Available online at www.sciencedirect.com
ScienceDirect
J. Differential Equations 256 (2014) 1157–1190
Journal of Differential Equations
www.elsevier.com/locate/jde

Melnikov theory for nonlinear implicit ODEs

Flaviano Battelli [a,∗,1], Michal Fečkan [b,c,2]

[a] *Department of Industrial Engineering and Mathematical Sciences, Marche Polytecnic University, Via Brecce Bianche 1, 60131 Ancona, Italy*
[b] *Department of Mathematical Analysis and Numerical Mathematics, Comenius University, Mlynská dolina, 842 48 Bratislava, Slovakia*
[c] *Mathematical Institute of Slovak Academy of Sciences, Štefánikova 49, 814 73 Bratislava, Slovakia*

Received 23 May 2013; revised 11 October 2013

Available online 4 November 2013

Abstract

We apply dynamical system methods and Melnikov theory to study small amplitude perturbation of some implicit differential equations. In particular we show persistence of such orbits connecting singularities in finite time provided a Melnikov like condition holds.

MSC: 34A09; 34C23; 37G99

Keywords: Implicit ode; Perturbation; Melnikov method

1. Introduction

Quasilinear implicit differential equations such as

$$A(x)\dot{x} = f(x) + \varepsilon g(x, t, \varepsilon, \kappa), \qquad \dot{} = \frac{d}{dt} \tag{1.1}$$

* Corresponding author.
E-mail addresses: battelli@dipmat.univpm.it (F. Battelli), Michal.Feckan@fmph.uniba.sk (M. Fečkan).
[1] Partially supported by PRIN-MURST *Equazioni Differenziali Ordinarie e Applicazioni*.
[2] Partially supported by the Grants VEGA-MS 1/0507/11, VEGA-SAV 2/0029/13 and APVV-0134-10.

http://dx.doi.org/10.1016/j.jde.2013.10.012

where $x \in \mathbb{R}^n$ and $A(x)$ is an $n \times n$-matrix of constant rank, find applications in a large number of physical sciences and have been studied by several authors [11,14–17,19–22]. As an example, motivated by [12,23], in [2] an equation modeling a nonlinear RLC circuit with nonlinear capacity has been studied. It has been shown there that such an equation exhibits I-singularities in the sense explained in [22] and that such points persist under perturbations. On the other hand, there are many other works on implicit differential equations [1,9,7,8,10,13] dealing with more general implicit differential systems by using analytical and topological methods.

Equations like (1.1) are usually handled by multiplying them by the adjugate matrix adj $A(x)$ (transpose of the matrix of cofactors). It is obtained, then, the implicit ODE (IODE)

$$\omega(x)\dot{x} = F(x) + \varepsilon G(x, t, \varepsilon, \kappa), \tag{1.2}$$

where

$$\begin{aligned} \omega(x) &= \det A(x), \\ F(x) &= \operatorname{adj} A(x) f(x), \\ G(x, t, \varepsilon, \kappa) &= \operatorname{adj} A(x) g(x, t, \varepsilon, \kappa). \end{aligned} \tag{1.3}$$

Assuming that $A \in C^2(\mathbb{R}^n, L(\mathbb{R}^n))$, $f \in C^2(\mathbb{R}^n, \mathbb{R}^n)$ and $g \in C^2(\mathbb{R}^{n+m+2}, \mathbb{R}^n)$ is 1-periodic in t, then $\omega \in C^2(\mathbb{R}^n, \mathbb{R})$, $F \in C^2(\mathbb{R}^n, \mathbb{R}^n)$ and $G \in C^2(\mathbb{R}^{n+m+2}, \mathbb{R}^n)$ is 1-periodic in t.

As this paper is a continuation of [2], we shall look directly at Eq. (1.2), without considering condition (1.3) assuming $\omega \in C^2(\mathbb{R}^n, \mathbb{R})$, $F \in C^2(\mathbb{R}^n, \mathbb{R}^n)$ and $G \in C^2(\mathbb{R}^{n+m+2}, \mathbb{R}^n)$ is 1-periodic in t. Our objective is to give Melnikov like conditions for the existence of homoclinic like solutions of (1.2) connecting a singular point in finite time. Thus we assume the following conditions hold.

(C1) The unperturbed (1.2):

$$\omega(x)\dot{x} = F(x) \tag{1.4}$$

possesses a noncritical singularity at x_0, i.e. $\omega(x_0) = 0$ and $\omega'(x_0) \neq 0$.

(C2) The ODE:

$$x' = F(x), \quad {}' = \frac{d}{ds} \tag{1.5}$$

has the hyperbolic equilibrium x_0 [i.e. $F(x_0) = 0$ and the spectrum $\sigma(DF(x_0))$ has no eigenvalues on the imaginary axis] and a solution $\gamma(s)$ homoclinic to it, that is $\lim_{s\to\pm\infty} \gamma(s) = x_0$, and $\omega(\gamma(s)) \neq 0$ for any $s \in \mathbb{R}$. Moreover $G(x_0, t, \varepsilon, \kappa) = 0$ for any $t \in \mathbb{R}$, $\kappa \in \mathbb{R}^m$ and ε sufficiently small. Without loss of generality, we may, and will, assume $\omega(\gamma(s)) > 0$ for any $s \in \mathbb{R}$.

(C3) It results

$$\lim_{s\to\infty} \frac{1}{s} \log\left|\gamma(s) - x_0\right| = \mu_-, \qquad \lim_{s\to-\infty} \frac{1}{s} \log\left|\gamma(s) - x_0\right| = \mu_+ \tag{1.6}$$

where $\mu_\mp$ are simple eigenvalues of $F'(x_0)$ with the corresponding eigenvectors $\gamma_\pm$ and all the other eigenvalues of $F'(x_0)$ have real parts either less than μ_- of greater than μ_+.

(C4) $\langle \nabla\omega(x_0), \gamma_\pm\rangle > 0$ (or else $\omega'(x_0)\gamma_\pm > 0$).

(C5) $\frac{\gamma'(s)}{\omega(\gamma(s))}$ is the unique bounded solution, up to a multiplicative constant, of the linear system

$$x' = \left[F'\big(\gamma(s)\big) - \frac{F(\gamma(s))}{\omega(\gamma(s))}\omega'\big(\gamma(s)\big)\right]x$$
$$= F'\big(\gamma(s)\big)x - \frac{\omega'(\gamma(s))x}{\omega(\gamma(s))}F\big(\gamma(s)\big). \tag{1.7}$$

Remark 1.1.

(1) There is no straightforward change of variable so that (1.4) can be changed into (1.5). However in Section 3 (see Eq. (3.2)) we will see that a change of time $t = \theta(s)$ exists that converts the solution $\gamma(s)$ of (1.5) into a solution of (1.4).

(2) As the reader may have guessed, throughout the paper we use the notation $\dot{x} = \frac{dx}{dt}$ and $x' = \frac{dx}{ds}$. We hope this won't lead any misunderstanding since we will also write $\omega'(x)$ and $F'(x)$ for the derivative of $\omega(x)$, $F(x)$ with respect to x.

(3) Since $\gamma'(s) = F(\gamma(s))$ we immediately infer that $\frac{\gamma'(s)}{\omega(\gamma(s))}$ solves (1.7). In Section 3 we will also prove that $\frac{\gamma'(s)}{\omega(\gamma(s))}$ has finite limits as $s \to \pm\infty$. Thus in condition (C5) only uniqueness matters.

(4) It would seem more natural to assume conditions on the linearization

$$x' = F'\big(\gamma(s)\big)x \tag{1.8}$$

of (1.5) along $\gamma(s)$ rather than on (1.7). However we will see in next Section 2 that assuming conditions on (1.7) is in some sense more natural. By the way it follows from remark (c) in Section 6 that, under conditions (C1)–(C4) assumption (C5) is actually equivalent to the fact that $\gamma'(s)$ is the unique (up to a multiplicative constant) solution of (1.8) which is bounded on $\mathbb{R}$.

The paper is organized as follows. Section 2 is devoted to deeper explanation of the main assumptions (C3), (C4) and (C5). The main existence result is derived in Section 3 by obtaining a Melnikov function. A relationship between the Melnikov function and an adjoint linear problem is explained in Section 4. Several roughness theorems for exponential dichotomies of linear ODE which are used in Section 4 are presented in Section 5. Finally, some remarks are collected in Section 6.

2. Comments on the assumptions

In this section we pause to comment on conditions (C3), (C4) and (C5). First, concerning (C3) we prove the following result.

Proposition 2.1. *Suppose the following holds*:

(D) *There is a positive function* $\varphi \in C^2(\mathbb{R}, \mathbb{R})$ *such that* $\lim_{s\to\pm\infty}\varphi(s) = 0$, $\lim_{s\to\pm\infty}\frac{\gamma(s)-x_0}{\varphi(s)} = \gamma_\pm \neq 0$ *and* $\lim_{s\to\pm\infty}\frac{\varphi'(s)}{\varphi(s)} = \mu_\mp \neq 0$.

Then μ_- (*resp.* μ_+) *is a negative* (*resp. positive*) *eigenvalue of* $F'(x_0)$ *with* γ_+ (*resp.* γ_-) *as eigenvector and* (1.6) *holds.*

Vice versa if Eq. (1.6) *holds then* μ_- (*resp.* μ_+) *is a negative* (*resp. positive*) *real part of an eigenvalue of* $F'(x_0)$. *Moreover if*

(C) $\mu_\pm$ *are eigenvalues of* $F'(x_0)$, *and no other eigenvalues of* $F'(x_0)$ *have* $\mu_\pm$ *as real parts*

then (D) *holds.*

Proof. Suppose (D) holds. Since $\varphi(s) > 0$ we certainly have $\mu_- < 0 < \mu_+$. Then setting

$$\eta(s) := \frac{\gamma(s) - x_0}{\varphi(s)}$$

one has $\lim_{s\to\pm\infty} \eta(s) = \gamma_\pm$ and

$$\eta'(s) = \frac{F(x_0 + \varphi(s)\eta(s))}{\varphi(s)} - \frac{\varphi'(s)}{\varphi(s)}\eta(s).$$

Then

$$\lim_{s\to\pm\infty} \eta'(s) = \lim_{s\to\pm\infty} \frac{F'(x_0)\varphi(s)\eta(s) + o(\varphi(s)\eta(s))}{\varphi(s)} - \frac{\varphi'(s)}{\varphi(s)}\eta(s) = F'(x_0)\gamma_\pm - \mu_\mp\gamma_\pm.$$

But, since $\eta(s)$ is bounded and the limits $\lim_{s\to\pm\infty} \eta'(s)$ exist, they must be zero. So

$$F'(x_0)\gamma_\pm = \mu_\mp\gamma_\pm$$

that is $\mu_\mp$ are eigenvalues of $F'(x_0)$ with $\gamma_\pm$ as corresponding eigenvectors.

Next, to prove (1.6), consider the function $\varphi_0(s) := \varphi(s)e^{-\mu_- s}$. We have

$$\frac{\varphi_0'(s)}{\varphi_0(s)} = \frac{\varphi'(s)}{\varphi(s)} - \mu_- \to 0 \quad \text{as } s \to \infty.$$

Hence for any $\delta > 0$ there exists $\bar{s} > 0$ such that

$$-\delta < \frac{\varphi_0'(s)}{\varphi_0(s)} < \delta$$

for any $s \geqslant \bar{s}$. Integrating in $[\bar{s}, s]$ we get

$$\varphi(\bar{s})e^{(\mu_- - \delta)(s-\bar{s})} < \varphi(s) < \varphi(\bar{s})e^{(\mu_- + \delta)(s-\bar{s})}$$

or

$$\varphi(\bar{s})e^{-\delta(s-\bar{s})+\mu_-\bar{s}} < \frac{\varphi(s)}{e^{\mu_- s}} < \varphi(\bar{s})e^{\delta(s-\bar{s})+\mu_-\bar{s}}.$$

As a consequence:

$$-\delta \leqslant \liminf_{s\to\infty} \frac{1}{s} \log \frac{\varphi(s)}{e^{\mu_- s}} \leqslant \limsup_{s\to\infty} \frac{1}{s} \log \frac{\varphi(s)}{e^{\mu_- s}} \leqslant \delta$$

from which we deduce

$$\lim_{s\to\infty} \frac{1}{s} \log \frac{\varphi(s)}{e^{\mu_- s}} = 0.$$

So, for $s > \bar{s}$ we get

$$\frac{1}{s} \log\bigl|\gamma(s) - x_0\bigr| - \mu_- = \frac{1}{s} \log \frac{|\gamma(s) - x_0|}{e^{\mu_- s}} = \frac{1}{s}\left[\log \frac{|\gamma(s) - x_0|}{\varphi(s)} + \log \frac{\varphi(s)}{e^{\mu_- s}}\right]$$

and then

$$\lim_{s\to\infty} \frac{1}{s} \log\bigl|\gamma(s) - x_0\bigr| = \mu_-.$$

Similarly we prove that

$$\lim_{s\to-\infty} \frac{1}{s} \log\bigl|\gamma(s) - x_0\bigr| = \mu_+.$$

Vice versa, suppose that (1.6) holds. Then $u(s) = \gamma(s) - x_0$ is a non-zero solution of the equation:

$$u' = F'(x_0)u + \bigl[F(x_0 + u) - F(x_0) - F'(x_0)u\bigr]$$

and $F(x_0 + u) - F(x_0) - F'(x_0)u = O(u^2)$. Moreover, since x_0 is hyperbolic and $u(s) \to 0$ as $s \to \infty$ we know that $|u'(s)| \leqslant K e^{-\mu s}$, for a suitable $\mu > 0$. Then

$$\bigl|u(s)\bigr| \leqslant \int_s^\infty \bigl|u'(s)\bigr|\,ds \leqslant K\mu^{-1} e^{-\mu s}$$

and

$$\lim_{s\to\infty} \frac{\log|u(s)|}{s} \leqslant -\mu < 0.$$

From [6, Theorem 4.3, p. 335] it follows that $\mu_- := \lim_{s\to\infty} \frac{\log|u(s)|}{s} \leqslant -\mu < 0$ is the real part of an eigenvalue of $F'(x_0)$ and from [6, Theorem 4.5, p. 338] it follows also that a positive number $\delta > 0$ and solution $v(t)$ of the equation $x' = F'(x_0)x$ exist such that

$$\bigl|u(s) - v(s)\bigr| = O\bigl(e^{(\mu_- - \delta)s}\bigr). \tag{2.1}$$

Moreover, from the proof of [6, Theorem 4.5, p. 338] and (C) it follows that

$$v(s) = q_-(s)e^{\mu_- s},$$

where $q_-(s)$ is a non-zero vector valued polynomial whose degree d_- is less than the algebraic multiplicity of the eigenvalue μ_-. So using also (2.1):

$$\lim_{s\to\infty}\frac{\gamma(s)-x_0}{s^{d_-}e^{\mu_- s}}=\gamma_+\neq 0. \tag{2.2}$$

