

Abundant optical soliton solutions for an integrable $(2 + 1)$ -dimensional nonlinear conformable Schrödinger system

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ARTICLE INFO

Keywords:

(G'/G) -expansion method
Generalized Riccati equation mapping method
Kudryashov method
Integrable $(2 + 1)$ -dimensional NLCS system
Conformable derivative

ABSTRACT

The analytical solutions of the integrable generalized $(2 + 1)$ -dimensional nonlinear conformable Schrödinger (NLCS) system of equations was explored in this paper with the aid of three novel techniques which consist of (G'/G) -expansion method, generalized Riccati equation mapping method and the Kudryashov method in the conformable sense. We have discovered a new and more general variety of exact traveling wave solutions by using the proposed methods with a variety of soliton solutions of several structures. With several plots illustrating the behavior of dynamic shapes of the solutions, the findings are highly applicable and detailed the physical dynamic of the considered nonlinear system.

Introduction

The search for an exact soliton solutions to the nonlinear models has been one of the most fascinating and exciting areas of research in the field of sciences and engineering for many years [1–22]. Nonlinear models are frequently used in a broad range of research fields and have been studied from a variety of perspectives. In the development of new theories in mathematical physics, the study of solitons, especially in the field of plasma physics, has a critical part to play, while the development of mathematical methods, that further give us more accurate results for the extraction of solitons, is a highly prominent area of applied physics research with promising features in several other fields of physics. Nonlinear fractional models (NFM)s which consist of non-integer order are the generalizations of classical nonlinear models of integer order. Many physical phenomena have been modeled by the utilization of fractional calculus due to modern fractional-order models are more suitable and flexible when compared with the traditionally used integer-order models. Some applications of fractional calculus can be found in [23–39]. An elegant way of seeking the soliton solutions of nonlinear classical and fractional system is to suggest a transformation to arrive at a solvable a nonlinear ordinary differential equations (NODEs) using analytical techniques such as iterative shehu transform method [40], variational iteration method [41], perturbation-iteration algorithm

[42,43], sine–Gordon expansion method [44], tanh method [45], residual power series method [46,47], δ -homotopy perturbation transform method [48], sub-equation method [49], modified simple equation [50], q-homotopy analysis method [51,52], new extended direct algebraic method [53,54], F -expansion method [55], fractional reduced differential transform method [56], homogeneous balance method [57], q-homotopy analysis transform method [58,59], extended modified auxiliary equation mapping method [60], simple equation method [61] and considerably more.

In [62], Radha and Lakshmanan examined the integrable $(2 + 1)$ -dimensional NLS system of equations defined below:

$$\begin{aligned} P_t &= P_{xy} + PQ, \\ Q_x &= 2\left(|P|^2\right)_y. \end{aligned} \quad (1)$$

Using the P-analysis, its bilinear form have been obtained that can then be employed to produce the soliton solutions. This system have been studied in [62,60,63]. In [62], a direct method have been proposed to study ghost solitons of the $(2 + 1)$ -dimensional NLCS system. Recently, In [60], newly method called the extended modified auxiliary equation mapping method (EMAEMM) was applied to retrieve its exact traveling wave solutions of the $(2 + 1)$ -dimensional NLCS system. Hosseini et.al. in [63] used Kudryashov method and its modified version to investigate the exact solutions of the proposed problem. The focus of

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the current research is to further complement the studies previously conducted on the system 1 and to extract its optical soliton solutions using three novel approaches consisting of the (G'/G) expansion method, the generalized Riccati equation mapping method and the Kudryashov method. For this purpose, we analysis the generalized $(2+1)$ -dimensional NLCS system of equations given as

$$\begin{aligned} i\mathcal{D}_t^\rho P + \beta_1 P_{xy} + \beta_2 PQ &= 0, \\ \beta_3 Q_x + \beta_4 \left(|P|^2 \right)_y &= 0, \end{aligned} \quad (2)$$

where $\beta_1, \beta_2, \beta_3$ and β_4 are arbitrary constants and ρ , ($0 < \rho \leq 1$) is the fractional order.

The layout of the present article is as follows: In Section “Preliminaries”, we present definitions and some basic properties of the conformable derivative. The general formulation of solutions to integrable $(2+1)$ -dimensional NLCS system of equations which includes the implementation of three efficient and reliable techniques, namely, (G'/G) -expansion method, generalized Riccati equation mapping method and the Kudryashov method in the conformable sense are presented in Section “Mathematical formulation of the integrable $(2+1)$ -dimensional NLCS system”. Finally, Section “Concluding remarks” presents the concluding remarks.

Preliminaries

Here, we present definitions and some basic properties of the conformable derivative which can be found in [64,65].

Definition 1. Let $P : [0, \infty) \rightarrow \mathbb{R}$. The conformable derivative of P of order ρ is given by

$$\mathcal{D}_t^\rho P(t) = \lim_{h \rightarrow 0} \frac{P(t + h t^{1-\rho}) - P(t)}{h}, \quad \forall t > 0, \rho \in (0, 1). \quad (3)$$

Furthermore, If P is ρ -differentiable in some interval $(0, \zeta)$ where $\zeta > 0$ and $\lim_{t \rightarrow 0} P_t^{(\rho)}(t)$ exists. We define

$$P_t^{(\rho)}(0) = \lim_{t \rightarrow 0^+} P_t^{(\rho)}(t). \quad (4)$$

Lemma 1. [64] Let $\rho \in (0, 1]$ and P_1, P_2 be ρ -differentiable at a point $t > 0$. Then

- (i.) $\mathcal{D}_t^\rho (A_1 P_1 + A_2 P_2) = A_1 \mathcal{D}_t^\rho P_1 + A_2 \mathcal{D}_t^\rho P_2, \forall A_1, A_2 \in \mathbb{R}$.
- (ii.) $\mathcal{D}_t^\rho (t^\sigma) = \sigma t^{\sigma-\rho}, \forall \sigma \in \mathbb{R}$.
- (iii.) $\mathcal{D}_t^\rho (P_1 P_2) = P_1 \mathcal{D}_t^\rho P_2 + P_2 \mathcal{D}_t^\rho P_1$.
- (iv.) $\mathcal{D}_t^\rho \left(\frac{P_1}{P_2} \right) = \frac{P_2 \mathcal{D}_t^\rho P_1 - P_1 \mathcal{D}_t^\rho P_2}{P_2^2}$, provided $P_2 \neq 0$.
- (v.) $\mathcal{D}_t^\rho (K) = 0$, where K is a constant.
- (vi.) $\mathcal{D}_t^\rho P_1(t) = t^{1-\rho} \frac{dP_1(t)}{dt}$, for a differentiable function P_1 .

Mathematical formulation of the integrable $(2+1)$ -dimensional NLCS system

Consider the generalized integrable $(2+1)$ -dimensional NLCS system of equations

$$\begin{aligned} i\mathcal{D}_t^\rho P + \beta_1 P_{xy} + \beta_2 PQ &= 0, \\ \beta_3 Q_x + \beta_4 \left(|P|^2 \right)_y &= 0. \end{aligned} \quad (5)$$

Here, $P \in \mathbb{C}$ while $Q \in \mathbb{R}$. The transformation is suggested by us as:

$$\begin{aligned} P(x, y, t) &= P(\xi) e^{i\eta}, \\ Q(x, y, t) &= Q(\xi), \quad \xi = a_1 x + b_1 y + \frac{c_1}{\rho} t^\rho, \quad \eta = a_2 x + b_2 y + \frac{c_2}{\rho} t^\rho, \end{aligned} \quad (6)$$

where a_i, b_i , and $c_i, i = 1, 2$ are respectively the speed of wave, wave number and frequency of the soliton. Using Eq. 6, we achieve the imaginary and real parts of Eq. 5 as follows:

$$(c_1 + \beta_1(a_1 b_2 + a_2 b_1)) P'(\xi) = 0, \quad (7)$$

$$-c_2 P(\xi) - \beta_1 a_2 b_2 P(\xi) + \beta_1 a_1 b_1 P'(\xi) + \beta_2 P(\xi) Q(\xi) = 0, \quad (8)$$

and

$$\beta_3 a_1 Q'(\xi) + \beta_4 b_1 (P(\xi)^2)' = 0. \quad (9)$$

Integrating Eq. 9 once and setting constant of integration to zero result in

$$Q(\xi) = -\frac{b_1 \beta_4}{a_1 \beta_3} P(\xi)^2. \quad (10)$$

After solving Eq. 7 leads to

$$c_1 = -\beta_1(a_1 b_2 + a_2 b_1). \quad (11)$$

Inserting Eq. 10 into Eq. 8, we get

$$-(c_2 + \beta_1 a_2 b_2) P(\xi) + \beta_1 a_1 b_1 P'(\xi) - \frac{b_1 \beta_2 \beta_4}{a_1 \beta_3} P(\xi)^3 = 0. \quad (12)$$

By balancing $P(\xi)^3$ and $P'(\xi)$, we have $\varpi = 1$.

The (G'/G) -expansion method

Implementing the (G'/G) expansion method [66,67], the solution of Eq. 12 can be express as:

$$P(\xi) = k_0 + \sum_{i=1}^{\varpi} k_i \left(\frac{G'(\xi)}{G(\xi)} \right)^i, \quad k_\varpi \neq 0. \quad (13)$$

Here, $G(\xi)$ satisfies an ODE:

$$G'(\xi) = -\mu_1 G'(\xi) - \mu_2 G(\xi), \quad (14)$$

where constants $k_i, i = 0, 1, 2, \dots, \varpi$ and $\mu_i, i = 1, 2$ can be determined subsequent. The generalized solutions to the above mentioned ODE are given as

$$\begin{aligned} \frac{G'(\xi)}{G(\xi)} &= \begin{cases} -\frac{\mu_1 + \sqrt{\mu_1^2 - 4\mu_2}}{2} \left(\frac{\Omega_1 \sinh \left(\frac{1}{2} \sqrt{\mu_1^2 - 4\mu_2} \xi \right) + \Omega_2 \cosh \left(\frac{1}{2} \sqrt{\mu_1^2 - 4\mu_2} \xi \right)}{\Omega_1 \cosh \left(\frac{1}{2} \sqrt{\mu_1^2 - 4\mu_2} \xi \right) + \Omega_2 \sinh \left(\frac{1}{2} \sqrt{\mu_1^2 - 4\mu_2} \xi \right)} \right), & \mu_1^2 - 4\mu_2 > 0, \\ -\frac{\mu_1 + \sqrt{-(\mu_1^2 - 4\mu_2)}}{2} \left(\frac{\Omega_1 \sin \left(\frac{1}{2} \sqrt{-(\mu_1^2 - 4\mu_2)} \xi \right) + \Omega_2 \cos \left(\frac{1}{2} \sqrt{-(\mu_1^2 - 4\mu_2)} \xi \right)}{\Omega_1 \cos \left(\frac{1}{2} \sqrt{-(\mu_1^2 - 4\mu_2)} \xi \right) + \Omega_2 \sin \left(\frac{1}{2} \sqrt{-(\mu_1^2 - 4\mu_2)} \xi \right)} \right), & \mu_1^2 - 4\mu_2 < 0, \\ -\frac{\mu_1}{2} + \frac{\Omega_2}{\Omega_1 + \Omega_2 \xi}, & \mu_1^2 - 4\mu_2 = 0, \end{cases} \\ & \quad (15) \end{aligned}$$

where Ω_1 and Ω_2 are arbitrary constants. Since $\varpi = 1$, then Eq. 13 reads:

$$P(\xi) = k_0 + k_1 \frac{G'(\xi)}{G(\xi)}, \quad k_1 \neq 0, \quad (16)$$

Insertion of Eqs. 14 and 16 into Eq. 12, considering the $(G'/G)^i, i = 0, 1, 2, 3$, coefficients to zero results in some algebraic equations in the form:

$$\begin{aligned} (G'/G)^0(\xi) : & a_1 b_1 \beta_1 k_1 \mu_1 \mu_2 - \frac{b_1 \beta_2 \beta_4 k_0^3}{a_1 \beta_3} - a_2 b_2 \beta_1 k_0 - c_2 k_0 = 0, \\ (G'/G)^1(\xi) : & a_1 b_1 \beta_1 k_1 \mu_1^2 + 2a_1 b_1 \beta_1 k_1 \mu_2 - \frac{3b_1 \beta_2 \beta_4 k_1 k_0^2}{a_1 \beta_3} - a_2 b_2 \beta_1 k_1 - c_2 k_1 = 0, \\ (G'/G)^2(\xi) : & 3a_1 b_1 \beta_1 k_1 \mu_1 - \frac{3b_1 \beta_2 \beta_4 k_0 k_1^2}{a_1 \beta_3} = 0, \\ (G'/G)^3(\xi) : & 2a_1 b_1 \beta_1 k_1 - \frac{b_1 \beta_2 \beta_4 k_1^3}{a_1 \beta_3} = 0. \end{aligned} \quad (17)$$

