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On super-linear Emden–Fowler type differential equations

Zuzana Došlá [a], Mauro Marini [b,*]

[a] Department of Mathematics and Statistics, Masaryk University, Kotlářská 2, CZ-61137 Brno, Czech Republic
[b] Department of Mathematics and Informatics "Ulisse Dini", University of Florence, I-50139 Florence, Italy

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A B S T R A C T

We study the second order Emden–Fowler type differential equation

$$\left(a(t)\left|x'\right|^{\alpha} \operatorname{sgn} x'\right)' + b(t)|x|^{\beta} \operatorname{sgn} x = 0$$

in the super-linear case $\alpha < \beta$. Using a Hölder-type inequality, we resolve the open problem on the possible coexistence on three possible types of nonoscillatory solutions (subdominant, intermediate, and dominant solutions). Jointly with this, sufficient conditions for the existence of globally positive intermediate solutions are established. Some of our results are new also for the Emden–Fowler equation.

1. Introduction

Consider the second order nonlinear differential equation

$$\left(a(t)\left|x'\right|^{\alpha} \operatorname{sgn} x'\right)' + b(t)|x|^{\beta} \operatorname{sgn} x = 0, \tag{1}$$

where $\alpha > 0$, $\beta > 0$ are constants and a, b are continuous functions on $[0, \infty)$ such that $a(t) > 0$, $b(t) \geqslant 0$, $\sup\{b(t)\colon t \geqslant T\} > 0$ for all $T > 0$ and

$$I_a = \int_0^\infty a^{-1/\alpha}(s)\,ds = \infty, \qquad I_b = \int_0^\infty b(s)\,ds < \infty.$$

Jointly with (1), we study the special case $a(t) \equiv 1$, that is equation

$$\left(\left|x'\right|^{\alpha} \operatorname{sgn} x'\right)' + b(t)|x|^{\beta} \operatorname{sgn} x = 0. \tag{2}$$

* Corresponding author.
E-mail addresses: dosla@math.muni.cz (Z. Došlá), mauro.marini@unifi.it (M. Marini).

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If $\alpha = \beta$, then (1) is called *half-linear equation*. If $\alpha \neq \beta$, then (1) is usually called *Emden–Fowler type equation*, because its prototype is the well-known Emden–Fowler equation

$$x'' + b(t)|x|^{\beta} \operatorname{sgn} x = 0, \quad \beta \neq 1, \tag{3}$$

widely studied in the literature; see, e.g., [18,22] or [3,12] for dynamic equations on time scales.

By a solution of (1) we mean a function x, defined on some ray $[\tau_x, \infty)$, $\tau_x \geqslant 0$, such that $a|x'|^{\alpha} \operatorname{sgn} x'$ is continuously differentiable and satisfies (1) for any $t \geqslant \tau_x$. Observe that if either $\alpha \geqslant \beta$ or a, b are sufficiently smooth and $\alpha < \beta$, then any solution of (1) is continuable up to infinity, see, e.g., [23]. As usual, a solution x of (1) is said to be *nonoscillatory* if $x(t) \neq 0$ for large t and *oscillatory* otherwise. Eq. (1) is said to be *nonoscillatory* if any solution is nonoscillatory. Notice that, if $\alpha \neq \beta$, nonoscillatory solutions of (1) may coexist with oscillatory ones, while if $\alpha = \beta$ this fact is impossible, see, e.g., [10].

For the sake of simplicity, in the sequel we will consider only nonoscillatory solutions of (1), which are positive for large t.

For any solution x of (1), denote by $x^{[1]}$ the *quasiderivative of* x, i.e. the function

$$x^{[1]}(t) = a(t)\left|x'(t)\right|^{\alpha} \operatorname{sgn} x'(t).$$

Clearly, any eventually positive solutions x of (1) is increasing and the quasiderivative $x^{[1]}$ is positive nonincreasing for large t. Moreover, the class $\mathbb{P}$ of all eventually positive solutions of (1) can be divided into the three subclasses, see, e.g., [11]:

$$\begin{aligned}
\mathbb{M}^{+}_{\infty,\ell} &= \left\{x \in \mathbb{P}\colon x(\infty) = \infty,\ x^{[1]}(\infty) = \ell_x,\ 0 < \ell_x < \infty\right\},\\
\mathbb{M}^{+}_{\infty,0} &= \left\{x \in \mathbb{P}\colon x(\infty) = \infty,\ x^{[1]}(\infty) = 0\right\},\\
\mathbb{M}^{+}_{\ell,0} &= \left\{x \in \mathbb{P}\colon x(\infty) = \ell_x,\ x^{[1]}(\infty) = 0,\ 0 < \ell_x < \infty\right\}.
\end{aligned} \tag{4}$$

The superscript symbol $+$ means that solutions are eventually positive increasing. Following [7,8,13], solutions in $\mathbb{M}^{+}_{\infty,\ell}$, $\mathbb{M}^{+}_{\infty,0}$, and $\mathbb{M}^{+}_{\ell,0}$ are called *dominant solutions, intermediate solutions* and *subdominant solutions*, respectively. Our main subject here will be intermediate solutions. Sometimes, they are called *weakly increasing solutions*, see [15].