A similar argument works as $s\to-\infty$ that is there exists $d_+\geqslant 0$ and a vector $\gamma_-\neq 0$ such that

$$\lim_{s\to-\infty}\frac{\gamma(s)-x_0}{s^{d_+}e^{\mu_+ s}}=\gamma_-. \tag{2.3}$$

To complete the proof of the proposition we set

$$\varphi(s)=\frac{1}{\varphi_-(s)^{-1}+\varphi_+(s)^{-1}}$$

where

$$\varphi_-(s)=\sqrt{s^{2d_-}+1}e^{\mu_- s}\quad\text{and}\quad\varphi_+(s)=\sqrt{s^{2d_+}+1}e^{\mu_+ s}.$$

As a matter of fact, using (2.2), (2.3) and

$$\lim_{s\to\infty}\frac{\varphi_-'(s)}{\varphi_-(s)}=\mu_-,\qquad\lim_{s\to-\infty}\frac{\varphi_+'(s)}{\varphi_+(s)}=\mu_+,$$
$$\lim_{s\to\infty}\frac{\varphi_-(s)}{\varphi(s)}=1,\qquad\lim_{s\to-\infty}\frac{\varphi_+(s)}{\varphi(s)}=1,$$
$$\frac{\varphi'(s)}{\varphi(s)}=\frac{\varphi_-'(s)}{\varphi_-(s)}\frac{1}{1+\frac{\varphi_-(s)}{\varphi_+(s)}}+\frac{\varphi_+'(s)}{\varphi_+(s)}\frac{1}{1+\frac{\varphi_+(s)}{\varphi_-(s)}}$$

we easily see that $\varphi(s)$ satisfies assumption (D) and then $\mu_\mp$ are eigenvalues of $F'(x_0)$ with eigenvectors $\gamma_\pm$. □

Of course condition (C3) is stronger than both conditions (C) and (D). It will be clear later why we need it.

Next, in the light of Proposition 2.1 we derive

$$\lim_{s\to\pm\infty}\frac{\omega(\gamma(s))}{\varphi(s)}=\lim_{s\to\pm\infty}\frac{\omega'(x_0)(\gamma(s)-x_0)+o(\gamma(s)-x_0)}{\varphi(s)}=\langle\nabla\omega(x_0),\gamma_\pm\rangle.$$

By (C2), we know $\omega(\gamma(s))>0$ for any $s\in\mathbb{R}$, so $\langle\nabla\omega(x_0),\gamma_\pm\rangle\geqslant 0$. Hence condition (C4) means that $\gamma(s)$ tends transversally to the singular manifold $\omega^{-1}(0)$ at x_0.

Next we look at assumption (C5). First we note that

$$\frac{\gamma'(s)}{\omega(\gamma(s))}=\frac{F(\gamma(s))}{\omega(\gamma(s))}=\frac{F'(x_0)(\gamma(s)-x_0)+o(\gamma(s)-x_0)}{\omega'(x_0)(\gamma(s)-x_0)+o(\gamma(s)-x_0)}$$

and then

$$\lim_{s\to\pm\infty}\frac{\gamma'(s)}{\omega(\gamma(s))}=\frac{F'(x_0)\gamma_\pm}{\omega'(x_0)\gamma_\pm}=\frac{\mu_\mp\gamma_\pm}{\omega'(x_0)\gamma_\pm}. \tag{2.4}$$

Hence $\frac{\gamma'(s)}{\omega(\gamma(s))}$ is bounded on $\mathbb{R}$ and it is easy to check that it solves (1.7). Next let

$$\theta(s):=\int_0^s\omega\big(\gamma(\tau)\big)\,d\tau$$

and set $\Gamma(t)=\gamma(\theta^{-1}(t))$. Then $\Gamma(t)$ satisfies $\omega(x)\dot x=F(x)$ whose linearization along $\Gamma(t)$ is

$$F'\big(\Gamma(t)\big)z=\dot\Gamma(t)\omega'\big(\Gamma(t)\big)z+\omega\big(\Gamma(t)\big)\dot z=F\big(\Gamma(t)\big)\frac{\omega'(\Gamma(t))z}{\omega(\Gamma(t))}+\omega\big(\Gamma(t)\big)\dot z$$

i.e.

$$\omega\big(\Gamma(t)\big)\dot z=F'\big(\Gamma(t)\big)z-F\big(\Gamma(t)\big)\frac{\omega'(\Gamma(t))z}{\omega(\Gamma(t))} \tag{2.5}$$

and (1.7) is derived from (2.5) taking $x(s)=z(\theta(s))$. This fact should clarify why we need to consider the linear system (1.7) instead of (1.8). We set

$$\varphi(s):=\frac{1}{\mathrm{e}^{-\mu_-s}+\mathrm{e}^{-\mu_+s}}$$

and note that, from assumption (C3) and the proof of Proposition 2.1 we have

$$\lim_{s\to\pm\infty}\frac{\gamma(s)-x_0}{\varphi(s)}=\gamma_\pm\neq 0. \tag{2.6}$$

We emphasize the fact that many of the results obtained in the next section depend on the fact that (2.6) holds with

$$\lim_{s\to\pm\infty}\frac{\varphi'(s)}{\varphi(s)}=\mu_\mp.$$

Remark 2.2. An alternative way for condition (C2) would be to assume that there is a bounded solution $\Gamma(t)$ of (1.4) on a finite open interval asymptotic to x_0 such that $\omega(\Gamma(t))\neq 0$. But then it would be awkward to define the hyperbolicity of (1.1) at x_0. For this reason, we follow our way by transforming the problem into common one on the infinite interval $\mathbb{R}$.

3. Existence of bounded solutions

In this section we construct a Melnikov like function to characterize the bifurcation to a solution of the perturbed equation (1.2) tending to x_0 in finite time. First, from (C1) we get

$$\omega(x) = \big\langle \nabla\omega(x_0), x - x_0\big\rangle + O\big(|x - x_0|^2\big) \tag{3.1}$$

for a scalar product $\langle\cdot,\cdot\rangle$ in $\mathbb{R}^n$. Then taking the shift and change of time

$$t = \alpha + \theta(s), \quad \theta(s) := \int_0^s \omega\big(\gamma(r)\big)\, dr \tag{3.2}$$

we get

$$\omega(z)z' = \omega\big(\gamma(s)\big)\big(F(z) + \varepsilon G\big(z, \alpha + \theta(s), \varepsilon, \kappa\big)\big) \tag{3.3}$$

with $z(s) = x(\alpha + \theta(s))$. By (3.1) we have

$$T_\pm := \lim_{s\to\pm\infty} \theta(s) = \int_0^{\pm\infty} \omega\big(\gamma(r)\big)\, dr < \infty.$$

We are looking for solutions $z(s)$ of Eq. (3.3) tending to x_0 at the same rate as $\gamma(s)$. Then $x(t) = z(\theta^{-1}(t - \alpha))$ is a solution of Eq. (1.2) tending to x_0 as $t \to \pm T + \alpha$ at the same rate as $\Gamma(t - \alpha) - x_0$.

So in (3.3) we make the change of variables

$$z(s) = \gamma(s) + \varphi(s)y(s) = x_0 + \varphi(s)\big(\eta(s) + y(s)\big) \tag{3.4}$$

where $\eta(s)$ is the bounded function $\frac{\gamma(s)-x_0}{\varphi(s)}$. First we note (see (3.1))

$$\begin{aligned} \omega\big(z(s)\big) &\geqslant \big\langle \nabla\omega(x_0), \varphi(s)\big(\eta(s) + y(s)\big)\big\rangle - K_1\big|\varphi(s)\big(\eta(s) + y(s)\big)\big|^2 \\ &= \varphi(s)\big(\big\langle \nabla\omega(x_0), \eta(s) + y(s)\big\rangle - K_1\varphi(s)\big|\big(\eta(s) + y(s)\big)\big|^2\big) \end{aligned} \tag{3.5}$$

for a constant $K_1 > 0$ and any $s \in \mathbb{R}$, $|y| \leqslant 1$. Then, using (C4), (3.5) implies

$$\omega\big(z(s)\big) \geqslant \frac{1}{2}\varphi(s)\big\langle \nabla\omega(x_0), \gamma_\pm\big\rangle > 0 \tag{3.6}$$

for $|s| > 0$ large and $|y|$ small. Then (3.6) and $\omega(\gamma(t)) > 0$ imply the existence of $M > 0$ and $\delta > 0$ so that

$$\omega\big(z(s)\big) \geqslant M\varphi(s)$$

for any $s \in \mathbb{R}$ and $|y| \leqslant \delta$. Now plugging (3.4) into (3.3) we derive the equation

$$
\begin{aligned}
y' = {} & \frac{\omega(\gamma)}{\varphi\omega(\gamma+\varphi y)} F(\gamma+\varphi y) - \frac{F(\gamma)}{\varphi} - \frac{\varphi'}{\varphi} y \\
& + \varepsilon \frac{\omega(\gamma)}{\varphi\omega(\gamma+\varphi y)} G\big(\gamma+\varphi y, \theta(s)+\alpha, \varepsilon, \kappa\big).
\end{aligned} \tag{3.7}
$$

Note that, from $G(x_0, t, \varepsilon, \kappa) = 0$ it follows that the quantity

$$
\frac{G(\gamma+\varphi y, \alpha+\theta(s), \varepsilon, \kappa)}{\varphi} = \frac{G(x_0+\varphi(\eta+y), \alpha+\theta(s), \varepsilon, \kappa)}{\varphi}
$$

is bounded uniformly in $s \in \mathbb{R}$, $\kappa \in \mathbb{R}^m$ and (y, ε) small.

Then the linearization of (3.7) at $y = 0$, $\varepsilon = 0$ is

$$
y' = \left[F'\big(\gamma(s)\big) - \frac{F(\gamma(s))\omega'(\gamma(s))}{\omega(\gamma(s))} - \frac{\varphi'(s)}{\varphi(s)}\mathbb{I}\right] y. \tag{3.8}
$$

We are interested in the limiting equation of (3.8) as $s \to \pm\infty$. To this end we need to evaluate the limits:

$$
\begin{aligned}
\lim_{s\to\pm\infty} \frac{F(\gamma(s))}{\omega(\gamma(s))}\omega'\big(\gamma(s)\big) y &= \lim_{s\to\pm\infty} \frac{F(\gamma(s))}{\omega(\gamma(s))}\omega'(x_0) y \\
&= \lim_{s\to\pm\infty} \frac{F(x_0+\varphi(s)\eta(s))}{\omega(x_0+\varphi(s)\eta(s))}\omega'(x_0) y \\
&= \lim_{s\to\pm\infty} \frac{F'(x_0)\eta(s)}{\omega'(x_0)\eta(s)}\omega'(x_0) y \\
&= \frac{F'(x_0)\gamma_\pm}{\omega'(x_0)\gamma_\pm}\omega'(x_0) y = \frac{\mu_\mp\gamma_\pm}{\omega'(x_0)\gamma_\pm}\omega'(x_0) y.
\end{aligned}
$$

So the limiting equation of (3.8) at $\pm\infty$ is respectively

$$
y' = \left[F'(x_0) - \frac{\mu_\mp\gamma_\pm}{\omega'(x_0)\gamma_\pm}\omega'(x_0) - \mu_\mp\mathbb{I}\right] y.
$$

Then we are lead to find the spectrum of the linear maps:

$$
y \mapsto F'(x_0) y - \frac{\mu_-\gamma_+}{\omega'(x_0)\gamma_+}\omega'(x_0) y - \mu_- y \tag{3.9}
$$

and

$$
y \mapsto F'(x_0) y - \frac{\mu_+\gamma_-}{\omega'(x_0)\gamma_-}\omega'(x_0) y - \mu_+ y. \tag{3.10}
$$

Setting $L = F'(x_0) - \mu_\mp\mathbb{I}$, $v_0 = \gamma_\pm$ and

$$
\omega_0^* = \frac{\mu_\mp}{\omega'(x_0)\gamma_\pm}\omega'(x_0),
$$

(3.9) and (3.10) have a form

$$v \mapsto \Upsilon v = Lv - (\omega_0^* v) v_0.$$

Note $Lv_0 = 0$. We prove the following.

Lemma 3.1. *Let V be a (real or complex) finite dimensional vector space and $L : V \to V$ be a linear map. Suppose that $Lv_0 = 0$ and that a subspace W of V exists such that $V = W \oplus [v_0]$ and $L : W \to W$. Let $\omega_0^* \in V^*$ be such that $\omega_0^* v_0 \neq 0$. Then for the spectrum of the linear map $\Upsilon : v \mapsto Lv - (\omega_0^* v)v_0$ the following hold:*

(a) *all the eigenvalues of L different from 0 and $-\omega_0^* v_0$ are also eigenvalues of Υ with the same algebraic and geometric multiplicities and vice versa;*
(b) *the geometric multiplicity of 0 as eigenvalue of Υ is one less of the geometric multiplicity of 0 as eigenvalue of L;*
(c) *$-\omega_0^* v_0$ is an eigenvalue of Υ and its geometric multiplicity is either equal to or one more than the geometric multiplicity of $-\omega_0^* v_0$ as eigenvalue of L.*

Moreover, in case (c) the second situation occurs if and only if all eigenvectors of L of the eigenvalue $-\omega_0^* v_0$ satisfy $\omega_0^* v = 0$.

Proof. Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ be the field of scalars, $\lambda_0 = -\omega_0^* v_0 \neq 0$ and $U = \{u \in V \mid \omega_0^* u = 0\}$. We split any $v = u + cv_0$, $u \in U$ and $c \in \mathbb{F}$. We have

$$\Upsilon v = Lu + c\lambda_0 v_0.$$

Then $\Upsilon v = 0$, for some $v \neq 0$ if and only if

(i) either $c = 0$ and $Lu = 0$, $u \neq 0$;
(ii) or $c \neq 0$ and $Lu = -c\lambda_0 v_0$.