By solving the above equations through Mathematica software and considering Eq. 11, the outcomes are:

$$\begin{aligned} c_1 &= -\beta_1(a_1 b_2 + a_2 b_1), \quad c_2 = -\frac{1}{2}\beta_1(a_1 b_1 \mu_1^2 - 4a_1 b_1 \mu_2 + 2a_2 b_2), \quad k_0 \\ &= \pm \sqrt{\frac{a_1^2 \mu_1^2 \beta_1 \beta_3}{2\beta_2 \beta_4}}, \quad k_1 = \pm \sqrt{\frac{2a_1^2 \beta_1 \beta_3}{\beta_2 \beta_4}}. \end{aligned} \quad (18)$$

From the above solutions with Eqs. 15 and 16, we obtain solutions of Eq. 5 as

$$\begin{aligned} P(x, y, t) &= \pm \sqrt{\frac{a_1^2 \beta_1 \beta_3 (\mu_1^2 - 4\mu_2)}{2\beta_2 \beta_4}} \tanh\left(\frac{1}{2} \sqrt{\mu_1^2 - 4\mu_2} \xi\right) e^{i(a_2 x + b_2 y + \frac{c_2}{\rho} \varphi)}, \\ Q(x, y, t) &= -\frac{a_1 b_1 \beta_1 (\mu_1^2 - 4\mu_2)}{2\beta_2} \tanh^2\left(\frac{1}{2} \sqrt{\mu_1^2 - 4\mu_2} \xi\right). \end{aligned} \quad (22)$$

When $\Omega_1 = 0$ and $\Omega_2 \neq 0$, in Eq. 19 reveals the singular soliton solutions of Eq. 5 as

$$\begin{aligned} P(x, y, t) &= \pm \sqrt{\frac{a_1^2 \beta_1 \beta_3 (\mu_1^2 - 4\mu_2)}{2\beta_2 \beta_4}} \coth\left(\frac{1}{2} \sqrt{\mu_1^2 - 4\mu_2} \xi\right) e^{i(a_2 x + b_2 y + \frac{c_2}{\rho} \varphi)}, \\ Q(x, y, t) &= -\frac{a_1 b_1 \beta_1 (\mu_1^2 - 4\mu_2)}{2\beta_2} \coth^2\left(\frac{1}{2} \sqrt{\mu_1^2 - 4\mu_2} \xi\right). \end{aligned} \quad (23)$$

Remark 2. A special case when $\Omega_1 \neq 0$ and $\Omega_2 = 0$, in Eq. 20 reveals the periodic solitary wave solutions of Eq. 5 as

$$\begin{aligned} P(x, y, t) &= \mp \sqrt{\frac{a_1^2 \beta_1 \beta_3 (4\mu_2 - \mu_1^2)}{2\beta_2 \beta_4}} \tan\left(\frac{1}{2} \sqrt{4\mu_2 - \mu_1^2} \xi\right) e^{i(a_2 x + b_2 y + \frac{c_2}{\rho} \varphi)}, \\ Q(x, y, t) &= -\frac{a_1 b_1 \beta_1 (4\mu_2 - \mu_1^2)}{2\beta_2} \tan^2\left(\frac{1}{2} \sqrt{\mu_1^2 - 4\mu_2} \xi\right). \end{aligned} \quad (24)$$

When $\Omega_1 = 0$ and $\Omega_2 \neq 0$, in Eq. 20 also reveals the periodic solitary wave solutions of Eq. 5 as

$$P_1(x, y, t) = \pm \sqrt{\frac{a_1^2 \beta_1 \beta_3 (\mu_1^2 - 4\mu_2)}{2\beta_2 \beta_4}} \left(\frac{\Omega_1 \sinh\left(\frac{1}{2} \sqrt{\mu_1^2 - 4\mu_2} \xi\right) + \Omega_2 \cosh\left(\frac{1}{2} \sqrt{\mu_1^2 - 4\mu_2} \xi\right)}{\Omega_1 \cosh\left(\frac{1}{2} \sqrt{\mu_1^2 - 4\mu_2} \xi\right) + \Omega_2 \sinh\left(\frac{1}{2} \sqrt{\mu_1^2 - 4\mu_2} \xi\right)} \right) e^{i(a_2 x + b_2 y + \frac{c_2}{\rho} \varphi)}, \quad (19)$$

$$Q_1(x, y, t) = -\frac{a_1 b_1 \beta_1 (\mu_1^2 - 4\mu_2)}{2\beta_2} \left(\frac{\Omega_1 \sinh\left(\frac{1}{2} \sqrt{\mu_1^2 - 4\mu_2} \xi\right) + \Omega_2 \cosh\left(\frac{1}{2} \sqrt{\mu_1^2 - 4\mu_2} \xi\right)}{\Omega_1 \cosh\left(\frac{1}{2} \sqrt{\mu_1^2 - 4\mu_2} \xi\right) + \Omega_2 \sinh\left(\frac{1}{2} \sqrt{\mu_1^2 - 4\mu_2} \xi\right)} \right)^2.$$

$$P_2(x, y, t) = \mp \sqrt{\frac{a_1^2 \beta_1 \beta_3 (4\mu_2 - \mu_1^2)}{2\beta_2 \beta_4}} \left(\frac{\Omega_1 \sin\left(\frac{1}{2} \sqrt{4\mu_2 - \mu_1^2} \xi\right) - \Omega_2 \cos\left(\frac{1}{2} \sqrt{4\mu_2 - \mu_1^2} \xi\right)}{\Omega_1 \cos\left(\frac{1}{2} \sqrt{4\mu_2 - \mu_1^2} \xi\right) + \Omega_2 \sin\left(\frac{1}{2} \sqrt{4\mu_2 - \mu_1^2} \xi\right)} \right) e^{i(a_2 x + b_2 y + \frac{c_2}{\rho} \varphi)}, \quad (20)$$

$$Q_2(x, y, t) = -\frac{a_1 b_1 \beta_1 (4\mu_2 - \mu_1^2)}{2\beta_2} \left(\frac{\Omega_1 \sin\left(\frac{1}{2} \sqrt{4\mu_2 - \mu_1^2} \xi\right) - \Omega_2 \cos\left(\frac{1}{2} \sqrt{4\mu_2 - \mu_1^2} \xi\right)}{\Omega_1 \cos\left(\frac{1}{2} \sqrt{4\mu_2 - \mu_1^2} \xi\right) + \Omega_2 \sin\left(\frac{1}{2} \sqrt{4\mu_2 - \mu_1^2} \xi\right)} \right)^2.$$

$$P_3(x, y, t) = \pm \sqrt{\frac{2\beta_1 \beta_3}{\beta_2 \beta_4}} \frac{a_1 \Omega_2}{(\Omega_1 + \xi \Omega_2)} e^{i(a_2 x + b_2 y + \frac{c_2}{\rho} \varphi)}, \quad (21)$$

$$Q_3(x, y, t) = -\frac{2a_1 b_1 \beta_1 \Omega_2^2}{\Omega_1 + \beta_2 (\xi \Omega_2)^2}.$$

$$\begin{aligned} P(x, y, t) &= \pm \sqrt{\frac{a_1^2 \beta_1 \beta_3 (4\mu_2 - \mu_1^2)}{2\beta_2 \beta_4}} \cot\left(\frac{1}{2} \sqrt{4\mu_2 - \mu_1^2} \xi\right) e^{i(a_2 x + b_2 y + \frac{c_2}{\rho} \varphi)}, \\ Q(x, y, t) &= -\frac{a_1 b_1 \beta_1 (4\mu_2 - \mu_1^2)}{2\beta_2} \cot^2\left(\frac{1}{2} \sqrt{4\mu_2 - \mu_1^2} \xi\right). \end{aligned} \quad (25)$$

Remark 1. A special case when $\Omega_1 \neq 0$ and $\Omega_2 = 0$, in Eq. 19 reveals the dark soliton solutions of Eq. 5 as

The GREM method

As regards to the GREM method [68,69], we presume that Eq. 5 solution is described by

$$P(\xi) = k_0 + \sum_{i=1}^{\varpi} k_i \mathcal{R}^i(\xi), \quad k_{\varpi} \neq 0. \quad (26)$$

where constants k_i , $i = 0, 1, 2, \dots, \varpi$ and μ_i , $i = 1, 2$ can parameters to be evaluated later. The function $\mathcal{R}(\xi)$ satisfies the generalized Riccati equation defined by

$$\mathcal{R}'(\xi) = \delta_0 + \delta_1 \mathcal{R}(\xi) + \delta_2 \mathcal{R}^2(\xi), \quad \delta_2 \neq 0, \quad (27)$$

where δ_0 , δ_1 and δ_2 are constants. The resulting solutions of Eq. 27 are categorized as follows:

Set 1. For $\varrho = \delta_1^2 - 4\delta_0\delta_2 > 0$, $\delta_1\delta_2 \neq 0$ (or $\delta_0\delta_2 \neq 0$) and non-zero M and N are real constants:

$$\mathcal{R}_1(\xi) = -\frac{1}{2\delta_2} \left(\delta_1 + \sqrt{\varrho} \tanh \left(\frac{\sqrt{\varrho}}{2} \xi \right) \right),$$

$$\mathcal{R}_2(\xi) = -\frac{1}{2\delta_2} \left(\delta_1 + \sqrt{\varrho} \coth \left(\frac{\sqrt{\varrho}}{2} \xi \right) \right),$$

$$\mathcal{R}_3(\xi) = -\frac{1}{2\delta_2} \left(\delta_1 + \sqrt{\varrho} \left(\tanh(\sqrt{\varrho} \xi) \pm i \operatorname{sech}(\sqrt{\varrho} \xi) \right) \right),$$

$$\mathcal{R}_4(\xi) = -\frac{1}{2\delta_2} \left(\delta_1 + \sqrt{\varrho} \left(\coth(\sqrt{\varrho} \xi) \pm \operatorname{csch}(\sqrt{\varrho} \xi) \right) \right),$$

$$\mathcal{R}_5(\xi) = -\frac{1}{4\delta_2} \left(2\delta_1 + \sqrt{\varrho} \left(\tanh \left(\frac{\sqrt{\varrho}}{4} \xi \right) + \coth \left(\frac{\sqrt{\varrho}}{4} \xi \right) \right) \right),$$

$$\mathcal{R}_6(\xi) = \frac{1}{2\delta_2} \left(-\delta_1 + \frac{\sqrt{\varrho(M^2 + N^2)} - M\sqrt{\varrho} \cosh(\sqrt{\varrho} \xi)}{M \sinh(\sqrt{\varrho} \xi) + N} \right),$$

$$\mathcal{R}_7(\xi) = \frac{1}{2\delta_2} \left(-\delta_1 - \frac{\sqrt{\varrho(M^2 + N^2)} + M\sqrt{\varrho} \cosh(\sqrt{\varrho} \xi)}{M \sinh(\sqrt{\varrho} \xi) + N} \right),$$

$$\mathcal{R}_8(\xi) = \frac{2\delta_0 \cosh \left(\frac{\sqrt{\varrho}}{2} \xi \right)}{\sqrt{\varrho} \sinh \left(\frac{\sqrt{\varrho}}{2} \xi \right) - \delta_1 \cosh \left(\frac{\sqrt{\varrho}}{2} \xi \right)},$$

$$\mathcal{R}_9(\xi) = \frac{-2\delta_0 \sinh \left(\frac{\sqrt{\varrho}}{2} \xi \right)}{\delta_1 \sinh \left(\frac{\sqrt{\varrho}}{2} \xi \right) - \sqrt{\varrho} \cosh \left(\frac{\sqrt{\varrho}}{2} \xi \right)},$$

$$\mathcal{R}_{10}(\xi) = \frac{2\delta_0 \cosh(\sqrt{\varrho} \xi)}{\sqrt{\varrho} \sinh(\sqrt{\varrho} \xi) - \delta_1 \cosh(\sqrt{\varrho} \xi) \pm i\sqrt{\varrho}},$$

$$\mathcal{R}_{11}(\xi) = \frac{2\delta_0 \sinh(\sqrt{\varrho} \xi)}{\sqrt{\varrho} \cosh(\sqrt{\varrho} \xi) - \delta_1 \sinh(\sqrt{\varrho} \xi) \pm \sqrt{\varrho}},$$

$$\mathcal{R}_{12}(\xi) = \frac{4\delta_0 \sinh \left(\frac{\sqrt{\varrho}}{4} \xi \right) \cosh \left(\frac{\sqrt{\varrho}}{4} \xi \right)}{2\sqrt{\varrho} \cosh^2 \left(\frac{\sqrt{\varrho}}{4} \xi \right) - 2\delta_1 \sinh \left(\frac{\sqrt{\varrho}}{4} \xi \right) \cosh \left(\frac{\sqrt{\varrho}}{4} \xi \right) - \sqrt{\varrho}}.$$

