{R}^N$); thus we shall use mainly the direct notations of Eqs. (2) and (4) rather than those in Eqs. (5)–(6).

A. Some more formal considerations

This work gives some formal tools for the study of "observables" of interest of a dynamical system, where often the only possible measurable quantities of the system are averages in the phase space of the system, using a DF as weight. Thus we make the following:

Assumption A. The state of the system is defined, at any given time, by a DF on the whole phase space, $\rho(p;\,t)$.

In practice we develop our approach considering to stay in a Riemannian manifold $\boldsymbol{M}$ given by a differentiable manifold of dimension N endowed with a Riemannian metric on $\boldsymbol{M}$ (actually, for the sake of simplicity, we shall consider $\boldsymbol{M} = \mathbb{R}^N$, i.e., the flat Euclidean space equipped with the ordinary Euclidean intrinsic metric). Therefore the DF measure of a set $\mathcal{X} \in \boldsymbol{M}$ is

$$\mu_\rho(\mathcal{X}) = \int_{\mathcal{X}} \rho(q;t)dq. \tag{7}$$

The usual Lebesgue measure of the set $\mathcal{X}$ is denoted by $\mu_L(\mathcal{X})$, and the density of the Lebesgue measure is the uniform density $1/\mu_L(\boldsymbol{M})$ for all points $q \in \boldsymbol{M}$. Thus $dq = \mu_L(dq)$ and we normalize the DF as $\mu_\rho(\boldsymbol{M}) = 1$.

Working on a differentiable manifold of dimension N endowed with a Riemannian metric, the next requirement is satisfied directly by Assumption A, Eqs. (2)–(4), and the continuity equation; however, for a more general approach we introduce the following:

Assumption B. We operate in a Banach space where a linear temporal evolution operator $\mathcal{L}$ of the DF $\rho(q;\,t)$ is defined as

$$\frac{\partial}{\partial t}\rho(q;t) = \mathcal{L}\rho(q;t), \tag{8}$$

with, again, $q \in \boldsymbol{M}$. We call $\mathcal{L}$ the Liouville operator or the Liouvillian.

Using this assumption we can introduce the dual space and the associated Liouville adjoint operator $\mathcal{L}^+$ in the following way:

$$\frac{d}{dt}\langle f(q)\rangle_\rho = \int_{\boldsymbol{M}} f(q)\,\mathcal{L}\rho(q;t)\,dq = \int_{\boldsymbol{M}} (\mathcal{L}^+ f(q))\,\rho(q;t)\,dq, \tag{9}$$

where the function $f(q)$ belongs to the Banach space and the symbol $\langle\ldots\rangle_\rho$ means the average with the DF $\rho(q;t)$ as weight.

Coming back for simplicity to $\boldsymbol{M} = \mathbb{R}^N$ (or a subset of $\mathbb{R}^N$), i.e., $q = \boldsymbol{x}$, and to the dynamical system described by the Lagrangian (or Langevin) picture in Eq. (2), because the time evolution is deterministic, using the continuity condition, we have that the Liouville operator is given by the following first order partial differential operator:

$$\mathcal{L}_{\boldsymbol{V}} = -\frac{\partial}{\partial x_i} V_i(\boldsymbol{x};\xi) \tag{10}$$

and

$$\frac{d}{dt} f(\boldsymbol{x}) = \left(\mathcal{L}^+_{\boldsymbol{V}} f(\boldsymbol{x})\right). \tag{11}$$

Assuming the DF vanishes at the boundary of the set of the possible states of the system (or making it to vanish at the boundary), we have

$$\mathcal{L}^+_{\boldsymbol{V}} = V_i(\boldsymbol{x};\xi)\frac{\partial}{\partial x_i}. \tag{12}$$

Comparing this with Eq. (4) we have that the adjoint Liouville operator is the vector field associated with the flux induced by the group (or semigroup) of transformation $\Phi^t_{\boldsymbol{V}}$ of our dynamical system. From Eq. (12) it follows that the time evolution by advection, for a time u, of any analytic function $s: \mathbb{R}^N \to \mathbb{R}$, starting from $s(\boldsymbol{x})$ at $u = 0$, is given by

$$s(\boldsymbol{x}_{\boldsymbol{V}}(t+u)) = \left(e^{\mathcal{L}^+_{\boldsymbol{V}} u} s(\boldsymbol{x})\right). \tag{13}$$

Notice that in the rhs of the above equation the time "t" does not enter; thus it should not be present also in the lhs of the same equation. Actually, for the lhs of Eq. (13), we should use the more rigorous notation $s(\boldsymbol{x}_{\boldsymbol{V}}(\boldsymbol{x};u))$, where the first argument of the function $\boldsymbol{x}_{\boldsymbol{V}}(\boldsymbol{a};u)$ is the initial condition, i.e., $\boldsymbol{x}_{\boldsymbol{V}}(\boldsymbol{a};0) = \boldsymbol{a}$. However, we think that the expression in Eq. (13) makes the notation less heavy, and it can be directly connected to the usual Lagrange (or Langevin) picture, where $\boldsymbol{x}(t) = \boldsymbol{x}$.

In the rhs of Eq. (13), we have used the big parentheses to emphasize the limits of application of the operator $e^{\mathcal{L}^+_{\boldsymbol{V}} u}$, namely, the rhs of Eq. (13) is a function. It is important to specify that, to avoid confusion in the notation, as we are going to work with the algebra of operators.

III. PERTURBATION APPROACHES AND THE LIE EVOLUTION OF DIFFERENTIAL OPERATORS

As we stated in the Introduction, we are interested in the case where the tangent vector $\boldsymbol{V}$ can be separated in two parts as in Eq. (3), where $-\epsilon\,\boldsymbol{I}(\boldsymbol{x})\xi$ is considered as a "force" that perturbs the system of interest. Therefore the Liouvillian in Eq. (10) can be written as

$$\mathcal{L}_{\boldsymbol{V}} = \mathcal{L}_0 + \xi\mathcal{L}_I, \tag{14}$$

where

$$\mathcal{L}_0 \equiv \frac{\partial}{\partial x_i} C_i(\boldsymbol{x}) \tag{15}$$

and

$$\mathcal{L}_I \equiv \epsilon\frac{\partial}{\partial x_i} I_i(\boldsymbol{x}), \tag{16}$$

from which Eq. (8) becomes

$$\frac{\partial}{\partial t}\rho(\boldsymbol{x};t) = \mathcal{L}_0\rho(\boldsymbol{x};t) + \xi\,\mathcal{L}_I\rho(\boldsymbol{x};t). \tag{17}$$

Usually, for a perturbation power expansion of the Liouville equation (17), one works in the interaction representation

$$\frac{\partial}{\partial t}\tilde{\rho}(\boldsymbol{x};t) = \xi\,\tilde{\mathcal{L}}_I(t)\tilde{\rho}(\boldsymbol{x};t), \tag{18}$$

where

$$\tilde{\rho}(\boldsymbol{x};t) \equiv e^{-\mathcal{L}_0 t}\rho(\boldsymbol{x};t) \tag{19}$$

and

$$\tilde{\mathcal{L}}_I(t) \equiv e^{-\mathcal{L}_0 t}\mathcal{L}_I e^{\mathcal{L}_0 t}. \tag{20}$$

As we already stated in the Introduction, Eq. (20) gives the central operator we focus on. It is clear from Eqs. (18) and (20) that it would be really useful to have, if possible, an explicit analytic expression of it, in terms of the basis $(\partial/\partial x_1, \ldots, \partial/\partial x_N)$ of the tangent manifold of the states of the system at any point $\boldsymbol{x}$. This is actually the issue that we address in this paper.

The specific case of perturbation approach we are interested in is where the points $\boldsymbol{x}$ of the system of Eq. (2) represent the states of a part of interest of a larger dynamical system (the dot stays for d/dt),

$$\begin{aligned}
\dot{\boldsymbol{x}} &= -\boldsymbol{C}(\boldsymbol{x}) - \epsilon\,\boldsymbol{I}(\boldsymbol{x})\xi,\\
\dot{\xi} &= F(\xi, \boldsymbol{\pi}, \epsilon h(\boldsymbol{x})),\\
\dot{\boldsymbol{\pi}} &= \boldsymbol{Q}(\xi, \boldsymbol{\pi}, \epsilon h(\boldsymbol{x})).
\end{aligned} \tag{21}$$

In this case, the parameter ξ is not fixed but it represents an extra variable of the system, which from now on we shall call *booster*, mimicking, usually, a more or less complex set of external variables interacting with the system of interest.[11,26–28] Thus, the booster variable ξ is part of a general "external environment" of which the dynamics is given by the velocity vector field $(F, \boldsymbol{Q})$. In turn, this dynamics is affected by the smooth function $\epsilon h(\boldsymbol{x})$, representing the "reaction force" of the system of interest on the perturbing environment. For the sake of simplicity, we also assume $h(0) = 0$. The equation of motion for the DF of the part of interest can be obtained using a Zwanzig-like formal projection approach in the perturbation version of Refs. 2, 3, 10, 12, and 29. In this approach, we hide the dynamics and the initial state of the "external" part $(\xi, \boldsymbol{\pi})$ by some system-specific average procedure. Consequently, the evolution of the DF of the part of interest is not anymore deterministic and, to the lowest non-vanishing order on $\mathcal{L}_I$, the time evolution of the reduced DF, obtained by this projection/perturbation approach, is governed by an integro-differential equation,

$$\begin{aligned}
\frac{\partial}{\partial t}\rho(\boldsymbol{x};t) &= \mathcal{L}_0\rho(\boldsymbol{x};t) + \left\{\mathcal{L}_I \int_0^t du\,\phi(u)\,e^{\mathcal{L}_0 u}\mathcal{L}_I e^{-\mathcal{L}_0 u}\right\}\rho(\boldsymbol{x};t)\\
&\quad + \left\{\mathcal{L}_I \int_0^t du\,\epsilon S(u)\,h(\boldsymbol{x}_0(t-u))\right\}\rho(\boldsymbol{x};t),
\end{aligned} \tag{22}$$

where the Liouvillians $\mathcal{L}_0$ and $\mathcal{L}_I$ are defined in Eqs. (15) and (16), respectively, $\phi(u)$ is the (unnormalized) unperturbed ($\epsilon = 0$) auto-correlation function of the booster variable ξ, and $S(u)$ is the response function of the average of ξ to the reaction term $h(\boldsymbol{x})$, having assumed that the average value of ξ is vanishing for $\epsilon = 0$. Moreover, in Eq. (22) we have used the notation introduced in Eq. (13) for the unperturbed (here backward) evolution of $\boldsymbol{x}$,

$$\boldsymbol{x}_0(t-u) \equiv \left(e^{-\mathcal{L}_0^+ u}\boldsymbol{x}\right). \tag{23}$$

Notice that the last term in the rhs of Eq. (22), resulting from the reaction of the system of interest *on* the external $(\xi, \boldsymbol{\pi})$ dynamical system, is a first order partial derivative operator, i.e., it induces an additional deterministic drift that, in certain cases, can be identified as a dissipation force.[3] However in the present work, the focus is on the second term in the rhs of Eq. (22) that contains the expression in Eq. (1) [changing u with $-t$, it is also the same as of Eq. (20)]. This operator is usually drastically simplified by using some approximations, involving time scale separation (see, for example, Ref. 29), or specific system arguments.[10] In Ref. 2, for the specific case we dealt with there, we have shown that this term gives rise exactly to a first order partial derivative; therefore, Eq. (22) results to be a FPE. Moreover, in the same paper, we were able to obtain the analytic expression of this first order partial differential operator.

Here we shall show how and when it is possible in general to get a formal expression of our central operator of Eq. (1), in terms of the basis $(\partial/\partial x_1, \ldots, \partial/\partial x_N)$ of the vector field of the fluxes on the phase space in the unperturbed ($\epsilon = 0$) velocities' vector field C_i of Eq. (3).

For further use, we introduce the *left* and the *right* representations of a differential operator of the vector field: in general a Liouvillian is written in the *left* representation, as for $\mathcal{L}_0$ and $\mathcal{L}_I$ in Eqs. (15) and (16), respectively, i.e., in the form $\partial_j f_j(\boldsymbol{x})$, with the components of the basis of the vector field put on the left of the velocity field. This is because it evolves the DF in the expectation calculus of observables of interest. However in standard differential calculus, a vector field is written in the *right* representation $g_j(\boldsymbol{x})\partial_j$, i.e., with the components of the basis of the vector field on the right. Of course we have $\partial_j f_j(\boldsymbol{x}) = f_j(\boldsymbol{x})\partial_j + (\partial_j f_j(\boldsymbol{x}))$; thus the *left* and the *right* representations coincide for solenoidal vector fields (as it is the case for Hamiltonian flows). Notice that in the Liouville framework to any Liouvillian $\mathcal{L} = \partial_j f_j(\boldsymbol{x})$ (*left* representation) is associated the adjoint one defined as $\mathcal{L}^{+} = -f_j(\boldsymbol{x})\partial_j$ (*right* representation) that, in the Lagrangian point of view, evolves directly the observable of interest inside the expectation calculus, instead of evolving the DF. In the following, we shall use also the definitions "*left* kind" and "*right* kind" for a first order differential operator to refer to a vector field that is written in the *left* representation $\partial_j f_j(\boldsymbol{x})$ and to vector field that is written in the *right* representation $f_j(\boldsymbol{x})\partial_j$, respectively.