The interesting problem is whether all three types of nonoscillatory solutions can simultaneously exist. Since there are well-known necessary and sufficient conditions for the existence of subdominant and dominant solutions (see below), the coexistence problem leads to the problem on the nonexistence of intermediate solutions.

This problem has a long history. For Eq. (3), it started sixty years ago by Atkinson [1], Moore and Nehari [19] in case $\beta > 1$ and Belohorec [5] in case $\beta < 1$. In particular, in [5,19] it is proved that this triple coexistence is impossible and intermediate solutions of (3) cannot coexist with dominant solutions or subdominant ones. Moreover, in [19] the question of existence of globally nonoscillatory solutions, that is solutions different from zero for any $t \geqslant 0$, is posed.

For the more general equation (1), this study was continued in nineties of the last century in [11,13,17] and in the recent years in [7,8,20]. In particular, the question of the possible triple coexistence has been solved in negative way in [7] when $\alpha = \beta$, in [20] in the sub-linear case, i.e. when $\alpha > \beta$, and partially in [8] in the super-linear case, i.e. when $\alpha < \beta$ and $0 < \alpha < 1$.

Hence the following two questions arise:

1. When $1 < \alpha < \beta$, can these three types of nonoscillatory solutions of (1) simultaneously coexist?
2. When $\alpha < \beta$, does (1) have globally nonoscillatory solutions, in particular intermediate solutions?

Observe that, in the super-linear case, necessary and sufficient conditions for existence of intermediate solutions are difficult to establish due to the problem to find sharp upper and lower bounds, as it is pointed out in [2, p. 241] and [15, p. 3].

Motivated by [20], here we give an answer to both these questions. In particular, we completely resolve the problem on the triple coexistence and, as regards properly nonoscillatory solutions, a sufficient condition for their existence is given. As a consequence, by means of a topological limit process, an existence result for intermediate solution is obtained too. Observe that, until now, the existence of intermediate solutions was an open problem also for Eq. (3) with $\beta > 1$.

Some auxiliary results are given in Section 2. The coexistence problem is considered in Section 3 for Eq. (2). The existence results for properly nonoscillatory (intermediate) solutions of (2) are presented in Section 4. The extension of these results to Eq. (1) is given in Section 5 and examples with some open problems conclude the paper.

2. Auxiliary results

The following integral relations play an important role in our approach. These are based on the following necessary and sufficient conditions for the existence of subdominant and dominant solutions of (1), as the following results summarize, see e.g. [11].

Define

$$J = \int_0^\infty \frac{1}{a^{1/\alpha}(s)} \left(\int_s^\infty b(r)\, dr \right)^{1/\alpha} ds, \qquad K = \int_0^\infty b(s) \left(\int_0^s \frac{1}{a^{1/\alpha}(r)}\, dr \right)^{\beta} ds.$$

For the particular case (2), integrals J, K read as

$$J_1 = \int_0^\infty \left(\int_s^\infty b(r)\, dr \right)^{1/\alpha} ds, \qquad K_1 = \int_0^\infty s^\beta b(s)\, ds.$$

Proposition 1. *The following hold for (1):*

- i_1) *The class $\mathbb{M}^+_{\infty,\ell}$ is nonempty if and only if $K < \infty$. Moreover, for any ℓ, $0 < \ell < \infty$, there exists $x \in \mathbb{M}^+_{\infty,\ell}$ such that $\lim_{t\to\infty} x^{[1]}(t) = \ell$.*
- i_2) *The class $\mathbb{M}^+_{\ell,0}$ is nonempty if and only if $J < \infty$. Moreover, for any ℓ, $0 < \ell < \infty$, there exists $x \in \mathbb{M}^+_{\ell,0}$ such that $\lim_{t\to\infty} x(t) = \ell$.*
- i_3) *Let $\alpha > \beta$. Eq. (1) is oscillatory if and only if $K = \infty$.*
- i_4) *Let $\alpha < \beta$. Eq. (1) is oscillatory if and only if $J = \infty$.*

Remark 1. Claims i_1), i_2) of Proposition 1 are slightly more general than those proved in [14]. Claims i_1), i_2) hold also in the half-linear case $\alpha = \beta$, see, e.g. [7, Theorem 6, Theorem 7]. Observe that these results are proved in [7] by assuming the positivity of the function b. Nevertheless, it is easy to verify that they continue to hold also in case $b(t) \geqslant 0$, and $\sup\{b(t)\colon t \geqslant T\} > 0$ for all $T > 0$.