Set 2. For $\varrho = \delta_1^2 - 4\delta_0\delta_2 < 0$, $\delta_1\delta_2 \neq 0$ (or $\delta_0\delta_2 \neq 0$), M and N are non-zero real constants that satisfy $M^2 - N^2 > 0$:

$$\mathcal{R}_{13}(\xi) = \frac{1}{2\delta_2} \left(-\delta_1 + \sqrt{-\varrho} \tan \left(\frac{\sqrt{-\varrho}}{2} \xi \right) \right),$$

$$\mathcal{R}_{14}(\xi) = -\frac{1}{2\delta_2} \left(\delta_1 + \sqrt{-\varrho} \cot \left(\frac{\sqrt{-\varrho}}{2} \xi \right) \right),$$

$$\mathcal{R}_{15}(\xi) = \frac{1}{2\delta_2} \left(-\delta_1 + \sqrt{-\varrho} (\tan(\sqrt{-\varrho} \xi) \pm \sec(\sqrt{-\varrho} \xi)) \right),$$

$$\mathcal{R}_{16}(\xi) = -\frac{1}{2\delta_2} (\delta_1 + \sqrt{-\varrho} (\cot(\sqrt{-\varrho} \xi) \pm \csc(\sqrt{-\varrho} \xi))),$$

$$\mathcal{R}_{17}(\xi) = \frac{1}{4\delta_2} \left(-2\delta_1 + \sqrt{-\varrho} \left(\tan \left(\frac{\sqrt{-\varrho}}{4} \xi \right) - \cot \left(\frac{\sqrt{-\varrho}}{4} \xi \right) \right) \right),$$

$$\mathcal{R}_{18}(\xi) = \frac{1}{2\delta_2} \left(-\delta_1 + \frac{\pm \sqrt{\varrho(N^2 - M^2)} \pm M\sqrt{-\varrho} \cos(\sqrt{-\varrho} \xi)}{M \sin(\sqrt{-\varrho} \xi) + N} \right),$$

$$\mathcal{R}_{19}(\xi) = -\frac{2\delta_0 \cos \left(\frac{\sqrt{-\varrho}}{2} \xi \right)}{\sqrt{-\varrho} \sin \left(\frac{\sqrt{-\varrho}}{2} \xi \right) + \delta_1 \cos \left(\frac{\sqrt{-\varrho}}{2} \xi \right)},$$

$$\mathcal{R}_{20}(\xi) = \frac{2\delta_0 \sin \left(\frac{\sqrt{-\varrho}}{2} \xi \right)}{\sqrt{-\varrho} \cos \left(\frac{\sqrt{-\varrho}}{2} \xi \right) - \delta_1 \sin \left(\frac{\sqrt{-\varrho}}{2} \xi \right)},$$

$$\mathcal{R}_{21}(\xi) = -\frac{2\delta_0 \cos(\sqrt{-\varrho} \xi)}{\sqrt{-\varrho} \sin(\sqrt{-\varrho} \xi) + \delta_1 \cos(\sqrt{-\varrho} \xi) \pm \sqrt{-\varrho}},$$

$$\mathcal{R}_{22}(\xi) = -\frac{2\delta_0 \sin(\sqrt{-\varrho} \xi)}{\sqrt{-\varrho} \cos(\sqrt{-\varrho} \xi) + \delta_1 \sin(\sqrt{-\varrho} \xi) \pm \sqrt{-\varrho}},$$

$$\mathcal{R}_{23}(\xi) = \frac{4\delta_0 \sin \left(\frac{\sqrt{-\varrho}}{4} \xi \right) \cos \left(\frac{\sqrt{-\varrho}}{4} \xi \right)}{2\sqrt{-\varrho} \cos^2 \left(\frac{\sqrt{-\varrho}}{4} \xi \right) - 2\delta_1 \sin \left(\frac{\sqrt{-\varrho}}{4} \xi \right) \cos \left(\frac{\sqrt{-\varrho}}{4} \xi \right) - \sqrt{-\varrho}}.$$

Set 3. For $\delta_0 = 0$ and $\delta_1\delta_2 \neq 0$:

$$\mathcal{R}_{24}(\xi) = -\frac{\delta_1 \xi_0}{\delta_2 (\xi_0 + \cosh(\delta_1 \xi) - \sinh(\delta_1 \xi))},$$

$$\mathcal{R}_{25}(\xi) = -\frac{\delta_1 (\cosh(\delta_1 \xi) - \sinh(\delta_1 \xi))}{\delta_2 (\xi_0 + \cosh(\delta_1 \xi) - \sinh(\delta_1 \xi))}.$$

$$\mathcal{R}_{26}(\xi) = -\frac{\delta_1 (\cosh(\delta_1 \xi) + \sinh(\delta_1 \xi))}{\delta_2 (\xi_0 + \cosh(\delta_1 \xi) + \sinh(\delta_1 \xi))}.$$

Set 4. For $\delta_2 \neq 0$ and $\delta_1 = \delta_0 = 0$:

$$\mathcal{R}_{27}(\xi) = -\frac{1}{\delta_2 \xi + \xi_0},$$

where ξ_0 is an arbitrary constant. Since $\varpi = 1$, from Eq. 26 we have

$$P(\xi) = k_0 + k_1 \mathcal{R}(\xi), \quad k_1 \neq 0. \quad (28)$$

Inserting Eqs. 27 and 28 into Eq. 12 with the coefficients of $\mathcal{R}^i(\xi)$, $i = 0, 1, 2, 3$, to zero, we obtain

$$\begin{aligned}
\mathcal{R}^0(\xi) : & k_0(-a_2 b_2 \beta_1 - c_2) + a_1 b_1 \beta_1 \delta_0 \delta_1 k_1 - \frac{b_1 \beta_2 \beta_4 k_0^3}{a_1 \beta_3} = 0, \\
\mathcal{R}^1(\xi) : & k_1(-a_2 b_2 \beta_1 - c_2) + a_1 b_1 \beta_1 \delta_1^2 k_1 + 2 a_1 b_1 \beta_1 \delta_0 \delta_2 k_1 - \frac{3 b_1 \beta_2 \beta_4 k_1 k_0^2}{a_1 \beta_3} = 0, \\
\mathcal{R}^2(\xi) : & 3 a_1 b_1 \beta_1 \delta_1 \delta_2 k_1 - \frac{3 b_1 \beta_2 \beta_4 k_0 k_1^2}{a_1 \beta_3} = 0, \\
\mathcal{R}^3(\xi) : & 2 a_1 b_1 \beta_1 \delta_2^2 k_1 - \frac{b_1 \beta_2 \beta_4 k_1^3}{a_1 \beta_3} = 0.
\end{aligned} \tag{29}$$

Considering Eq. 11 and the solution of the above equations, we have the following cases:

Case 1.

$$\begin{aligned}
c_1 = & -\beta_1(a_1 b_2 + a_2 b_1), \quad c_2 = -\frac{1}{2}\beta_1(a_1 b_1 \delta_1^2 - 4 a_1 b_1 \delta_0 \delta_2 + 2 a_2 b_2), \quad k_0 \\
= & -a_1 \delta_1 \sqrt{\frac{\beta_1 \beta_3}{2 \beta_2 \beta_4}}, \quad k_1 = -a_1 \delta_2 \sqrt{\frac{2 \beta_1 \beta_3}{\beta_2 \beta_4}}.
\end{aligned} \tag{30}$$

According to the above values with the use of Eq. 28 and the already defined solutions sets, we have the following results:

For $\varrho = \delta_1^2 - 4 \delta_0 \delta_2 > 0$ and $\delta_1 \delta_2 \neq 0$ (or $\delta_0 \delta_2 \neq 0$):

$$\begin{aligned}
P_4(x, y, t) &= \sqrt{\frac{a_1^2 \beta_1 \beta_3 \varrho}{2 \beta_2 \beta_4}} \tanh\left(\frac{\sqrt{\varrho}}{2} \xi\right) e^{i\left(a_2 x + b_2 y + \frac{c_2 \varrho^0}{\rho}\right)}, \\
Q_4(x, y, t) &= -\frac{a_1 b_1 \beta_1 \varrho}{2 \beta_2} \tanh^2\left(\frac{\sqrt{\varrho}}{2} \xi\right),
\end{aligned} \tag{31}$$

$$\begin{aligned}
P_5(x, y, t) &= \sqrt{\frac{a_1^2 \beta_1 \beta_3 \varrho}{2 \beta_2 \beta_4}} \coth\left(\frac{\sqrt{\varrho}}{2} \xi\right) e^{i\left(a_2 x + b_2 y + \frac{c_2 \varrho^0}{\rho}\right)}, \\
Q_5(x, y, t) &= -\frac{a_1 b_1 \beta_1 \varrho}{2 \beta_2} \coth^2\left(\frac{\sqrt{\varrho}}{2} \xi\right),
\end{aligned} \tag{32}$$

$$\begin{aligned}
P_6(x, y, t) &= \sqrt{\frac{a_1^2 \beta_1 \beta_3 \varrho}{2 \beta_2 \beta_4}} \left(\tanh(\sqrt{\varrho} \xi) \pm i \operatorname{sech}(\sqrt{\varrho} \xi) \right) e^{i\left(a_2 x + b_2 y + \frac{c_2 \varrho^0}{\rho}\right)}, \\
Q_6(x, y, t) &= -\frac{a_1 b_1 \beta_1 \varrho}{2 \beta_2} \left(\tanh(\sqrt{\varrho} \xi) \pm i \operatorname{sech}(\sqrt{\varrho} \xi) \right)^2,
\end{aligned} \tag{33}$$

$$\begin{aligned}
P_7(x, y, t) &= \sqrt{\frac{a_1^2 \beta_1 \beta_3 \varrho}{2 \beta_2 \beta_4}} \left(\coth(\sqrt{\varrho} \xi) \pm \operatorname{csch}(\sqrt{\varrho} \xi) \right) e^{i\left(a_2 x + b_2 y + \frac{c_2 \varrho^0}{\rho}\right)}, \\
Q_7(x, y, t) &= -\frac{a_1 b_1 \beta_1 \varrho}{2 \beta_2} \left(\coth(\sqrt{\varrho} \xi) \pm \operatorname{csch}(\sqrt{\varrho} \xi) \right)^2,
\end{aligned} \tag{34}$$

$$\begin{aligned}
P_8(x, y, t) &= \frac{1}{2} \sqrt{\frac{a_1^2 \beta_1 \beta_3 \varrho}{2 \beta_2 \beta_4}} \left(\tanh\left(\frac{\sqrt{\varrho}}{4} \xi\right) + \coth\left(\frac{\sqrt{\varrho}}{4} \xi\right) \right) e^{i\left(a_2 x + b_2 y + \frac{c_2 \varrho^0}{\rho}\right)}, \\
Q_8(x, y, t) &= -\frac{a_1 b_1 \beta_1 \varrho}{8 \beta_2} \left(\tanh\left(\frac{\sqrt{\varrho}}{4} \xi\right) + \coth\left(\frac{\sqrt{\varrho}}{4} \xi\right) \right)^2,
\end{aligned} \tag{35}$$