IV. THE LIE DERIVATIVE

Let $\mathbb{O}$ be the set of operators $f \to s$, where f and s are smooth functions on a generic set $\mathcal{S}$ (here we do not need to identify it better) to $\mathbb{R}$, and let us choose an operator $\mathcal{A} \in \mathbb{O}$. For any $\mathcal{B} \in \mathbb{O}$, we define the adjoint-Lie operator $\mathcal{A}^{\times}[\mathcal{B}]$, associated with the operator $\mathcal{A}$ and applied to $\mathcal{B}$ as the standard antisymmetric compound of operators (the commutator),

$$\mathcal{A}^{\times}[\mathcal{B}]f \equiv [\mathcal{A}, \mathcal{B}]f \equiv \mathcal{A}\mathcal{B}f - \mathcal{B}\mathcal{A}f \equiv (\mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A})f. \tag{24}$$

The adjoint-Lie operator so defined introduces a Lie algebra structure in the set $\mathbb{O}$. It is well known that, having a general Lie algebra (for example, a vector field), the adjoint-Lie operator of the algebra is a derivative-like operator. In fact, for any $\mathcal{A}, \mathcal{B}, \mathcal{D} \in \mathbb{O}$, if we apply the definition of Eq. (24), the usual Leibniz rule holds (very easy to demonstrate),

$$\mathcal{A}^{\times}[\mathcal{B}\mathcal{D}]\cdots = \mathcal{B}\,\mathcal{A}^{\times}[\mathcal{D}]\cdots + \mathcal{A}^{\times}[\mathcal{B}]\mathcal{D}\cdots \tag{25}$$

and, likewise,

$$(\mathcal{A}\mathcal{B})^{\times}[\mathcal{D}]\cdots = \mathcal{A}\,\mathcal{B}^{\times}[\mathcal{D}]\cdots + \mathcal{A}^{\times}[\mathcal{D}]\mathcal{B}\cdots. \tag{26}$$

Thus we can say that the definition of the adjoint-Lie operator in Eq. (24) corresponds to the Lie derivative "along $\mathcal{A}$" of the generic operator $\mathcal{B}$ as usually introduced in differential geometry. With the introduced Lie algebra structure of the set $\mathbb{O}$, we can obtain a Lie group defining the exponential adjoint-Lie operator as

$$\mathfrak{B}(u)\cdots \equiv e^{\mathcal{A}^{\times}u}[\mathcal{B}]\cdots \equiv \sum_{k=0}^{\infty} \frac{u^k}{k!}\mathcal{A}^{\times k}[\mathcal{B}]\cdots, \tag{27}$$

where $u \in \mathbb{R}$. Thus, we shall refer to the exponential adjoint-Lie operator as the Lie evolution induced by $\mathcal{A}$ (or along $\mathcal{A}$). From the definitions given in the previous equation, it is clear that the following equation holds true:

$$\frac{d}{du}\mathfrak{B}(u)\cdots = \mathcal{A}^{\times}[\mathfrak{B}(u)]\cdots. \tag{28}$$

Notice that, using in Eq. (27) the Leibniz rule of Eq. (26), it is straightforward to obtain the following result (Hadamard Lemma):

$$e^{\mathcal{A}^{\times}u}[\mathcal{B}]\cdots = e^{\mathcal{A}u}\,\mathcal{B}e^{-\mathcal{A}u}\cdots. \tag{29}$$

From the Hadamard lemma, we have the trivial but very important property,

$$e^{\mathcal{A}^{\times}u}[\mathcal{B}\mathcal{D}]\cdots = e^{\mathcal{A}^{\times}u}[\mathcal{B}]\, e^{\mathcal{A}^{\times}u}[\mathcal{D}]\cdots, \tag{30}$$

which means that the Lie evolution of the composition of many operators is equal to the composition of the Lie evolution of each operator, exactly as it happens for the standard time evolution of product of functions. Using this property we have that for any function $s(X)$ that can be represented as a power series over its general argument X, the following important equations hold:

$$\begin{aligned} e^{\mathcal{A}^{\times}u}[s(X)]\cdots &= e^{\mathcal{A}^{\times}u}\left[\sum_{k=1}^{\infty} a_k X^k\right]\cdots \\ &= \sum_{k=1}^{\infty} a_k (e^{\mathcal{A}^{\times}u}[X])^k \cdots = s\left(e^{\mathcal{A}^{\times}u}[X]\right)\cdots . \end{aligned} \tag{31}$$

It is worthwhile stressing that in the previous equation the argument X can be a point of the space $\mathbb{R}^N$ (or $\boldsymbol{M}$) of the dynamical system of Eq. (2) or it can also be a generic operator $\in \mathbb{O}$, for example, a differential operator.

Coming back to our specific case, comparing Eqs. (29)–(30) with Eqs. (15), (16), (20), and (22), we see that we can identify the unperturbed Liouvillian $\mathcal{L}_0 = \partial_i C_i$ with $\mathcal{A}$, ∂_j (i.e., the natural basis of the tangent vector field on the manifold of the system) with $\mathcal{B}$ and the (smooth) functions $I(\boldsymbol{x})$ with $\mathcal{D}$. Thus, we are lead to conclude that to evaluate the Lie evolution of the interaction Liouville operator $\mathcal{L}_I$ along the unperturbed Liouvillian $\mathcal{L}_0$, we have to solve the Lie evolution of both the function $I(\boldsymbol{x})$ and the elements ∂_j of the basis of the vector field.

Notice that using Eq. (31) this result can be generalized also to the cases where the perturbation Liouville operator $\mathcal{L}_I$ is not a first order partial derivative, but it is a general analytic function of differential operators (for example, when the perturbation is given by a diffusion process).

To evaluate the Lie evolution, induced by the unperturbed Liouvillian $\mathcal{L}_0$, of the elements ∂_j of the basis of the vector field, it is useful to have in mind the following basic results ($1 \le i, j \le N, f$ and s are enough smooth functions):

$$\partial_i^{\times}\left[\partial_j\right]\cdots = 0, \tag{32}$$

$$\partial_i^{\times}[s(\boldsymbol{x})]\cdots = (\partial_i s(\boldsymbol{x}))\cdots, \tag{33}$$

$$f(\boldsymbol{x})^{\times}[s(\boldsymbol{x})]\cdots = 0, \tag{34}$$

$$f(\boldsymbol{x})^{\times}\left[\partial_j\right]\cdots = -\left(\partial_j f(\boldsymbol{x})\right)\cdots . \tag{35}$$

Equations (33) and (35) are equivalent to each other because of the antisymmetric property of the elements of the Lie algebra, and they mean that the Lie derivative along the basis of the tangent vector field of a given function is not an operator, but just the ordinary derivative of the same function, along the direction of the basis of the tangent vector field. More in general we have the following fact:

Proposition 1. If $\mathcal{A}$ and $\mathcal{B}$ are generic partial differential operators with derivatives up to order m and l, respectively (including the case $m = 0$ or/and $l = 0$ when $\mathcal{A}$ or/and $\mathcal{B}$ are just functions of the variables of the system), then the Lie evolution of $\mathcal{B}$ along $\mathcal{A}$, defined as $e^{\mathcal{A}^{\times}u}[\mathcal{B}]$, is generally a partial differential operator with all order, up to infinity, partial derivatives, apart from the cases m = 0, where it is a function, and m = 1 where it is a partial differential operator of the same maximum order of $\mathcal{B}$.

Proof. To demonstrate Proposition 1, let us start from the case $m = 0$, i.e., where the operator $\mathcal{A}$ is just a function: $\mathcal{A} = f(\boldsymbol{x})$. In this case, it is convenient to notice that if $\mathcal{B} = s(\boldsymbol{x})\partial_j^l$, with $l, j \in \mathbb{N} : (0 \le j \le N)$, exploiting Eq. (30), we have

$$e^{f(\boldsymbol{x})^{\times}u}\left[s(\boldsymbol{x})\partial_j^l\right]\cdots = e^{f(\boldsymbol{x})^{\times}u}[s(\boldsymbol{x})]\left(e^{f(\boldsymbol{x})^{\times}u}\left[\partial_j\right]\right)^l\cdots, \tag{36}$$

but $e^{f(\boldsymbol{x})^\times u}[s(\boldsymbol{x})] = s(\boldsymbol{x})$ and expanding $e^{f(\boldsymbol{x})^\times u}\left[\partial_j\right]$ in power series as in Eq. (27), we see that the first order term of the series is given by $-\left(\partial_j f(\boldsymbol{x})\right)$ while the higher orders vanish; thus, we obtain

$$e^{f(\boldsymbol{x})^\times u}\left[s(\boldsymbol{x})\partial_j^l\right]\cdots = s(\boldsymbol{x})\left(1-\left(\partial_j f(\boldsymbol{x})\right)^l\right)\cdots, \tag{37}$$

which demonstrates Proposition 1 for $m = 0$. For $m \geq 1$, we leave to the reader to check directly that the nth term of the series $e^{\mathcal{A}^\times u}\left[s(\boldsymbol{x})\partial_j^l\right]$ contains derivatives of order up to $k = l + (m-1)\,n$ [including the trivial case $l = 0$; hint: as in Eq. (37), exploiting Eq. (30), we can reduce the demonstration to the simpler case $l = 1$]. Thus, for $n \to \infty$, we have that $k \to \infty$, apart from the case where $m = 1$, for which $k = l$. □

Notice that in some particular cases, also for $m > 1$ the maximum order of the partial derivative operator, corresponding to the Lie evolution of $\mathcal{B}$ along $\mathcal{A}$, can be finite, let says $= \bar{k}$, because the coefficients of the differential operators with $k \geq \bar{k}$ vanish for any n. The simplest case is when $\mathcal{A}^\times[\mathcal{B}] = 0$, i.e., when $\mathcal{A}$ and $\mathcal{B}$ commute to each other for which $e^{\mathcal{A}^\times u}[\mathcal{B}] = \mathcal{B}$. A less trivial case is considered by Corollary 1 at the end of the present section.

Here, as previously done in this section, we have to identify the partial differential operator $\mathcal{A}$ with the unperturbed Liouvillian $\mathcal{L}_0$ of Eq. (15) that drives the *deterministic* (but not necessarily reversible) time evolution of the DF of the system of interest in Eqs. (2)–(3). Thus $\mathcal{A} = \mathcal{L}_0$ is a first order partial differential operator (i.e., $m = 1$) of the *left* kind of the manifold of the states of the system. Therefore, from Proposition 1, we have that the corresponding exponential adjoint-Lie operator *preserves* the order of the differential operator to which it is applied. In turn, this means that $e^{\mathcal{L}_0^\times u}[\partial_i]$ is a first order partial differential operator (it is also straightforward to show that it is of the same *left* kind of $\mathcal{L}_0$), i.e., $e^{\mathcal{L}_0^\times u}[\ldots]$ is a Lie group *on the vector field* of the manifold of the states of the system (or, better, on the tangent bundle of the manifold).

This fact is almost trivial from a mathematical point of view, but we think worthwhile stressing its important role in the physics problems we are dealing with here.

From Proposition 1, we also have that the exponential adjoint-Lie operator $e^{\mathcal{L}_0^\times u}[\ldots]$ applied to a function of the variables of interest (i.e., $l = 0$ and $m = 1$) *is not a differential operator*, but it is still a function. Actually, we can state something more.

Proposition 2. Let $f(\boldsymbol{x}) = (f_1(\boldsymbol{x}), \ldots, f_N(\boldsymbol{x}))$ *be a vector of N analytic functions* $f_j(\boldsymbol{x}) \in C^\infty(\mathbb{R}^N)$: $\mathbb{R}^N \to \mathbb{R}$, $1 \leq j \leq N$. *Let* $\mathcal{A}$ *be a generic linear differential operator* $C^\infty(\mathbb{R}^N) \to C^\infty(\mathbb{R}^N)$ *of the form*

$$\mathcal{A}\cdots \equiv \partial_i f_i(\boldsymbol{x})\cdots \tag{38}$$

defined in a proper Banach or Hilbert space according to the specific case such that

$$\mathcal{A}^+\cdots \equiv -f_i(\boldsymbol{x})\,\partial_i\cdots \tag{39}$$

is the adjoint of $\mathcal{A}$. *For any analytic function* $s(\boldsymbol{x}) : \mathbb{R}^n \to \mathbb{R}$, *the following equalities hold true (we recall that we use* $\boldsymbol{x}(t) \equiv \boldsymbol{x}$*):*

$$e^{\mathcal{A}^\times u}[s(\boldsymbol{x})]\cdots = e^{(-\mathcal{A}^+)^\times u}[s(\boldsymbol{x})]\cdots = \left(e^{-\mathcal{A}^+ u}s(\boldsymbol{x})\right)\cdots = s(\boldsymbol{x}_{\mathcal{A}}(t-u))\cdots, \tag{40}$$

where $\boldsymbol{x}_{\mathcal{A}}(t-u) \equiv \left(e^{-\mathcal{A}^+ u}\boldsymbol{x}\right)$.

Taking the limit $u \to 0$ in the previous result, we have the following lemma:

$$\mathcal{A}^\times[s(\boldsymbol{x})]\cdots = (-\mathcal{A}^+)^\times[s(\boldsymbol{x})]\cdots = (-\mathcal{A}^+ s(\boldsymbol{x}))\cdots. \tag{41}$$

Note that Proposition 2 means that the Lie evolution along the operator $\mathcal{A}$ of Eq. (39) of any analytic *function* $s(\boldsymbol{x})$ is identical to the Lie evolution along the (opposite sign) adjoint operator $-\mathcal{A}^+$, and *it is not an operator*, but is the standard back time evolution of the function $s(\boldsymbol{x})$ along the flux induced by $\mathcal{A}^+$.