Results in Proposition 1 follow also from other papers, in which more general equations than (1) are considered, see, e.g., [2, Theorems 3.13.11, 3.13.12], [16, Sections 18, 19], [18, Theorems 17.1, 17.2].

Remark 2. The relations between integrals J and K when $\alpha \neq \beta$ have been proved in [6]. From here we get the following possible cases of mutual behavior of integrals J, K:

C_1) $J = \infty$, $K = \infty$;
C_2) $J = \infty$, $K < \infty$ and $\alpha > \beta$;
C_3) $J < \infty$, $K = \infty$ and $\alpha < \beta$;
C_4) $J < \infty$, $K < \infty$.

In case C_1), all continuable solutions are oscillatory, see, e.g., [18, Theorems 11.3, 11.4]. Clearly, this fact is not true in the half-linear case, as the Euler equation illustrates, see, e.g., [10, Theorem 1.4.4]. Moreover, the case C_2) occurs only if $\alpha > \beta$, and the case C_3) only for $\alpha < \beta$. Finally, in view of Proposition 1, if any of the case C_i), $i = 2, 3, 4$, holds, then the class $\mathbb{P}^+$ is nonempty. By the quoted result [20], in the sub-linear case $\alpha > \beta$ Eq. (1) has intermediate solutions if and only if the case C_2) occurs. Hence, in the sub-linear case, by Proposition 1 the triple coexistence of nonoscillatory solutions is impossible also for (1).

We close this section with a Hölder-type inequality, which is needed in the following.

Lemma 1. *Let λ, μ be such that $\mu > 1, \lambda\mu > 1$ and let f, g be nonnegative continuous functions for $t \geqslant T$. Then*

$$\left(\int_T^t g(s)\left(\int_s^t f(\tau)\,d\tau\right)^{\lambda} ds\right)^{\mu}$$
$$\leqslant \lambda^{\mu}\left(\frac{\mu-1}{\lambda\mu-1}\right)^{\mu-1}\left(\int_T^t f(\tau)\left(\int_T^{\tau} g(s)\,ds\right)^{\mu} d\tau\right)\left(\int_T^t f(\tau)\,d\tau\right)^{\lambda\mu-1}.$$

Proof. First, assume f is positive for $t \geqslant T$. Integrating by parts, we have

$$\int_T^t f(s)\left(\int_s^t f(\tau)\,d\tau\right)^{\lambda-1}\left(\int_T^s g(\sigma)\,d\sigma\right) ds$$
$$= -\frac{1}{\lambda}\int_T^t \frac{d}{ds}\left(\int_s^t f(\tau)\,d\tau\right)^{\lambda}\left(\int_T^s g(\sigma)\,d\sigma\right) ds = \frac{1}{\lambda}\int_T^t g(s)\left(\int_s^t f(\tau)\,d\tau\right)^{\lambda} ds.$$

Thus, using the Hölder inequality, we obtain

$$\int_T^t g(s)\left(\int_s^t f(\tau)\,d\tau\right)^{\lambda} ds$$
$$= \lambda\int_T^t f^{1/p}(s)f^{1/q}\left(\int_s^t f(\tau)\,d\tau\right)^{\lambda-1}\left(\int_T^s g(\sigma)\,d\sigma\right) ds$$
$$\leqslant \lambda\left(\int_T^t f(s)\left(\int_T^s g(\sigma)\,d\sigma\right)^{p} ds\right)^{1/p}\left(\int_T^t f(s)\left(\int_s^t f(\tau)\,d\tau\right)^{(\lambda-1)q} ds\right)^{1/q},$$

where $p > 1$, $p^{-1} + q^{-1} = 1$. Choosing

$$p = \mu, \qquad q = \frac{\mu}{\mu-1},$$

we get

$$\int_T^t g(s)\left(\int_s^t f(\tau)\,d\tau\right)^{\lambda} ds$$
$$\leqslant \lambda\left(\int_T^t f(s)\left(\int_T^s g(\sigma)\,d\sigma\right)^{\mu} ds\right)^{1/\mu}\left(\int_T^t f(s)\left(\int_s^t f(\tau)\,d\tau\right)^{\nu} ds\right)^{(\mu-1)/\mu}, \tag{5}$$

where $\nu = (\lambda-1)\mu/(\mu-1) > -1$. Moreover, we have

$$\int_T^t f(s)\left(\int_s^t f(\tau)\,d\tau\right)^{\nu} ds = \frac{1}{\nu+1}\left(\int_T^t f(s)\,ds\right)^{\nu+1}.$$

Hence, from (5) we obtain

$$\int_T^t g(s)\left(\int_s^t f(\tau)\,d\tau\right)^{\lambda} ds \leqslant h\left(\int_T^t f(s)\left(\int_T^s g(\sigma)\,d\sigma\right)^{\mu} ds\right)^{1/\mu}\left(\int_T^t f(\tau)\,d\tau\right)^{(\lambda\mu-1)/\mu},$$

where

$$h = \lambda\left(\frac{\mu-1}{\lambda\mu-1}\right)^{(\mu-1)/\mu},$$

i.e. the assertion.