$$\begin{aligned}
P_9(x, y, t) &= -\sqrt{\frac{a_1^2 \beta_1 \beta_3}{2 \beta_2 \beta_4}} \left(\frac{\sqrt{\varrho(M^2 + N^2)} - M \sqrt{\varrho} \cosh(\sqrt{\varrho} \xi)}{M \sinh(\sqrt{\varrho} \xi) + N} \right) e^{i\left(a_2 x + b_2 y + \frac{c_2 \varrho^0}{\rho}\right)}, \\
Q_9(x, y, t) &= -\frac{a_1 b_1 \beta_1}{2 \beta_2} \left(\frac{\sqrt{\varrho(M^2 + N^2)} - M \sqrt{\varrho} \cosh(\sqrt{\varrho} \xi)}{M \sinh(\sqrt{\varrho} \xi) + N} \right)^2,
\end{aligned} \tag{36}$$

$$\begin{aligned}
P_{10}(x, y, t) &= \sqrt{\frac{a_1^2 \beta_1 \beta_3}{2 \beta_2 \beta_4}} \left(\frac{\sqrt{\varrho(M^2 + N^2)} + M \sqrt{\varrho} \cosh(\sqrt{\varrho} \xi)}{M \sinh(\sqrt{\varrho} \xi) + N} \right) e^{i\left(a_2 x + b_2 y + \frac{c_2 \varrho^0}{\rho}\right)}, \\
Q_{10}(x, y, t) &= -\frac{a_1 b_1 \beta_1}{2 \beta_2} \left(\frac{\sqrt{\varrho(M^2 + N^2)} + M \sqrt{\varrho} \cosh(\sqrt{\varrho} \xi)}{M \sinh(\sqrt{\varrho} \xi) + N} \right)^2,
\end{aligned} \tag{37}$$

$$P_{11}(x, y, t) = -\sqrt{\frac{a_1^2 \delta_1^2 \beta_1 \beta_3}{2 \beta_2 \beta_4}} \left(1 + \frac{4 \delta_0 \delta_2 \cosh\left(\frac{\sqrt{\varrho}}{2} \xi\right)}{\delta_1 \left(\sqrt{\varrho} \sinh\left(\frac{\sqrt{\varrho}}{2} \xi\right) - \delta_1 \cosh\left(\frac{\sqrt{\varrho}}{2} \xi\right) \right)} \right) e^{i\left(a_2 x + b_2 y + \frac{c_2 \varrho^0}{\rho}\right)}, \tag{38}$$

$$Q_{11}(x, y, t) = -\frac{a_1 b_1 \beta_1 \delta_1^2}{2 \beta_2} \left(1 + \frac{4 \delta_0 \delta_2 \cosh\left(\frac{\sqrt{\varrho}}{2} \xi\right)}{\delta_1 \left(\sqrt{\varrho} \sinh\left(\frac{\sqrt{\varrho}}{2} \xi\right) - \delta_1 \cosh\left(\frac{\sqrt{\varrho}}{2} \xi\right) \right)} \right)^2,$$

$$P_{12}(x, y, t) = -\sqrt{\frac{a_1^2 \delta_1^2 \beta_1 \beta_3}{2 \beta_2 \beta_4}} \left(1 - \frac{4 \delta_0 \delta_2 \sinh\left(\frac{\sqrt{\varrho}}{2} \xi\right)}{\delta_1 \left(\delta_1 \sinh\left(\frac{\sqrt{\varrho}}{2} \xi\right) - \sqrt{\varrho} \cosh\left(\frac{\sqrt{\varrho}}{2} \xi\right) \right)} \right) e^{i\left(a_2 x + b_2 y + \frac{c_2 \varrho^0}{\rho}\right)}, \tag{39}$$

$$Q_{12}(x, y, t) = -\frac{a_1 b_1 \beta_1 \delta_1^2}{2 \beta_2} \left(1 - \frac{4 \delta_0 \delta_2 \sinh\left(\frac{\sqrt{\varrho}}{2} \xi\right)}{\delta_1 \left(\delta_1 \sinh\left(\frac{\sqrt{\varrho}}{2} \xi\right) - \sqrt{\varrho} \cosh\left(\frac{\sqrt{\varrho}}{2} \xi\right) \right)} \right)^2,$$

$$P_{13}(x, y, t) = -\sqrt{\frac{a_1^2 \delta_1^2 \beta_1 \beta_3}{2 \beta_2 \beta_4}} \left(1 + \frac{4 \delta_0 \delta_2 \cosh(\sqrt{\varrho} \xi)}{\delta_1 (\sqrt{\varrho} \sinh(\sqrt{\varrho} \xi) - \delta_1 \cosh(\sqrt{\varrho} \xi) \pm i \sqrt{\varrho})} \right) e^{i(a_2 x + b_2 y + \frac{c_2}{\rho} \varrho)}, \quad (40)$$

$$Q_{13}(x, y, t) = -\frac{a_1 b_1 \beta_1 \delta_1^2}{2 \beta_2} \left(1 + \frac{4 \delta_0 \delta_2 \cosh(\sqrt{\varrho} \xi)}{\delta_1 (\sqrt{\varrho} \sinh(\sqrt{\varrho} \xi) - \delta_1 \cosh(\sqrt{\varrho} \xi) \pm i \sqrt{\varrho})} \right)^2, \quad (41)$$

$$P_{14}(x, y, t) = -\sqrt{\frac{a_1^2 \delta_1^2 \beta_1 \beta_3}{2 \beta_2 \beta_4}} \left(1 + \frac{4 \delta_0 \delta_2 \sinh(\sqrt{\varrho} \xi)}{\delta_1 (\sqrt{\varrho} \cosh(\sqrt{\varrho} \xi) - \delta_1 \sinh(\sqrt{\varrho} \xi) \pm \sqrt{\varrho})} \right) e^{i(a_2 x + b_2 y + \frac{c_2}{\rho} \varrho)}, \quad (41)$$

$$Q_{14}(x, y, t) = -\frac{a_1 b_1 \beta_1 \delta_1^2}{2 \beta_2} \left(1 + \frac{4 \delta_0 \delta_2 \sinh(\sqrt{\varrho} \xi)}{\delta_1 (\sqrt{\varrho} \cosh(\sqrt{\varrho} \xi) - \delta_1 \sinh(\sqrt{\varrho} \xi) \pm \sqrt{\varrho})} \right)^2, \quad (41)$$

$$P_{15}(x, y, t) = -\sqrt{\frac{a_1^2 \delta_1^2 \beta_1 \beta_3}{2 \beta_2 \beta_4}} \left(1 + \frac{8 \delta_0 \delta_2 \sinh\left(\frac{\sqrt{\varrho}}{4} \xi\right) \cosh\left(\frac{\sqrt{\varrho}}{4} \xi\right)}{\delta_1 \left(2 \sqrt{\varrho} \cosh^2\left(\frac{\sqrt{\varrho}}{4} \xi\right) - 2 \delta_1 \sinh\left(\frac{\sqrt{\varrho}}{4} \xi\right) \cosh\left(\frac{\sqrt{\varrho}}{4} \xi\right) - \sqrt{\varrho} \right)} \right) e^{i(a_2 x + b_2 y + \frac{c_2}{\rho} \varrho)}, \quad (42)$$

$$Q_{15}(x, y, t) = -\frac{a_1 b_1 \beta_1 \delta_1^2}{2 \beta_2} \left(1 + \frac{8 \delta_0 \delta_2 \sinh\left(\frac{\sqrt{\varrho}}{4} \xi\right) \cosh\left(\frac{\sqrt{\varrho}}{4} \xi\right)}{\delta_1 \left(2 \sqrt{\varrho} \cosh^2\left(\frac{\sqrt{\varrho}}{4} \xi\right) - 2 \delta_1 \sinh\left(\frac{\sqrt{\varrho}}{4} \xi\right) \cosh\left(\frac{\sqrt{\varrho}}{4} \xi\right) - \sqrt{\varrho} \right)} \right)^2. \quad (42)$$

For $\varrho = \delta_1^2 - 4 \delta_0 \delta_2 < 0$ and $\delta_1 \delta_2 \neq 0$ (or $\delta_0 \delta_2 \neq 0$):

$$P_{16}(x, y, t) = -\sqrt{\frac{a_1^2 \beta_1 \beta_3 \varrho}{2 \beta_2 \beta_4}} \tan\left(\frac{\sqrt{-\varrho}}{2} \xi\right) e^{i(a_2 x + b_2 y + \frac{c_2}{\rho} \varrho)}, \quad (43)$$

$$Q_{16}(x, y, t) = \frac{a_1 b_1 \beta_1 \varrho}{2 \beta_2} \tan^2\left(\frac{\sqrt{-\varrho}}{2} \xi\right), \quad (43)$$

$$P_{17}(x, y, t) = -\sqrt{\frac{a_1^2 \beta_1 \beta_3 \varrho}{2 \beta_2 \beta_4}} \cot\left(\frac{\sqrt{-\varrho}}{2} \xi\right) e^{i(a_2 x + b_2 y + \frac{c_2}{\rho} \varrho)}, \quad (44)$$

$$Q_{17}(x, y, t) = \frac{a_1 b_1 \beta_1 \varrho}{2 \beta_2} \cot^2\left(\frac{\sqrt{-\varrho}}{2} \xi\right)^2, \quad (44)$$

$$P_{19}(x, y, t) = \sqrt{-\frac{a_1^2 \beta_1 \beta_3 \varrho}{2 \beta_2 \beta_4}} (\cot(\sqrt{-\varrho} \xi) \pm \csc(\sqrt{-\varrho} \xi)) e^{i(a_2 x + b_2 y + \frac{c_2}{\rho} \varrho)}, \quad (46)$$

$$Q_{19}(x, y, t) = \frac{a_1 b_1 \beta_1 \varrho}{2 \beta_2} (\cot(\sqrt{-\varrho} \xi) \pm \csc(\sqrt{-\varrho} \xi))^2, \quad (46)$$

$$P_{20}(x, y, t) = -\sqrt{-\frac{a_1^2 \beta_1 \beta_3 \varrho}{2 \beta_2 \beta_4}} \left(\tan\left(\frac{\sqrt{-\varrho}}{4} \xi\right) - \cot\left(\frac{\sqrt{-\varrho}}{4} \xi\right) \right) e^{i(a_2 x + b_2 y + \frac{c_2}{\rho} \varrho)}, \quad (47)$$

$$Q_{20}(x, y, t) = \frac{a_1 b_1 \beta_1 \varrho}{8 \beta_2} \left(\tan\left(\frac{\sqrt{-\varrho}}{4} \xi\right) - \cot\left(\frac{\sqrt{-\varrho}}{4} \xi\right) \right)^2, \quad (47)$$

$$P_{21}(x, y, t) = -\sqrt{\frac{a_1^2 \beta_1 \beta_3}{2 \beta_2 \beta_4}} \left(\frac{\pm \sqrt{\varrho(N^2 - M^2)} \pm M \sqrt{-\varrho} \cos(\sqrt{-\varrho} \xi)}{M \sin(\sqrt{-\varrho} \xi) + N} \right) e^{i(a_2 x + b_2 y + \frac{c_2}{\rho} \varrho)}, \quad (48)$$

$$Q_{21}(x, y, t) = -\frac{a_1 b_1 \beta_1}{2 \beta_2} \left(\frac{\pm \sqrt{\varrho(N^2 - M^2)} \pm P \sqrt{-\varrho} \cos(\sqrt{-\varrho} \xi)}{M \sin(\sqrt{-\varrho} \xi) + N} \right)^2, \quad (48)$$

$$P_{18}(x, y, t) = -\sqrt{\frac{a_1^2 \beta_1 \beta_3 \varrho}{2 \beta_2 \beta_4}} (\tan(\sqrt{-\varrho} \xi) \pm \sec(\sqrt{-\varrho} \xi)) e^{i(a_2 x + b_2 y + \frac{c_2}{\rho} \varrho)}, \quad (45)$$

$$Q_{18}(x, y, t) = \frac{a_1 b_1 \beta_1 \varrho}{2 \beta_2} (\tan(\sqrt{-\varrho} \xi) \pm \sec(\sqrt{-\varrho} \xi))^2, \quad (45)$$

$$P_{22}(x, y, t) = -\sqrt{\frac{a_1^2 \delta_1^2 \beta_1 \beta_3}{2 \beta_2 \beta_4}} \left(1 - \frac{4 \delta_0 \delta_2 \cos\left(\frac{\sqrt{-Q}}{2} \xi\right)}{\delta_1 \left(\sqrt{-Q} \sin\left(\frac{\sqrt{-Q}}{2} \xi\right) + \delta_1 \cos\left(\frac{\sqrt{-Q}}{2} \xi\right) \right)} \right) e^{i \left(a_2 x + b_2 y + \frac{c_2}{\rho} t^\rho \right)}, \quad (49)$$