Proof of Proposition 2. To demonstrate that Eq. (40) holds for the exponential adjoint-Lie operator, we start from the first order term of the series that defines the exponential operator, i.e., we start demonstrating Eq. (41). The equalities in Eq. (41) can be obtained directly using the Leibniz rule in Eq. (26),

$$\mathcal{A}^{\times}[s(\boldsymbol{x})]\cdots \equiv (\partial_i f_i(\boldsymbol{x}))^{\times}[s(\boldsymbol{x})]\cdots = (\partial_i\, s(\boldsymbol{x}))f_i(\boldsymbol{x})\cdots\,, \tag{42}$$

$$(-\mathcal{A}^{+})^{\times}[s(\boldsymbol{x})]\cdots \equiv (f_i(\boldsymbol{x})\,\partial_i)^{\times}[s(\boldsymbol{x})]\cdots = f_i(\boldsymbol{x})(\partial_i\, s(\boldsymbol{x}))\cdots\,. \tag{43}$$

Then, it is straightforward to check that the principle of mathematical induction can be used to shift this property to all the terms of the series that defines the exponential adjoint-Lie operator. □

From the physicist point of view, the importance of Proposition 2 is clear when we set $f_i(x) = C_i$, i.e., $\mathcal{A} = \mathcal{L}_0$ as usual in the present work. In fact in this case we have

$$\mathcal{L}_0^{\times}[s(\boldsymbol{x})]\cdots = \left(-\mathcal{L}_0^{+}s(\boldsymbol{x})\right)\cdots = \left(-\frac{d}{dt}s(\boldsymbol{x})\right)\cdots \tag{44}$$

and

$$e^{\mathcal{L}_0^{\times}u}[s(\boldsymbol{x})]\cdots = e^{(-\mathcal{L}_0^{+})^{\times}u}[s(\boldsymbol{x})]\cdots = \left(e^{-\mathcal{L}_0^{+}u}s(\boldsymbol{x})\right)\cdots = s(\boldsymbol{x}_0(t-u))\cdots\,. \tag{45}$$

It is important to note that Eq. (45) implies that *as it regards the Lie evolution of functions, the Liouvillian* $\mathcal{L}_0 = \partial_i\, C_i$ *and the opposite of its adjoint* $-\mathcal{L}_0^{+} = C_i\,\partial_i$ *generate the same flux, corresponding to the simple backward evolution of the function itself* (i.e., the *left* and *right* representations of the Liouvillian are, in this sense, equivalent, despite the fact that the velocity vector field is not solenoidal). In some way, this equivalence generalizes the case of symplectic or co-symplectic fluxes, where we have the identity $\mathcal{L}_0 = -\mathcal{L}_0^{+}$. Unfortunately, when the Lie evolution along the Liouvillian is applied to a differential operator, we do not have, in general, the equivalence between $\mathcal{L}_0$ and $-\mathcal{L}_0^{+}$. However this equivalence is desirable because it simplifies a lot the calculus and it makes possible a direct connection with some classical results on Lie Algebras, where the differential operators are usually written in the *right* representation instead of the *left* one. Now we shall show *in which condition the equivalence between* $\mathcal{L}_0 = \partial_i\, C_i$ *and* $-\mathcal{L}_0^{+} = C_i\,\partial_i$ *holds also for the Lie evolution of operators.*

Proposition 3. If $\mathcal{L}_0 = \partial_i\, C_i$*, the following conditions are equivalent:*

1. $\mathcal{L}_0^{\times}\left[\mathcal{L}_0^{+}\right] = (\mathcal{L}_0^{+})^{\times}\left[\mathcal{L}_0^{+}\right] = 0$, i.e., *the Liouvillian commutes with its adjoint,*
2. $\forall j | 1 \le j \le N$, $(\partial_j \partial_i C_i) = 0$, *i.e., we have constant divergence of the velocity vector field* C_i*,*
3. *for the Lie evolution of vector fields of the tangent bundle,* $\mathcal{L}_0$ *is equivalent to* $-\mathcal{L}_0^{+}$*, i.e., for any element* $\alpha_j(\boldsymbol{x})\partial_j$ *of the vector space, we have*

$$e^{\mathcal{L}_0^{\times}u}\left[\alpha_j(\boldsymbol{x})\partial_j\right] = e^{(-\mathcal{L}_0^{+})^{\times}u}\left[\alpha_j(\boldsymbol{x})\partial_j\right], \tag{46}$$

from which, taking the first order in u,

$$\mathcal{L}_0^{\times}\left[\alpha_j(\boldsymbol{x})\partial_i\right] = (-\mathcal{L}_0^{+})^{\times}\left[\alpha_j(\boldsymbol{x})\partial_j\right]. \tag{47}$$

Thus we introduce the following:

Assumption C. The Liouvillian $\mathcal{L}_0$ *of the unperturbed system of interest in Eq. (15) commutes with its adjoint* $\mathcal{L}_0^{+}$ *(or one of the three equivalent conditions of Proposition 3 is satisfied).*

Hereafter, unless explicitly stated, to stay in a more general context, we do not assume that the above assumption is fulfilled; however, the specific application examples that we shall consider later ahead shall share the validity of this assumption.

Proof of Proposition 3. We start the demonstration showing that condition (2) implies condition (1). This is very easy because $\mathcal{L}_0^{\times}\left[\mathcal{L}_0^{+}\right] = (C_k \partial_k \partial_i C_i)$, which is vanishing if condition (2) holds. The reversal is not so obvious, but it comes out considering $\mathcal{L}_0$ as the generator of the flux in the manifold of $\boldsymbol{x}$: $0 = \mathcal{L}_0^{\times}\left[\mathcal{L}_0^{+}\right] = -(\mathcal{L}_0^{+}\partial_i C_i) = -d(\partial_i C_i)/dt \to$ the divergence of the velocity vector field C_i is constant. Now we demonstrate that from condition (3) condition (1) follows. Actually this is easy; in fact, it is enough to choose $\alpha_j = C_j$ in Eq. (47) to have directly $\mathcal{L}_0^{\times}\left[\mathcal{L}_0^{+}\right] = (-\mathcal{L}_0^{+})^{\times}\left[\mathcal{L}_0^{+}\right] = 0$. For the reverse, we use the equivalence between conditions (1) and (2) and we demonstrate that (2) implies (3).

As in the cases of the previous propositions, to demonstrate Eq. (46) assuming condition (2) to hold, we start demonstrating the equation for the first order term, corresponding to Eq. (47); then it is easy, by induction, to shift the same property to all the terms of the power series that defines the adjoint-Lie evolution in Eq. (46). For this purpose, we first use the Leibniz rule: $\mathcal{L}_0^{\times}\left[\alpha_j(\boldsymbol{x})\partial_j\right] = \mathcal{L}_0^{\times}\left[\alpha_j(\boldsymbol{x})\right]\partial_j + \alpha_j(\boldsymbol{x})\,\mathcal{L}_0^{\times}\left[\partial_j\right]$. In the first term of the rhs of this equation, Proposition 2 allows us to substitute $\mathcal{L}_0$ with $-\mathcal{L}_0^{+}$. Thus it remains to show that the same holds for the second term, but that is easy to do $\mathcal{L}_0^{\times}\left[\partial_j\right] = -\partial_i\,(\partial_j C_i) = -(\partial_j C_i)\partial_i - (\partial_j\partial_i C_i)$ that, assuming condition (2) is satisfied, ends the demonstration. □

Notice that under Assumption C, using the Lie time evolution (that, thanks to Proposition 2, for functions is equivalent to the standard time evolution), we can substitute $\mathcal{L}_0 = \partial_j\, C_j$ with $(\mathcal{L}_0 - \mathcal{L}_0^{+})/2$ and we recover the antisymmetric property of the Liouvillian operator also for *non*-Hamiltonian (symplectic or co-symplectic) fluxes. This fact could be used for an eigenvalue/eigenvector approach to the Lie evolution of operators, but it is beyond the scope of the present work.

Now we consider an important non-trivial case that is an exception to Proposition 1. In fact, in this case the operator $\mathcal{A}$ is a field of differential operators of order $m > 1$, but $e^{\mathcal{A}^{\times}u}[\mathcal{B}]$ is still a finite order tensor field of differential operators. This is specified and demonstrated by the following:

Corollary 1. Let the set $\{\mathfrak{e}_1, \mathfrak{e}_2, \ldots, \mathfrak{e}_{N^l}\}$ *be a basis of the tangent tensor field of order* l *on the manifold of the system. If* $\mathcal{A}$ *can be divided in two parts* $\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2$, *where* $\mathcal{A}_1$ *is a differential operator of order* $m_1 = 1$ *with coefficients that are linear functions of the vector* $\boldsymbol{x}$ *of the phase space of the system* ($\mathcal{A}_1 = \partial_j a_{jk} x_k \mid a_{jk}$ *constants), while* $\mathcal{A}_2$ *is a differential operator of order* $m_2 > 1$, *commutating with all the* N^l *elements of the set* $\{\mathfrak{e}_i\}$: $\mathcal{A}_2^{\times}[\mathfrak{e}_i] = 0$, $i = 1, \ldots, N^l$, *then, for any* $i = 1, \ldots, N^l$, *the Lie evolution of* $\mathfrak{e}_i$ *along* $\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2$ *is the same differential operator of order* l *as for the case* $\mathcal{A} = \mathcal{A}_1$.

Proof. For the sake of simplicity, we assume that $l = 1$, namely, that $\mathfrak{e}_i$ is a basis of the tangent vector field and, without loss of generality, we can assume that it is the natural basis: $\mathfrak{e}_i = \partial_i$, $i = 1, \ldots, N$. As usual, using Eq. (30) it is straightforward to generalize the demonstration for the case where $l > 1$. Then we define the set $\mathfrak{E}_i^{\mathcal{O}}(u)$, $i = 1, \ldots, N$, as the Lie evolution of the basis of $\mathfrak{e}_i$ along the generic operator $\mathcal{O}$; $\mathfrak{E}_i^{\mathcal{O}}(u) \equiv e^{\mathcal{O}^{\times}u}[\mathfrak{e}_i]$. From Proposition 1, we know that $\mathfrak{E}_i^{\mathcal{A}_1}(u)$ *is a first order differential operator* and, from $\mathcal{A}_1 = \partial_j a_{jk} x_k$, we have $\mathcal{A}_1^{\times}[\mathfrak{e}_i] = \mathcal{A}_1^{\times}[\partial_i] = -\partial_j a_{ji}$. Thus, exploiting Eq. (28), we get

$$\begin{aligned}\frac{d}{du}\mathfrak{E}_i^{\mathcal{A}_1}(u) &= \mathcal{A}_1^{\times}\left[\mathfrak{E}_i^{\mathcal{A}_1}(u)\right] \\ &= e^{\mathcal{A}_1^{\times}u}\left[\mathcal{A}_1^{\times}[\partial_i]\right] = -e^{\mathcal{A}_1^{\times}u}\left[\partial_j a_{ji}\right] = -\mathfrak{E}_j^{\mathcal{A}_1}(u)\,a_{ji}.\end{aligned} \tag{48}$$

Concerning $\mathfrak{E}_i^{\mathcal{A}}(u) = e^{\mathcal{A}^{\times}u}[\mathfrak{e}_i]$, we have

$$\begin{aligned}\frac{d}{du}\mathfrak{E}_i^{\mathcal{A}}(u) &= (\mathcal{A}_1 + \mathcal{A}_2)^{\times}\left[\mathfrak{E}_i^{\mathcal{A}}(u)\right] = e^{\mathcal{A}^{\times}u}\left[(\mathcal{A}_1 + \mathcal{A}_2)^{\times}[\partial_i]\right] \\ &= e^{\mathcal{A}^{\times}u}\left[\mathcal{A}_1^{\times}[\partial_i]\right] = -e^{\mathcal{A}^{\times}u}\left[\partial_j a_{ji}\right] = -\mathfrak{E}_j^{\mathcal{A}}(u)\,a_{ji}.\end{aligned} \tag{49}$$

Equation (49) is formally identical to (48); therefore, since $\mathfrak{E}_i^{\mathcal{A}}(u)$ and $\mathfrak{E}_i^{\mathcal{A}_1}(u)$ satisfy the same equation with the same initial conditions ($\mathfrak{E}_i^{\mathcal{A}}(0) = \mathfrak{E}_i^{\mathcal{A}_1}(0) = \partial_i$), these operators are equal to each other for any u. □

The above corollary is useful, for instance, to easily deal with the projection approach also when the unperturbed system of interest includes second order differential operators as standard diffusive processes,

$$\mathcal{A} = \mathcal{L}_0 = \partial_i\, a_{ik} x_k + D_i\, \partial_i^2, \tag{50}$$

where the diffusion coefficients D_i are constants. This is done as sketched in the following. First, we notice that in this case Proposition 2 does not apply, and the Lie evolution along $\mathcal{L}_0$ of a function of $\boldsymbol{x}$ is not a function, but a differential operator: $\mathcal{L}_0^{\times}\left[x_j\right] = a_{jk} x_k + 2D_j\partial_j$. More precisely, using Corollary

1 it is possible to show that a linear function of $\boldsymbol{x}$ is "Lie evolved" to a first order partial differential operator, while a quadratic function is "Lie evolved" to a second order partial differential operator and so on. This fact, if exploited in Eq. (22) with $\mathcal{L}_I$ given in Eq. (16), leads to the result that when the unperturbed Liouvillian is of the kind in Eq. (50), the FPE structure breaks down and the master equation of Eq. (22) is an infinite series of differential operators of any order, apart from the case where the interaction Liouvillian $\mathcal{L}_I$ depends linearly on the variables of the system of interest. In fact, if $I(\boldsymbol{x}) \propto \boldsymbol{x}$, the master equation is solvable and it is a third-order differential operator. For the proof of this and for its consequences from a physical point of view, we refer the reader to a next paper we are working on.