If f has zeros for $t \geqslant T$, for any $s \in [T,t)$ consider the subset of the interval (s,t) in which f is positive and let $I(s)$ its closure, that is $I(s) = cl\{r \in (s,t)\colon f(r) > 0\}$. Since $\int_s^t f(\tau)\,d\tau = \int_{I(s)} f(\tau)\,d\tau$, the assertion follows using the same argument as before. □

3. The coexistence problem

In this section we consider the problem whether intermediate solutions may coexist with subdominant and dominant solutions. In view of Proposition 1, necessarily both integrals J, K must be convergent. As claimed above, the recent result [20, Theorem 1.2.] states that this triple coexistence is not possible in the sub-linear case $\alpha > \beta$.

Now, we prove that intermediate solutions of (2) do not exist when $\alpha < \beta$ and both integrals J_1, K_1 are convergent. For sub-linear and super-linear equations (3) this result has been proved in [5] and [19], respectively, using the property that intermediate solutions of (3) satisfy

$$\lim_{t\to\infty} \frac{tx'(t)}{x(t)} = 1.$$

When $\alpha \neq 1$, this approach cannot be used and we use the following property of intermediate solutions.

Lemma 2. *Let $1 < \alpha < \beta$ and assume*

$$\int_0^\infty s^\beta b(s)\,ds < \infty.$$

Then for any intermediate solution x of (2) we have

$$\liminf_{t\to\infty} \frac{tx'(t)}{x(t)} > 0.$$

Proof. Since

$$\left(x'(t)\right)^{\alpha} = \int_{t}^{\infty} b(s)x^{\beta}(s)\,ds,$$

we can choose t_1 large so that $x(t) > 0, x'(t) > 0$ for $t \geqslant t_1$ and

$$\int_{t_1}^{\infty} r^{\beta} b(r)\,dr < 1. \tag{6}$$

Let t_2 be large so that

$$\widetilde{k}\left(\int_{t_2}^{\infty} r^{\beta} b(r)\,dr\right)^{(\beta-\alpha)/\alpha} < 1, \tag{7}$$

where

$$\widetilde{k} = \frac{1}{\alpha}\left(\frac{\alpha(\beta-1)}{\beta-\alpha}\right)^{(\beta-1)/\beta}$$

and set $T = \max\{t_1, t_2\}$. From (2) we obtain for $t \geqslant T$

$$x(t) - x(T) = \int_{T}^{t}\left(\int_{s}^{t} b(r)x^{\beta}(r)\,dr + \int_{t}^{\infty} b(r)x^{\beta}(r)\,dr\right)^{1/\alpha} ds.$$

Thus, from the inequality

$$(X+Y)^{1/\alpha} \leqslant X^{1/\alpha} + Y^{1/\alpha},$$

where X, Y are two positive numbers, we obtain

$$x(t) - x(T) \leqslant \int_{T}^{t}\left(\int_{s}^{t} b(r)x^{\beta}(r)\,dr\right)^{1/\alpha} ds + t\left(\int_{t}^{\infty} b(r)x^{\beta}(r)\,dr\right)^{1/\alpha}.$$

Using Lemma 1 with $f(r) = b(r)x^{\beta}(r)$, $g(r) \equiv 1$, $\lambda = 1/\alpha$ and $\mu = \beta$ we have

$$x(t) - x(T) \leqslant \widetilde{k}\left(\int_{T}^{t} r^{\beta} b(r)x^{\beta}(r)\,dr\right)^{1/\beta}\left(\int_{T}^{t} b(\tau)x^{\beta}(\tau)\,d\tau\right)^{(\beta-\alpha)/\alpha} + t\left(\int_{t}^{\infty} b(r)x^{\beta}(r)\,dr\right)^{1/\alpha}$$

or, in virtue of (7),

$$\begin{aligned} x(t) - x(T) &\leqslant \left(\int_{T}^{t} r^{\beta} b(r)x^{\beta}(r)\,dr\right)^{1/\beta} + t\left(\int_{t}^{\infty} b(r)x^{\beta}(r)\,dr\right)^{1/\alpha} \\ &= \left(\int_{T}^{t} r^{\beta} b(r)x^{\beta}(r)\,dr\right)^{1/\beta} + tx'(t) \end{aligned}$$

Since

$$\left(\int_T^t r^\beta b(r) x^\beta(r)\,d\tau\right)^{1/\beta} \leqslant x(t)\left(\int_T^\infty r^\beta b(r)\,d\tau\right)^{1/\beta},$$

we obtain

$$1-\frac{x(T)}{x(t)} \leqslant \left(\int_T^\infty r^\beta b(r)\,d\tau\right)^{1/\beta} + \frac{tx'(t)}{x(t)}$$

or

$$\frac{tx'(t)}{x(t)} \geqslant 1-\frac{x(T)}{x(t)} - \left(\int_T^\infty r^\beta b(r)\,d\tau\right)^{1/\beta}$$

which, in view of (6), gives the assertion. □

Now, we can resolve the coexistence problem for (2).