$$Q_{22}(x, y, t) = -\frac{a_1 b_1 \beta_1 \delta_1^2}{2 \beta_2} \left(1 - \frac{4 \delta_0 \delta_2 \cos\left(\frac{\sqrt{-Q}}{2} \xi\right)}{\delta_1 \left(\sqrt{-Q} \sin\left(\frac{\sqrt{-Q}}{2} \xi\right) + \delta_1 \cos\left(\frac{\sqrt{-Q}}{2} \xi\right) \right)} \right)^2,$$

$$P_{23}(x, y, t) = -\sqrt{\frac{a_1^2 \delta_1^2 \beta_1 \beta_3}{2 \beta_2 \beta_4}} \left(1 + \frac{4 \delta_0 \delta_2 \sin\left(\frac{\sqrt{-Q}}{2} \xi\right)}{\delta_1 \left(\sqrt{-Q} \cos\left(\frac{\sqrt{-Q}}{2} \xi\right) - \delta_1 \sin\left(\frac{\sqrt{-Q}}{2} \xi\right) \right)} \right) e^{i \left(a_2 x + b_2 y + \frac{c_2}{\rho} t^\rho \right)}, \quad (50)$$

$$Q_{23}(x, y, t) = -\frac{a_1 b_1 \beta_1 \delta_1^2}{2 \beta_2} \left(1 + \frac{4 \delta_0 \delta_2 \sin\left(\frac{\sqrt{-Q}}{2} \xi\right)}{\delta_1 \left(\sqrt{-Q} \cos\left(\frac{\sqrt{-Q}}{2} \xi\right) - \delta_1 \sin\left(\frac{\sqrt{-Q}}{2} \xi\right) \right)} \right)^2,$$

$$P_{24}(x, y, t) = -\sqrt{\frac{a_1^2 \delta_1^2 \beta_1 \beta_3}{2 \beta_2 \beta_4}} \left(1 - \frac{4 \delta_0 \delta_2 \cos(\sqrt{-Q} \xi)}{\delta_1 (\sqrt{-Q} \sin(\sqrt{-Q} \xi) + \delta_1 \cos(\sqrt{-Q} \xi) \pm \sqrt{-Q})} \right) e^{i \left(a_2 x + b_2 y + \frac{c_2}{\rho} t^\rho \right)}, \quad (51)$$

$$Q_{24}(x, y, t) = -\frac{a_1 b_1 \beta_1 \delta_1^2}{2 \beta_2} \left(1 - \frac{4 \delta_0 \delta_2 \cos(\sqrt{-Q} \xi)}{\delta_1 (\sqrt{-Q} \sin(\sqrt{-Q} \xi) + \delta_1 \cos(\sqrt{-Q} \xi) \pm \sqrt{-Q})} \right)^2,$$

$$P_{25}(x, y, t) = -\sqrt{\frac{a_1^2 \delta_1^2 \beta_1 \beta_3}{2 \beta_2 \beta_4}} \left(1 - \frac{4 \delta_0 \delta_2 \sin(\sqrt{-Q} \xi)}{\delta_1 (\sqrt{-Q} \cos(\sqrt{-Q} \xi) + \delta_1 \sin(\sqrt{-Q} \xi) \pm \sqrt{-Q})} \right) e^{i \left(a_2 x + b_2 y + \frac{c_2}{\rho} t^\rho \right)}, \quad (52)$$

$$Q_{25}(x, y, t) = -\frac{a_1 b_1 \beta_1 \delta_1^2}{2 \beta_2} \left(1 - \frac{4 \delta_0 \delta_2 \sin(\sqrt{-Q} \xi)}{\delta_1 (\sqrt{-Q} \cos(\sqrt{-Q} \xi) + \delta_1 \sin(\sqrt{-Q} \xi) \pm \sqrt{-Q})} \right)^2,$$

$$P_{26}(x, y, t) = -\sqrt{\frac{a_1^2 \delta_1^2 \beta_1 \beta_3}{2 \beta_2 \beta_4}} \left(1 + \frac{8 \delta_0 \delta_2 \sin\left(\frac{\sqrt{-Q}}{4} \xi\right) \cos\left(\frac{\sqrt{-Q}}{4} \xi\right)}{\delta_1 \left(2 \sqrt{-Q} \cos^2\left(\frac{\sqrt{-Q}}{4} \xi\right) - 2 \delta_1 \sin\left(\frac{\sqrt{-Q}}{4} \xi\right) \cos\left(\frac{\sqrt{-Q}}{4} \xi\right) - \sqrt{-Q} \right)} \right) e^{i \left(a_2 x + b_2 y + \frac{c_2}{\rho} t^\rho \right)}, \quad (53)$$

$$Q_{26}(x, y, t) = -\frac{a_1 b_1 \beta_1 \delta_1^2}{2 \beta_2} \left(1 + \frac{8 \delta_0 \delta_2 \sin\left(\frac{\sqrt{-Q}}{4} \xi\right) \cos\left(\frac{\sqrt{-Q}}{4} \xi\right)}{\delta_1 \left(2 \sqrt{-Q} \cos^2\left(\frac{\sqrt{-Q}}{4} \xi\right) - 2 \delta_1 \sin\left(\frac{\sqrt{-Q}}{4} \xi\right) \cos\left(\frac{\sqrt{-Q}}{4} \xi\right) - \sqrt{-Q} \right)} \right)^2,$$

For $\delta_0 = 0$ and $\delta_1 \delta_2 \neq 0$:

$$P_{27}(x, y, t) = -\sqrt{\frac{a_1^2 \delta_1^2 \beta_1 \beta_3}{2 \beta_2 \beta_4}} \left(1 - \frac{2 \xi_0}{(\xi_0 + \cosh(\delta_1 \xi) \pm \sinh(\delta_1 \xi))} \right) e^{i \left(a_2 x + b_2 y + \frac{c_2}{\rho} t^\rho \right)},$$

$$Q_{27}(x, y, t) = -\frac{a_1 b_1 \beta_1 \delta_1^2}{2 \beta_2} \left(1 - \frac{2 \xi_0}{(\xi_0 + \cosh(\delta_1 \xi) \pm \sinh(\delta_1 \xi))} \right)^2, \quad (54)$$

$$P_{28}(x, y, t) = -\sqrt{\frac{a_1^2 \delta_1^2 \beta_1 \beta_3}{2 \beta_2 \beta_4}} \left(1 - \frac{2(\cosh(\delta_1 \xi) - \sinh(\delta_1 \xi))}{(\xi_0 + \cosh(\delta_1 \xi) - \sinh(\delta_1 \xi))} \right) e^{i \left(a_2 x + b_2 y + \frac{c_2}{\rho} t^\rho \right)},$$

$$Q_{28}(x, y, t) = -\frac{a_1 b_1 \beta_1 \delta_1^2}{2 \beta_2} \left(1 - \frac{2(\cosh(\delta_1 \xi) - \sinh(\delta_1 \xi))}{(\xi_0 + \cosh(\delta_1 \xi) - \sinh(\delta_1 \xi))} \right)^2, \quad (55)$$

$$P_{29}(x, y, t) = -\sqrt{\frac{a_1^2 \delta_1^2 \beta_1 \beta_3}{2 \beta_2 \beta_4}} \left(1 - \frac{2(\cosh(\delta_1 \xi) + \sinh(\delta_1 \xi))}{(\xi_0 + \cosh(\delta_1 \xi) + \sinh(\delta_1 \xi))} \right) e^{i \left(a_2 x + b_2 y + \frac{c_2}{\rho} t^\rho \right)},$$

$$Q_{29}(x, y, t) = -\frac{a_1 b_1 \beta_1 \delta_1^2}{2 \beta_2} \left(1 - \frac{2(\cosh(\delta_1 \xi) + \sinh(\delta_1 \xi))}{(\xi_0 + \cosh(\delta_1 \xi) + \sinh(\delta_1 \xi))} \right)^2, \quad (56)$$

For $\delta_2 \neq 0$ and $\delta_1 = \delta_0 = 0$:

$$P_{30}(x, y, t) = \frac{a_1 \delta_2 \sqrt{2 \beta_1 \beta_3}}{\sqrt{\beta_2 \beta_4} (\delta_2 \xi + \xi_0)} e^{i \left(a_2 x + b_2 y + \frac{c_2}{\rho} t^\rho \right)}, \quad (57)$$

$$Q_{30}(x, y, t) = -\frac{2 a_1 b_1 \beta_1 \delta_2^2}{\beta_2 (\delta_2 \xi + \xi_0)^2},$$

where $\xi = a_1 x + b_1 y + \frac{c_1}{\rho} t^\rho$, $c_2 = -\frac{1}{2} \beta_1 (a_1 b_1 \delta_1^2 - 4 a_1 b_1 \delta_0 \delta_2 + 2 a_2 b_2)$, and ξ_0 is arbitrary constant.

Case 2. $c_1 = -\beta_1(a_1b_2 + a_2b_1)$, $c_2 = -\frac{1}{2}\beta_1(a_1b_1\delta_1^2 - 4a_1b_1\delta_0\delta_2 + 2a_2b_2)$, $k_0 = a_1\delta_1\sqrt{\frac{\beta_1\beta_3}{2\beta_2\beta_4}}$, $k_1 = a_1\delta_2\sqrt{\frac{2\beta_1\beta_3}{\beta_2\beta_4}}$.

$$(58)$$

According to the above values with the use of Eq. 28 and the already defined solutions sets, we have the following results:

For $\varrho = \delta_1^2 - 4\delta_0\delta_2 > 0$ and $\delta_1\delta_2 \neq 0$ (or $\delta_0\delta_2 \neq 0$):

$$P_{31}(x, y, t) = -\sqrt{\frac{a_1^2\beta_1\beta_3\varrho}{2\beta_2\beta_4}} \tanh\left(\frac{\sqrt{\varrho}}{2}\xi\right) e^{i\left(a_2x+b_2y+\frac{c_2\varrho}{\rho}t\right)},$$

$$(59)$$

$$Q_{31}(x, y, t) = -\frac{a_1b_1\beta_1\varrho}{2\beta_2} \tanh^2\left(\frac{\sqrt{\varrho}}{2}\xi\right),$$

$$P_{32}(x, y, t) = -\sqrt{\frac{a_1^2\beta_1\beta_3\varrho}{2\beta_2\beta_4}} \coth\left(\frac{\sqrt{\varrho}}{2}\xi\right) e^{i\left(a_2x+b_2y+\frac{c_2\varrho}{\rho}t\right)},$$

$$(60)$$

$$Q_{32}(x, y, t) = -\frac{a_1b_1\beta_1\varrho}{2\beta_2} \coth^2\left(\frac{\sqrt{\varrho}}{2}\xi\right),$$

$$P_{33}(x, y, t) = -\sqrt{\frac{a_1^2\beta_1\beta_3\varrho}{2\beta_2\beta_4}} \left(\tanh(\sqrt{\varrho}\xi) \pm i \operatorname{sech}(\sqrt{\varrho}\xi) \right) e^{i\left(a_2x+b_2y+\frac{c_2\varrho}{\rho}t\right)},$$

$$Q_{33}(x, y, t) = -\frac{a_1b_1\beta_1\varrho}{2\beta_2} \left(\tanh(\sqrt{\varrho}\xi) \pm i \operatorname{sech}(\sqrt{\varrho}\xi) \right)^2,$$