Case (i) holds if and only if 0 is an eigenvalue of L with geometric multiplicity greater than 1. Indeed if $Lv_1 = 0$ for some v_1 independent of v_0, then its projection u on U along v_0 solves $Lu = 0$. Next, case (ii) holds if and only if $v_0 \in \mathcal{R}L$. Indeed from (ii) it immediately follows that $v_0 \in \mathcal{R}L$ (since $c, \lambda_0 \neq 0$), vice versa if $v_0 = Lv_1$ then the same equality holds for the projection u of v_1 on U along v_0 so (ii) holds with $c = -\lambda_0^{-1} \neq 0$. Finally, if (i) does not hold then v_0 is the unique eigenvector of the 0 eigenvalue of L but then

$$v_0 \in \mathcal{R}L \quad \Leftrightarrow \quad \mathcal{N}L^2 \neq \mathcal{N}L$$

since $\mathcal{N}L = \operatorname{span}\{v_0\}$. So $\Upsilon v = 0$, for some $v \neq 0$ if and only if either 0 is an eigenvalue of L with geometric multiplicity greater than 1 or 0 is an eigenvalue of L with geometric multiplicity equal to 1 but algebraic multiplicity greater than 1, i.e.

$$0 \notin \sigma(\Upsilon) \quad \text{if and only if} \quad 0 \text{ is a simple eigenvalue of } L.$$

Now consider the splitting $V = W \oplus \operatorname{span}\{v_0\}$ with $L : W \to W$. Let $\lambda \in \mathbb{F}$, $\lambda \neq 0$ and write $v = w + cv_0$. Then $Lv - \lambda v = 0$ if and only if $Lw - \lambda w = \lambda c v_0 \Rightarrow c = 0$ since $W \cap \operatorname{span}\{v_0\} = \{0\}$ and $Lw = \lambda w$. As a consequence for any $\lambda \neq 0$ it results

$$\mathcal{N}(L - \lambda\mathbb{I}) \cap W = \mathcal{N}(L - \lambda\mathbb{I}). \tag{3.11}$$

Next from Grassman formula, and $v_0 \in \mathcal{N}L$, $v_0 \notin W$ we easily get

$$\dim[\mathcal{N}L \cap W] = \dim \mathcal{N}L - 1, \tag{3.12}$$

hence we may suppose that W contains all the eigenvectors of the 0 eigenvalue of L that are independent of v_0.

We split any vector $v \in \mathbb{C}^n$ as $v = w + cv_0$, $w \in W$, $c \in \mathbb{C}$. Then, for any $0 \neq \lambda \in \mathbb{C}$, we have

$$\begin{aligned}(\Upsilon - \lambda\mathbb{I})v &= Lv - (\omega_0^* v)v_0 - \lambda v = (L - \lambda\mathbb{I})w - \omega_0^*(w + cv_0)v_0 - \lambda c v_0 \\ &= (L - \lambda\mathbb{I})w - (\omega_0^* w)v_0 - c(\lambda - \lambda_0)v_0 \\ &= (L - \lambda\mathbb{I})\left[w + \frac{1}{\lambda}\left[\omega_0^* w + c(\lambda - \lambda_0)\right]v_0\right] = (L - \lambda\mathbb{I})T_\lambda v,\end{aligned} \tag{3.13}$$

where

$$T_\lambda : w + cv_0 \to w + \frac{1}{\lambda}\left[\omega_0^* w + c(\lambda - \lambda_0)\right]v_0$$

is a linear isomorphism on V for any $\lambda \notin \{\lambda_0, 0\}$. So

$$\sigma(\Upsilon) \setminus \{\lambda_0, 0\} = \sigma(L) \setminus \{\lambda_0, 0\}, \tag{3.14}$$

and both the geometric and algebraic multiplicities of eigenvalues in (3.14) are preserved since T_λ is an isomorphism and $\det(\Upsilon - \lambda\mathbb{I}) = \det(L - \lambda\mathbb{I}) \det T_\lambda$. This proves (a).

Next the equality $(\Upsilon - \lambda\mathbb{I})v = (L - \lambda\mathbb{I})w - (\omega_0^* w)v_0 - c(\lambda - \lambda_0)v_0$ with $\lambda = 0$ reads

$$\Upsilon v = Lw - \left[\omega_0^* w - c\lambda_0\right]v_0$$

and hence $\Upsilon v = 0$ if and only if $Lw = 0$ and $c = \frac{\omega_0^* w}{\lambda_0}$, so from (3.12) we deduce that the geometric multiplicity of 0 as eigenvalue of Υ is one less than its geometric multiplicity of 0 as eigenvalue of L. This proves (b).

Finally if $\lambda = \lambda_0$ then from (3.13) it follows

$$(\Upsilon - \lambda_0\mathbb{I})v = (L - \lambda_0\mathbb{I})w - (\omega_0^* w)v_0.$$

Hence v_0 is an eigenvector of Υ with eigenvalue λ_0 and $(\Upsilon - \lambda_0\mathbb{I})v = 0$ if only if $(L - \lambda_0\mathbb{I})w = 0$ and $\omega_0^* w = 0$ that is $w \in \mathcal{N}(L - \lambda_0\mathbb{I}) \cap W \cap U = \mathcal{N}(L - \lambda_0\mathbb{I}) \cap U$ (see (3.11)). Since

$$\dim\left[\mathcal{N}(L - \lambda_0\mathbb{I}) \cap U\right] = \dim \mathcal{N}(L - \lambda_0\mathbb{I}) + \dim U - \dim\left[\mathcal{N}(L - \lambda_0\mathbb{I}) + U\right]$$

and $\dim U = n - 1$ we have the following possibilities:

(i) $\dim[\mathcal{N}(L-\lambda_0\mathbb{I})\cap U] = \dim\mathcal{N}(L-\lambda_0\mathbb{I})$,
(ii) $\dim[\mathcal{N}(L-\lambda_0\mathbb{I})\cap U] = \dim\mathcal{N}(L-\lambda_0\mathbb{I}) - 1$.

In the first case the geometric multiplicity of λ_0 as an eigenvalue of Υ is one more than the geometric multiplicity of λ_0 as an eigenvalue of L while in the second, the geometric multiplicity of λ_0 as eigenvalue of Υ equals to the geometric multiplicity of λ_0 as eigenvalue of L. This proves point (c). □

Let $m_a^L(\lambda)$, $m_g^L(\lambda)$ be the algebraic and geometric multiplicities of the eigenvalue λ of L. Lemma 3.1 implies that

$$m_a^L(0) + m_a^L(\lambda_0) = m_a^\Upsilon(0) + m_a^\Upsilon(\lambda_0) \tag{3.15}$$

since, according to Lemma 3.1 (a) $\sum_{\lambda\neq 0,\lambda_0} m_a^L(\lambda) = \sum_{\lambda\neq 0,\lambda_0} m_a^\Upsilon(\lambda)$. Furthermore $m_a^L(0) \geqslant 1$, $m_a^\Upsilon(\lambda_0) \geqslant 1$.

Now, suppose that 0 is a semi-simple eigenvalue of L i.e. $m_a^L(0) = m_g^L(0)$ and that λ_0 is not an eigenvalue of L i.e. $m_a^L(\lambda_0) = 0$. Applying Lemma 3.1 (b)–(c), to (3.15) we obtain

$$0 \leqslant m_a^\Upsilon(\lambda_0) - 1 = m_g^\Upsilon(0) - m_a^\Upsilon(0) \leqslant 0$$

that is

$$m_a^\Upsilon(\lambda_0) = 1, \qquad m_a^\Upsilon(0) = m_g^\Upsilon(0) = m_g^L(0) - 1.$$

So we proved the following.

Corollary 3.2. *Suppose the conditions of Lemma 3.1 hold and moreover, that 0 is a semi-simple eigenvalue of L and $-\omega_0^* v_0$ is not an eigenvalue of L. Then*

$$\sigma(\Upsilon) = \sigma(L) \cup \{-\omega_0^* v_0\} \setminus \{0\}.$$

Moreover 0 is a semi-simple eigenvalue of Υ of multiplicity one less than the multiplicity of 0 as eigenvalue of L, $-\omega_0^* v_0$ is a simple eigenvalue of Υ and the multiplicities of all the other eigenvalues of Υ are the same as eigenvalues of L.

When applying Lemma 3.1 to (3.9), (3.10) we note that $\lambda_0 = -\omega_0^* v_0 = -\mu_\mp$ and then λ_0 is an eigenvalue of $F'(x_0) - \mu_\mp\mathbb{I}$ if and only if the equation:

$$F'(x_0)v - \mu_\mp v = -\mu_\mp v \quad \Leftrightarrow \quad F'(x_0)v = 0$$

has a non-zero solution. But this contradicts the hyperbolicity of x_0. So, according to Corollary 3.2, from assumption (C3) we conclude that the spectrum of (3.9) is $\{\mu - \mu_- \mid \mu \in \sigma(F'(x_0)),\ \mu \neq \mu_-\} \cup \{-\mu_-\}$ and, similarly, that the spectrum of (3.10) is $\{\mu - \mu_+ \mid \mu \in \sigma(F'(x_0)),\ \mu \neq \mu_+\} \cup \{-\mu_+\}$.

As a consequence (3.9) has $k_- - 1$ eigenvalues with negative real parts, counted with multiplicities, and $n - k_- + 1$ with positive real parts and (3.10) has $k_- + 1$ eigenvalues with negative real parts, counted with multiplicities, and $n - k_- - 1$ with positive real parts. Here k_- is the number of eigenvalues of $F'(x_0)$ with negative real parts counted with multiplicities.

Then the linear system:

$$y' = F'(x_0)y - \frac{\mu_-\gamma_+}{\omega'(x_0)\gamma_+}\omega'(x_0)y - \mu_- y \tag{3.16}$$

has an exponential dichotomy on $\mathbb{R}$ with projection P_+^0 with rank $P_+^0 = k_- - 1$, and

$$y' = F'(x_0)y - \frac{\mu_+\gamma_-}{\omega'(x_0)\gamma_-}\omega'(x_0)y - \mu_+ y$$

has an exponential dichotomy on $\mathbb{R}$ with projection P_-^0 with rank $P_-^0 = k_- + 1$. As a consequence (see [3]) the linear system (3.8) has an exponential dichotomy on both $\mathbb{R}_+$ and $\mathbb{R}_-$ with projections $P_\pm$ such that

$$\operatorname{rank} P_+ = k_- - 1 \quad \text{and} \quad \operatorname{rank} P_- = k_- + 1$$

or $\dim \mathcal{R}P_+ = k_- - 1$, $\dim \mathcal{N}P_- = n - k_- - 1$. Using

$$\dim \mathcal{R}P_+ + \dim \mathcal{N}P_- = \dim[\mathcal{R}P_+ + \mathcal{N}P_-] + \dim[\mathcal{R}P_+ \cap \mathcal{N}P_-]$$

we get then

$$\dim[\mathcal{R}P_+ + \mathcal{N}P_-] + \dim[\mathcal{R}P_+ \cap \mathcal{N}P_-] = n - 2.$$

We want to show that, if (C5) holds, then $\mathcal{R}P_+ \cap \mathcal{N}P_- = \{0\}$. As a matter of facts it is easy to check that Eq. (3.8) is obtained from (1.7) by the change $x = \varphi(s)y$. Indeed plugging this change into Eq. (1.7) we get

$$\varphi(s)y' + \varphi'(s)y = x' = \left[F'\bigl(\gamma(s)\bigr) - \frac{F(\gamma(s))}{\omega(\gamma(s))}\omega'\bigl(\gamma(s)\bigr)\right]\varphi(s)y$$

that gives (3.8). So, if $x(0) \in \mathcal{R}P_+$ we have $y(0) \in \mathcal{R}P_+$ and hence $y(s)$ is a bounded solution of the linear equation (3.8) and then the solution of (1.7) starting from $\varphi(0)y(0)$ would tend to zero exponentially as $s \to \infty$. Similarly, if $y(0) \in \mathcal{N}P_-$ then the solution of (1.7) starting from $\varphi(0)y(0)$ would tend to zero exponentially as $s \to -\infty$. As a consequence if $\mathcal{R}P_+ \cap \mathcal{N}P_- \neq \{0\}$ Eq. (1.7) should have a non-zero solution tending to zero exponentially as $|s| \to \infty$ contradicting assumption (C5).

Hence $\dim[\mathcal{R}P_+ + \mathcal{N}P_-] = n - 2$ and the space of bounded solutions of the equation

$$y' = -\left[F'\bigl(\gamma(s)\bigr)^* - \frac{\omega'(\gamma(s))^* F(\gamma(s))^*}{\omega(\gamma(s))} - \frac{\varphi'(s)}{\varphi(s)}\mathbb{I}\right]y \tag{3.17}$$

adjoint to (3.8) (i.e. the space of solutions of (3.17) with initial values in the space $[\mathcal{R}P_+ + \mathcal{N}P_-]^\perp = \mathcal{N}P_+^* \cap \mathcal{R}P_-^*$) has dimension 2. Let the space of bounded solutions of (3.17) be spanned by $\psi_1(s)$, $\psi_2(s)$. Then from the general theory (see e.g. [5]) we deduce the following result.