V. SOLVING THE LIE EVOLUTION OF DIFFERENTIAL OPERATORS

With the Assumptions A-B and the Propositions 1-3 of Secs. II–IV we are now in the right position to solve the Lie evolution of differential operators. From Proposition 1, we have that the Lie evolution of vector fields along the Liouvillian $\mathcal{L}_0$, as, for example, that in Eq. (20), is still a vector field. Now we want to find the coefficients of this Lie-evolved vector field. Notice that if the differential operator that we want to evolve is given by $\alpha_j(\boldsymbol{x})\partial_j$, from Proposition 2 we have

$$e^{\mathcal{L}_0^\times u}\left[\alpha_j(\boldsymbol{x})\partial_j\right] = \alpha_j(\boldsymbol{x}_0(t-u))\, e^{\mathcal{L}_0^\times u}\left[\partial_j\right]. \tag{51}$$

Thus all we need to evaluate is the Lie evolution of the basis of the vector field:

$$\mathfrak{E}_j(u) \equiv e^{\mathcal{L}_0^\times u}\left[\partial_j\right]. \tag{52}$$

Taking the derivative with respect to u of the previous equation, we obtain

$$\frac{d}{du}\mathfrak{E}_j(u) = \mathcal{L}_0^\times\left[e^{\mathcal{L}_0^\times u}[\partial_j]\right] = e^{\mathcal{L}_0^\times u}\left[\mathcal{L}_0^\times[\partial_j]\right], \tag{53}$$

from which, using $\mathcal{L}_0^\times\left[\partial_j\right] = -\partial_i\,(\partial_j C_i)$, Eq. (30), and Proposition 2, we get

$$\begin{aligned}\frac{d}{du}\mathfrak{E}_j(u) &= -\mathfrak{E}_i(u)(\partial_j C_i)_{\boldsymbol{x}=\boldsymbol{x}_0(t-u)}\\ &\quad\left[= -(\partial_j C_i)_{\boldsymbol{x}=\boldsymbol{x}_0(t-u)}\mathfrak{E}_i(u) \text{ under the validity of Assumption C}\right],\end{aligned} \tag{54}$$

where $(\partial_j C_i)_{\boldsymbol{x}=\boldsymbol{x}_0(t-u)} \equiv (e^{-\mathcal{L}_0^+ u}\partial_j C_i)$ is the unperturbed backward evolution of the partial derivatives of the unperturbed velocity vector field C_i. Equation (54) represents a set of N linear ordinary non-autonomous differential equations (ODE) that can be solved by standard methods, in terms of the initial conditions $\mathfrak{E}_j(0) = \partial_j$ for $j = 1, 2, \ldots, N$. Of course, to solve this linear ODE, we need to know the coefficients $(\partial_j C_i)_{\boldsymbol{x}=\boldsymbol{x}_0(t-u)}$, namely, we must be able to integrate (analytically, numerically, under some case-specific approximations, etc.) the unperturbed equation of motion for $\boldsymbol{x}(t)$ in Eq. (2) with $\epsilon = 0$. But that is our starting point, as we stated in the Introduction and in Sec. II. However there is a way to obtain directly the explicit solution of Eq. (54). As we noted above, from Proposition 1, we know that the operator $\mathfrak{E}_j(u)$ in Eq. (52) is a vector field, i.e., it contains first order partial derivatives with some position dependent coefficients,

$$\begin{aligned}\mathfrak{E}_j(u) &\equiv e^{\mathcal{L}_0^\times u}\left[\partial_j\right] = \partial_k\, \beta_{j\,k}(\boldsymbol{x};u)\\ &\qquad\left[= \beta_{j\,k}(\boldsymbol{x};u)\,\partial_k \quad \text{under Assumption C}\right]\\ \beta_{j\,k}(\boldsymbol{x};0) &= \delta_{jk},\end{aligned} \tag{55}$$

where δ_{jk} is the Kronecker tensor. Thus $\beta_{j\,k}(\boldsymbol{x};u)$ are the components on the dual space of the vector field of the operator $\mathfrak{E}_j(u)$. On one hand, the u derivative of the last equation gives

$$\begin{aligned}\frac{d}{du}\mathfrak{E}_j(u) &= \partial_k\,(\frac{d}{du}\beta_{j\,k}(\boldsymbol{x};u))\\ &\quad\left[= (\frac{d}{du}\beta_{j\,k}(\boldsymbol{x};u))\,\partial_k \;\text{ under Assumption C}\right]\end{aligned} \tag{56}$$

but, from the other hand, exploiting Eq. (28) we also have

$$\frac{d}{du}\mathfrak{E}_j(u) = \partial_k\, \mathcal{L}_0^{\times}\left[\beta_{j\,k}(\boldsymbol{x};u)\right] + \mathcal{L}_0^{\times}[\partial_k]\,\beta_{j\,k}(\boldsymbol{x};u)$$
$$\left[= \mathcal{L}_0^{\times}\left[\beta_{j\,k}(\boldsymbol{x};u)\right]\partial_k + \beta_{j\,k}(\boldsymbol{x};u)\mathcal{L}_0^{\times}[\partial_k]\ \text{under Assumption C}\right]. \tag{57}$$

Comparing to each other the expressions of Eqs. (56)–(57) for the time evolution of $\mathfrak{E}(u)$ and considering again that $\mathcal{L}_0^{\times}[\partial_k] = -\partial_h\,(\partial_k C_h)$, we obtain the following system of equations for the evolution of the coefficients $\beta_{j\ k}(\boldsymbol{x}, u)$:

$$\frac{d}{du}\beta_{j\,k}(\boldsymbol{x};u) = -(\mathcal{L}_0^{+}\beta_{j\,k}(\boldsymbol{x};u)) - (\partial_h C_k)\beta_{j\,h}(\boldsymbol{x};u),$$
$$\beta_{j\,k}(\boldsymbol{x};0) = \delta_{jk}. \tag{58}$$

Equation (58) is a system of linear partial differential equations (PDEs) for the coefficients of the operator $\mathfrak{E}(u)$ that results from the Lie evolution of the jth element of the basis of the vector field on the tangent bundle. Thus, with respect to the linear ODE in Eq. (56) one could think that Eq. (58) does not simplify our task of finding the analytical expression for $\mathfrak{E}(u)$. However that is not true because the partial derivative operator in Eq. (58) is just the unperturbed Liouvillian of the system, i.e., it is the time evolution operator that generates the unperturbed flux in the manifold of the states of the system. We recall that we assume we are able to solve in some way the unperturbed equation of motion of the system. Taking into account this fact, we have the following main result of the present paper:

Proposition 4. Under the validity of Assumptions A-B, the coefficients of the Lie-evolution of the basis of the vector field, as defined in Eq. (55), are given by

$$\beta_{j\,k}(\boldsymbol{x};u) = \left(e^{-\mathcal{L}_0^{+}u}\partial_j(x_{0_k}(t+u))\right) \equiv \left(\partial_j(x_{0_k}(t+u))\right)_{\boldsymbol{x}=\boldsymbol{x}_0(t-u)}, \tag{59}$$

where $1 \le j, k \le N$, *and*

$$x_{0_k}(t+u) \equiv \left(e^{\mathcal{L}_0^{+}u}x_k\right) \tag{60}$$

in which $\mathcal{L}_0$ *is the Liouville operator of Eq. (15).*

Proof. The proof can be carried out in two different ways. The first one is finding the formal solution of Eq. (58). For that we exploit the interaction picture defining the variable $\tilde{\beta}_{j\,k}(\boldsymbol{x};u) \equiv \exp(\mathcal{L}_0^{+}u)\beta_{j\,k}(\boldsymbol{x};u)$. Using this definition in Eq. (58) we have

$$\frac{d}{du}\tilde{\beta}_{j\,k}(\boldsymbol{x};u) = -(\partial_h C_k)_{\boldsymbol{x}=\boldsymbol{x}_0(t+u)}\tilde{\beta}_{j\,h}(\boldsymbol{x};u),$$
$$\tilde{\beta}_{j\,k}(\boldsymbol{x};0) = \delta_{jk}, \tag{61}$$

where $-(\partial_h C_k)_{\boldsymbol{x}=\boldsymbol{x}_0(t+u)} \equiv -(e^{\mathcal{L}_0^{+}u}\partial_h C_k)$ is the unperturbed forward time evolution of the partial derivatives of the velocity field $-C_k$. By inspection, considering that $dx_i(t+u)/du = -C_i(t+u)$, we can easily verify that the solution of the system of Eq. (61) is given by

$$\tilde{\beta}_{j\,k}(\boldsymbol{x};u) = \partial_j(x_{0_k}(t+u)). \tag{62}$$

Getting out of the interaction picture the first proof ends. The second way to demonstrate Proposition 4 exploits directly the definition of the Lie-evolution of the basis of the vector field of Eq. (52), applied to a generic analytic function $s(\boldsymbol{x}) : \mathbb{R}^n \to \mathbb{R}$ (we recall that the parentheses indicate the limits of application of the included operator, namely, the result is a function and not an operator),

$$\begin{aligned}
\left(e^{\mathcal{L}_0^\times u}\left[\partial_j\right]s(\boldsymbol{x})\right) &\equiv \left(e^{\mathcal{L}_0 u}\partial_j e^{-\mathcal{L}_0 u}s(\boldsymbol{x})\right)\\
&= \left(e^{\mathcal{L}_0 u}\partial_j e^{-\mathcal{L}_0^\times u}[s(\boldsymbol{x})]e^{-\mathcal{L}_0 u}\right)\\
&= \left(e^{\mathcal{L}_0^\times u}\left[\partial_j s(\boldsymbol{x}_0(t+u))\right]\right)\\
&= \left(e^{\mathcal{L}_0^\times u}\left[\left(\partial_j s(\boldsymbol{x}_0(t+u))\right) + s(\boldsymbol{x}_0(t+u))\partial_j\right]\right)\\
&= \left(e^{\mathcal{L}_0^\times u}\left[\left(\partial_j x_{0_k}(t+u)\right)(\partial_k s(\boldsymbol{x}))_{\boldsymbol{x}=\boldsymbol{x}_0(t+u)}\right] + s(\boldsymbol{x})e^{\mathcal{L}_0^\times u}\left[\partial_j\right]\right)\\
&= \left(e^{\mathcal{L}_0^\times u}\left[\left(\partial_j x_{0_k}(t+u)\right)\right]e^{\mathcal{L}_0^\times u}\left[(\partial_k s(\boldsymbol{x}))_{\boldsymbol{x}=\boldsymbol{x}_0(t+u)}\right]\right) + \left(s(\boldsymbol{x})e^{\mathcal{L}_0^\times u}\left[\partial_j\right]\right)\\
&= \left(e^{\mathcal{L}_0^\times u}\left[\left(\partial_j x_{0_k}(t+u)\right)\right]\partial_k\, s(\boldsymbol{x})\right) + \left(s(\boldsymbol{x})e^{\mathcal{L}_0^\times u}\left[\partial_j\right]\right),
\end{aligned} \tag{63}$$

where we have repeatedly used Eq. (40) of Proposition 2. Now, using Eq. (55) in the lhs and in the last term of the last line of the above equation and from the fact that the function $s(\boldsymbol{x})$ is generic, the proof is completed. □

Proposition 4 looks quite formal but actually gives rise to expressions that can be evaluated in many important cases when the unperturbed equation of motion of the system is "known" (i.e., it can be solved analytically) or when it enters in convolution expressions with fast decaying kernels, where it can be approximated by Taylor's power expansion (see Sec. VI for details).

To give an intuitive interpretation of the meaning of Eq. (59), let us start from the definition of the operator $\mathfrak{E}_j(u)$, representing the Lie evolution of the vector field along the Liouvillian and let us apply it to a generic function $s(\boldsymbol{x})$ as in Eq. (63). In the third line of this equation, we clearly see that the operator $\mathfrak{E}_j(u)$ evolves for a time u the function to which it is applied, then it takes its derivative in the "j" direction, and, finally, it evolves the result back to the initial time. As the partial derivative does not commute with the time evolution (driven by the operator $\mathcal{L}_0$), at the end we have a different result with respect to the same partial derivative ∂_j applied to the function $s(\boldsymbol{x})$ at the initial time. However, Proposition 1 states that the result is again proportional to the differential of $s(\boldsymbol{x})$, but, in general, along a different direction. Proposition 4 asserts that this direction and the rescaling factor depend on the time parameter u and are given by the vector $\beta_{j\ k}(\boldsymbol{x}; u)$ of Eq. (59). Now, looking better to the evolution of the vector field given in Eq. (59), we see that it is again given by the same process of forward evolution, partial derivative in the j direction and backward evolution, but now applied to the trajectory $\boldsymbol{x}(t+u)$. A simple unidimensional case should make clearer the meaning of the β-coefficients: let us assume that $N = 1$, i.e., the phase space is given by $x \in \mathbb{R}$ (or a subset of $\mathbb{R}$), and that the vector field for the flux is defined by the quite trivial differential equation: $\dot{x} = -\gamma\, x$, from which $x_0(t+u) = x\, e^{-\gamma u}$. The third line of Eq. (63) says that the action of the operator $\mathfrak{E}(u)$ on the function $s(x)$ can be decomposed in the following three steps (see Fig. 1):

1. $s(x) \to g(x; u) \equiv s(x_0(t+u)) = s\left(x\, e^{-\gamma u}\right)$,
2. $dg(x; u) \equiv (\partial_x g(x; u))$,
3. $(\mathfrak{E}(u)s(x)) = dg(x\, e^{\gamma u}; u)$,

while Eq. (59) says that (note that in this case Assumption C holds true) $\beta(x; u)$ is given by the following three steps equivalent to those given above:

1. $x \to x_0(t+u) = x\, e^{-\gamma u}$,
2. $dx(x; u) \equiv (\partial_x x_0(t+u)) = e^{-\gamma u}$,
3. $\beta(x; u) = dx(x\, e^{\gamma u}; u) = e^{-\gamma u}$.

Thus, in this unidimensional linear dissipative case, the Lie evolution does not change the shape of the partial derivative of $s(x)$ (namely, the β coefficient does not depend on x), but it rescales it by a decaying factor. In a multidimensional linear case, as in Sec. VI A, the vector field is also rotated by the Lie evolution [see Eq. (70)].

Note that the second proof of Proposition 4, represented by the lines of Eq. (63), follows quite straightforward from the definition of the Lie-evolution of the basis of the vector field; thus both the

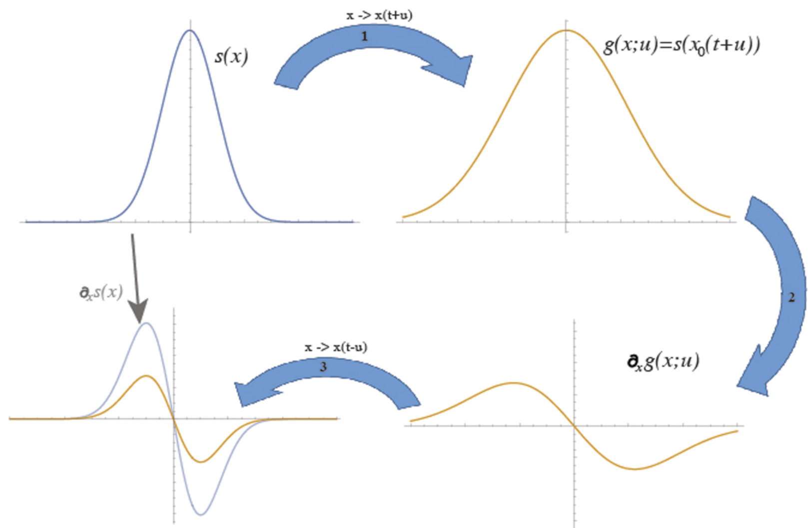

FIG. 1. A graphical representation of the three steps corresponding to the Lie evolution of the differential operator ∂_x applied to the function $s(x) = \exp[-x/2]$ for the one-dimensional flux defined by the equation $\dot{x} = -\gamma\, x$ (see text for detail).

first proof that exploit Eq. (58) and the previous results of Eq. (54) would seem useless. However, we have decided to show explicitly the relationships expressed by these equations because often, in practical cases, it happens that the explicit expressions for some coefficients $\beta_{j\ k}(\boldsymbol{x}; u)$ (or for the Lie evolution of some components of the vector field) are known (e.g., from general conservation laws). In such cases, Eq. (58) [or Eq. (54)] allow us to obtain the expressions also for the other components (see Secs. VII A and VII B for examples about that).