(61)

$$P_{34}(x, y, t) = -\sqrt{\frac{a_1^2\beta_1\beta_3\varrho}{2\beta_2\beta_4}} \left(\coth(\sqrt{\varrho}\xi) \pm \operatorname{csch}(\sqrt{\varrho}\xi) \right) e^{i\left(a_2x+b_2y+\frac{c_2\varrho}{\rho}t\right)},$$

$$Q_{34}(x, y, t) = -\frac{a_1b_1\beta_1\varrho}{2\beta_2} \left(\coth(\sqrt{\varrho}\xi) \pm \operatorname{csch}(\sqrt{\varrho}\xi) \right)^2,$$

$$(62)$$

$$P_{35}(x, y, t) = -\frac{1}{2}\sqrt{\frac{a_1^2\beta_1\beta_3\varrho}{2\beta_2\beta_4}} \left(\tanh\left(\frac{\sqrt{\varrho}}{4}\xi\right) + \coth\left(\frac{\sqrt{\varrho}}{4}\xi\right) \right) e^{i\left(a_2x+b_2y+\frac{c_2\varrho}{\rho}t\right)},$$

$$Q_{35}(x, y, t) = -\frac{a_1b_1\beta_1\varrho}{8\beta_2} \left(\tanh\left(\frac{\sqrt{\varrho}}{4}\xi\right) + \coth\left(\frac{\sqrt{\varrho}}{4}\xi\right) \right)^2,$$

$$(63)$$

$$P_{36}(x, y, t) = \sqrt{\frac{a_1^2\beta_1\beta_3}{2\beta_2\beta_4} \left(\frac{\sqrt{\varrho(M^2+N^2)} - M\sqrt{\varrho} \cosh(\sqrt{\varrho}\xi)}{M \sinh(\sqrt{\varrho}\xi) + N} \right)} e^{i\left(a_2x+b_2y+\frac{c_2\varrho}{\rho}t\right)},$$

$$Q_{36}(x, y, t) = -\frac{a_1b_1\beta_1}{2\beta_2} \left(\frac{\sqrt{\varrho(M^2+N^2)} - M\sqrt{\varrho} \cosh(\sqrt{\varrho}\xi)}{M \sinh(\sqrt{\varrho}\xi) + N} \right)^2,$$

$$(64)$$

$$P_{37}(x, y, t) = -\sqrt{\frac{a_1^2\beta_1\beta_3}{2\beta_2\beta_4} \left(\frac{\sqrt{\varrho(M^2+N^2)} + M\sqrt{\varrho} \cosh(\sqrt{\varrho}\xi)}{M \sinh(\sqrt{\varrho}\xi) + N} \right)} e^{i\left(a_2x+b_2y+\frac{c_2\varrho}{\rho}t\right)},$$

$$Q_{37}(x, y, t) = -\frac{a_1b_1\beta_1}{2\beta_2} \left(\frac{\sqrt{\varrho(M^2+N^2)} + M\sqrt{\varrho} \cosh(\sqrt{\varrho}\xi)}{M \sinh(\sqrt{\varrho}\xi) + N} \right)^2,$$

$$(65)$$

$$P_{38}(x, y, t) = \sqrt{\frac{a_1^2\delta_1^2\beta_1\beta_3}{2\beta_2\beta_4}} \left(1 + \frac{4\delta_0\delta_2 \cosh\left(\frac{\sqrt{\varrho}}{2}\xi\right)}{\delta_1 \left(\sqrt{\varrho} \sinh\left(\frac{\sqrt{\varrho}}{2}\xi\right) - \delta_1 \cosh\left(\frac{\sqrt{\varrho}}{2}\xi\right) \right)} \right) e^{i\left(a_2x+b_2y+\frac{c_2\varrho}{\rho}t\right)},$$

$$(66)$$

$$Q_{38}(x, y, t) = -\frac{a_1b_1\beta_1\delta_1^2}{2\beta_2} \left(1 + \frac{4\delta_0\delta_2 \cosh\left(\frac{\sqrt{\varrho}}{2}\xi\right)}{\delta_1 \left(\sqrt{\varrho} \sinh\left(\frac{\sqrt{\varrho}}{2}\xi\right) - \delta_1 \cosh\left(\frac{\sqrt{\varrho}}{2}\xi\right) \right)} \right)^2,$$

$$P_{39}(x, y, t) = \sqrt{\frac{a_1^2\delta_1^2\beta_1\beta_3}{2\beta_2\beta_4}} \left(1 - \frac{4\delta_0\delta_2 \sinh\left(\frac{\sqrt{\varrho}}{2}\xi\right)}{\delta_1 \left(\delta_1 \sinh\left(\frac{\sqrt{\varrho}}{2}\xi\right) - \sqrt{\varrho} \cosh\left(\frac{\sqrt{\varrho}}{2}\xi\right) \right)} \right) e^{i\left(a_2x+b_2y+\frac{c_2\varrho}{\rho}t\right)},$$

$$Q_{39}(x, y, t) = -\frac{a_1b_1\beta_1\delta_1^2}{2\beta_2} \left(1 - \frac{4\delta_0\delta_2 \sinh\left(\frac{\sqrt{\varrho}}{2}\xi\right)}{\delta_1 \left(\delta_1 \sinh\left(\frac{\sqrt{\varrho}}{2}\xi\right) - \sqrt{\varrho} \cosh\left(\frac{\sqrt{\varrho}}{2}\xi\right) \right)} \right)^2,$$

$$P_{40}(x, y, t) = \sqrt{\frac{a_1^2\delta_1^2\beta_1\beta_3}{2\beta_2\beta_4}} \left(1 + \frac{4\delta_0\delta_2 \cosh(\sqrt{\varrho}\xi)}{\delta_1 \left(\sqrt{\varrho} \sinh(\sqrt{\varrho}\xi) - \delta_1 \cosh(\sqrt{\varrho}\xi) \pm i\sqrt{\varrho} \right)} \right) e^{i\left(a_2x+b_2y+\frac{c_2\varrho}{\rho}t\right)},$$

$$Q_{40}(x, y, t) = -\frac{a_1b_1\beta_1\delta_1^2}{2\beta_2} \left(1 + \frac{4\delta_0\delta_2 \cosh(\sqrt{\varrho}\xi)}{\delta_1 \left(\sqrt{\varrho} \sinh(\sqrt{\varrho}\xi) - \delta_1 \cosh(\sqrt{\varrho}\xi) \pm i\sqrt{\varrho} \right)} \right)^2,$$

$$(68)$$

$$P_{41}(x, y, t) = \sqrt{\frac{a_1^2 \delta_1^2 \beta_1 \beta_3}{2 \beta_2 \beta_4}} \left(1 + \frac{4 \delta_0 \delta_2 \sinh(\sqrt{\varrho} \xi)}{\delta_1 (\sqrt{\varrho} \cosh(\sqrt{\varrho} \xi) - \delta_1 \sinh(\sqrt{\varrho} \xi) \pm \sqrt{\varrho})} \right) e^{i(a_2 x + b_2 y + \frac{c_2}{\varrho} t^\rho)}, \quad (69)$$

$$Q_{41}(x, y, t) = -\frac{a_1 b_1 \beta_1 \delta_1^2}{2 \beta_2} \left(1 + \frac{4 \delta_0 \delta_2 \sinh(\sqrt{\varrho} \xi)}{\delta_1 (\sqrt{\varrho} \cosh(\sqrt{\varrho} \xi) - \delta_1 \sinh(\sqrt{\varrho} \xi) \pm \sqrt{\varrho})} \right)^2, \quad (70)$$

$$P_{42}(x, y, t) = \sqrt{\frac{a_1^2 \delta_1^2 \beta_1 \beta_3}{2 \beta_2 \beta_4}} \left(1 + \frac{8 \delta_0 \delta_2 \sinh\left(\frac{\sqrt{\varrho}}{4} \xi\right) \cosh\left(\frac{\sqrt{\varrho}}{4} \xi\right)}{\delta_1 \left(2 \sqrt{\varrho} \cosh^2\left(\frac{\sqrt{\varrho}}{4} \xi\right) - 2 \delta_1 \sinh\left(\frac{\sqrt{\varrho}}{4} \xi\right) \cosh\left(\frac{\sqrt{\varrho}}{4} \xi\right) - \sqrt{\varrho} \right)} \right) e^{i(a_2 x + b_2 y + \frac{c_2}{\varrho} t^\rho)}, \quad (70)$$

$$Q_{42}(x, y, t) = -\frac{a_1 b_1 \beta_1 \delta_1^2}{2 \beta_2} \left(1 + \frac{8 \delta_0 \delta_2 \sinh\left(\frac{\sqrt{\varrho}}{4} \xi\right) \cosh\left(\frac{\sqrt{\varrho}}{4} \xi\right)}{\delta_1 \left(2 \sqrt{\varrho} \cosh^2\left(\frac{\sqrt{\varrho}}{4} \xi\right) - 2 \delta_1 \sinh\left(\frac{\sqrt{\varrho}}{4} \xi\right) \cosh\left(\frac{\sqrt{\varrho}}{4} \xi\right) - \sqrt{\varrho} \right)} \right)^2. \quad (70)$$

For $\varrho = \delta_1^2 - 4 \delta_0 \delta_2 < 0$ and $\delta_1 \delta_2 \neq 0$ (or $\delta_0 \delta_2 \neq 0$):

$$P_{43}(x, y, t) = \sqrt{\frac{a_1^2 \beta_1 \beta_3 \varrho}{2 \beta_2 \beta_4}} \tan\left(\frac{\sqrt{-\varrho}}{2} \xi\right) e^{i(a_2 x + b_2 y + \frac{c_2}{\varrho} t^\rho)}, \quad (71)$$

$$Q_{43}(x, y, t) = \frac{a_1 b_1 \beta_1 \varrho}{2 \beta_2} \tan^2\left(\frac{\sqrt{-\varrho}}{2} \xi\right),$$

$$P_{44}(x, y, t) = \sqrt{\frac{a_1^2 \beta_1 \beta_3 \varrho}{2 \beta_2 \beta_4}} \cot\left(\frac{\sqrt{-\varrho}}{2} \xi\right) e^{i(a_2 x + b_2 y + \frac{c_2}{\varrho} t^\rho)}, \quad (72)$$

$$Q_{44}(x, y, t) = \frac{a_1 b_1 \beta_1 \varrho}{2 \beta_2} \cot^2\left(\frac{\sqrt{-\varrho}}{2} \xi\right)^2,$$

$$P_{45}(x, y, t) = \sqrt{\frac{a_1^2 \beta_1 \beta_3 \varrho}{2 \beta_2 \beta_4}} (\tan(\sqrt{-\varrho} \xi) \pm \sec(\sqrt{-\varrho} \xi)) e^{i(a_2 x + b_2 y + \frac{c_2}{\varrho} t^\rho)},$$

$$Q_{45}(x, y, t) = \frac{a_1 b_1 \beta_1 \varrho}{2 \beta_2} (\tan(\sqrt{-\varrho} \xi) \pm \sec(\sqrt{-\varrho} \xi))^2, \quad (73)$$

$$P_{46}(x, y, t) = -\sqrt{\frac{a_1^2 \beta_1 \beta_3 \varrho}{2 \beta_2 \beta_4}} (\cot(\sqrt{-\varrho} \xi) \pm \csc(\sqrt{-\varrho} \xi)) e^{i(a_2 x + b_2 y + \frac{c_2}{\varrho} t^\rho)},$$

$$Q_{46}(x, y, t) = \frac{a_1 b_1 \beta_1 \varrho}{2 \beta_2} (\cot(\sqrt{-\varrho} \xi) \pm \csc(\sqrt{-\varrho} \xi))^2, \quad (74)$$

$$P_{47}(x, y, t) = \sqrt{\frac{a_1^2 \beta_1 \beta_3 \varrho}{2 \beta_2 \beta_4}} \left(\tan\left(\frac{\sqrt{-\varrho}}{4} \xi\right) - \cot\left(\frac{\sqrt{-\varrho}}{4} \xi\right) \right) e^{i(a_2 x + b_2 y + \frac{c_2}{\varrho} t^\rho)},$$

$$Q_{47}(x, y, t) = \frac{a_1 b_1 \beta_1 \varrho}{8 \beta_2} \left(\tan\left(\frac{\sqrt{-\varrho}}{4} \xi\right) - \cot\left(\frac{\sqrt{-\varrho}}{4} \xi\right) \right)^2, \quad (75)$$

$$P_{48}(x, y, t) = \sqrt{\frac{a_1^2 \beta_1 \beta_3}{2 \beta_2 \beta_4}} \left(\frac{\pm \sqrt{\varrho(N^2 - M^2)} \pm M \sqrt{-\varrho} \cos(\sqrt{-\varrho} \xi)}{M \sin(\sqrt{-\varrho} \xi) + N} \right) e^{i(a_2 x + b_2 y + \frac{c_2}{\varrho} t^\rho)}, \quad (76)$$

$$Q_{48}(x, y, t) = -\frac{a_1 b_1 \beta_1}{2 \beta_2} \left(\frac{\pm \sqrt{\varrho(N^2 - M^2)} \pm P \sqrt{-\varrho} \cos(\sqrt{-\varrho} \xi)}{M \sin(\sqrt{-\varrho} \xi) + N} \right)^2,$$