Theorem 1. *Let $\alpha < \beta$ and $K_1 < \infty$. Then (2) does not have intermediate solutions.*

Proof. If $0 < \alpha \leqslant 1$, as claimed, the assertion follows from [8, Theorem 4.1.]. Now, assume $\alpha > 1$ and, by contradiction, let x be an intermediate solution of (2) such that $x(t) > 0, x'(t) > 0$ for large t. In virtue of Lemma 2, there exists $t_1 \geqslant 0$ such that $x(t) > 0, x'(t) > 0$ for $t \geqslant t_1$ and

$$tx'(t) > mx(t). \tag{8}$$

Without loss of generality, suppose

$$\int_{t_1}^\infty s^\beta b(s)\,ds < m/2. \tag{9}$$

Since x is an intermediate solution, we have $\lim_{t\to\infty} t^{-1}x(t) = 0$ and so there exists $t_2 \geqslant 0$ such that for $t \geqslant t_2$

$$x(t) < t. \tag{10}$$

Put $T = \max\{t_1, t_2\}$. Integrating (2) on (T, t) we have

$$\big(x'(t)\big)^\alpha - \big(x'(T)\big)^\alpha = -\int_T^t b(s) x^\beta(s)\,ds$$

or, in view of (8), (9) and (10)

$$\begin{aligned}\big(x'(T)\big)^\alpha - \big(x'(t)\big)^\alpha &= \int_T^t b(s) x^{\beta-\alpha}(s) x^\alpha(s)\,ds < \frac{1}{m}\int_T^t s^\alpha b(s) x^{\beta-\alpha}(s)\big(x'(s)\big)^\alpha\,ds \\ &\leqslant \frac{(x'(T))^\alpha}{m}\int_T^t s^\beta b(s)\,ds \leqslant \frac{(x'(T))^\alpha}{2}.\end{aligned}$$

Thus

$$\frac{(x'(T))^\alpha}{2} < \left(x'(t)\right)^\alpha,$$

which gives a contradiction as $t \to \infty$, since $\lim_{t\to\infty} x'(t) = 0$. □

From Theorem 1 we get the following corollaries.

Corollary 1. *Let $\alpha < \beta$. If (2) has a dominant solution, then it has also subdominant solution and all other nonoscillatory solutions are either dominant or subdominant.*

Proof. Since (2) has a dominant solution, from Proposition 1 and Remark 2 we get $K_1 < \infty$, $J_1 < \infty$. Again from Proposition 1, Eq. (2) has subdominant solution. Hence, the assertion follows from Theorem 1. □

Corollary 2. *Eq. (2) does not have simultaneously subdominant, intermediate and dominant solutions.*

Proof. The necessary and sufficient conditions for the existence of subdominant and dominant solutions are given in Proposition 1. If $\alpha > \beta$, then the assertion follows from [20, Theorem 1.2.] and from the relation between integrals J, K, see Remark 2. If $\alpha < \beta$, then the only possible cases are cases $C_3), C_4)$. Hence, the assertion immediately follows from Theorem 1 and Proposition 1. Finally, if $\alpha = \beta$, the assertion follows from [7, Theorems 4, 6 and 7]. □

4. Existence of intermediate solutions

In this section we study the existence of intermediate solutions of (2) in the super-linear case $\alpha < \beta$. By Remark 2, intermediate solutions can exist only when $J < \infty$ and $K = \infty$. First, we state conditions under which all subdominant solutions of (2) are properly nonoscillatory on the whole interval $[T, \infty)$, $T > 0$. Using this property of subdominant solutions, we construct intermediate solutions for (2).

Define the function

$$F(t) = t^\gamma b(t) \tag{11}$$

where

$$\gamma = \frac{1 + \alpha\beta + 2\alpha}{\alpha + 1}.$$

If F is nondecreasing for large t, say $t \geqslant T > 0$, then eventually positive subdominant solutions of (2) are properly nonoscillatory and equibounded on $[T, \infty)$, as the following result shows.

Theorem 2. *Let $\alpha < \beta$ and F be nondecreasing on $[T, \infty)$, $T > 0$. Then any eventually positive subdominant solution x of (2) satisfies on the whole interval $[T, \infty)$*

$$x(t) > 0, \qquad x'(t) \geqslant 0 \tag{12}$$

and

$$0 < x(t) \leqslant \varphi(t) \tag{13}$$

where

$$\varphi(t) = c\big(F(t)\big)^{-1/(\beta-\alpha)} t^{\alpha/(\alpha+1)} \tag{14}$$

and c is a suitable positive constant which depends on α and β.