VI. APPLYING PROPOSITION 4 TO PROJECTION APPROACHES

Coming back to the main task of the present work, we consider the general dynamical system of Eq. (21) where we are interested in the statistical behavior of the variables of interest $\boldsymbol{x}$, induced by the interaction with the "chaotic" (or uncorrelated, to some extent) variable ξ. To hide the unobserved variable ξ, we use the projection approach that lead us to Eq. (22), where we insert the explicit expressions for the unperturbed Liouvillian $\mathcal{L}_0$ and the perturbation Liouvillian $\mathcal{L}_I$ given in Eqs. (15) and (16), respectively. Then, exploiting Proposition 2 we get [we use $\partial_j I_j = I_j \partial_j + (\partial_k I_k)$]

$$\begin{aligned}\frac{\partial}{\partial t}\rho(\boldsymbol{x};t) &= \mathcal{L}_0\rho(\boldsymbol{x};t)\\ &\quad + \epsilon^2\partial_i I_i(\boldsymbol{x})\left\{\int_0^\infty du\,\phi(u)\,I_j(\boldsymbol{x}_0(t-u))e^{\mathcal{L}_0^\times u}\left[\partial_j\right]\right\}\rho(\boldsymbol{x};t)\\ &\quad + \epsilon^2\partial_i I_i(\boldsymbol{x})\left\{\int_0^\infty du\,\phi(u)(\partial_k I_k(\boldsymbol{x}))_{\boldsymbol{x}=\boldsymbol{x}_0(t-u)}\right.\\ &\quad \left. + \int_0^\infty du\,S(u)h(\boldsymbol{x}_0(t-u))\right\}\rho(\boldsymbol{x};t), \end{aligned} \tag{64}$$

where we have replaced with infinity the time "t" of integration because we assume that we are observing the system for times much longer than the relaxation time of the autocorrelation function $\phi(u)$ and of the response function $S(u)$ of the booster.

The last line of Eq. (64) contains only first order partial derivatives and corresponds to the deterministic drift (for example, a friction) induced by the feedback of the system of interest on the booster.[3] The second-to-last line is a part of the "noise-induced drift" that appears in Stratonovich systems because the correlation time of the "noise" (here the booster ξ variable) is not vanishing. Concerning the previous line (the second one) of the above equation, from Proposition A it contains second order derivatives; thus it gives rise to a diffusion process (plus other Stratonovich-like drift terms for the average evolution of the system of interest variables, of course). Thus, using the definitions in Eqs. (52) and (55) and the main result of Eq. (59), the above equation becomes the following generalized FPE:

$$\frac{\partial}{\partial t}\rho(\boldsymbol{x};t) = \partial_i(C_i + \Gamma_i + \mathsf{D}_{ik}\partial_k)\rho(\boldsymbol{x};t), \tag{65}$$

where the diffusion matrix D and the drift vector Γ, induced by the interaction with the external variable ξ, are given by

$$\mathsf{D}_{ik} = \epsilon^2 I_i(\boldsymbol{x})\int_0^\infty du\,\phi(u)\,I_j(\boldsymbol{x}_0(t-u))\Big(\partial_j(x_{0_k}(t+u))\Big)_{\boldsymbol{x}=\boldsymbol{x}_0(t-u)} \tag{66}$$

and

$$\Gamma_i = \epsilon^2 I_i(\boldsymbol{x})\left\{\int_0^\infty du\,\phi(u)(\partial_k I_k(\boldsymbol{x}))_{\boldsymbol{x}=\boldsymbol{x}_0(t-u)} + \int_0^\infty du\,S(u)h(\boldsymbol{x}_0(t-u))\right\}, \tag{67}$$

respectively. Equations (65)–(67) are important results because they are the formal transport coefficients of the generalized FPE stemming from the perturbation projection approach applied to the very large class of dynamical systems given in Eq. (21). At the same time, from Eqs. (66)–(67) it is possible to establish which are the conditions for standard statistics (Gaussian, canonical, Levy, thermodynamics, linear Onsager regressions,[3,30,31] etc.) to emerge for the system of interest. Thus Eqs. (66)–(67) can be used to identify the partition of the set of systems into equivalence classes for the emergence of different kinds of regular and universal statistical properties.

Looking at these expressions for the transport coefficients, we see that they are expressed in terms of convolutions between the autocorrelation function, for the diffusion coefficients, and the response function, for the drift coefficients, of the perturbation ξ, with the *unperturbed* back-time evolution of functions of the variables of interest. Thus these quantities are all related to the dynamical features of the booster variable that directly interacts with the system of interest and to the dynamics of the unperturbed system of interest too, of course. No *ad hock* external constraints are introduced, as a given temperature or the canonical or Gaussian equilibrium DF.

Now the question is as follows: How can we solve the expressions in Eqs. (66)–(67) in practical cases? Usually the information about the perturbing system, (ξ, $\boldsymbol{\pi}$) are limited to experimental, observational, or numerical data. The first case refers to laboratory experiments; thus the correlation function and the response function of the perturbing system can be obtained as a fit of the data from targeted experiments. The second case concerns, for example, climatological or geophysical phenomena, as El Niño/La Niña,[32] that are large scale oceanic events induced mainly by the interaction with the fast atmosphere. Here the correlation function can still be obtained from data observation, but the response function of the fast atmosphere (for example) can be only obtained by simplified analytical or numerical models.

The last case refers to formal tractable low order models (LOM), usually derived starting from complex fundamental differential equations (for example, fluid dynamical "building blocks" equations) of complex processes,[33] reduced to a simplified set of ordinary differential equations for few variables (for example, the Lorenz models of atmospheric circulation[34–38]), through a series of approximations and hypotheses. In this case, the autocorrelation function and the response function are obtained analytically or by numerical simulations.

It is clear that in general it is not possible to manage so easily the expressions in Eqs. (66)–(67). Below we present how we can work with them in two representative cases.

A. The linear velocity field case

Let us assume that the unperturbed velocity field can be approximated as being linear: $C_i = \mathsf{C}_{ik}x_k$, where $1 \le i, k \le N$, and C_{ik} are constants, components of a $N \times N$ matrix that we name C. In this case, the unperturbed time evolution of the system of interest is a linear function of the initial position,

$$x_{0_k}(t+u) = \mathsf{B}_{kl}(u)x_l, \qquad 1 \le l \le N,$$
$$\mathsf{B}_{kl}(0) = \delta_{kl}, \tag{68}$$

where $\mathsf{B}_{kl}(u)$ are the components of the matrix $\mathsf{B}(u)$ given by

$$\mathsf{B}(u) \equiv e^{-\mathsf{C}u}, \tag{69}$$

from which

$$\left(\partial_j(x_{0_k}(t+u))\right)_{\boldsymbol{x}=\boldsymbol{x}_0(t-u)} = \mathsf{B}_{kj}(u). \tag{70}$$

Just for the sake of simplicity, we also assume that the reaction function $h(\boldsymbol{x})$ is a linear function of the system of interest variables: $h(\boldsymbol{x}) = h_k x_k$ (it is straightforward to generalize the results releasing this assumption). Using Eqs. (68)–(70) in Eq. (65), we obtain a more standard FPE,

$$\frac{\partial}{\partial t}\rho(\boldsymbol{x};t) = \partial_i(R_i(\boldsymbol{x}) + \mathsf{G}_{ik}(\boldsymbol{x})x_k + \mathsf{D}_{ik}(\boldsymbol{x})\partial_k)\rho(\boldsymbol{x}), \tag{71}$$

with

$$R_i(\boldsymbol{x}) \equiv \epsilon^2 I_i(\boldsymbol{x}) \int_0^\infty du\, \phi(u)(\partial_l I_l(\boldsymbol{x}))_{\boldsymbol{x}=\boldsymbol{x}_0(t-u)}, \tag{72}$$

$$\mathsf{G}_{ik}(\boldsymbol{x}) \equiv \mathsf{C}_{ik} + \epsilon^2 I_i(\boldsymbol{x}) \int_0^\infty du\, S(u) h_l\, \mathsf{B}_{lk}(-u), \tag{73}$$

$$\mathsf{D}_{ik}(\boldsymbol{x}) \equiv \epsilon^2 I_i(\boldsymbol{x}) \int_0^\infty du\, \phi(u)\, I_j(\boldsymbol{x}_0(t-u))\mathsf{B}_{kj}(u). \tag{74}$$

From Eq. (69), i.e., from the fact that the equation of motion of the system of interest is a system of linear ODE, it follows that in the above equations the functions $\mathsf{B}_{lk}(u)$ can be expressed as linear combinations, with constant coefficients, of the real and imaginary parts of $\exp(\lambda_i u)$, where λ_i, $i = 1, \ldots, N$, are the eigenvalues of the matrix C [see Eqs. (78)–(79) for the case $N = 2$].

The FPE of Eq. (71) is written in the conservative form

$$\frac{\partial}{\partial t}\rho(\boldsymbol{x};t) = \partial_i j_i, \tag{75}$$

where the current $\boldsymbol{j}$ is given by $j_i \equiv (R_i(\boldsymbol{x}) + \mathsf{G}_{ik}x_k + \mathsf{D}_{ik}\partial_k)\rho(\boldsymbol{x};t)$. If the perturbation vector field $\boldsymbol{I}$ does not depend on $\boldsymbol{x}$, it is easy to study the conditions for which the divergence of $\boldsymbol{j}$ goes to 0 for $t \to \infty$. If these conditions hold true, in particular, defining D and G as the $N \times N$ matrix with elements D_{ik} and G_{ik}, respectively, we have the following Gaussian function as the stationary DF:

$$\rho_{eq}(\boldsymbol{x}) = \sqrt{\frac{\pi^N}{\det \mathsf{A}}}\, \exp\left(-\frac{\boldsymbol{x}^T \mathsf{A}\boldsymbol{x}}{2}\right), \tag{76}$$

where the $N \times N$ matrix A is defined by the following equation (superscript "T" denotes transposition):

$$\mathsf{G} \cdot \mathsf{A}^{-1} + \mathsf{A}^{-1} \cdot \mathsf{G}^T = 2\mathsf{D} \tag{77}$$

as it can be directly verified substituting the Gaussian in Eq. (76) in Eq. (71) and using Eq. (77).

To show the importance of the result in Eqs. (71)–(73), we consider a specific application, namely, the Recharge Oscillator Model (ROM)[15,16,39–45] mimicking the El Niño Southern Oscillations (ENSO),

$$\dot{\zeta} = -\omega\, T - \gamma_\zeta \zeta,$$
$$\dot{T} = \omega\, \zeta - \gamma_T T, \tag{78}$$

where the $\zeta = x_1$ and $T = x_2$ variables refer to the depth of the thermocline of the West Equatorial Pacific Ocean and to the Sea Surface Temperature (SST) of the East Equatorial Pacific Ocean, respectively. In this case, the eigenvalues of the matrix C are $\lambda_\pm = -\Gamma/2 \pm i\Omega$, where $\Gamma \equiv \gamma_\zeta + \gamma_T$ and $\Omega = \sqrt{\omega^2 + \gamma_\zeta\gamma_T - \frac{\Gamma^2}{4}}$, from which

$$
\begin{aligned}
\mathsf{B}_{\zeta\zeta}(u) &= e^{-\frac{\Gamma}{2}u}\left[\cos(\Omega\, u) + \frac{\Gamma}{2}\frac{\sin(\Omega\, u)}{\Omega}\right],\\
\mathsf{B}_{\zeta T}(u) &= -e^{-\frac{\Gamma}{2}u}\sqrt{\Omega^2 + \frac{\Gamma^2}{4}}\,\frac{\sin(\Omega\, u)}{\Omega},\\
\mathsf{B}_{T\zeta}(u) &= -\mathsf{B}_{\zeta T},\\
\mathsf{B}_{TT}(u) &= e^{-\frac{\Gamma}{2}u}\left[\cos(\Omega\, u) - \frac{\Gamma}{2}\frac{\sin(\Omega\, u)}{\Omega}\right].
\end{aligned}
\tag{79}
$$

Note that if $\Gamma^2/4 > \omega^2 + \gamma_\zeta\gamma_T$, in the above equations we must substitute the trigonometric functions with the corresponding hyperbolic ones. Both the thermocline depth and the SST are subjected to a fast forcing by the interaction with the atmosphere, mainly due to the Madden Julian Oscillations (MJO) and the Westerly Wind Burst (WWB).[33,46–49] These can be represented as multiplicative perturbations to the ROM.[32,50,51] Considering the dynamics of the atmosphere as the generic “rest of the system” identified by the booster variable ξ and the other “irrelevant” variables $\boldsymbol{\pi}$ as in Eq. (21), we can write

$$
\begin{aligned}
\dot{\zeta} &= -\omega\, T - \gamma_\zeta \zeta - \epsilon\, I_\zeta(T)\xi,\\
\dot{T} &= \omega\, \zeta - \gamma_T T - \epsilon\, I_T(T)\xi,\\
\dot{\xi} &= F(\xi, \boldsymbol{\pi}),\\
\dot{\boldsymbol{\pi}} &= \boldsymbol{Q}(\xi, \boldsymbol{\pi}),
\end{aligned}
\tag{80}
$$

where the interaction vector field $(I_\zeta(T), I_T(T))$ depends on the anomalous temperature T as

$$
\begin{aligned}
I_\zeta(T) &= -(K_\zeta + \alpha_\zeta T),\\
I_T(T) &= -(K_T + \alpha_T T),
\end{aligned}
\tag{81}
$$

in the emulation of an enhanced air-sea coupling as SST anomalies increase (see, for example, Ref. 17). Comparing the above equation with that of Eq. (21), we see that here the reaction term $h(\boldsymbol{x})$ is vanishing.