$$P_{49}(x, y, t) = \sqrt{\frac{a_1^2 \delta_1^2 \beta_1 \beta_3}{2 \beta_2 \beta_4}} \left(1 - \frac{4 \delta_0 \delta_2 \cos\left(\frac{\sqrt{-\varrho}}{2} \xi\right)}{\delta_1 \left(\sqrt{-\varrho} \sin\left(\frac{\sqrt{-\varrho}}{2} \xi\right) + \delta_1 \cos\left(\frac{\sqrt{-\varrho}}{2} \xi\right) \right)} \right) e^{i(a_2 x + b_2 y + \frac{c_2}{\varrho} t^\rho)}, \quad (77)$$

$$Q_{49}(x, y, t) = -\frac{a_1 b_1 \beta_1 \delta_1^2}{2 \beta_2} \left(1 - \frac{4 \delta_0 \delta_2 \cos\left(\frac{\sqrt{-\varrho}}{2} \xi\right)}{\delta_1 \left(\sqrt{-\varrho} \sin\left(\frac{\sqrt{-\varrho}}{2} \xi\right) + \delta_1 \cos\left(\frac{\sqrt{-\varrho}}{2} \xi\right) \right)} \right)^2,$$

$$P_{50}(x, y, t) = \sqrt{\frac{a_1^2 \delta_1^2 \beta_1 \beta_3}{2 \beta_2 \beta_4}} \left(1 + \frac{4 \delta_0 \delta_2 \sin\left(\frac{\sqrt{-Q}}{2} \xi\right)}{\delta_1 \left(\sqrt{-Q} \cos\left(\frac{\sqrt{-Q}}{2} \xi\right) - \delta_1 \sin\left(\frac{\sqrt{-Q}}{2} \xi\right) \right)} \right) e^{i \left(a_2 x + b_2 y + \frac{c_2}{\rho} t^\rho \right)}, \quad (78)$$

$$Q_{50}(x, y, t) = -\frac{a_1 b_1 \beta_1 \delta_1^2}{2 \beta_2} \left(1 + \frac{4 \delta_0 \delta_2 \sin\left(\frac{\sqrt{-Q}}{2} \xi\right)}{\delta_1 \left(\sqrt{-Q} \cos\left(\frac{\sqrt{-Q}}{2} \xi\right) - \delta_1 \sin\left(\frac{\sqrt{-Q}}{2} \xi\right) \right)} \right)^2,$$

$$P_{51}(x, y, t) = \sqrt{\frac{a_1^2 \delta_1^2 \beta_1 \beta_3}{2 \beta_2 \beta_4}} \left(1 - \frac{4 \delta_0 \delta_2 \cos(\sqrt{-Q} \xi)}{\delta_1 (\sqrt{-Q} \sin(\sqrt{-Q} \xi) + \delta_1 \cos(\sqrt{-Q} \xi) \pm \sqrt{-Q})} \right) e^{i \left(a_2 x + b_2 y + \frac{c_2}{\rho} t^\rho \right)}, \quad (79)$$

$$Q_{51}(x, y, t) = -\frac{a_1 b_1 \beta_1 \delta_1^2}{2 \beta_2} \left(1 - \frac{4 \delta_0 \delta_2 \cos(\sqrt{-Q} \xi)}{\delta_1 (\sqrt{-Q} \sin(\sqrt{-Q} \xi) + \delta_1 \cos(\sqrt{-Q} \xi) \pm \sqrt{-Q})} \right)^2,$$

$$P_{52}(x, y, t) = \sqrt{\frac{a_1^2 \delta_1^2 \beta_1 \beta_3}{2 \beta_2 \beta_4}} \left(1 - \frac{4 \delta_0 \delta_2 \sin(\sqrt{-Q} \xi)}{\delta_1 (\sqrt{-Q} \cos(\sqrt{-Q} \xi) + \delta_1 \sin(\sqrt{-Q} \xi) \pm \sqrt{-Q})} \right) e^{i \left(a_2 x + b_2 y + \frac{c_2}{\rho} t^\rho \right)}, \quad (80)$$

$$Q_{52}(x, y, t) = -\frac{a_1 b_1 \beta_1 \delta_1^2}{2 \beta_2} \left(1 - \frac{4 \delta_0 \delta_2 \sin(\sqrt{-Q} \xi)}{\delta_1 (\sqrt{-Q} \cos(\sqrt{-Q} \xi) + \delta_1 \sin(\sqrt{-Q} \xi) \pm \sqrt{-Q})} \right)^2,$$

$$P_{53}(x, y, t) = \sqrt{\frac{a_1^2 \delta_1^2 \beta_1 \beta_3}{2 \beta_2 \beta_4}} \left(1 + \frac{8 \delta_0 \delta_2 \sin\left(\frac{\sqrt{-Q}}{4} \xi\right) \cos\left(\frac{\sqrt{-Q}}{4} \xi\right)}{\delta_1 \left(2 \sqrt{-Q} \cos^2\left(\frac{\sqrt{-Q}}{4} \xi\right) - 2 \delta_1 \sin\left(\frac{\sqrt{-Q}}{4} \xi\right) \cos\left(\frac{\sqrt{-Q}}{4} \xi\right) - \sqrt{-Q} \right)} \right) e^{i \left(a_2 x + b_2 y + \frac{c_2}{\rho} t^\rho \right)}, \quad (81)$$

$$Q_{53}(x, y, t) = -\frac{a_1 b_1 \beta_1 \delta_1^2}{2 \beta_2} \left(1 + \frac{8 \delta_0 \delta_2 \sin\left(\frac{\sqrt{-Q}}{4} \xi\right) \cos\left(\frac{\sqrt{-Q}}{4} \xi\right)}{\delta_1 \left(2 \sqrt{-Q} \cos^2\left(\frac{\sqrt{-Q}}{4} \xi\right) - 2 \delta_1 \sin\left(\frac{\sqrt{-Q}}{4} \xi\right) \cos\left(\frac{\sqrt{-Q}}{4} \xi\right) - \sqrt{-Q} \right)} \right)^2,$$

For $\delta_0 = 0$ and $\delta_1 \delta_2 \neq 0$:

$$P_{54}(x, y, t) = \sqrt{\frac{a_1^2 \delta_1^2 \beta_1 \beta_3}{2 \beta_2 \beta_4}} \left(1 - \frac{2 \xi_0}{(\xi_0 + \cosh(\delta_1 \xi) \pm \sinh(\delta_1 \xi))} \right) e^{i \left(a_2 x + b_2 y + \frac{c_2}{\rho} t^\rho \right)},$$

$$Q_{54}(x, y, t) = -\frac{a_1 b_1 \beta_1 \delta_1^2}{2 \beta_2} \left(1 - \frac{2 \xi_0}{(\xi_0 + \cosh(\delta_1 \xi) \pm \sinh(\delta_1 \xi))} \right)^2, \quad (82)$$

$$P_{55}(x, y, t) = \sqrt{\frac{a_1^2 \delta_1^2 \beta_1 \beta_3}{2 \beta_2 \beta_4}} \left(1 - \frac{2(\cosh(\delta_1 \xi) - \sinh(\delta_1 \xi))}{(\xi_0 + \cosh(\delta_1 \xi) - \sinh(\delta_1 \xi))} \right) e^{i \left(a_2 x + b_2 y + \frac{c_2}{\rho} t^\rho \right)},$$

$$Q_{55}(x, y, t) = -\frac{a_1 b_1 \beta_1 \delta_1^2}{2 \beta_2} \left(1 - \frac{2(\cosh(\delta_1 \xi) - \sinh(\delta_1 \xi))}{(\xi_0 + \cosh(\delta_1 \xi) - \sinh(\delta_1 \xi))} \right)^2, \quad (83)$$

$$P_{56}(x, y, t) = \sqrt{\frac{a_1^2 \delta_1^2 \beta_1 \beta_3}{2 \beta_2 \beta_4}} \left(1 - \frac{2(\cosh(\delta_1 \xi) + \sinh(\delta_1 \xi))}{(\xi_0 + \cosh(\delta_1 \xi) + \sinh(\delta_1 \xi))} \right) e^{i \left(a_2 x + b_2 y + \frac{c_2}{\rho} t^\rho \right)},$$

$$Q_{56}(x, y, t) = -\frac{a_1 b_1 \beta_1 \delta_1^2}{2 \beta_2} \left(1 - \frac{2(\cosh(\delta_1 \xi) + \sinh(\delta_1 \xi))}{(\xi_0 + \cosh(\delta_1 \xi) + \sinh(\delta_1 \xi))} \right)^2, \quad (84)$$

For $\delta_2 \neq 0$ and $\delta_1 = \delta_0 = 0$:

$$P_{57}(x, y, t) = -\frac{a_1 \delta_2 \sqrt{2 \beta_1 \beta_3}}{\sqrt{\beta_2 \beta_4} (\delta_2 \xi + \xi_0)} e^{i \left(a_2 x + b_2 y + \frac{c_2}{\rho} t^\rho \right)}, \quad (85)$$

$$Q_{57}(x, y, t) = -\frac{2 a_1 b_1 \beta_1 \delta_2^2}{\beta_2 (\delta_2 \xi + \xi_0)^2},$$

where $\xi = a_1 x + b_1 y + \frac{c_1}{\rho} t^\rho$, $c_2 = -\frac{1}{2}$

$$\beta_1 (a_1 b_1 \delta_1^2 - 4 a_1 b_1 \delta_0 \delta_2 + 2 a_2 b_2), \text{ and } \xi_0$$

is arbitrary constant.

The Kudryashov method

With regard to the Kudryashov method [70–74], we assume that the Eq. 5 solution is defined as

$$P(\xi) = k_0 + \sum_{i=1}^m k_i \mathcal{R}^i(\xi), \quad k_m \neq 0. \quad (86)$$

The function $\mathcal{R}(\xi)$ satisfies an ODE express as

$$\mathcal{R}'(\xi) = \sqrt{\mathcal{R}^2(\xi) (1 - \Omega \mathcal{R}^2(\xi))}. \quad (87)$$

The solution to the ODE referred to above is given as

$$\mathcal{R}(\xi) = \frac{4B_1}{(4B_1^2 - \Omega) \sinh(\xi) + (4B_1^2 + \Omega) \cosh(\xi)}, \quad \Omega = 4B_1 B_2, \quad (88)$$

where B_1 and B_2 are arbitrary constants. Again, since $m = 1$, from Eq. 26

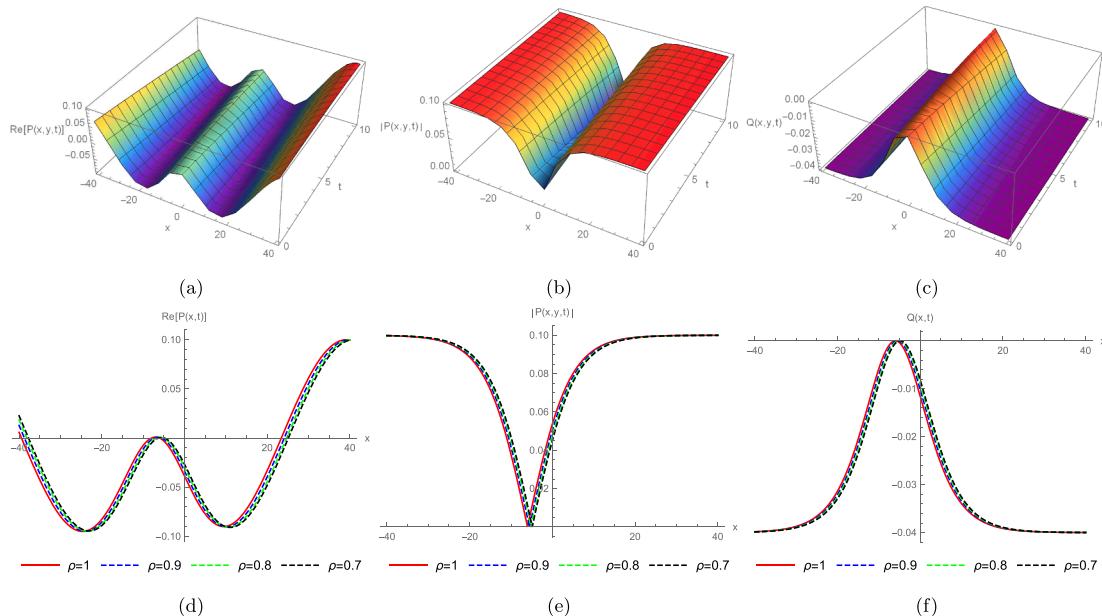


Fig. 1. The dark soliton solution Eq. 22, (a)-(c) 3D plots at $\rho = 1$, (d)-(f) 2D plots at $t = 10$, with $\beta_1 = \beta_2 = -1$, $\beta_3 = 1$, $\beta_4 = -2$, $a_1 = a_2 = 0.1$, $b_1 = b_2 = 0.2$, $\mu_1 = \sqrt{8}$, $\mu_2 = 1$, $y = 1$, $\Omega_1 = 1$, and $\Omega_2 = 0$.