Theorem 3.3. *Suppose conditions* (C1)–(C5) *hold and* $\kappa \in \mathbb{R}$. *Then Eq.* (3.17) *has a two-dimensional space of bounded solution. Let* $\psi_1(s)$ *and* $\psi_2(s)$ *be a basis for the space of bounded solutions of* (3.17). *If the Poincaré–Melnikov function*:

$$\begin{pmatrix} \mathcal{M}_1(\alpha,\kappa) \\ \mathcal{M}_2(\alpha,\kappa) \end{pmatrix} := \begin{pmatrix} \int_{-\infty}^{\infty} \frac{\psi_1^*(s)}{\varphi(s)} G(\gamma(s),\alpha+\theta(s),0,\kappa)\,ds \\ \int_{-\infty}^{\infty} \frac{\psi_2^*(s)}{\varphi(s)} G(\gamma(s),\alpha+\theta(s),0,\kappa)\,ds \end{pmatrix} \tag{3.18}$$

has a simple zero at (α_0,κ_0), *then there exist* $\rho > 0$, $\bar{\varepsilon} > 0$ *and* $\bar{\kappa} > 0$ *such that for any* $|\varepsilon| < \varepsilon_0$ *and* $|\kappa| < \bar{\kappa}$ *Eq.* (3.7) *has a unique bounded solution* $y(s,\varepsilon,\kappa)$ *such that* $\sup_{s\in\mathbb{R}} |y(s,\varepsilon,\kappa)| < \rho$. *Moreover*

$$\lim_{(\varepsilon,\kappa)\to(0,0)} \sup_{s\in\mathbb{R}} \bigl|y(s,\varepsilon,\kappa)\bigr| = 0.$$

Recall that $\frac{G(\gamma(s),\alpha+\theta(s),0,\kappa)}{\varphi(s)}$ is bounded on $s \in \mathbb{R}$, and, because of the exponential dichotomy on both $\mathbb{R}_+$ and $\mathbb{R}_-$ of system (3.8), $\psi_1(s)$, $\psi_2(s)$ tend to zero exponentially as $s \to \pm\infty$.

Remark 3.4.

(1) It follows from the discussion at the beginning of this section that, if assumptions of Theorem 3.3 hold, the perturbed equation (1.2) has a solution $x(t)$ on the open interval $(T_- + \alpha, T_+ + \alpha)$ tending to x_0 in finite time and at the same rate as $\Gamma(t-\alpha) - x_0$.

(2) Let us look at the spectrum of the linear maps:

$$\mathcal{L}_+ : y \mapsto F'(x_0)y - \frac{\mu_-\gamma_+}{\omega'(x_0)\gamma_+}\omega'(x_0)y - \mu_- y$$

and

$$\mathcal{L}_- : y \mapsto F'(x_0)y - \frac{\mu_+\gamma_-}{\omega'(x_0)\gamma_-}\omega'(x_0)y - \mu_+ y,$$

when $\omega(x)$ and $F(x)$ are as in (1.3), i.e. when Eq. (1.4) is obtained from (1.1) with $\varepsilon = 0$ multiplying by the adjugate matrix. According to Section 6, generically, $F'(x_0)$ has an $(n-2)$-dimensional kernel. But from (C2), it also has the eigenvectors $\gamma_\pm$ with (simple) eigenvalues $\mu_\mp$. Hence

$$\sigma\bigl(F'(x_0)\bigr) = \{0, \mu_\pm\}$$

where 0 is a semi-simple eigenvalue of multiplicity $n-2$ and $\mu_\pm$ are simple. Then the spectrum of

$$L_- : y \mapsto F'(x_0)y - \mu_+ y$$

is $\sigma(L_-) = \{-\mu_+, 0, \mu_- - \mu_+\}$ where $-\mu_+ < 0$ is semi-simple with multiplicity $n-2$, and 0, $\mu_- - \mu_+ < 0$ are simple. Similarly the spectrum of

$$L_+ : y \mapsto F'(x_0)y - \mu_- y$$

is $\sigma(L_+) = \{-\mu_-, 0, \mu_+ - \mu_-\}$ where $-\mu_- > 0$ is semi-simple with multiplicity $n-2$, and 0, $\mu_+ - \mu_+ > 0$ are simple. Then both $L_\pm$ satisfy the assumptions of Lemma 3.1 with $\omega_0^* = \frac{\mu_\mp}{\omega'(x_0)\gamma_\pm}\omega'(x_0)$, respectively.
Hence, according to Lemma 3.1 the spectrum of $\mathcal{L}_+$ is

$$\sigma(\mathcal{L}_+) = \{\mu_+ - \mu_-, -\mu_-\}$$

with $\mu_+ - \mu_- > 0$ of multiplicity 1 and the geometric multiplicity of $-\mu_- > 0$ is either equal to or one more than $n-2$ and hence, equal to $n-1$. Similarly

$$\sigma(\mathcal{L}_-) = \{\mu_- - \mu_+, -\mu_+\}$$

with $\mu_- - \mu_+ < 0$ of multiplicity 1 and the geometric multiplicity of $-\mu_+ < 0$ equals $n-1$. So $\mathcal{L}_+$ has only positive eigenvalues (unstable) and $\mathcal{L}_-$ has only negative eigenvalues (stable). As a consequence the linear system:

$$y' = \left[F'\big(\gamma(s)\big) - \frac{F(\gamma(s))\omega'(\gamma(s))}{\omega(\gamma(s))} - \frac{\varphi'(s)}{\varphi(s)}\mathbb{I}\right]y$$

has an exponential dichotomy on $\mathbb{R}_+$ with projection $P_+ = 0$ and on $\mathbb{R}_-$ with projection $P_- = \mathbb{I}$. This implies that

$$\mathcal{N}P_+^* \cap \mathcal{R}P_-^* = \mathbb{R}^n$$

that is all solutions of the adjoint system

$$y' = -\left[F'\big(\gamma(s)\big)^* - \frac{\omega'(\gamma(s))^* F(\gamma(s))^*}{\omega(\gamma(s))} - \frac{\varphi'(s)}{\varphi(s)}\mathbb{I}\right]y$$

are bounded on $\mathbb{R}$. Summarizing, when $\omega(x)$ and $F(x)$ are as in (1.3) we have a codimension n problem.

4. Melnikov function and the original equation

In this section we want to express the Melnikov function (3.18) in terms of the solutions of the equation adjoint to (2.5) and time t. Passing to time $t = \theta(s)$ we get

$$\begin{pmatrix} \mathcal{M}_1(\alpha,\kappa) \\ \mathcal{M}_2(\alpha,\kappa) \end{pmatrix} := \begin{pmatrix} \int_{T_-}^{T_+} \frac{\psi_1^*(\theta^{-1}(t))}{\varphi(\theta^{-1}(t))\omega(\Gamma(t))} G(\Gamma(t), t+\alpha, 0, \kappa)\,dt \\ \int_{T_-}^{T_+} \frac{\psi_2^*(\theta^{-1}(t))}{\varphi(\theta^{-1}(t))\omega(\Gamma(t))} G(\Gamma(t), t+\alpha, 0, \kappa)\,dt \end{pmatrix}$$

where $\Gamma(t) = \gamma(\theta^{-1}(t))$ is a solution of $\omega(x)\dot{x} = F(x)$ such that $\lim_{t\to\pm T}\Gamma(t) = x_0$. We recall that $\psi_{1,2}(s)$ are (bounded) solutions of the adjoint system (3.17) and hence

$$\psi_{1,2}(0) \in \mathcal{R}P_-^* \cap \mathcal{N}P_+^*.$$

We already observed that $\psi_{1,2}(s) \to 0$ as $|s| \to \infty$ exponentially fast. As a matter of fact we can determine the exponential rate of $\psi_{1,2}(s)$. Indeed in Theorems 5.1 and 5.3 of the next section we will prove that projections $P_\pm$ of ranks $\operatorname{rank} P_+ = k_- - 1$, $\operatorname{rank} P_- = k_- + 1$ exist such that the fundamental matrix $X(t)$ of (3.8) satisfies

$$\begin{aligned}
\big\| X(s'') P_+ X(s')^{-1} \big\| &\leqslant k e^{\alpha_-(s''-s')}, \quad 0 \leqslant s' \leqslant s'', \\
\big\| X(s'') (\mathbb{I} - P_+) X(s')^{-1} \big\| &\leqslant k e^{-\mu_-(s''-s')}, \quad 0 \leqslant s'' \leqslant s', \\
\big\| X(s'') P_- X(s')^{-1} \big\| &\leqslant k e^{-\mu_+(s''-s')}, \quad s' \leqslant s'' \leqslant 0, \\
\big\| X(s'') (\mathbb{I} - P_-) X(s')^{-1} \big\| &\leqslant k e^{\alpha_+(s''-s')}, \quad s'' \leqslant s' \leqslant 0
\end{aligned} \tag{4.1}$$

for some $k > 0$, where

$$\begin{aligned}
&\max\big\{\Re\mu - \mu_- \;\big|\; \mu \in \sigma\big(F'(x_0)\big),\ \Re\mu < \mu_-\big\} \\
&\quad < \alpha_- < \max\big\{\Re\mu - \mu_- \;\big|\; \mu \in \sigma\big(F'(x_0)\big),\ \Re\mu < \mu_-\big\} + \delta < 0, \\
&0 < \min\big\{\Re\mu - \mu_+ \;\big|\; \mu \in \sigma\big(F'(x_0)\big),\ \Re\mu > \mu_+\big\} - \delta \\
&\quad < \alpha_+ < \min\big\{\Re\mu - \mu_+ \;\big|\; \mu \in \sigma\big(F'(x_0)\big),\ \Re\mu > \mu_+\big\}
\end{aligned}$$

and $\delta > 0$ can be taken as close to 0 as we like. We observe that the exponents $-\mu_\pm$ in the second and third rows are the best we can expect. Indeed we have already observed that the eigenvalues of the linear map (3.9) are

$$\big\{\mu - \mu_- \;\big|\; \mu \in \sigma\big(F'(x_0)\big),\ \mu \neq \mu_-\big\} \cup \{-\mu_-\}$$

and hence the limiting equation for $s \to \infty$ (3.16) has and exponential dichotomy with exponents $\alpha_0 < 0$ (larger than $\max\{\Re\mu - \mu_- \mid \mu \in \sigma(F'(x_0)),\ \Re\mu < \mu_-\}$, but as close to it as we like) and $\mu_- > 0$ i.e. the fundamental matrix $X_0(s)$ of (3.16) satisfies

$$\begin{aligned}
\big\| X_0(s'') P_+^0 X_0(s')^{-1} \big\| &\leqslant k e^{\alpha_0(s''-s')}, \quad 0 \leqslant s' \leqslant s'', \\
\big\| X_0(s'') \big(\mathbb{I} - P_+^0\big) X_0(s')^{-1} \big\| &\leqslant k e^{-\mu_-(s''-s')}, \quad 0 \leqslant s'' \leqslant s'
\end{aligned}$$

where P_+^0 is the projections onto the space of generalized eigenvectors of the eigenvalues with negative real parts of (3.9) and $k > 0$ is a suitable constant. Hence because of roughness of exponential dichotomies we can only conclude that the second inequality of (4.1) holds with another exponents $\tilde{\mu}_-$ that in general is strictly larger than $-\mu_-$. Similarly, the standard roughness property implies that the third inequality of (4.1) holds with another exponents $\tilde{\mu}_+$, in general strictly smaller than $-\mu_+$.

From the second estimate in (4.1) with $s'' = 0$ and $s' = s > 0$, we get

$$\big\| X^{-1}(s)^* \big(\mathbb{I} - P_+^*\big) \big\| \leqslant k e^{\mu_- s}$$

whilst from the third with $s'' = 0$ and $s' = s < 0$ we get

$$\big\| X^{-1}(s)^* P_-^* \big\| \leqslant k e^{\mu_+ s}.$$

Hence the solutions of the adjoint system (3.17) starting from $\mathcal{N}P_+^*$ tend to zero as $s \to +\infty$ at the exponential rate $e^{\mu_- s}$ and the solutions of the adjoint system (3.17) starting from $\mathcal{R}P_-^*$ tend to zero as $s \to -\infty$ at the exponential rate $e^{\mu_+ s}$. This implies the claim that

$$\psi_{1,2}(s) = O\big(e^{\mu_\mp s}\big) \quad \text{as } s \to \pm\infty.$$

But, since $\mu_\pm$ are simple eigenvalues of $F'(x_0)$ we have

$$\varphi(s) = \big(e^{-\mu_- s} + e^{-\mu_+ s}\big)^{-1} = O\big(e^{\mu_\mp s}\big),$$

as $s \to \pm\infty$. Hence

$$\psi_{1,2}(s) = O\big(\varphi(s)\big) \quad \text{as } s \to \pm\infty.$$

Setting $y = \varphi(s)x$ in (3.17) we obtain the system:

$$x' = -\left[F'\big(\gamma(s)\big)^* - \frac{\omega'(\gamma(s))^* F(\gamma(s))^*}{\omega(\gamma(s))}\right]x. \tag{4.2}$$

As a consequence, the solutions of Eq. (4.2) starting from points in $\mathcal{R}P_-^* \cap \mathcal{N}P_+^*$ are bounded above by a constant both as $s \to +\infty$ and as $s \to -\infty$. In other words:

$$x(0) \in \mathcal{R}P_-^* \cap \mathcal{N}P_+^* \quad \Rightarrow \quad \big\|x(s)\big\| = O(1) \quad \text{as } |s| \to \infty$$

for any solution of Eq. (4.2). Next note that, changing s with $t = \theta(s)$ in (4.2) and recalling that $\Gamma(t) = \gamma(\theta^{-1}(t))$ we get the equation:

$$\omega\big(\Gamma(t)\big)\dot{v} = \frac{\omega'(\Gamma(t))^*}{\omega(\Gamma(t))} F\big(\Gamma(t)\big)^* v - F'\big(\Gamma(t)\big)^* v \tag{4.3}$$

(here we write $v(t)$ for $x(s)$ with $t = \theta(s)$). Note that (4.3) is the equation adjoint to the linearization of $\omega(x)\dot{x} = F(x)$ along its solution $\Gamma(t)$. Setting $y(s) = \varphi(s)v(\theta(s))$ and $t = \theta(s)$, in (4.2) then we see that y satisfies (3.17). As a consequence $\psi_j(s) = \varphi(s)v_j(\theta(s))$ where $v_j(t)$ satisfies (4.3) and $\psi_j(0) \in \mathcal{R}P_-^* \cap \mathcal{N}P_+^*$ if and only if $v_j(0) \in \mathcal{R}P_-^* \cap \mathcal{N}P_+^*$. So we conclude that

$$\begin{pmatrix} \mathcal{M}_1(\alpha,\kappa) \\ \mathcal{M}_2(\alpha,\kappa) \end{pmatrix} := \begin{pmatrix} \int_{T_-}^{T_+} v_1^*(t) \frac{G(\Gamma(t),t+\alpha,0,\kappa)}{\omega(\Gamma(t))}\,dt \\ \int_{T_-}^{T_+} v_2^*(t) \frac{G(\Gamma(t),t+\alpha,0,\kappa)}{\omega(\Gamma(t))}\,dt \end{pmatrix}$$

where $v_{1,2}(t)$ are two independent solutions of Eq. (4.3) such that

$$\big|v_j\big(\theta(s)\big)\big| = O(1) \quad \text{as } |s| \to \infty$$

that is

$$\big|v_j(t)\big| = O(1) \quad \text{as } t \to T_\pm$$

since

$$\lim_{s\to\pm\infty}\frac{\theta(s)\mp T_\pm}{e^{\mu_\mp s}}=\lim_{s\to\pm\infty}\frac{\omega(\gamma(s))}{\mu_\mp e^{\mu_\mp s}}=\mu_\mp^{-1}\lim_{s\to\pm\infty}\frac{\omega'(x_0)(\gamma(s)-x_0)}{e^{\mu_\mp s}}$$
$$=\mu_\mp^{-1}\lim_{s\to\pm\infty}\frac{\omega'(x_0)(\gamma(s)-x_0)}{\varphi(s)}\frac{\varphi(s)}{e^{\mu_\mp s}}=\mu_\mp^{-1}\omega'(x_0)\gamma_\pm\neq 0.$$

We conclude this section noting that from the preceding discussion it follows that Eq. (4.3) has a two-dimensional space of bounded solutions as $t\to T_\pm$ and $\{v_1(t),v_2(t)\}$ is a basis of such a space.