The goal is to describe the statistics of the ENSO, represented by the variables (ζ, T) perturbed by the atmosphere as in the above equation. Usually, for the sake of simplicity, the perturbation $\epsilon\, I_\zeta(\zeta, T)\xi$ of the thermocline depth is not considered,[32,50] but thanks to the results of this paper it is easy to take into account also this contribution. From Eqs. (80)–(81), we have

$$
\begin{aligned}
\mathcal{L}_0 &= \omega\partial_\zeta T + \gamma_\zeta\partial_\zeta\zeta - \omega\partial_T\zeta + \gamma_T\partial_T T,\\
\mathcal{L}_I &= -\partial_\zeta\epsilon(K_\zeta + \alpha_\zeta T) - \partial_T\epsilon\,(K_T + \alpha_T T).
\end{aligned}
\tag{82}
$$

Through the Zwanzig projection approach we arrive to the result of Eqs. (71)–(74) in which $\partial_i \mathsf{G}_{ik}(\boldsymbol{x})x_k = \mathcal{L}_0$ [because $h(\boldsymbol{x}) = 0$], $I(\boldsymbol{x})$ is in Eq. (81) and $\zeta_0(t-u) = \mathsf{B}_{\zeta\zeta}(-u)\zeta + \mathsf{B}_{\zeta T}(-u)T$, $T_0(t-u) = \mathsf{B}_{T\zeta}(-u)\zeta + \mathsf{B}_{TT}(-u)T$. Thus, we get

$$
\begin{aligned}
\frac{\partial}{\partial t}\rho(\zeta, T; t) = \Big\{\mathcal{L}_0 &+ \partial_\zeta\Big(R_\zeta(T) + \mathsf{D}_{\zeta\zeta}(t)\partial_\zeta + \mathsf{D}_{\zeta T}(t)\partial_T\Big)\\
&+ \partial_T\Big(R_T(T) + \mathsf{D}_{T\zeta}(t)\partial_\zeta + \mathsf{D}_{TT}(t)\partial_T\Big)\Big\}\rho(\zeta, T; t),
\end{aligned}
\tag{83}
$$

where

$$
R_\zeta \equiv \epsilon^2(\alpha_\zeta + \alpha_T)(K_\zeta + \alpha_\zeta T)\int_0^\infty du\, \phi(u), \tag{84}
$$

$$
R_T \equiv \epsilon^2(\alpha_\zeta + \alpha_T)(K_T + \alpha_T T)\int_0^\infty du\, \phi(u),
$$

$$
\begin{aligned}
\mathsf{D}_{\zeta\zeta} \equiv{}& \epsilon^2 (K_\zeta + \alpha_\zeta T) \int_0^\infty du\, \phi(u) \\
&\times \Big\{ \Big[K_T + \alpha_T (\mathsf{B}_{T\zeta}(-u)\zeta + \mathsf{B}_{TT}(-u)T)\Big] \mathsf{B}_{\zeta\zeta}(u) \\
&+ \Big[K_\zeta + \alpha_\zeta (\mathsf{B}_{T\zeta}(-u)\zeta + \mathsf{B}_{TT}(-u)T)\Big] \mathsf{B}_{\zeta T}(u) \Big\}, \\
\mathsf{D}_{\zeta T} \equiv{}& \epsilon^2 (K_\zeta + \alpha_\zeta T) \int_0^\infty du\, \phi(u) \\
&\times \Big\{ \Big[K_T + \alpha_T (\mathsf{B}_{T\zeta}(-u)\zeta + \mathsf{B}_{TT}(-u)T)\Big] \mathsf{B}_{T\zeta}(u) \\
&+ \Big[K_\zeta + \alpha_\zeta (\mathsf{B}_{T\zeta}(-u)\zeta + \mathsf{B}_{TT}(-u)T)\Big] \mathsf{B}_{TT}(u) \Big\}, \\
\mathsf{D}_{T\zeta} \equiv{}& \epsilon^2 (K_T + \alpha_T T) \int_0^\infty du\, \phi(u) \\
&\times \Big\{ \Big[K_T + \alpha_T (\mathsf{B}_{T\zeta}(-u)\zeta + \mathsf{B}_{TT}(-u)T)\Big] \mathsf{B}_{\zeta\zeta}(u) \\
&+ \Big[K_\zeta + \alpha_\zeta (\mathsf{B}_{T\zeta}(-u)\zeta + \mathsf{B}_{TT}(-u)T)\Big] \mathsf{B}_{\zeta T}(u) \Big\}, \\
\mathsf{D}_{TT} \equiv{}& \epsilon^2 (1 + \alpha_T T) \int_0^\infty du\, \phi(u) \\
&\times \Big\{ \Big[K_T + \alpha_T (\mathsf{B}_{T\zeta}(-u)\zeta + \mathsf{B}_{TT}(-u)T)\Big] \mathsf{B}_{T\zeta}(u) \\
&+ \Big[K_\zeta + \alpha_\zeta (\mathsf{B}_{Th}(-u)\zeta + \mathsf{B}_{TT}(-u)T)\Big] \mathsf{B}_{TT}(u) \Big\}.
\end{aligned}
\tag{85}
$$

From the above expressions, we deduce that the diffusion terms of the FPE are second order polynomials in ζ, T with constant coefficients (the subscripts i, j stand for the subscripts ζ or T),

$$
\mathsf{D}_{ij} = \mathsf{D}_{ij}^{(0)} + \mathsf{D}_{ij}^{(1)} h + \mathsf{D}_{ij}^{(2)} T + \mathsf{D}_{ij}^{(3)} hT + \mathsf{D}_{ij}^{(4)} T^2. \tag{86}
$$

Using then in Eqs. (85)–(86) the explicit results for the B_{ij} coefficients given in Eq. (79), we get that the terms $\mathsf{D}_{ij}^{(k)}$ are linear combination of the real and of the imaginary parts of the Laplace transform of the auto-correlation function $\phi(t)$ evaluated at the point, in the complex plane, corresponding to the eigenvalues $\lambda_\pm = -\Gamma/2 \pm i\Omega$ of the unperturbed ROM. It is clear that in general, with second order polynomials as diffusion coefficients of the FPE, the stationary DF shall not be Gaussian, but (if it exists) typically it will be skewed with some kurtosis values and power law tails. Kurtosis and skewness of the histogram of the frequency of the SST anomalies during El Niño/La Niña phenomena are actually obtained from observations.[32] Moreover, in general, it shall not be possible to get the analytic expression of such stationary DF. However, just from the fact that the diffusion coefficients are second order polynomials of the system of interest variables, we can get closed first order ODE for the moments. In fact, it is straightforward to verify that from the FPE of Eqs. (83)–(86), we get a system of first order ODE for the nth moments that does not involve moments of higher order. For example, the equation of motion for the average of ζ and T is given by a couple of first order linear ODE, i.e., it corresponds to a forced dumped oscillator. This fact can be used to get easily all the relevant statistical information of the ENSO, but this is beyond the scope of the present paper.

There are of course many other cases of systems that can be treated in a similar way, for example, connected RLC electric circuits, mechanical rotors supported by magnets,[52] just to cite a couple of them.

B. The large time scale separation case

If there is a large time scale separation between the slow dynamics of the system of interest and that of the fast perturbing $(\xi, \boldsymbol{\pi})$ system; there are methods that allows us to obtain a FPE for the part of interest, usually by averaging the fast variables with some adiabatic procedure that introduces, by hand, decorrelation assumptions among high order cross moments of the slow and fast systems (see, for example, Refs. 53–55). Because the results in Eqs. (65)–(67) hold true regardless of the time scale assumption, they can be used for a rigorous and systematic power expansion procedure in terms of the time scale of the booster.

For fast booster we mean that the autocorrelation function $\phi(t)$ and the response function $S(t)$ of ξ decay in a short time, compared to the typical time scale of the dynamics of x. Thus, in Eqs. (66) and (67), we can use the Taylor expansions in the parameter u of $\left(\partial_j(x_{0_k}(t+u))\right)_{\boldsymbol{x}=\boldsymbol{x}_0(t-u)}$ and of $x_{0_k}(t-u)$ around $u = 0$ and take only the first power terms,

$$\begin{aligned}\left(\partial_j(x_{0_k}(t+u))\right)_{\boldsymbol{x}=\boldsymbol{x}_0(t-u)} &= \delta_{jk} - (\partial_j C_k)u + C_h(\partial_h\partial_j C_k)u^2 \\ &- \Big((\partial_j C_h)(\partial_h C_k) + C_h(\partial_j\partial_h C_k)\Big)\frac{u^2}{2} + \ldots . \\ &= \delta_{jk} - (\partial_j C_k)u + \Big(C_h(\partial_j\partial_h C_k) - 2(\partial_j C_h)(\partial_h C_k)\Big)\frac{u^2}{4} + \ldots\end{aligned} \tag{87}$$

and

$$x_{0_k}(t-u) = x_k + C_k\, u - (C_h(\partial_h C_k))\frac{u^2}{2} + \cdots . \tag{88}$$

Inserting these expressions in Eqs. (66)–(67) we get [for the sake of simplicity we assume that, as it is the case for linear Hamiltonian interactions, the perturbation vector field $\boldsymbol{I}$ does not depend on $\boldsymbol{x}$ and the reaction term $h(\boldsymbol{x})$ is a homogeneous linear function of the variables of interest, namely, $h(\boldsymbol{x}) = h_k\, x_k$]

$$\mathsf{D}_{ik} = \epsilon^2 I_i\, I_j\left(\delta_{jk}\tau - (\partial_j C_k)\,\eta^2 + \Big(C_h(\partial_j\partial_h C_k) - 2(\partial_j C_h)(\partial_h C_k)\Big)\frac{\kappa^3}{4}\right) \tag{89}$$

and

$$\Gamma_i = \epsilon^2 I_i\, h_k\left(x_k\,\vartheta + C_k\,\beta^2 - (C_h(\partial_h C_k))\frac{\theta^3}{2}\right), \tag{90}$$

where the decay time τ of the autocorrelation function is defined as

$$\tau \equiv \frac{1}{\phi(0)}\int_0^\infty \phi(u)du, \tag{91}$$

the decay time ϑ of the susceptibility $\chi(t) \equiv \int_0^t S(u)du$ is defined as

$$\vartheta \equiv \int_0^\infty \left(1 - \frac{\chi(u)}{\chi(\infty)}\right)du, \tag{92}$$

and the times (with possibly complex imaginary values) η, β, κ, and θ are defined as

$$\eta^2 \equiv \frac{1}{\phi(0)}\int_0^\infty \phi(u)u\,du, \quad \beta^2 \equiv \int_0^\infty \left(1 - \frac{\chi(u)}{\chi(\infty)}\right)u\,du, \tag{93}$$

$$\kappa^3 \equiv \frac{1}{\phi(0)}\int_0^\infty \phi(u)u^2\,du, \quad \theta^3 \equiv \int_0^\infty \left(1 - \frac{\chi(u)}{\chi(\infty)}\right)u^2\,du, \tag{94}$$

respectively. We see in Eq. (90) that the first term of the power expansion of the drift coefficient Γ_i is linear on the variables of the system; thus it introduces oscillations and additional frictions (positive or negative) on the system of interest, while the contribution of the second and third terms is more complicate and strongly system-dependent. Concerning the diffusion coefficients in Eq. (89) that stem from our main result on the Lie evolution of differential operators, we see that the first order in the power expansion looks quite trivial; it is a standard diffusion term as that of a white noise forcing, while the other terms of the series can contribute in locally enhancing or depressing the diffusion, depending on whether their signs are positive or negative, respectively. For example, if the perturbation field $\boldsymbol{I}$ has only one non-vanishing component, the second order contributions to the diagonal elements of the diffusion matrix vanish for both symplectic and cosymplectic systems of interest while the off-diagonal terms are very easy to calculate and depend mainly on the symmetric property of the system (or on the structure constant of the Lie algebra).

VII. THE SPECIAL CASES OF HAMILTONIAN (SYMPLECTIC OR CO-SYMPLECTIC) SYSTEMS WITH ONE PARAMETER-DISSIPATION

Our result in Eq. (59) can be simplified for some particular classes of systems that, for their fundamental importance, deserve to be separately treated. One of them is the set of the Hamiltonian co-symplectic systems, possibly also subjected to a linear non-conservative force (e.g., friction), such as Volterra gyrostats,[20–22] spin systems interacting with magnetic fields, the inviscid or the dissipative Euler equation, or some low order model approximations of the Navier-Stokes (NS) equations,[19–22] including, for example, the famous Lorenz system.[23,56,57] Another very important and large class is that of the ordinary symplectic two-dimensional systems subjected to some linear friction or explosive force, as, just to quote a few examples among so many, the "ubiquitous" Duffing oscillator, the celebrated recharge oscillator mimicking the El Niño Southern Oscillation (ENSO),[14–16,32] or the usual picture of a chemical reaction process in a solvent, where a particle (the reactant) reacts escaping from a potential well by jumping over a barrier of height E_b.[6,11,58,59]

In both these special cases, to solve the unperturbed Lie evolution of the basis ($\partial_1, \ldots, \partial_N$) of the vector space, it is convenient to use a slightly different procedure with respect to that of Sec. V, taking advantage of the energy conservation of the co-symplectic/symplectic part of the flux.

A co-symplectic/symplectic structure of the unperturbed velocity field, where linear non-Hamiltonian terms are also present, means $C_i = -J_{i,l}\,(\partial_l H) + \gamma_i x_i$, namely,

$$\dot{x}_i = J_{i,l}\,(\partial_l H) - \gamma_i x_i = \{x_i, H\}_{PB} - \gamma_i x_i \quad 1 \le i \le N, \tag{95}$$

where

$$H \equiv \alpha \frac{x_N^2}{2} + U(x_1, x_3, \ldots, x_{N-1}) \tag{96}$$

is the Hamiltonian, which here we assume to be quadratic at least for one of the variables (here labeled as x_N) and $J_{i,l}$ is a skew-symmetric tensor that defines the following generalized Poisson Brackets (f, g are smooth enough functions $\mathbb{R}^N \to \mathbb{R}$),

$$\{f, g\}_{PB} = (\partial_i f) J_{il} (\partial_l g), \tag{97}$$

for which the Jacobi identity can be written as[19]

$$J_{il}\partial_l J_{jk} + J_{jl}\partial_l J_{ki} + J_{kl}\partial_l J_{ij} = 0. \tag{98}$$

Notice that in a right tensor notation we should distinguish between contravariant and covariant parts of the tensor, using superscript Latin indexes and subscript indexes, respectively, but here, for the sake of simplicity, we do not make this distinction.