we have

$$P(\xi) = k_0 + k_1 \mathcal{R}(\xi), \quad k_1 \neq 0. \quad (89)$$

Putting Eqs. 86 and 87 into Eq. 12 accordingly, collecting all the coefficient of $\mathcal{R}^i(\xi)$, $i = 0, 1, 2, 3$, and setting them to zero, we have

$$\begin{aligned} \mathcal{R}^0(\xi) : & -\frac{b_1 \beta_2 \beta_4 k_0^3}{a_1 \beta_3} - a_2 b_2 \beta_1 k_0 - c_2 k_0 = 0, \\ \mathcal{R}^1(\xi) : & -\frac{3b_1 \beta_2 \beta_4 k_0 k_1^2}{a_1 \beta_3} = 0, \\ \mathcal{R}^2(\xi) : & -\frac{3b_1 \beta_2 \beta_4 k_1 k_0^2}{a_1 \beta_3} + a_1 b_1 \beta_1 k_1 - a_2 b_2 \beta_1 k_1 - c_2 k_1 = 0, \\ \mathcal{R}^3(\xi) : & -2a_1 b_1 \beta_1 k_1 \Omega - \frac{b_1 \beta_2 \beta_4 k_1^3}{a_1 \beta_3} = 0. \end{aligned} \quad (90)$$

The solutions of the equations obtained above along with Eq. 11 yields:

$$\begin{aligned} c_1 &= -\beta_1(a_2 b_1 + a_1 b_2), \quad c_2 = \beta_1(a_1 b_1 - a_2 b_2), \quad k_0 = 0, \quad k_1 \\ &= \pm i a_1 \sqrt{\frac{2\beta_1 \beta_3 \Omega}{\beta_2 \beta_4}}. \end{aligned}$$

By incorporating these parameters into Eq. 89 and consider Eq. 88, we obtain the solutions

$$\begin{aligned} P_{58}(x,y,t) &= \pm \frac{4ia_1 B_1 \sqrt{2\beta_1 \beta_3 \Omega}}{\sqrt{\beta_2 \beta_4} ((4B_1^2 - \Omega) \sinh(\xi) + (4B_1^2 + \Omega) \cosh(\xi))} e^{i(a_2 x + b_2 y + \frac{c_2}{\rho} t^\rho)}, \\ Q_{58}(x,y,t) &= -\frac{\beta_4 b_1}{a_1 \beta_3} \left(-\frac{4ia_1 A \sqrt{2\beta_1 \beta_3 \Omega}}{\sqrt{\beta_2 \beta_4} ((4A^2 - \Omega) \sinh(\xi) + (4A^2 + \Omega) \cosh(\xi))} \right)^2. \end{aligned} \quad (91)$$

Remark 3. Letting $B_1 = B_2 = 1$ in Eq. 91 yields the bright soliton solutions of Eq. 5 as follows:

$$\begin{aligned} P(x,t) &= \pm i \sqrt{\frac{2a_1^2 \beta_1 \beta_3}{\beta_2 \beta_4}} \operatorname{sech} \left(a_1 x + b_1 y - \frac{\beta_1(a_2 b_1 + a_1 b_2)}{\rho} t^\rho \right) e^{i \left(a_2 x + b_2 y + \frac{\beta_1(a_1 b_1 - a_2 b_2)}{\rho} t^\rho \right)}, \\ Q(x,t) &= \frac{2a_1 b_1 \beta_1}{\beta_2} \operatorname{sech}^2 \left(a_1 x + b_1 y - \frac{\beta_1(a_2 b_1 + a_1 b_2)}{\rho} t^\rho \right). \end{aligned} \quad (92)$$

Remark 4. Letting $B_1 = 1$ and $B_2 = -1$ in Eq. 91 yields the singular soliton solutions of Eq. 5 as:

Remark 5. It should be noted that all the solutions presented in this

$$\begin{aligned} P(x, t) &= \pm i \sqrt{\frac{2a_1^2 \beta_1 \beta_3}{\beta_2 \beta_4}} \operatorname{csch} \left(a_1 x + b_1 y - \frac{\beta_1(a_2 b_1 + a_1 b_2)}{\rho} t^\rho \right) e^{i \left(a_2 x + b_2 y + \frac{\beta_1(a_1 b_1 - a_2 b_2)}{\rho} t^\rho \right)}, \\ Q(x, t) &= -\frac{2a_1 b_1 \beta_1}{\beta_2} \operatorname{csch}^2 \left(a_1 x + b_1 y - \frac{\beta_1(a_2 b_1 + a_1 b_2)}{\rho} t^\rho \right). \end{aligned} \quad (93)$$

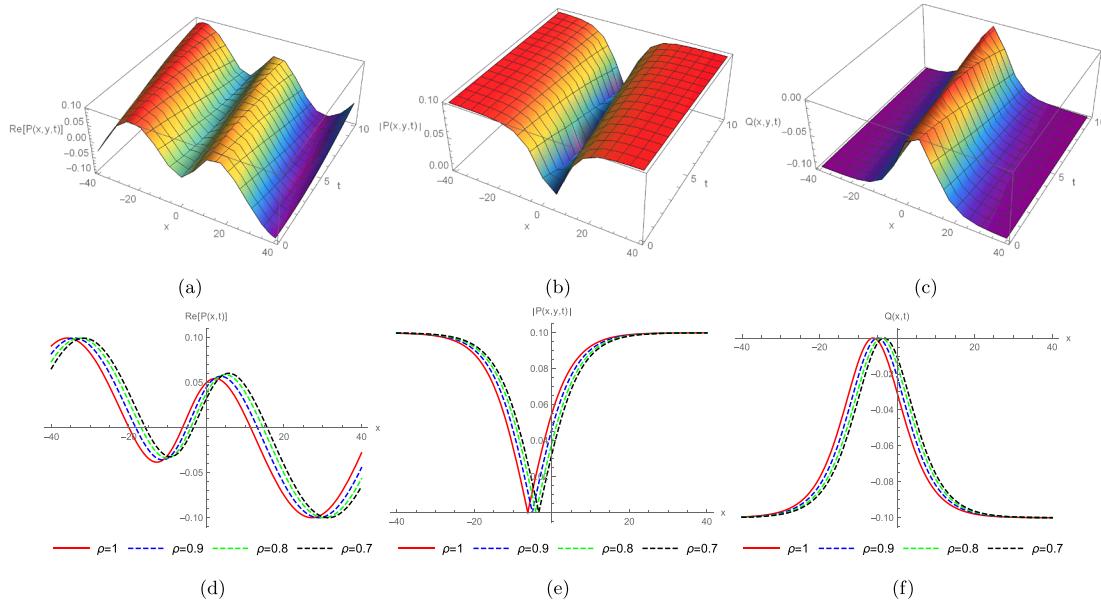


Fig. 2. The optical soliton solution Eq. 38 (a)-(c) 3D plots at $\rho = 1$, (d)-(f) 2D plots at $t = 10$, $\beta_1 = -1$, $\beta_2 = \beta_3 = 1$, $\beta_4 = -2$, $a_1 = a_2 = 0.1$, $b_1 = b_2 = 0.5$, $y = 1$, $\delta_1 = \sqrt{8}$, $\delta_2 = \delta_0 = -1$, $M = 2$, and $N = 1$.

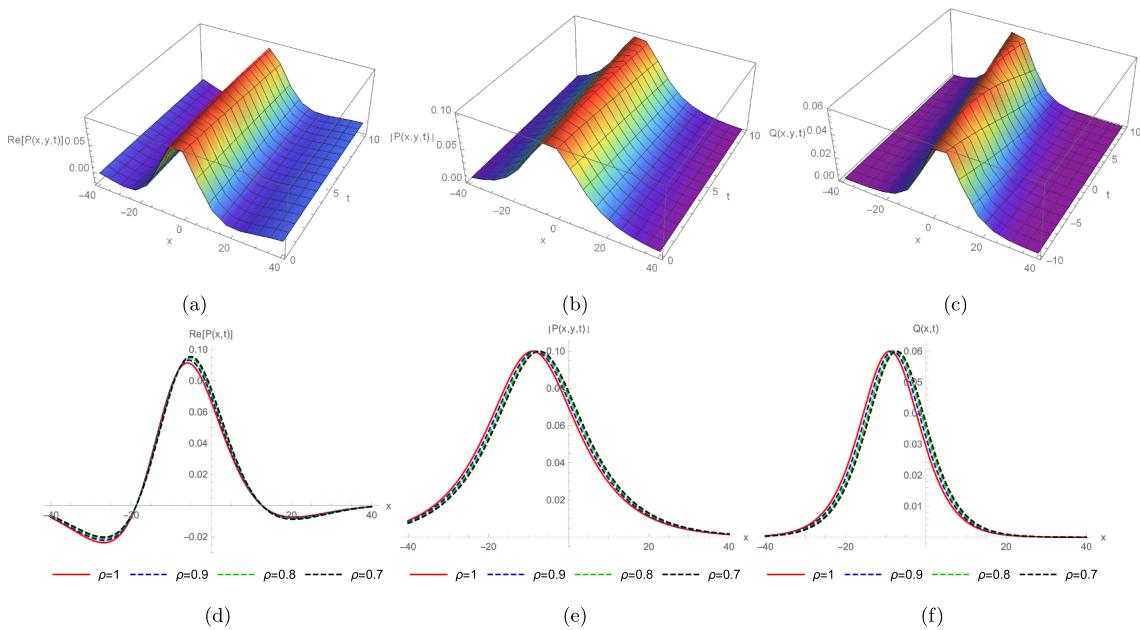


Fig. 3. The bright soliton solution Eq. 22, (a)-(c) 3D plots at $\rho = 1$, (d)-(f) 2D plots at $t = 10$, with $\beta_1 = \beta_2 = -1$, $\beta_3 = 1$, $\beta_4 = -2$, $a_1 = a_2 = 0.1$, $b_1 = b_2 = 0.3$, $y = 1$, $B_1 = 1$ and $B_2 = 1$.

work have been verified by the help of Mathematica package software.

Concluding remarks

We successfully study the integrable generalized $(2+1)$ -dimensional nonlinear conformable Schrödinger system with nonlinear optical applications. Three different schemes, specifically, (G'/G) -expansion method, generalized Riccati equation mapping method and the Kudryashov method have been implemented to construct several optical soliton solutions to the proposed integrable system. The 3D and 2D (with distinct fractional order ρ) plots are depicted in Fig. 1–3 to examine the dynamical characteristics of the soliton solutions. The advantages of choosing the conformable derivative over the the most common used fractional derivatives, the Caputo and Riemann–Liouville fractional derivative in this present investigation are as follows:

- (i.) The conformable derivative provide us more flexibility when applying to the FPDEs, in view of the fact that the FPDEs of conformable type are more convenient to compute analytically or approximately when compared to the Caputo or Riemann–Liouville fractional derivative.
- (ii.) This definition is natural and it fulfills almost all of the properties that the classical derivative and integral possesses, particularly, the product rule, power rule, quotient rule, integration by parts, chain rule, linearity, derivatives with constant functions is zero, mean value theorem, and Rolle's theorem [65].

Our results show that the proposed methods are powerful, reliable, and efficient tools for constructing new wave solutions to this nonlinear system and potentially be extended to other kinds of nonlinear integer and non-integer order systems arising in applied physics. Finally, in developing new theories in plasma physics, nuclear physics, fluid dynamics, quantum mechanics, soliton dynamics, optical physics, electromagnetism, biomedical concerns, industrial studies, mathematical physics, and many other natural and physical sciences, these findings can be highly applicable.

CRediT authorship contribution statement

Lanre Akinyemi: Conceptualization, Validation. **Mehmet Senol:** Formal analysis, Validation. **Hadi Rezazadeh:** Data curation, Writing - original draft. **Hijaz Ahmad:** Writing - original draft, Writing - review & editing. **Hao Wang:** Resources, Funding.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgment

This work was supported by Programs of Henan Polytechnic University (No. B2017-57).

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