5. Roughness

In this section we want to prove (4.1). The basic results to obtain such inequalities are the roughness Theorems 5.1 and 5.2. However we complete the treatment adding other similar results in this respect since we believe that these results concerning exponential dichotomies are interesting themselves. We also note that a related result has been proved in [4].

To start with we note that the function $u(s):=\frac{\gamma'(s)}{\omega(\gamma(s))\varphi(s)}$ is a solution of the linear equation (3.8) that satisfies

$$\frac{c_1}{\varphi(s)}\leqslant|u(s)|\leqslant\frac{c_2}{\varphi(s)}$$

where

$$0<c_1=\inf_{s\in\mathbb{R}}\frac{\gamma'(s)}{\omega(\gamma(s))}\quad\text{and}\quad c_2=\sup_{s\in\mathbb{R}}\frac{\gamma'(s)}{\omega(\gamma(s))}.$$

For $0\leqslant s\leqslant t$ we have

$$\frac{\varphi(s)}{\varphi(t)}=\frac{e^{-\mu_-t}+e^{-\mu_+t}}{e^{-\mu_-s}+e^{-\mu_+s}}=\frac{1+e^{(\mu_--\mu_+)t}}{1+e^{(\mu_--\mu_+)s}}e^{\mu_-(s-t)}$$

while for $s\leqslant t\leqslant 0$ we have

$$\frac{\varphi(s)}{\varphi(t)}=\frac{1+e^{(\mu_+-\mu_-)t}}{1+e^{(\mu_+-\mu_-)s}}e^{\mu_+(s-t)}.$$

Hence

$$e^{\mu_-(s-t)}\leqslant\frac{\varphi(s)}{\varphi(t)}\leqslant 2e^{\mu_-(s-t)}$$

for $0\leqslant s\leqslant t$, and

$$e^{\mu_+(s-t)}\leqslant\frac{\varphi(s)}{\varphi(t)}\leqslant 2e^{\mu_+(s-t)}$$

for $s\leqslant t\leqslant 0$. As a consequence

$$\frac{c_1}{c_2}\left|u(s)\right| \leqslant \left|u(t)\right|e^{\mu_-(t-s)} \leqslant \frac{2c_2}{c_1}\left|u(s)\right|$$

for $0 \leqslant s \leqslant t$ and

$$\frac{c_1}{c_2}\left|u(s)\right| \leqslant \left|u(t)\right|e^{\mu_+(t-s)} \leqslant \frac{2c_2}{c_1}\left|u(s)\right| \tag{5.1}$$

for $s \leqslant t \leqslant 0$. Note that $-\mu_-$ (resp. $-\mu_+$) is the eigenvalue with the least positive (resp. largest negative) real part of the linear map (3.9) (resp. (3.10)) and is simple. So the first two inequalities in (4.1) are a consequence of the following.

Theorem 5.1. *Let $A(t)$ be a bounded matrix and suppose the following hold*:

(a) $\lim_{t\to\infty} A(t) = A$;
(b) *there exist a negative number $\alpha < 0$ and a real simple eigenvalue $\mu^* > 0$ of A such that any other eigenvalue of A satisfies either $\Re\mu < \alpha$ or $\Re\mu > \mu^*$*;
(c) *there exists a non-zero solution $u(t)$ of the linear equation $\dot{x} = A(t)x$ such that*

$$k_1\left|u(s)\right| \leqslant \left|u(t)\right|e^{-\mu^*(t-s)} \leqslant k_2\left|u(s)\right|,$$

for some $k_1, k_2 > 0$ and $0 \leqslant s \leqslant t$.

Then the linear equation $\dot{x} = A(t)x$ has an exponential dichotomy on $\mathbb{R}_+$ with exponents α and μ^* i.e. there exists a projection P_+ such that the fundamental matrix $X(t)$ of equation $\dot{x} = A(t)x$, with $X(0) = \mathbb{I}$ satisfies

$$\begin{aligned} \left\|X(t)P_+X^{-1}(s)\right\| &\leqslant k e^{\alpha(t-s)}, \quad 0 \leqslant s \leqslant t, \\ \left\|X(t)(\mathbb{I} - P_+)X^{-1}(s)\right\| &\leqslant k e^{\mu^*(t-s)}, \quad 0 \leqslant t \leqslant s. \end{aligned}$$

Proof. Let $\bar{\mu} > \mu^*$ and $\delta > 0$ be such that the following hold

$$\begin{gathered} \mu^* < \bar{\mu} < \bar{\mu} + 2\delta < \min\left\{\Re\mu \mid \mu \in \sigma(A),\ \Re\mu > \mu^*\right\}, \\ \max\left\{\Re\mu \mid \mu \in \sigma(A),\ \Re\mu < 0\right\} < \alpha - \delta. \end{gathered}$$

From classical theory we know that there exist projections Q_s, Q_c, Q_u in $\mathbb{R}^n$ with rank $Q_c = 1$ and $Q_s + Q_c + Q_u = \mathbb{I}$ such that the fundamental matrix $X_0(t)$ of $\dot{x} = Ax$, with $X_0(0) = \mathbb{I}$, satisfies

$$\begin{aligned} \left\|X_0(t)Q_sX_0^{-1}(s)\right\| &\leqslant k_s e^{(\alpha-\delta)(t-s)}, \quad s \leqslant t, \\ \left\|X_0(t)Q_cX_0^{-1}(s)\right\| &\leqslant k_s e^{\mu^*(t-s)}, \quad t \leqslant s, \\ \left\|X_0(t)Q_uX_0^{-1}(s)\right\| &\leqslant k_s e^{(\bar{\mu}+2\delta)(t-s)}, \quad t \leqslant s. \end{aligned}$$

Setting $Y_0(t) = X_0(t)e^{-(\mu^*+2\delta)t}$ we get

$$\begin{aligned}
&\left\|Y_0(t)Q_sY_0^{-1}(s)\right\| \leqslant k_s\mathrm{e}^{(\alpha-\mu^*-3\delta)(t-s)} \leqslant k_s\mathrm{e}^{-2\delta(t-s)}, \quad s\leqslant t,\\
&\left\|Y_0(t)Q_cY_0^{-1}(s)\right\| \leqslant k_s\mathrm{e}^{-2\delta(t-s)}, \quad t\leqslant s,\\
&\left\|Y_0(t)Q_uY_0^{-1}(s)\right\| \leqslant k_s\mathrm{e}^{(\bar\mu-\mu^*)(t-s)}, \quad t\leqslant s.
\end{aligned}$$

Now, $Y_0(t)$ is the fundamental matrix of the linear system $\dot y = [A-(\mu^*+2\delta)\mathbb{I}]y$. Hence because of roughness of exponential dichotomies there exist a projection $\widetilde{Q}:\mathbb{R}^n\to\mathbb{R}^n$ with $\operatorname{rank}\widetilde{Q} = \operatorname{rank}[Q_s+Q_c]$ and a constant k such that the fundamental matrix $Y(t)$ of $\dot y = [A(t)-(\mu^*+2\delta)\mathbb{I}]y$ satisfies

$$\begin{aligned}
&\left\|Y(t)\widetilde{Q}Y^{-1}(s)\right\| \leqslant k\mathrm{e}^{-\delta(t-s)}, \quad 0\leqslant s\leqslant t,\\
&\left\|Y(t)(\mathbb{I}-\widetilde{Q})Y^{-1}(s)\right\| \leqslant k\mathrm{e}^{(\bar\mu-\mu^*-\delta)(t-s)}, \quad 0\leqslant t\leqslant s.
\end{aligned} \tag{5.2}$$

Since the fundamental matrix $X(t)$ of $\dot x = A(t)x$ is $X(t)=Y(t)\mathrm{e}^{(\mu^*+2\delta)t}$ we obtain

$$\begin{aligned}
&\left\|X(t)\widetilde{Q}X^{-1}(s)\right\| \leqslant k\mathrm{e}^{(\mu^*+\delta)(t-s)}, \quad 0\leqslant s\leqslant t,\\
&\left\|X(t)(\mathbb{I}-\widetilde{Q})X^{-1}(s)\right\| \leqslant k\mathrm{e}^{(\bar\mu+\delta)(t-s)}, \quad 0\leqslant t\leqslant s.
\end{aligned} \tag{5.3}$$

Next, again from roughness of exponential dichotomies, we know that a projection P_+ with $\operatorname{rank}P_+ = \operatorname{rank}Q_s = \operatorname{rank}\widetilde{Q}-1$ exists such that

$$\begin{aligned}
&\left\|X(t)P_+X^{-1}(s)\right\| \leqslant k\mathrm{e}^{\alpha(t-s)}, \quad 0\leqslant s\leqslant t,\\
&\left\|X(t)(\mathbb{I}-P_+)X^{-1}(s)\right\| \leqslant k\mathrm{e}^{(\mu^*-\delta)(t-s)}, \quad 0\leqslant t\leqslant s.
\end{aligned} \tag{5.4}$$

Without loss of generality we may assume that the constants k in (5.3) and (5.4) are the same. Next

$$\begin{aligned}
&\left\|Y(t)P_+Y^{-1}(s)\right\| \leqslant k\mathrm{e}^{(-\mu^*-2\delta+\alpha)(t-s)}, \quad 0\leqslant s\leqslant t,\\
&\left\|Y(t)(\mathbb{I}-P_+)Y^{-1}(s)\right\| \leqslant k\mathrm{e}^{-3\delta(t-s)}, \quad 0\leqslant t\leqslant s.
\end{aligned} \tag{5.5}$$

From (5.2) and (5.5) it follows that

$$\begin{aligned}
&\mathcal{R}\widetilde{Q} = \left\{y_0\in\mathbb{R}^n \;\middle|\; \left|Y(t)y_0\right| \leqslant k\left|y(0)\right|\mathrm{e}^{-\delta t}\right\},\\
&\mathcal{R}P_+ = \left\{y_0\in\mathbb{R}^n \;\middle|\; \left|Y(t)y_0\right| \leqslant k\left|y(0)\right|\mathrm{e}^{(-\mu^*-2\delta+\alpha)t}\right\}.
\end{aligned}$$

Hence

$$\mathcal{R}P_+\subset\mathcal{R}\widetilde{Q}$$

and has codimension 1 in $\mathcal{R}\widetilde{Q}$. Moreover we can take as $\mathcal{N}P_+$ (resp. $\mathcal{N}\widetilde{Q}$) any complement of the range. Now, $u(0)\notin\mathcal{R}P_+$ since otherwise, for $t\geqslant 0$:

$$k_1\left|u(0)\right|\mathrm{e}^{\mu^*t} \leqslant \left|u(t)\right| = \left|X(t)P_+X^{-1}(0)u(0)\right| \leqslant k\mathrm{e}^{\alpha t}\left|u(0)\right| \to 0$$

as $t \to \infty$. But $u(0) \in \mathcal{R}\widetilde{Q}$, indeed we have, for $s \geqslant 0$:

$$\begin{aligned}\left|(\mathbb{I}-\widetilde{Q})u(0)\right| &= \left|X(0)(\mathbb{I}-\widetilde{Q})X^{-1}(s)u(s)\right| \leqslant k\mathrm{e}^{-(\bar{\mu}+\delta)s}\left|u(s)\right| \\ &\leqslant kk_2\mathrm{e}^{-(\bar{\mu}-\mu^*+\delta)s}\left|u(0)\right| \to 0,\end{aligned}$$

as $s \to \infty$. So $u(0) \in \mathcal{R}\widetilde{Q}$ and we may choose P_+ so that

$$\mathcal{R}P_+ \subset \mathcal{R}\widetilde{Q} \quad \text{and} \quad \mathcal{N}P_+ = \mathcal{N}\widetilde{Q} \oplus \operatorname{span}\{u(0)\}.$$

Now, we take the projection $S : \mathcal{N}P_+ \to \mathcal{N}P_+$ such that $\mathcal{N}S = \mathcal{N}\widetilde{Q}$ and $\mathcal{R}S = \operatorname{span}\{u(0)\}$ and set $P_1 := S(\mathbb{I} - P_+)$, $P_2 := (\mathbb{I} - S)(\mathbb{I} - P_+)$. We have

$$P_1^2 = S(\mathbb{I}-P_+)S(\mathbb{I}-P_+) = S^2(\mathbb{I}-P_+) = S(\mathbb{I}-P_+) = P_1$$

since $(\mathbb{I} - P_+)S(\mathbb{I} - P_+) = S(\mathbb{I} - P_+)$. Similarly, using $P_+S = 0$:

$$P_2^2 = (\mathbb{I}-S)(\mathbb{I}-P_+)(\mathbb{I}-S)(\mathbb{I}-P_+) = (\mathbb{I}-S)(\mathbb{I}-S-P_+)(\mathbb{I}-P_+) = (\mathbb{I}-S)(\mathbb{I}-P_+).$$