In these cases, the divergence of the vector field of velocities is equal to the sum of the coefficients of the non-Hamiltonian terms: $\partial_i C_i = \sum_i \gamma_i$ (see Subsection VII B for a discussion about that), which means that we are under Assumption C.

The symplectic/co-symplectic structure of the equation of motion makes conservative (in the sense that preserves the energy value) the corresponding flux; in fact, $\dot{H} = \{H, H\}_{PB} = 0$. As we stated above, this is actually the key ingredient of the present procedure and it is exploited with the following change of variables,

$$\begin{aligned} \tilde{x}_1 &= x_1 \\ &\vdots \\ \tilde{x}_{N-1} &= x_{N-1}, \\ E &= \alpha \frac{x_N^2}{2} + U(x_1, x_2, \ldots, x_{N-1}), \end{aligned} \tag{99}$$

i.e., we replace the last variable (namely, the one that enters in the Hamiltonian through a quadratic term) with the energy E. From this we have the reverse relations,

$$\begin{aligned} x_1 &= \tilde{x}_1 \\ &\vdots \\ x_{N-1} &= \tilde{x}_{N-1}, \\ x_N &= \sqrt{2(E - U(x_1, x_2, \ldots, x_{N-1}))/\alpha}, \end{aligned} \tag{100}$$

and $(1 \le i \le N-1)$

$$\begin{aligned} \partial_i &= \tilde{\partial}_i + (\tilde{\partial}_i U(\tilde{x}_1, \ldots, \tilde{x}_{N-1}))\partial_E, \\ \partial_N &= \alpha\, x_N(\tilde{x}_1, \ldots, \tilde{x}_{N-1}, E)\partial_E, \end{aligned} \tag{101}$$

where $x_N(\tilde{x}_1, \ldots, \tilde{x}_{N-1}, E) \equiv \sqrt{2(E - U(\tilde{x}_1, \ldots, \tilde{x}_{N-1}))/\alpha}$ and $\tilde{\partial}_i \equiv \frac{\partial}{\partial \tilde{x}_i}$.

We call $\mathcal{G}$ the Liouvillian operator $\mathcal{L}_0 = \partial_i C_i$ after this transformation of variables. The counterpart of Eq. (52), i.e., the Lie evolution of the new basis of the vector field along the Liouville operator, after the above change of variables, is

$$\begin{aligned} \mathfrak{E}_1(u) &\equiv e^{\mathcal{L}_0^\times u}[\partial_1] = e^{\mathcal{G}^\times u}\left[\tilde{\partial}_1 + (\tilde{\partial}_1 U(\tilde{x}_1, \ldots, \tilde{x}_{N-1}))\partial_E\right] \\ &\vdots \\ \mathfrak{E}_{N-1}(u) &\equiv e^{\mathcal{L}_0^\times u}[\partial_{N-1}] = e^{\mathcal{G}^\times u}\left[\tilde{\partial}_{N-1} + (\tilde{\partial}_{N-1} U(\tilde{x}_1, \ldots, \tilde{x}_{N-1}))\partial_E\right], \\ \mathfrak{E}_N(u) &\equiv e^{\mathcal{L}_0^\times u}[\partial_N] = e^{\mathcal{G}^\times u}[x_N(\tilde{x}_1, \ldots, \tilde{x}_{N-1}, E)\partial_E] = x_N(\tilde{x}_1, \ldots, \tilde{x}_{N-1}, E; t-u)\, e^{\mathcal{G}^\times u}[\partial_E], \end{aligned} \tag{102}$$

where $x_N(\tilde{x}_1, \ldots, \tilde{x}_{N-1}, E; t-u) \equiv (e^{-\mathcal{G}^+ u} x_N(\tilde{x}_1, \ldots, \tilde{x}_{N-1}, E))$ is, as usual, the unperturbed back-time evolution of the x_N variable. From now on, for simplicity, we shall omit the "tilde" on the new variables $\tilde{x}_i$.

Here we shall discuss separately the two main cases cited at the beginning of the present section. We shall start from the second one, i.e., where the dimension N of the space of the states of the system is 2 and then we shall address the case where N is odd. In general, if N is even, an old theorem that is credited to Darboux says that a transformation of variables (at least locally) that makes $J = J_c$ exists, with

$$J_c \equiv \begin{pmatrix} 0_{N/2} & \mathcal{I}_{N/2} \\ -\mathcal{I}_{N/2} & 0_{N/2} \end{pmatrix}, \tag{103}$$

where $0_{N/2}$ is an $N/2 \times N/2$ matrix of zeros and $\mathcal{I}_{N/2}$ is the $N/2 \times N/2$ unit matrix. The subscript c of J_c indicates that the system is written in terms of canonical coordinates. Thus for N even the co-symplectic dynamics is equivalent to just symplectic.

A. The two-dimensional dissipative oscillator

In the two-dimensional standard Hamiltonian case with friction, we have $J = J_c$ with $\mathcal{I}_1 = 1$, and from Eqs. (96)–(95) we see that $x_2 = v$ is the impulse, $x_1 = x$ is the coordinate of the system, and $U(x)$ is the potential [we set $\alpha = 1$ and we define $U' \equiv (\partial_x U)$],

$$\begin{aligned} \dot{x} &= -C_x = v, \\ \dot{v} &= -C_v = -U'(x) - \gamma v. \end{aligned} \tag{104}$$

Thus $\mathcal{L}_0 = -v\partial_x + U'(x)\partial_v + \gamma\partial_v v$ that, after the change of variables in Eq. (99), becomes

$$\mathcal{G} \equiv -v(E, x)\partial_x + \gamma v(E, x)\partial_E\, v(E, x), \tag{105}$$

with $x_2(E, x_1) = v(E, x) \equiv \sqrt{2(E - U(x))}$. This particular case is not so trivial as it would look at first sight because, as we have already stressed in the Introduction, standard Zwanzig projection approaches as those in Refs. 3, 11, and 55 do not directly face the problem of solving the Lie evolution of differential operator as it is developed in the present paper. For that they are limited to Hamiltonian or nearly Hamiltonian cases, i.e., where γ is vanishing or almost vanishing. The general case was recently successfully faced for the first time in Ref. 2, where the Lie evolution of the differential operator ∂_v along the unperturbed Liouvillian was introduced and solved by using an original procedure. Here we report the main steps of this approach. In terms of the (x, E) coordinates, Eq. (102) is rewritten as

$$e^{\mathcal{L}_0^\times u}[\partial_v] = v_0(t-u)\,\mathfrak{E}_E(u), \tag{106}$$

where the operator $\mathfrak{E}_E(u)$ is defined as the Lie evolution of the operator ∂_E,

$$\mathfrak{E}_E(u) \equiv e^{\mathcal{G}^\times u}[\partial_E]. \tag{107}$$

By a direct calculation we obtain

$$\mathcal{G}^\times[\partial_E] = -\frac{1}{v(E,x)^2}(\mathcal{G}-\gamma) - \gamma\,\partial_E. \tag{108}$$

Taking the time (u) derivative of Eq. (107) and using Eq. (108) we obtain

$$\frac{d}{du}\mathfrak{E}_E(u) = -\gamma\mathfrak{E}_E(u) - \frac{1}{v_0^2(E,x;t-u)}(\mathcal{G}-\gamma), \tag{109}$$

where $v_0^2(E,x;t-u) \equiv (v_0(E,x;t-u))^2$. The formal solution of Eq. (109) is straightforward,

$$\mathfrak{E}_E(u) = e^{-\gamma u}\,\partial_E - e^{-\gamma u}\left(\int_0^u du' e^{\gamma u'}\frac{1}{v_0^2(E,x;t-u')}\right)(\mathcal{G}-\gamma). \tag{110}$$

Thus, using Eq. (100) to go back to the original $\boldsymbol{x} = (x, v)$ variables, we have

$$\begin{aligned}&\mathfrak{E}_v(u) \equiv e^{\mathcal{L}_0^\times u}[\partial_v] = \\ &e^{-\gamma u}\frac{v_0(t-u)}{v}\partial_v - e^{-\gamma u}v_0(t-u)\left(\int_0^u du' e^{\gamma u'}\frac{1}{v_0^2(t-u')}\right)(\mathcal{L}_0-\gamma).\end{aligned} \tag{111}$$

Equation (111) is an interesting result because in the rhs the coefficients of the vector field do not involve partial derivatives of the trajectories as in the general result of Eq. (59). However, as it is illustrated in Appendix A, it is also possible to put this result in a more compact form that, depending of the specific case, could be easier to deal with,

$$\mathfrak{E}_v(u) \equiv e^{\mathcal{L}_0^\times u}[\partial_v] = -e^{-\gamma u}(\partial_v\, x_0(t-u))\partial_x + e^{-\gamma u}(\partial_x\, x_0(t-u))\partial_v. \tag{112}$$

This is precisely the result we can find in Ref. 2. Now, to obtain the corresponding expression for the Lie evolution of ∂_x, we exploit the system of ODE for the $\mathfrak{E}_i$ operators in Eq. (54), for the present specific case of Hamiltonian oscillator, defined by Eq. (104). Thus we have

$$\frac{d}{du}\mathfrak{E}_x(u) = -\,(U''(x))_{x_0(t-u)}\mathfrak{E}_v(u), \tag{113}$$

$$\frac{d}{du}\mathfrak{E}_v(u) = \gamma\mathfrak{E}_v(u) - \mathfrak{E}_x(u). \tag{114}$$

Using just Eq. (114), we obtain $\mathfrak{E}_x(u) = \gamma\mathfrak{E}_v(u) - \frac{d}{du}\mathfrak{E}_v(u)$ that, with Eq. (112), gives the explicit result for $\mathfrak{E}_x(u)$,

$$\mathfrak{E}_x(u) \equiv e^{\mathcal{L}_0^\times u}[\partial_x] = e^{-\gamma u}(\partial_v\, v_0(t-u))\partial_x - e^{-\gamma u}(\partial_x\, v_0(t-u))\partial_v. \tag{115}$$

Comparing Eqs. (112) and (115) with Eq. (55) we have

$$\begin{aligned}&\beta_{x\,x}(x,v;u) = e^{-\gamma u}(\partial_v\, v_0(t-u)), &&\beta_{x\,v}(x,v;u) = -e^{-\gamma u}(\partial_x\, v_0(t-u)),\\ &\beta_{v\,x}(x,v;u) = -e^{-\gamma u}(\partial_v\, x_0(t-u)), &&\beta_{v\,v}(x,v;u) = e^{-\gamma u}(\partial_x\, x_0(t-u)).\end{aligned} \tag{116}$$

Equation (116) is a simplified result, with respect to the more general one in Eq. (59), which applies to the two-dimensional Hamiltonian, with friction, case (for example, the Duffing oscillator).

Notice that, using directly the general result in Eq. (59) applied to the present two-dimensional system in Eq. (104), without passing to the energy variable as done here above, we get

$$\begin{aligned} \beta_{x\,x}(x, v; u) &= (\partial_x\, x_0(t+u))_{\boldsymbol{x}=\boldsymbol{x}_0(t-u)}, & \beta_{x\,v}(x, v; u) &= (\partial_x\, v_0(t+u))_{\boldsymbol{x}=\boldsymbol{x}_0(t-u)},\\ \beta_{v\,x}(x, v; u) &= (\partial_v\, x_0(t+u))_{\boldsymbol{x}=\boldsymbol{x}_0(t-u)}, & \beta_{v\,v}(x, v; u) &= (\partial_v\, v_0(t+u))_{\boldsymbol{x}=\boldsymbol{x}_0(t-u)}. \end{aligned} \tag{117}$$

These are expressions much more complicate than that in Eq. (116), at least formally. Of course it is also possible to start from the general expressions of Eq. (117) and arrive directly to the simplified ones of Eq. (116) (see Appendix B).