Proof. The argument is similar to the one in [4, Lemma 5 and Theorem 3]. Firstly, let us show that any eventually positive subdominant solution x of (2) is positive on the whole interval $[T, \infty)$. By contradiction, assume there exists $t_1 \geqslant T$ such that $x(t_1) = 0$ and, without loss of generality, suppose $x(t) > 0$ for $t > t_1$. Since $\alpha < \beta$ and the uniqueness of solutions of (2) with respect to the initial data holds, we have

$$x'(t_1) = x_1 \neq 0.$$

Hence, in virtue of [4, Lemma 5], we have that there exists $t_2 > t_1$ such that for any $t \geqslant t_2$

$$x(t) \geqslant \frac{|x_1|(t_1)^{1/(\alpha+1)}}{2} t^{\alpha/(\alpha+1)}.$$

Thus, x is unbounded as t tends to infinity, which is a contradiction. Hence $x(t) > 0$ for $t \geqslant T$.

Let us prove that $x'(t) \geqslant 0$ for $t \geqslant T$. By contradiction, suppose there exists $\tau \geqslant T$ such that $x'(\tau) < 0$. Since x' is nonincreasing on $[T, \infty)$, we get for $t > \tau$

$$x(t) \leqslant x(\tau) + \big(x'(\tau)\big)^{1/\alpha}(t - \tau)$$

and so x should be negative for large t, which is a contradiction. Then (12) is proved.

Now we prove that (13) holds for any subdominant solution x of (2) satisfying (12) for $t \geqslant T > 0$. Integrating (2) we have

$$\begin{aligned}\big(x'(t)\big)^{\alpha} &= \int_t^{\infty} b(s)x^{\beta}(s)\,ds = \int_t^{\infty} s^{\gamma}b(s)x^{\beta}(s)s^{-\gamma}\,ds \\ &\geqslant b(t)t^{\gamma}x^{\beta}(t)\int_t^{\infty} s^{-\gamma}\,ds = \frac{1}{\gamma-1}tb(t)x^{\beta}(t)\end{aligned}$$

or

$$\frac{x'(t)}{x^{\beta/\alpha}(t)} \geqslant k\big(t^{\gamma}b(t)\big)^{1/\alpha}t^{(1-\gamma)/\alpha},$$

where $k = (\gamma-1)^{-1/\alpha}$. Hence, again integrating on (t, ∞) we obtain

$$\frac{\alpha}{\beta-\alpha}\left(\frac{1}{x(t)}\right)^{(\beta-\alpha)/\alpha} \geqslant \frac{\alpha}{\beta-\alpha}\left(\frac{1}{x_{\infty}}\right)^{(\beta-\alpha)/\alpha} + k\frac{\alpha+1}{\beta-\alpha}\big(t^{\gamma}b(t)\big)^{1/\alpha}t^{(\alpha-\beta)/(\alpha+1)},$$

where $x_{\infty} = \lim_{t\to\infty} x(t)$. Thus for $t \geqslant T$ we have

$$\frac{1}{x(t)} \geqslant k_1\big(t^{\gamma}b(t)\big)^{1/(\beta-\alpha)}t^{-\alpha/(\alpha+1)} = k_1F^{1/(\beta-\alpha)}(t)t^{-\alpha/(\alpha+1)},$$

where

$$k_1 = \left(k\frac{\alpha+1}{\alpha}\right)^{\alpha/(\beta-\alpha)},$$

which yields (13). $\square$

Using Theorem 2, we obtain the following existence result for intermediate solutions of (2).

Theorem 3. *Let $\alpha < \beta$ and F be nondecreasing on $[T,\infty)$, $T > 0$. If $J_1 < \infty$, $K_1 = \infty$ and*

$$\int_T^\infty b(t)\varphi^\beta(t)\,dt < \infty, \tag{15}$$

where φ is given by (14). Then (2) has intermediate solutions x such that

$$0 < x(t) \leqslant \varphi(t) \leqslant kt^{\frac{\alpha}{\alpha+1}}$$

for $t \geqslant T$ and some $k > 0$.