So P_1 and P_2 are projections and trivially satisfy $P_1 + P_2 = \mathbb{I} - P_+$. For any given projection P (for example P_+, P_1, $\widetilde{Q}$ etc.) we set

$$P(s) := X(s)PX^{-1}(s).$$

We also set

$$S(s) = X(s)SX^{-1}(s)_{|\mathcal{N}P_+(s)}.$$

Then note that

$$\mathbb{I} - P_+(s) := X(s)(\mathbb{I}-P_+)X^{-1}(s) = P_1(s) + P_2(s).$$

It is clear that $\mathcal{N}P_1 = \mathcal{R}P_+ \oplus \mathcal{N}\widetilde{Q}$, $\mathcal{R}P_1 = \operatorname{span}\{u(0)\}$, $\mathcal{N}P_2 = \mathcal{R}P_+ \oplus \operatorname{span}\{u(0)\}$ and $\mathcal{R}P_2 = \mathcal{N}\widetilde{Q}$ and then for any $s \in \mathbb{R}$

$$\begin{aligned}&\mathcal{N}P_1(s) = \mathcal{R}P_+(s) \oplus \mathcal{N}\widetilde{Q}(s), \qquad \mathcal{R}P_1(s) = \operatorname{span}\{u(s)\},\\ &\mathcal{N}P_2(s) = \mathcal{R}P_+(s) \oplus \operatorname{span}\{u(s)\}, \qquad \mathcal{R}P_2(s) = \mathcal{N}\widetilde{Q}(s),\\ &\mathcal{N}P_+(s) = \mathcal{N}\widetilde{Q}(s) \oplus \operatorname{span}\{u(s)\}.\end{aligned}$$

Let $x \in \mathcal{N}P_+(s)$. Then $x = x_1 + cu(s)$, where $x_1 \in \mathcal{N}\widetilde{Q}(s)$ (note, in general, c, x_1 may depend on s) or, using $u(s) \in \mathcal{R}\widetilde{Q}(s)$:

$$x = \left(\mathbb{I}-\widetilde{Q}(s)\right)x + cu(s) = \left(\mathbb{I}-\widetilde{Q}(s)\right)x + \widetilde{Q}(s)x$$

where $\widetilde{Q}(s)x = cu(s)$. We have

$$S(s)x = S(s)\big[\big(\mathbb{I} - \widetilde{Q}(s)\big)x + cu(s)\big] = S(s)cu(s)$$
$$= cX(s)Su(0) = cX(s)u(0) = cu(s) = \widetilde{Q}(s)x.$$

Hence (see (5.3)):

$$\big\|S(s)\big\| \leqslant \big\|\widetilde{Q}(s)\big\| \leqslant k$$

and then

$$\big\|P_1(s)\big\| = \big\|S(s)\big(\mathbb{I} - P_+(s)\big)\big\| \leqslant k^2.$$

Next

$$\big\|X(t)(\mathbb{I} - P_+)X^{-1}(s)\big\| \leqslant \big\|X(t)P_1X^{-1}(s)\big\| + \big\|X(t)P_2X^{-1}(s)\big\|.$$

So for $0 \leqslant t \leqslant s$ we have, using $(\mathbb{I} - \widetilde{Q})P_2 = P_2$:

$$\big\|X(t)P_2X^{-1}(s)\big\| = \big\|X(t)(\mathbb{I} - \widetilde{Q})X^{-1}(s)X(s)P_2X^{-1}(s)\big\| \leqslant k\mathrm{e}^{(\bar{\mu}+\delta)(t-s)}\big\|P_2(s)\big\|.$$

Next, for any $x \in \mathbb{R}^n$ there exists $c(s) \in \mathbb{R}$ such that $P_1X^{-1}(s)x = c(s)u(0)$. Then, for $0 \leqslant t \leqslant s$:

$$\begin{aligned}\big|X(t)P_1X^{-1}(s)x\big| &= \big|X(t)c(s)u(0)\big| = \big|c(s)u(t)\big| \leqslant \big|c(s)\big|\big|u(t)\big| \\ &\leqslant \big|c(s)\big|k_1^{-1}\big|u(s)\big|\mathrm{e}^{\mu^*(t-s)} = k_1^{-1}\big|X(s)c(s)u(0)\big|\mathrm{e}^{\mu^*(t-s)} \\ &= k_1^{-1}\big|P_1(s)x\big|\mathrm{e}^{\mu^*(t-s)}\end{aligned}$$

that is

$$\big\|X(t)P_1X^{-1}(s)\big\| \leqslant k_2\big\|P_1(s)\big\|\mathrm{e}^{\mu^*(t-s)}$$

where $P_1(s) = X(s)P_1X^{-1}(s)$. As a consequence we obtain, for $0 \leqslant t \leqslant s$:

$$\big\|X(t)(\mathbb{I} - P_+)X^{-1}(s)\big\| \leqslant \big[k\mathrm{e}^{(\bar{\mu}-\mu^*+\delta)(t-s)}\big\|P_2(s)\big\| + k_2\big\|P_1(s)\big\|\big]e^{\mu^*(t-s)}.$$

Since $\mu^* < \bar{\mu}$ the thesis follows provided we prove that $P_1(s)$ and $P_2(s)$ are bounded in $s \geqslant 0$ and this easily follows from $\|\mathbb{I} - P_+(s)\| \leqslant k$, $\|P_1(s)\| \leqslant k^2$ and the equality $P_1(s) + P_2(s) = \mathbb{I} - P_+(s)$. The proof is complete. □

Changing s, t with $-s, -t$ in Theorem 5.1 and noting that the conditions

$$k_1\big|u(s)\big| \leqslant \big|u(t)\big|\mathrm{e}^{-\mu^*(t-s)} \leqslant k_2\big|u(s)\big| \quad \text{for } 0 \leqslant s \leqslant t$$

and

$$k_2^{-1}\big|u(s)\big| \leqslant \big|u(t)\big|\mathrm{e}^{-\mu^*(t-s)} \leqslant k_1^{-1}\big|u(s)\big| \quad \text{for } 0 \leqslant t \leqslant s$$

are obtained one from the other changing s with t, we obtain the following.

Theorem 5.2. *Let $A(t)$ be a bounded matrix and suppose the following hold*:

(a) $\lim_{t\to-\infty} A(t) = A$;
(b) *there exist a positive number $\beta > 0$ and a real simple eigenvalue $\mu_* < 0$ of A such that any other eigenvalue of A satisfies either $\Re\mu > \beta$ or $\Re\mu < \mu_*$*;
(c) *there exists a non-zero solution $u(t)$ of the linear equation $\dot{x} = A(t)x$ such that*

$$k_1\big|u(s)\big| \leqslant \big|u(t)\big|e^{-\mu_*(t-s)} \leqslant k_2\big|u(s)\big|$$

for some $k_1, k_2 > 0$ and $s \leqslant t \leqslant 0$.

Then the linear equation $\dot{x} = A(t)x$ has an exponential dichotomy on $\mathbb{R}_-$ with exponents μ_* and β i.e. there exists a projection P_- such that the fundamental matrix $X(t)$ of equation $\dot{x} = A(t)x$, with $X(0) = \mathbb{I}$ satisfies

$$\big\|X(t)P_-X^{-1}(s)\big\| \leqslant ke^{\mu_*(t-s)}, \quad s \leqslant t \leqslant 0,$$
$$\big\|X(t)(\mathbb{I} - P_-)X^{-1}(s)\big\| \leqslant ke^{\beta(t-s)}, \quad t \leqslant s \leqslant 0.$$

The third and fourth inequalities in (4.1) follow from Theorem 5.2 and Eq. (5.1).

Since we believe that this kind of results concerning roughness of exponential dichotomies are interesting themselves we state for completeness the following result whose proof is very similar to that of Theorem 5.1.

Theorem 5.3. *Let $A(t)$ be a bounded matrix and suppose the following hold*:

(a) $\lim_{t\to\infty} A(t) = A$;
(b) *there exist a positive number $\beta > 0$ and a real and simple eigenvalue $\mu_* < 0$ of A such that any other eigenvalue of A satisfies either $\Re\mu < \mu_*$ or $\Re\mu > \beta$*;
(c) *there exists a non-zero solution $u(t)$ of the linear equation $\dot{x} = A(t)x$ such that*

$$k_1\big|u(s)\big| \leqslant \big|u(t)\big|e^{-\mu_*(t-s)} \leqslant k_2\big|u(s)\big|$$

for some $k_1, k_2 > 0$ and $0 \leqslant s \leqslant t$.

Then the linear equation $\dot{x} = A(t)x$ has an exponential dichotomy on $\mathbb{R}_+$ with exponents μ_* and β i.e. there exists a projection P_+ such that the fundamental matrix $X(t)$ of equation $\dot{x} = A(t)x$, with $X(0) = \mathbb{I}$ satisfies

$$\big\|X(t)P_+X^{-1}(s)\big\| \leqslant ke^{\mu_*(t-s)}, \quad 0 \leqslant s \leqslant t,$$
$$\big\|X(t)(\mathbb{I} - P_+)X^{-1}(s)\big\| \leqslant ke^{\beta(t-s)}, \quad 0 \leqslant t \leqslant s. \tag{5.6}$$

Proof. Let $\delta > 0$ be such that $\Re\mu > \beta + 3\delta$ for any eigenvalue μ of A with positive real part and $\Re\mu < \mu_* - 3\delta$ for any eigenvalue μ of A with negative real part. From classical theory we know that there exist projections Q_s, Q_c and Q_u on $\mathbb{R}^n$ such that $Q_s + Q_c + Q_u = \mathbb{I}$ and a constant $k_0 \geqslant 1$ such that the fundamental matrix $X_0(t)$ of the linear system $\dot{x} = Ax$, satisfies

$$\begin{aligned}
&\left\|X_0(t)Q_sX_0^{-1}(s)\right\| \leqslant k_0\mathrm{e}^{(\mu_*-3\delta)(t-s)}, \quad s \leqslant t,\\
&\left\|X_0(t)Q_cX_0^{-1}(s)\right\| \leqslant k_0\mathrm{e}^{\mu_*(t-s)}, \quad s \geqslant t,\\
&\left\|X_0(t)Q_uX_0^{-1}(s)\right\| \leqslant k_0\mathrm{e}^{(\beta+3\delta)(t-s)}, \quad s \geqslant t.
\end{aligned}$$

Then the fundamental matrix $Y_0(t) = X_0(t)\mathrm{e}^{(\delta-\mu_*)t}$ of $\dot{y} = [A + (\delta - \mu_-)\mathbb{I}]y$ satisfies

$$\begin{aligned}
&\left\|Y_0(t)Q_sY_0^{-1}(s)\right\| \leqslant k_0\mathrm{e}^{-2\delta(t-s)}, \quad s \leqslant t,\\
&\left\|Y_0(t)Q_cY_0^{-1}(s)\right\| \leqslant k_0\mathrm{e}^{\delta(t-s)}, \quad s \geqslant t,\\
&\left\|Y_0(t)Q_uY_0^{-1}(s)\right\| \leqslant k_0\mathrm{e}^{(\beta-\mu_*+4\delta)(t-s)}, \quad s \geqslant t.
\end{aligned}$$

From the roughness of exponential dichotomies we deduce that the linear equation $\dot{y} = [A(t) + (\delta - \mu_-)\mathbb{I}]y$ has an exponential dichotomy on $\mathbb{R}_+$ with projections Q_+ such that

$$\begin{aligned}
&\left\|Y(t)Q_+Y^{-1}(s)\right\| \leqslant c_s\mathrm{e}^{-\delta(t-s)}, \quad 0 \leqslant s \leqslant t,\\
&\left\|Y(t)(\mathbb{I} - Q_+)Y^{-1}(s)\right\| \leqslant c_u\mathrm{e}^{\frac{\delta}{2}(t-s)}, \quad 0 \leqslant t \leqslant s.
\end{aligned}$$

Note that, from [18] it follows $Y(t)Q_+Y^{-1}(t) \to Q_s$ and then $Y(t)(\mathbb{I} - Q_+)Y^{-1}(t) = \mathbb{I} - Y(t)Q_+Y^{-1}(t) \to Q_c + Q_u$ as $t \to \infty$. We note that $\mathcal{R}Q_+$ is uniquely determined and indeed:

$$\mathcal{R}Q_+ = \left\{y \in \mathbb{R}^n \;\middle|\; \left|Y(t)y\right| \leqslant c_s|y|\mathrm{e}^{-\delta t}\right\}$$

but its kernel $\mathcal{N}Q_+$ can be taken as any complement of the range. Using $Y(t) = X(t)\mathrm{e}^{(\delta-\mu_*)t}$ we get

$$\begin{aligned}
&\left\|X(t)Q_+X^{-1}(s)\right\| \leqslant c_s\mathrm{e}^{(\mu_*-2\delta)(t-s)}, \quad 0 \leqslant s \leqslant t,\\
&\left\|X(t)(\mathbb{I} - Q_+)X^{-1}(s)\right\| \leqslant c_u\mathrm{e}^{(\mu_*-\frac{\delta}{2})(t-s)}, \quad 0 \leqslant t \leqslant s
\end{aligned} \tag{5.7}$$

and also $X(t)Q_+X^{-1}(t) \to Q_s$ and $X(t)(\mathbb{I} - Q_+)X^{-1}(t) \to Q_c + Q_u$ as $t \to \infty$. Now, from the roughness of exponential dichotomies we know that a projection P_+ exists such that Eq. (5.6) holds with a $\tilde{\mu}_*$, slightly bigger than μ_*, instead of μ_* i.e.

$$\begin{aligned}
&\left\|X(t)P_+X^{-1}(s)\right\| \leqslant k\mathrm{e}^{\tilde{\mu}_-(t-s)}, \quad 0 \leqslant s \leqslant t,\\
&\left\|X(t)(\mathbb{I} - P_+)X^{-1}(s)\right\| \leqslant k\mathrm{e}^{\beta(t-s)}, \quad 0 \leqslant t \leqslant s.
\end{aligned} \tag{5.8}$$

Now, from (5.7), (5.8) it follows that $\mathcal{R}Q_+ \subset \mathcal{R}P_+$. Moreover $u(0) \notin \mathcal{R}Q_+$ since otherwise, for $t \geqslant 0$:

$$0 < k_1\left|u(0)\right| \leqslant \left|u(t)\right|\mathrm{e}^{-\mu_*t} = \left|X(t)Q_+X^{-1}(0)u(0)\right|\mathrm{e}^{-\mu_*t} \leqslant c_s\left|u(0)\right|\mathrm{e}^{-2\delta t} \to 0$$

as $t \to \infty$. Since $\mathcal{N}Q_+$ can be any complement of $\mathcal{R}Q_+$ we may assume that $u(0) \in \mathcal{N}Q_+$. Next $u(0) \in \mathcal{R}P_+$ since, for $t \geqslant 0$:

$$\begin{aligned}\left|(\mathbb{I}-P_+)u(0)\right| &= \left|X(0)(\mathbb{I}-P_+)X^{-1}(t)u(t)\right| \leqslant k\mathrm{e}^{-\beta t}\left|u(t)\right| \\ &\leqslant k\mathrm{e}^{-\beta t}k_2\left|u(0)\right|\mathrm{e}^{\mu_* t} = kk_2\left|u(0)\right|\mathrm{e}^{(\mu_*-\beta)t} \to 0,\end{aligned}$$

as $t \to \infty$. Then, because of dimensions: $\mathcal{R}P_+ = \mathcal{R}Q_+ \oplus V$, where $V := \operatorname{span}\{u(0)\}$ and, because of invariance:

$$\mathcal{R}P_+(s) = \mathcal{R}Q_+(s) \oplus \operatorname{span}\{u(s)\}$$

where, for any projection P we set $P(s) = X(s)PX^{-1}(s)$. Note, then, that $u(s) \in \mathcal{N}Q_+(s)$ for any $s \in \mathbb{R}$.