B. The co-symplectic odd-dimensional case

For odd N we assume that J depends linearly on $\boldsymbol{x}$: $J_{il} = c_{ilm}x_m$, where the coefficients c_{ilm} must satisfy the following requirements ($1 \le k, i, j, m, l \le N$):

$$c_{ilm} = -c_{lim} \tag{118}$$

$$0 = c_{klm}c_{ijl} + c_{jlm}c_{kil} + c_{ilm}c_{jkl} \tag{119}$$

to make J skew-symmetric and for the Jacobi identity to hold, respectively. These coefficients are the structure constants of the Lie algebra generated by the generalized Poisson Brackets given in Eq. (97),

$$\{f(\boldsymbol{x}), g(\boldsymbol{x})\}_{PB} \equiv c^k_{il}\, x_k(\partial_i f(\boldsymbol{x}))\,(\partial_l g(\boldsymbol{x})). \tag{120}$$

Note that Assumption C means $(\partial_j\partial_i C_i) = 0$, i.e., the divergence of the velocity vector field is constant on the phase space of the system. From the dynamical systems defined in Eq. (95), we have that if the co-symplectic part of the flux preserves the volume in the phase space, thus Assumption C is satisfied; in fact, we have $\partial_i C_i = \sum_i \gamma_i = constant$. However, unlike in the case of standard symplectic fluxes, where the Liouville theorem always holds true, for co-symplectic fluxes that is not obvious, in fact, using the antisymmetric properties of c_{ilm} with respect to the first two indexes [see Eq. (118)], the request of vanishing divergence of the velocity vector field gives $(\partial_j\, c_{ili}\partial_l H) = 0$, i.e., involves also the third index of the structure constants. As it is clear from Eqs. (118)–(119), in general, the structure constants do not necessarily possess defined symmetry properties upon the exchange of all indexes, i.e., including the third one. However in general semi-simple Lie algebras can, by a coordinate change, be brought into a form in which the structure constants are completely antisymmetric,[60] such as for SO(3) symmetric systems or fluid theories in Eulerian variables. Therefore, we assume that we are working with the right coordinates for which the structure constants are completely antisymmetric and thus the Liouville theorem holds true for the co-symplectic part of the flux: $c_{ili}(\partial_l H) = 0$. Namely, for $\gamma_i = 0$, we have $\mathcal{L}_0 = \partial_i C_i = C_i\partial_i = -\mathcal{L}_0^+$. Given that, to simplify the discussion, without loss of generality, we limit ourself to the SO(3) case, i.e., $N = 3$, where the structure constants correspond to the values of the Levi–Cività symbol: $c_{klm} = -\epsilon_{klm}$. Thus, we have

$$C_i = \epsilon_{ikl}x_l(\partial_k H) + \gamma_i x_i, \tag{121}$$

i.e., more explicitly, using the Hamiltonian in Eq. (96)

$$\begin{aligned} \dot{x}_1 &= -C_1 = -\,x_3\,(\partial_2 U(x_1, x_2)) + x_3\,\alpha x_2 - \gamma_1 x_1,\\ \dot{x}_2 &= -C_2 = -\,x_3\,\alpha x_1 + x_3\,(\partial_1 U(x_1, x_2)) - \gamma_2 x_2,\\ \dot{x}_3 &= -C_3 = -\,(\partial_1 U(x_1, x_2))\,x_2 + (\partial_2 U(x_1, x_2))x_1 - \gamma_3 x_3, \end{aligned} \tag{122}$$

from which (I recall that here, if we set $\gamma_i = 0$, we have $\mathcal{L}_0 = \partial_i C_i = C_i\partial_i$)

$$\begin{aligned}\mathcal{L}_0 = & x_3((\partial_2 U(x_1,x_2)) - \alpha x_2))\partial_1 \\ & + x_3(\alpha x_1 - (\partial_1 U(x_1,x_2)))\partial_2 \\ & + ((\partial_1 U(x_1,x_2))\, x_2 - (\partial_2 U(x_1,x_2))x_1)\partial_3 \\ & + \gamma_1\partial_1\, x_1 + \gamma_2\partial_2\, x_2 + \gamma_3\partial_3\, x_3 \end{aligned} \tag{123}$$

that, after the change of variables in Eq. (99) (still omitting the "tilde" on the new variables $\tilde{x}_i$), becomes

$$\begin{aligned}\mathcal{G} = & \mathcal{G}_c + \gamma_1\partial_1\, x_1 + \gamma_2\partial_2\, x_2 + +\gamma_1(\partial_1 U(x_1,x_2))\partial_E\, x_1 + \gamma_2(\partial_2 U(x_1,x_2))\partial_E\, x_2 \\ & + \gamma_3\alpha\, x_3(x_1,x_2,E)\partial_E\, x_3(x_1,x_2,E), \end{aligned} \tag{124}$$

where $\mathcal{G}_c$ is the Liouvillian operator of the conservative part of the flux, written in the new $(x_1, \ldots, x_{N-1}, E)$ coordinates, i.e.,

$$\begin{aligned}\mathcal{G}_c \equiv \mathcal{G}|_{\gamma_i=0} = & x_3(x_1,x_2,E)((\partial_2 U(x_1,x_2)) - \alpha\, x_2))\partial_1 \\ & + x_3(x_1,x_2,E)(\alpha x_1 - (\partial_1 U(x_1,x_2)))\partial_2 \\ & + ((\partial_1 U(x_1,x_2))\, x_2 - (\partial_2 U(x_1,x_2))x_1)\partial_3. \end{aligned} \tag{125}$$

By a simple calculation, we obtain the analogous of Eq. (108),

$$\mathcal{G}^{\times}[\partial_E] = -\frac{1}{x_3(x_1,x_2,E)^2}(\mathcal{G}_c + \gamma_3\alpha\, x_3(x_1,x_2,E)\partial_E\, x_3(x_1,x_2,E) - \gamma_3) - \gamma_3\,\partial_E. \tag{126}$$

From the previous equation we see that we can repeat the steps of Subsection VII A [Eqs. (108)–(111)] only if $\gamma_2 = \gamma_3 = 0$; thus we make this assumption and we arrive to a result that is the counterpart of Eq. (111),

$$\begin{aligned} & \mathfrak{E}_3(u) \equiv e^{\mathcal{L}_0^{\times} u}[\partial_3] = \\ & e^{-\gamma_3 u}\frac{\alpha\, x_3(t-u)}{x_3}\partial_3 - e^{-\gamma_3 u}x_3(t-u)\left(\int_0^u du' e^{\gamma_3 u'}\frac{1}{\alpha\, x_3^2(t-u')}\right)(\mathcal{L}_0 - \gamma_3). \end{aligned} \tag{127}$$

The explicit expressions for $\mathfrak{E}_1(u)$ and $\mathfrak{E}_2(u)$ can be found, as in the case of the standard dissipative oscillator of Subsection VII A, inserting the above result in the simple ODE for the $\mathfrak{E}_i$ operators given in Eq. (54), where the velocity field C_i is given in Eq. (121). We do not go ahead in that here.

VIII. CONCLUSIONS

In this paper, we show how and in which cases, using a perturbation projective approach, we can obtain a generalized FPE (second order differential operator) for the DF of the subpart of interest of a general class of dynamical systems. The result is based on some propositions, here specified and demonstrated, about the Lie evolution of first order differential operators, along general Liouville operators. In particular, the diffusion-like coefficients of this generalized FPE are here obtained and given in terms of analytic expressions thanks to the central result in Eq. (59) that expresses the Lie-evolution of the vector field in the tangent bundle of the state space of the system.

Thus, with this work, we can shed some light on the problem of the emergence of regular regression laws and standard statistical mechanics on nature. Actually, the explicit formal results for the transport coefficients of this generalized FPE allow us to establish which are the conditions for the emergence of standard statistics (Gaussian, canonical, Levy, thermodynamics, linear Onsager regressions,[3,30,31] etc.) for the system of interest.

Notice that, while usually the projection procedure is applied to Hamiltonian "microscopic" systems, where dissipation and diffusion are related by the fluctuation dissipation theorem and stem from hiding the many and fast degrees of freedom,[7,9,10,54,61–64] the results of the present paper allow us to generalize the procedure to the much broader class of non-Hamiltonian systems. Moreover we show that under Assumption C, for the Lie time evolution along generic Liouvillians, we recover an effective antisymmetric property of the Liouvillian operator, i.e., $\mathcal{L}$ is equivalent to $-\mathcal{L}^+$ as for Hamiltonian fluxes, a fact that could be used also for an eigenvalue/eigenvector approach to the Lie evolution of operators.

Thus the applications of our results are not limited to the field of foundation of Statistical Mechanics and Thermodynamics. Some examples are classic or quantum dissipative rotators, spin systems interacting with magnetic fields,[65,66] as Landau-Lifshitz systems in micromagnetism,[67] and many others among which an important class belongs to the field of geophysical fluid dynamics. In fact, while the Navier-Stokes (NS) equations in the simplest inviscid flow approximation become the Euler equation of motion, i.e., a pure conservative co-symplectic dynamical system, more in general, the NS equations contain dissipative terms and thus cannot have a pure Hamiltonian representation. Actually, they can be approximated by a set of coupled nonlinear ordinary differential equations, called low-order models (LOM) in the form of coupled Volterra gyrostats.[20–22] These systems belong to the class treated in Sec. VII B. An interesting large scale Geophysical phenomenon that can be approached in this way is the El Niño-Southern Oscillation (ENSO),[32] a naturally occurring event in the tropical Pacific that has global impacts of great relevance to society. The ENSO stems form the weak interaction between the dissipative Pacific Equatorial Ocean and the forcing atmosphere.

The application of the results of the present paper to some of the cases above cited is an issue on which we are currently working.

APPENDIX A: TWO DIFFERENT EXPRESSIONS FOR THE LIE EVOLUTION OF ∂_v

Here we demonstrate how to arrive to Eq. (112) from Eq. (111).

In the first term of the rhs of Eq. (111) we use the following identity:

$$\begin{aligned} v_0(t-u) &= \mathcal{L}_0^{+} x_0(t-u) \\ &= v\,\partial_x x_0(t-u) - U'(x)(\partial_v x_0(t-u)) - \gamma v(\partial_v x_0(t-u)), \end{aligned} \tag{A1}$$

and we obtain

$$\begin{aligned} e^{\mathcal{L}_0^{\times} u}[\partial_v] = e^{-\gamma u}\frac{1}{v}\{ & v(\partial_x x_0(t-u)) \\ -(U'(x)+\gamma v)&\left[(\partial_v x_0(t-u)) + v\, v_0(t-u)\left(\int_0^u du' e^{\gamma u'}\frac{1}{v_0^2(t-u')}\right)\right]\Bigg\}\frac{\partial}{\partial v} \\ + e^{-\gamma u}&\left\{v\, v_0(t-u)\left(\int_0^u du' e^{\gamma u'}\frac{1}{v_0^2(t-u')}\right)\right\}\frac{\partial}{\partial x}. \end{aligned} \tag{A2}$$

To simplify the above expression, we prove the following equality:

$$(\partial_v x_0(t-u)) + v\, v_0(t-u)\left(\int_0^u du' e^{\gamma u'}\frac{1}{v_0^2(t-u')}\right) = 0. \tag{A3}$$

For non-energy conserving systems of interest, i.e., for $\gamma \neq 0$, this is a not trivial result. To this aim, we start noticing that the functions $f_1(u) \equiv v_0(t+u)$ and $f_2(u) \equiv (\partial_v x_0(t+u))$ are independent solutions of the following ordinary second order differential equation (SODE):

$$\ddot{f}(u) + \gamma\dot{f}(u) + U''(x_0(t+u))f(u) = 0, \tag{A4}$$

where the dot over the function f means here a derivative respect to the time variable u. From the solutions $f_1(u)$ and $f_2(u)$ of this SODE, we get the Wronskian

$$\begin{aligned} W(u) &\equiv f_1(u)\dot{f}_2(u) - \dot{f}_1(u)f_2(u) \\ &= v_0(t+u)(\partial_v v_0(t+u)) - \dot{v}_0(t+u)(\partial_v x_0(t+u)). \end{aligned} \tag{A5}$$

The Abel's theorem for the Wronskian applied to Eq. (A4) gives

$$\dot{W}(u) = -\gamma W(u). \tag{A6}$$

Then, using the obvious initial condition $W(0) = v$, we get directly the explicit expression for the Wronskian

$$W(u) = v\, e^{-\gamma u}. \tag{A7}$$

Introducing the function $z(u) \equiv f_2(u)/f_1(u) = (\partial_v x_0(t+u))/v_0(t+u)$ we have

$$\dot{z}(u) = \frac{W(u)}{f_1^2(u)} = \frac{e^{-\gamma u}}{v_0^2(t+u)}, \tag{A8}$$

i.e.,

$$\frac{(\partial_v x_0(t+u))}{v_0(t+u)} = z(u) = v\left(\int_0^u du' e^{-\gamma u'} \frac{1}{v_0^2(t+u')}\right). \tag{A9}$$

Making the change $u \to -u$ in the previous equation we arrive to Eq. (A3). At the end, the expression for $\mathfrak{E}_v(u)$ in Eqs. (112) is obtained by inserting Eq. (A3) into Eq. (A2). □

APPENDIX B: A SIMPLIFIED EXPRESSION FOR THE COEFFICIENTS OF THE LIE EVOLUTION OF THE BASIS OF THE VECTOR FIELD

To show that from Eq. (117) we can directly arrive to the simplified Eq. (116), we start rewriting the latter as

$$e^{\gamma u}(\partial_v\, x_0(t+u)) = -e^{\mathcal{L}_0^+ u}(\partial_v\, x_0(t-u)), \tag{B1a}$$

$$e^{\gamma u}(\partial_v\, v_0(t+u)) = e^{\mathcal{L}_0^+ u}(\partial_x\, x_0(t-u)). \tag{B1b}$$

Now, if we apply the exponential Liouville operator $\exp(\mathcal{L}_0^+ u)$ to Eq. (A3), we get

$$\begin{aligned} -e^{\mathcal{L}_0^+ u}(\partial_v x_0(t-u)) &= e^{\mathcal{L}_0^+ u}\left[v\, v_0(t-u) \int_0^u \mathrm{d}u' e^{\gamma u'} \tfrac{1}{v_0^2(t-u')}\right] \\ &= v_0(t+u) v \int_0^u \mathrm{d}u' e^{\gamma u'} \tfrac{1}{v_0^2(t-u'+u)} \\ &= e^{\gamma u} v_0(t+u) v \int_0^u d\theta e^{-\gamma\theta} \tfrac{1}{v_0^2(t+\theta)}, \end{aligned} \tag{B2}$$

where in the last integral term of the above equation we make the change of variable $\theta = u - u'$. If we compare the above result with Eq. (A9), Eq. (B1a) is demonstrated.

We have now to show that also Eq. (B1b) holds true. For that we take into account the following obvious identity:

$$v_0(t-u) = \mathcal{L}_0^+ x_0(t-u) \equiv v\,(\partial_x x_0(t-u)) + \dot{v}\,(\partial_v x_0(t-u)) \tag{B3}$$

from which

$$(\partial_x x_0(t-u)) = \frac{v_0(t-u)}{v} - \frac{\dot{v}}{v}\,(\partial_v x_0(t-u)), \tag{B4}$$

and let us apply the time shift operator $\exp(\mathcal{L}_0^+ u)$ to the above equation,

$$\begin{aligned} e^{\mathcal{L}_0^+ u}(\partial_x x_0(t-u)) &= \frac{v}{v_0(t+u)} - \frac{\dot{v}_0(t+u)}{v_0(t+u)}\, e^{\mathcal{L}_0^+ u}(\partial_v x_0(t-u)) \\ &= \frac{v}{v_0(t+u)} + \frac{\dot{v}_0(t+u)}{v_0(t+u)}\, e^{\gamma u}(\partial_v x_0(t+u)), \end{aligned} \tag{B5}$$

where, in the last row of the above equation, we have exploited Eq. (B1a). From Eq. (B5), using both $\exp(\gamma u) = v/W(u)$ [see Eq. (A7)] and the definition of the Wronskian in Eq. (A5), we get Eq. (B1b). □

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