Proof. Since $J_1 < \infty$, in view of Proposition 1, for any $n > 0$ Eq. (2) has a subdominant solution x_n such that

$$\lim_{t\to\infty} x_n(t) = n,$$

In virtue of Theorem 2, we have $0 < x_n(t) \leqslant \varphi(t)$ for $t \in [T,\infty)$, and so the sequence $\{x_n\}$ is equibounded on every finite subinterval of $[T,\infty)$. Again in view of Theorem 2, from (2) we get for $t \geqslant T$

$$0 \leqslant \left(x_n'(t)\right)^\alpha = \int_t^\infty b(r)x_n^\beta(r)\,dr \leqslant \int_t^\infty b(r)\varphi^\beta(r)\,dr. \tag{16}$$

Thus, the sequence $\{x_n\}$ is also equicontinuous on every finite subinterval of $[T,\infty)$. Moreover, from (2) and (16) also the sequence $\{x_n'\}$ is equibounded and equicontinuous on every finite subinterval of $[T,\infty)$. Hence there exists a converging subsequence $\{x_{n_j}^{(i)}\}$, $i = 0, 1$, which uniformly converges to a function $x^{(i)}$ on every finite subinterval of $[T,\infty)$. Clearly, x is an unbounded solution of (2). Since $J_2 = \infty$, then x is an intermediate solution of (2) and the proof is complete. □

Observe that the assumption "F nondecreasing for large t" is related with the existence of oscillatory solutions. More precisely, the following well-known result holds, see, e.g., [16, Theorem 18.4], [4, Theorem 1].

Theorem A. *If F is nondecreasing on $[T,\infty)$, $T > 0$ and b is locally of bounded variation on $[T,\infty)$, then (2) has oscillatory solutions.*

Hence, if b is sufficiently smooth, assumptions in Theorem 3 give not only the existence of two types of nonoscillatory solutions for (2), namely subdominant and intermediate solutions, but also the existence of oscillatory solutions.

Theorem 3 is new also for the Emden–Fowler equation (3) and reads as follows.

Corollary 3. *Let $\beta > 1$, the function $F_1(t) = t^{(\beta+3)/2}b(t)$ is nondecreasing on $[T,\infty)$, $T > 0$ and*

$$\int_0^\infty tb(t)dt < \infty, \quad \int_0^\infty t^\beta b(t)\,dt = \infty. \tag{17}$$

If

$$\int_T^\infty b(t)\varphi_1^\beta(t)\,dt < \infty, \tag{18}$$

where

$$\varphi_1(t) = c\big(F_1(t)\big)^{-1/(\beta-1)} t^{1/2}, \tag{19}$$

then (3) has intermediate solutions x such that

$$0 < x(t) \leqslant \varphi_1(t) \leqslant k t^{1/2} \tag{20}$$

for $t \geqslant T$ and some $k > 0$.

5. Extension to the general weight a

In this section we extend our main results on (non)existence of intermediate solutions to Eq. (1). Set

$$A(t) = \int_0^t a^{-1/\alpha}(\sigma)\, d\sigma.$$

In view of $I_a = \infty$, the change of variable

$$s = A(t), \qquad X(s) = x(t), \quad t \in [0,\infty),\ s \in [0,\infty) \tag{21}$$

transforms (1), $t \in [0,\infty)$, into

$$\frac{d}{ds}\big(\big|\dot{X}(s)\big|^{\alpha} \operatorname{sgn} \dot{X}(s)\big) + c(s) X^{\beta}(s) = 0, \quad s \in [0,\infty), \tag{22}$$

whereby $t(s)$ is the inverse function of $s(t)$, the function c is given by

$$c(s) = a^{1/\alpha}\big(t(s)\big) b\big(t(s)\big) \tag{23}$$

and the symbol $\dot{}$ denotes the derivative with respect to the variable s. Denote by G and Φ the functions

$$G(t) = A^{\gamma}(t) a^{1/\alpha}(t) b(t) \tag{24}$$

and

$$\Phi(t) = c\left(\frac{1}{G(t)}\right)^{1/(\beta-\alpha)} A^{\alpha/(\alpha+1)}(t). \tag{25}$$

The change of variable (21) transforms subdominant, intermediate and dominant solutions of (1) into subdominant, intermediate and dominant solutions of (22), respectively. From here, Theorem 1 and Corollary 2 we get the following.

Theorem 4. *Let $\alpha < \beta$. If $K < \infty$, then (1) does not have intermediate solutions.*

Corollary 4. *Eq. (1) does not admit simultaneously dominant, intermediate and subdominant solutions.*

Similarly, from Theorem 3 we obtain the following.

Theorem 5. *Let $\alpha < \beta$ and assume that the function G, given by (24), is nondecreasing on $[\tau, \infty), \tau > 0$. If $J < \infty$, $K = \infty$, and*

$$\int_{\tau}^{\infty} b(t)\Phi^{\beta}(t)\,dr < \infty,$$

where Φ is given by (25), then (1) has intermediate solutions x such that for $t \geqslant \tau$

$$0 < x(t) \leqslant \Phi(t) \leqslant kA^{\frac{\alpha}{\alpha+1}}(t).$$

Proof. Since G is non decreasing on $[\tau, \infty)$ and $\tau > 0$, in view of (23) the function $s^{\gamma}c(s)$ is nondecreasing on $[S, \infty)$, where $S = A(\tau)$. Hence, since $A(\tau) > 0$, the assertion follows applying Theorem 3 to (22). □

6. Examples and concluding remarks

The following examples illustrate our results in Section 4.