Now, we take the projection $S : \mathcal{R}P_+ \to \mathcal{R}P_+$ such that $\mathcal{R}S = V$ and $\mathcal{N}S = \mathcal{R}Q_+$. For any $x \in \mathcal{R}P_+(s)$ we write $x = x_1 + cu(s)$, $x_1 \in \mathcal{R}Q_+(s)$, and note that $Q_+(s)x = Q_+(s)x_1 + cQ_+(s)u(s) = Q_+x_1 = x_1$, since $u(s) \in \mathcal{N}Q_+(s)$ and $x_1 \in \mathcal{R}Q_+(s)$. Similarly, $[\mathbb{I} - Q_+(s)]x = cu(s)$. Then, for any $x \in \mathcal{R}P_+$ we have

$$S(s)x = S(s)\left[\mathbb{I} - Q_+(s)\right]x + S(s)Q_+(s)x = cS(s)u(s) = cu(s) = \left[\mathbb{I} - Q_+(s)\right]x.$$

So

$$S(s) = \left[\mathbb{I} - Q_+(s)\right]_{|\mathcal{R}P_+(s)}$$

and then

$$\left\|S(s)\right\| \leqslant \left\|\mathbb{I} - Q_+(s)\right\| \leqslant k.$$

Next set $P_1 := (\mathbb{I} - S)P_+$, $P_2 := SP_+$. We have

$$P_1^2 = (\mathbb{I} - S)P_+(\mathbb{I} - S)P_+ = (\mathbb{I} - S)P_+(P_+ - SP_+) = (\mathbb{I} - S)P_+ = P_1$$

since $P_+SP_+ = SP_+$. Similarly

$$P_2^2 = SP_+SP_+ = SSP_+ = SP_+ = P_2.$$

So P_1 and P_2 are projections. Next, $\mathcal{R}P_1 = \mathcal{R}Q_+$, $\mathcal{N}P_1 = V \oplus \mathcal{N}P_+$, $\mathcal{R}P_2 = V$, $\mathcal{N}P_2 = \mathcal{R}Q_+ \oplus \mathcal{N}P_+$ and $P_+ = P_1 + P_2$. Hence

$$\left\|X(t)P_+X^{-1}(s)\right\| \leqslant \left\|X(t)P_1X^{-1}(s)\right\| + \left\|X(t)P_2X^{-1}(s)\right\|$$

and, using $\mathcal{R}P_1 = \mathcal{R}Q_+$:

$$\left\|X(t)P_1X^{-1}(s)\right\| = \left\|X(t)Q_+P_1X^{-1}(s)\right\| \leqslant c_s\mathrm{e}^{(\mu_*-2\delta)(t-s)}\left\|P_1(s)\right\|.$$

Moreover, for any $x \in \mathbb{R}^n$ one has $P_2X^{-1}(s)x = c(s)u(0)$ and then

$$\left|X(t)P_2X^{-1}(s)x\right| = \left|c(s)u(t)\right| \leqslant k_2\left|c(s)u(s)\right|\mathrm{e}^{\mu_-(t-s)} = k_2\left|P_2(s)x\right|\mathrm{e}^{\mu_-(t-s)}$$

that is

$$\left\| X(t)P_2X^{-1}(s)\right\| \leqslant k_2\left\| P_2(s)\right\| e^{\mu_-(t-s)}.$$

As a consequence

$$\left\| X(t)P_+X^{-1}(s)\right\| \leqslant \left(c_s e^{-2\delta(t-s)}\left\| P_1(s)\right\| + k_2\left\| P_2(s)\right\|\right)e^{\mu_-(t-s)}$$

from which the thesis follows provided $P_1(s)$ and $P_2(s)$ are bounded. Since

$$P_1(s) + P_2(s) = P_+(s)$$

and $\|P_+(s)\| \leqslant k$, for $P_+(s) := X(s)P_+X^{-1}(s)$, it follows that $P_1(s)$ is bounded if and only if so is $P_2(s) = X(s)SP_+X^{-1}(s)$. But

$$\left\| P_2(s)\right\| = \left\| \left[\mathbb{I} - Q_+(s)\right]_{|\mathcal{R}P_+(s)} P_+(s)\right\| \leqslant \left\|\mathbb{I} - Q_+(s)\right\| \left\| P_+(s)\right\| \leqslant k^2.$$

The proof is complete. □

Changing s, t with $-s, -t$ in Theorem 5.3 we obtain the following.

Theorem 5.4. *Let $A(t)$ be a bounded matrix and suppose the following hold*:

(a) $\lim_{t\to-\infty} A(t) = A$;
(b) *there exist a real simple eigenvalue $\mu^* > 0$ of A and $\alpha < 0$ such that any other eigenvalue of A satisfies either $\Re\mu < \alpha$ or $\Re\mu > \mu^*$*;
(c) *there exists a non-zero solution $u(t)$ of the linear equation $\dot{x} = A(t)x$ such that*

$$k_1\left|u(s)\right| \leqslant \left|u(t)\right|e^{-\mu^*(t-s)} \leqslant k_2\left|u(s)\right|$$

for some $k_1, k_2 > 0$ and $s \leqslant t \leqslant 0$.

Then the linear equation $\dot{x} = A(t)x$ has an exponential dichotomy on $\mathbb{R}_-$ with exponents α and μ^* i.e. there exists a projection P_- such that the fundamental matrix $X(t)$ of equation $\dot{x} = A(t)x$, with $X(0) = \mathbb{I}$ satisfies

$$\left\| X(t)P_-X^{-1}(s)\right\| \leqslant k e^{\alpha(t-s)}, \quad s \leqslant t \leqslant 0,$$
$$\left\| X(t)(\mathbb{I} - P_-)X^{-1}(s)\right\| \leqslant k e^{\mu^*(t-s)}, \quad t \leqslant s \leqslant 0.$$

Theorems 5.1 and 5.3 can be applied simultaneously to obtain the following.

Theorem 5.5. *Let $A(t)$ be a bounded matrix and suppose the following hold*:

(a) $\lim_{t\to\infty} A(t) = A$;
(b) *there exist two real simple eigenvalues $\mu^* > 0$ and $\mu_* < 0$ of A such that any other eigenvalue of A satisfies either $\Re\mu < \mu_*$ or $\Re\mu > \mu^*$*;

(c) *there exist two non-zero solutions* $u_1(t)$ *and* $u_2(t)$ *of the linear equation* $\dot{x} = A(t)x$ *such that*

$$k_1|u_1(s)| \leqslant |u_1(t)|e^{-\mu^*(t-s)} \leqslant k_2|u_1(s)|,$$
$$k_1|u_2(s)| \leqslant |u_2(t)|e^{-\mu_*(t-s)} \leqslant k_2|u_2(s)|$$

for some $k_1, k_2 > 0$ *and* $0 \leqslant s \leqslant t$.

Then the linear equation $\dot{x} = A(t)x$ *has an exponential dichotomy on* $\mathbb{R}_-$ *with exponents* μ_* *and* μ^* *i.e. there exists a projection* P_+ *such that the fundamental matrix* $X(t)$ *of equation* $\dot{x} = A(t)x$*, with* $X(0) = \mathbb{I}$ *satisfies*

$$\|X(t)P_+X^{-1}(s)\| \leqslant ke^{\mu_*(t-s)}, \quad 0 \leqslant s \leqslant t,$$
$$\|X(t)(\mathbb{I} - P_+)X^{-1}(s)\| \leqslant ke^{\mu^*(t-s)}, \quad 0 \leqslant t \leqslant s.$$

Proof. Let $\delta > 0$ sufficiently small. From Theorem 5.1 we know that there exist a constant k and a projection $P_1 : \mathbb{R}^n \to \mathbb{R}^n$ such that the fundamental matrix of $\dot{x} = A(t)x$ satisfies

$$\|X(t)P_1X^{-1}(s)\| \leqslant ke^{(\mu_*+\delta)(t-s)}, \quad 0 \leqslant s \leqslant t,$$
$$\|X(t)(\mathbb{I} - P_1)X^{-1}(s)\| \leqslant ke^{\mu^*(t-s)}, \quad 0 \leqslant t \leqslant s.$$

On the other hand from Theorem 5.3 we see that a projection $P_2 : \mathbb{R}^n \to \mathbb{R}^n$ exists such that

$$\|X(t)P_2X^{-1}(s)\| \leqslant ke^{\mu_*(t-s)}, \quad 0 \leqslant s \leqslant t,$$
$$\|X(t)(\mathbb{I} - P_2)X^{-1}(s)\| \leqslant ke^{(\mu^*-\delta)(t-s)}, \quad 0 \leqslant t \leqslant s.$$

Then we see that $\mathcal{R}P_2 \subset \mathcal{R}P_1$ and then $\mathcal{R}P_2 = \mathcal{R}P_1$ because they have the same dimension. So $P_2P_1 = P_1$ and $P_1P_2 = P_2$. We get then, for $0 \leqslant s \leqslant t$:

$$\|X(t)P_1X^{-1}(s)\| = \|X(t)P_2X^{-1}(s)P_1(s)\| \leqslant k^2e^{\mu_*(t-s)}$$

where $P_1(s) = X(s)P_2X^{-1}(s)$. The thesis follows taking k^2 instead of k and $P_+ = P_1$. □

Similarly, using Theorems 5.2 and 5.4 we obtain:

Theorem 5.6. *Let* $A(t)$ *be a bounded matrix and suppose the following hold*:

(a) $\lim_{t\to-\infty} A(t) = A$;
(b) *there exist two real simple eigenvalues* $\mu^* > 0$ *and* $\mu_* < 0$ *of* A *such that any other eigenvalue of* A *satisfies either* $\Re\mu < \mu_*$ *or* $\Re\mu > \mu^*$;
(c) *there exist two non-zero solutions* $u_1(t)$ *and* $u_2(t)$ *of the linear equation* $\dot{x} = A(t)x$ *such that*

$$k_1|u_1(s)| \leqslant |u_1(t)|e^{-\mu^*(t-s)} \leqslant k_2|u_1(s)|,$$
$$k_1|u_2(s)| \leqslant |u_2(t)|e^{-\mu_*(t-s)} \leqslant k_2|u_2(s)|$$

for some $k_1, k_2 > 0$ *and* $s \leqslant t \leqslant 0$.

Then the linear equation $\dot{x} = A(t)x$ has an exponential dichotomy on $\mathbb{R}_-$ with exponents μ_* and μ^* i.e. there exists a projection P_- such that the fundamental matrix $X(t)$ of equation $\dot{x} = A(t)x$, with $X(0) = \mathbb{I}$ satisfies

$$\left\|X(t)P_-X^{-1}(s)\right\| \leqslant k\mathrm{e}^{\mu_*(t-s)}, \quad s \leqslant t \leqslant 0,$$
$$\left\|X(t)(\mathbb{I} - P_-)X^{-1}(s)\right\| \leqslant k\mathrm{e}^{\mu^*(t-s)}, \quad t \leqslant s \leqslant 0.$$

We conclude this section with a remark concerning homoclinic solutions of nonlinear systems.