Example 1. Consider the Emden–Fowler equation

$$x'' + \frac{1}{4(t+1)^{(\beta+3)/2}}|x|^{\beta}\operatorname{sgn} x = 0, \quad \beta > 1,\ t \geqslant 0. \tag{26}$$

We have

$$F_1(t) = \frac{1}{4}\left(\frac{t}{t+1}\right)^{(\beta+3)/2}.$$

Hence F_1 is increasing for $t \geqslant 0$ and a standard calculation shows that conditions (17), (18) are satisfied. Thus, by Corollary 3, Eq. (26) has intermediate solutions satisfying $0 < x(t) \leqslant c\sqrt{t}$ for $t > 0$ and some $c > 0$. Clearly, $x(t) = \sqrt{t+1}$ is such solution. Moreover, by Theorem A, (26) has also oscillatory solutions, and by Proposition 1 and Theorem 2 also subdominant solutions which are positive for $t > 0$.

Observe that the translation $s = t + 1$ transforms (26) into

$$\frac{d^2}{ds^2}y(s) + \frac{1}{4s^{(\beta+3)/2}}|y|^{\beta}\operatorname{sgn} y = 0, \quad \beta > 1,\ s \geqslant 1,$$

which has been deeply investigated in [19, pp. 33–35], where the following picture of solutions y passing through the point $y(1) = \ell > 0$ was obtained. Denote by $\dot{y}$ the derivative of y with respect to the variable s. For $\dot{y}(1) = \ell/2 \pm d$ (d is a suitable positive constant), the solutions y are positive and tend to a finite limit as $t \to \infty$, for $\ell/2 - d < \dot{y}(1) < \ell/2 + d$, $\dot{y}(1) \neq 1/2$, the solutions are positive and intersect the solution $\sqrt{t}$ an infinity of times and for $\dot{y}(1) < \ell/2 - d$ and $\dot{y}(1) > \ell/2 + d$, the solutions are oscillatory. This observation shows that Eq. (3) can have infinitely many intermediate solutions.

Example 2. Consider the Emden–Fowler equation

$$x'' + \frac{1}{(t+2)^2\log^2(t+2)}|x|^2\operatorname{sgn} x = 0, \quad t \geqslant 0. \tag{27}$$

We have $\beta = 2$ and

$$F_1(t) = \left(\frac{t}{t+2}\right)^2\frac{\sqrt{t}}{\log^2 t}, \qquad \varphi_1(t) = 4c\left(\frac{t+2}{t}\right)^2\log^2 t.$$

Thus, condition (18) is satisfied. Since also (17) is valid, using Proposition 1, Theorem 2, Corollary 3 and Theorem A, we get that (27) has oscillatory solutions, subdominant and intermediate positive solutions for $t > 0$. Obviously, the function $x(t) = \log(t + 2)$ is such solution.

Concluding remarks.

(1) Consider (1) and the opposite case $I_a < \infty$, $I_b = \infty$. The same coexistence problem (Sections 3, 5) for the sub-linear case $\alpha > \beta$ has been treated in the recent paper [14], for the super-linear case $\alpha < \beta$ is considered in a forthcoming paper [9].

(2) The existence of intermediate solutions of (2), given by Theorem 3, requires in particular that $J_1 < \infty$, $K_1 = \infty$ and (15). In view of Proposition 1 and Theorem 1, the conditions $J_1 < \infty$, $K_1 = \infty$ are also necessary. It is a question whether the assumption "$b(\cdot)\varphi^\beta(\cdot) \in L^1$ in a neighborhood of infinity" is necessary for the validity of Theorem 3.

(3) Theorem 3 ensures the existence of intermediate solutions when the function F is nondecreasing. Does it hold the existence of intermediate solutions of (1) when F is eventually decreasing? The following example illustrates this problem.

Example 3. Consider the Emden–Fowler equation

$$x'' + \frac{1}{4(t+2)^{5/2}\log(t+2)}|x|^2 \operatorname{sgn} x = 0, \quad t \geqslant 0. \tag{28}$$

We have $\beta = 2$ and

$$F_1(t) = \left(\frac{t}{t+2}\right)^{5/2} \frac{1}{\log(t+2)}.$$

Thus, F_1 is eventually decreasing and Corollary 3 cannot be applied. Nevertheless, $x(t) = (t+2)^{1/2}\log(t+2)$ is an intermediate solution of (28) satisfying $x(t) \leqslant kF_1^{-1}(t)\sqrt{t}$ for $t > 0$ and some $k > 0$.

7. Note added in proof

After this paper was written, an existence result for intermediate solution of (1), where the function b is, roughly speaking, close to the function $t^{-\nu}, \nu > 0$, and $a \equiv 1$, is given in [21].

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