Suppose that $\gamma(t)$ is a solution of a nonlinear equation $\dot{x} = g(x)$ asymptotic $|t| \to \infty$ to a hyperbolic fixed point x_0 and that the first part of condition (C3) holds, i.e.

$$\lim_{s\to\infty} \frac{1}{s} \log\left|\gamma(s) - x_0\right| = \mu_- < 0.$$

Assume, further, that μ_- is a real eigenvalue of $g'(x_0)$ and no other eigenvalues of $g'(x_0)$ have μ_- as real part. From the proof of Proposition 2.1 we know that, for some $d > 0$:

$$\frac{\gamma(s) - x_0}{s^d \mathrm{e}^{\mu_- s}} = \gamma_+ + O\left(\mathrm{e}^{-\delta s}\right) \quad \text{as } s \to \infty$$

with $g'(x_0)\gamma_+ = \mu_-\gamma_+$ and $\gamma_+ \neq 0$. Let $\mu_- < -\delta < 0$. We have

$$\begin{aligned}\frac{\dot{\gamma}(s)}{s^d \mathrm{e}^{\mu_- s}} &= \frac{g(\gamma(s))}{s^d \mathrm{e}^{\mu_- s}} = \frac{g'(x_0)(\gamma(s) - x_0) + O(|\gamma(s) - x_0|^2)}{s^d \mathrm{e}^{\mu_- s}}\\ &= g'(x_0)\left[\gamma_+ + O\left(\mathrm{e}^{-\delta s}\right)\right] + O\left(s^d \mathrm{e}^{\mu_- s}\right) = \mu_-\gamma_+ + O\left(\mathrm{e}^{-\delta s}\right).\end{aligned}$$

Then

$$\begin{aligned}\frac{\ddot{\gamma}(s)}{|\dot{\gamma}(s)|} &= \frac{g'(\gamma(s))\dot{\gamma}(s)}{s^d \mathrm{e}^{\mu_- s}} \frac{s^d \mathrm{e}^{\mu_- s}}{|\dot{\gamma}(s)|} = g'\left(\gamma(s)\right) \frac{\mu_-\gamma_+ + O(\mathrm{e}^{-\delta s})}{|\mu_-\gamma_+| - O(\mathrm{e}^{-\delta s})}\\ &= g'\left(\gamma(s)\right)\left[\frac{\mu_-\gamma_+}{|\mu_-\gamma_+|} + O\left(\mathrm{e}^{-\delta s}\right)\right] = \left[g'(x_0) + O\left(s^d \mathrm{e}^{\mu_- s}\right)\right]\left[-\frac{\gamma_+}{|\gamma_+|} + O\left(\mathrm{e}^{-\delta s}\right)\right]\\ &= -\mu_- v_+ + O\left(\mathrm{e}^{-\delta s}\right)\end{aligned}$$

where $v_+ := \frac{\gamma_+}{|\gamma_+|}$ is a unitary eigenvector of $g'(x_0)$ with eigenvalue μ_+. Now

$$\frac{d}{ds} \log\left|\dot{\gamma}(s)\right| = \frac{\langle\dot{\gamma}(s), \ddot{\gamma}(s)\rangle}{|\dot{\gamma}(s)|^2} = \left\langle -v_+ + O\left(\mathrm{e}^{-\delta s}\right), -\mu_- v_+ + O\left(\mathrm{e}^{-\delta s}\right)\right\rangle = \mu_- + O\left(\mathrm{e}^{-\delta s}\right),$$

since

$$\frac{\dot{\gamma}(s)}{|\dot{\gamma}(s)|} = \frac{\dot{\gamma}(s)}{s^d \mathrm{e}^{\mu_- s}} \frac{s^d \mathrm{e}^{\mu_- s}}{|\dot{\gamma}(s)|} = \frac{\mu_-\gamma_+}{|\mu_-\gamma_+|} + O\left(\mathrm{e}^{-\delta s}\right) = -v_+ + O\left(\mathrm{e}^{-\delta s}\right).$$

We get then

$$\frac{d}{ds}\log\left|\dot{\gamma}(s)\right| = \mu_- + r(s)$$

where $0 \leqslant |r(s)| = O(\mathrm{e}^{-\delta s})$ as $s \to \infty$. Integrating on $[s, t]$ we get

$$\left|\dot{\gamma}(t)\right|\mathrm{e}^{-\mu_-(t-s)} = \left|\dot{\gamma}(s)\right|\mathrm{e}^{\int_s^t r(\sigma)\,d\sigma}.$$

Since $k_2 := \int_0^\infty |r(s)|\,ds < \infty$ we get

$$k_2^{-1}\left|\dot{\gamma}(s)\right| \leqslant \left|\dot{\gamma}(t)\right|\mathrm{e}^{-\mu_-(t-s)} \leqslant k_2\left|\dot{\gamma}(s)\right| \quad \text{for } 0 \leqslant s \leqslant t.$$

So if the first part of condition (C3) holds, μ_- is a real eigenvalue of $g'(x_0)$ and no other eigenvalues of $g'(x_0)$ have μ_- as real part, then $\dot{\gamma}(t)$ satisfies condition (c) of Theorem 5.3. Similarly, if the second part of condition (C3) holds, μ_+ is a real eigenvalue of $g'(x_0)$ and no other eigenvalues of $g'(x_0)$ have μ_+ as real part, then $\dot{\gamma}(t)$ satisfies condition (c) of Theorem 5.1.

6. Concluding remarks

(a) Condition (C1) for (1.1) means that

(C1)′ The unperturbed (1.1):

$$A(x)\dot{x} = f(x)$$

possesses a noncritical 0-singularity at x_0 [22], i.e. $\det A(x_0) = 0$ and $(\det A)'(x_0) \neq 0$.

(b) Let $\mathcal{R}L$ and $\mathcal{N}L$ be (resp.) the range and the kernel of a linear map L. Consider Eq. (1.4) derived from (1.1) with $\varepsilon = 0$, i.e. with the condition (1.3). From [16, p. 430], [22] it follows that $F(x_0) = 0$ is equivalent to $f(x_0) \in \mathcal{R}A(x_0)$ and (C1)′ implies $\dim \mathcal{N}A(x_0) = 1$, $\mathcal{R}(\operatorname{adj} A(x_0)) = \mathcal{N}A(x_0)$ and $\mathcal{N}(\operatorname{adj} A(x_0)) = \mathcal{R}A(x_0)$. Hence $f(x_0) = A(x_0)g_1$ and $\mathcal{R}\operatorname{adj} A(x_0) = \operatorname{span}\{g_2\}$ for some $g_1, g_2 \in \mathbb{R}^n$. Next, we derive

$$F'(x_0)v = \left(\left[\operatorname{adj} A(x_0)\right]'v\right)f(x_0) + \operatorname{adj} A(x_0)\left[f'(x_0)v\right]. \tag{6.1}$$

Differentiating $\det A(x)\mathbb{I} = \operatorname{adj} A(x)A(x)$, we get

$$\left(\left[\det A(x_0)\right]'v\right)\mathbb{I} = \left(\left[\operatorname{adj} A(x_0)\right]'v\right)A(x_0) + \operatorname{adj} A(x_0)\left[A'(x_0)v\right],$$

which implies

$$\begin{aligned}\left(\left[\operatorname{adj} A(x_0)\right]'v\right)f(x_0) &= \left(\left[\operatorname{adj} A(x_0)\right]'v\right)A(x_0)g_1\\ &= \left(\left[\det A(x_0)\right]'v\right)g_1 - \operatorname{adj} A(x_0)\left[A'(x_0)v\right]g_1.\end{aligned}$$

Then from (6.1) we get

$$F'(x_0)v = \left(\left[\det A(x_0)\right]'v\right)g_1 + \operatorname{adj} A(x_0)\left(\left[f'(x_0)v\right] - \left[A'(x_0)v\right]g_1\right) \in \operatorname{span}\{g_1, g_2\}.$$

Hence $\dim \mathcal{R}F'(x_0) \leqslant 2$ and then for $n > 2$, $F'(x_0)$ is singular, so it cannot be hyperbolic. So, (1.4) with $\omega(x)$ and $F(x)$ as in (1.3) is degenerate for $n > 2$ (see also Remark 3.4). It seems that passing from (1.1) to (1.2) with (1.3) is not effective for $n > 2$. On the other hand, in [2] we do study the case $n = 2$ when assumptions (C1)–(C5) are satisfied and this paper is a continuation and generalization of [2].

(c) In this remark we prove that under assumptions (C1)–(C4), the space of bounded solutions of equations $x' = F'(\gamma(s))x$ and $x' = F'(\gamma(s))x - \frac{\omega'(\gamma(s))x}{\omega(\gamma(s))}F(\gamma(s))$ have the same dimension. As a consequence assuming (C5) is equivalent to assuming that $x' = F'(\gamma(s))x$ has only $\gamma'(s)$ as solution bounded on $\mathbb{R}$ (up to a multiplicative constant). Setting

$$u_1(s) = \gamma'(s), \qquad \vartheta(s) = -\frac{\nabla\omega(\gamma(s))}{\omega(\gamma(s))}, \qquad B(s) = F'\big(\gamma(s)\big) \tag{6.2}$$

we want look for conditions assuring that the two systems

$$x' = B(s)x \tag{6.3}$$

and

$$x' = B(s)x + \big[\vartheta(s)^* x\big]u_1(s) \tag{6.4}$$

have the same number of independent solutions bounded on $\mathbb{R}$. Let $u_i(t) = (u_{i1}(t), u_{i2}(t), \dots, u_{in}(t))^*$, $i = 1, \dots, d$ be a basis of all bounded solutions of (6.3). We look for a solution of (6.4) in the form

$$x(s) = \psi(s)u_1(s) + \sum_{i=2}^{d} x_{0i}u_i(s), \qquad \psi(0) = x_{01}. \tag{6.5}$$

Then (6.4) gives

$$\begin{aligned} x'(s) &= \psi'(s)u_1(s) + \psi(s)u_1'(s) + \sum_{i=2}^{d} x_{0i}u_i'(s) \\ &= \psi(s)u_1(s) + \psi(s)B(s)u_1(s) + \sum_{i=2}^{d} x_{0i}B(s)u_i(s) \\ &= \psi(s)B(s)u_1(s) + \sum_{i=2}^{d} x_{0i}B(s)u_i(s) \\ &\quad + \left(\psi(s)\big(\vartheta(s)^*u_1(s)\big) + \sum_{i=2}^{d} x_{0i}\big(\vartheta(s)^*u_i(s)\big)\right)u_1(s). \end{aligned}$$

So ψ is determined by the equation

$$\psi'(s) = \psi(s)\big(\vartheta(s)^* u_1(s)\big) + \sum_{i=2}^{d} x_{0i}\big(\vartheta(s)^* u_i(s)\big), \qquad \psi(0) = x_{01}. \tag{6.6}$$

Its solution is

$$\psi(s) = x_{01} e^{\int_0^s \vartheta(z)^* u_1(z)\,dz} + \sum_{i=2}^{d} x_{0i} \int_0^s e^{\int_r^s \vartheta(z)^* u_1(z)\,dz}\big(\vartheta(r)^* u_i(r)\big)\,dr.$$

So, if

$$\sup_{s\in\mathbb{R}}\left(\big|u_1(s)\big| e^{\int_0^s \vartheta(z)^* u_1(z)\,dz} + \sum_{i=2}^{d}\big|u_1(s)\big| \int_0^s e^{\int_r^s \vartheta(z)^* u_1(z)\,dz}\big|\vartheta(r)^* u_i(r)\big|\,dr\right) < \infty, \tag{6.7}$$

then $\psi(s)u_1(s)$ is bounded on $\mathbb{R}$. Varying the point $(x_{01}, \ldots, x_{0d}) \in \mathbb{R}^d$, we get d linearly independent bounded solutions on $\mathbb{R}$ of (6.4). Hence, if condition (6.7) holds, the dimension of the space of solutions of (6.4) bounded on $\mathbb{R}$ is at least d (i.e. is greater than or equal to the dimension of the space of solutions of (6.3) bounded on $\mathbb{R}$). In our case we have

$$\int_r^s \vartheta(z)^* u_1(z)\,dz = -\int_r^s \frac{\omega'(\gamma(z))\gamma'(z)}{\omega(\gamma(z))}\,dz = \log\left(\frac{\omega(\gamma(s))}{\omega(\gamma(r))}\right)$$

then

$$\big|u_1(s)\big| e^{\int_0^s \vartheta(z)^* u_1(z)\,dz} = \gamma'(s)\frac{\omega(\gamma(s))}{\omega(\gamma(0))}$$

and

$$\vartheta(s)^* u_i(s) = -\omega'\big(\gamma(s)\big)\frac{u_i(s)}{\omega(\gamma(s))}.$$

Since $\omega(\gamma(s)) \simeq \gamma(s) - x_0 \simeq \mathrm{e}^{\mu_\pm s}$ and $\mu_\pm$ are simple eigenvalues any solution of (6.3) bounded on $\mathbb{R}$ and different from $\gamma'(s)$ tends to zero as $s \to \pm\infty$ faster than $\omega(\gamma(s))$. Thus from (C3) it follows that, for any solution $u_i(s)$ of (1.8) bounded on $\mathbb{R}$, $\vartheta(s)^* u_i(s) \to 0$ at an exponential rate as $s \to \pm\infty$. It follows then that condition (6.7) is satisfied and hence Eq. (1.7) has at least as many independent solutions bounded on $\mathbb{R}$ as (1.8).

Now we prove the converse. Let

$$\widetilde{B}(s)x := B(s)x + \big(\vartheta(s)^* x\big)u_1(s)$$

and consider the two equations

$$x' = \widetilde{B}(s)x \tag{6.8}$$

and

$$x' = \widetilde{B}(s)x + \big(\tilde{\vartheta}(s)^* x\big)\tilde{u}_1(s) \tag{6.9}$$

with

$$\tilde{\vartheta}(s) := -e^{-\int_0^s \vartheta(z)^* u_1(z)\,dz}\vartheta(s),$$
$$\tilde{u}_1(s) := e^{\int_0^s \vartheta(z)^* u_1(z)\,dz} u_1(s).$$

From (6.7) we deduce that $\tilde{u}_1(s)$ is a bounded solution on $\mathbb{R}$ of (6.8). Note

$$\widetilde{B}(s)x + \big(\tilde{\vartheta}(s)^* x\big)\tilde{u}_1(s) = B(s)x.$$

Let $\{\tilde{u}_1(s), \tilde{u}_2(s), \ldots, \tilde{u}_{\tilde{d}}(s)\}$ be a basis for the space of solutions of (6.8), bounded on $\mathbb{R}$. We compute

$$\big|\tilde{u}_1(s)\big| e^{\int_0^s \tilde{\vartheta}(z)^* \tilde{u}_1(z)\,dz} = \big|u_1(s)\big|,$$
$$\big|\tilde{u}_1(s)\big| \int_0^s e^{\int_r^s \tilde{\vartheta}(z)^* \tilde{u}_1(z)\,dz} \big|\tilde{\vartheta}(r)^* \tilde{u}_i(r)\big|\,dr = \big|u_1(s)\big| \int_0^s \big|\vartheta(z)^* \tilde{u}_i(z)\big|\,dz.$$

So, if

$$\big|u_1(s)\big| \int_0^s \big|\vartheta(z)^* \tilde{u}_i(z)\big|\,dz \tag{6.10}$$

is bounded on $\mathbb{R}$ we deduce, from the previous part that the number of independent solutions of (6.9), i.e. $x' = B(s)x$, bounded on $\mathbb{R}$ is greater than the number of solutions of the same kind of (6.8). From the previous part we see that, if (6.7) holds together with (6.10), then (6.3) and (6.4) have the same number of independent solutions bounded on $\mathbb{R}$.

To conclude the proof we show that, taking $u_1(s)$ and $\vartheta(s)$ as in (6.2), the expression in (6.10) is bounded. Assuming (6.2) and using the fact that $\tilde{u}_i(z)$ is bounded on $\mathbb{R}$ we see that

$$\big|u_1(s)\big|\big|\vartheta(z)^* \tilde{u}_i(z)\big| \simeq \big|\gamma'(s)\big| \left|\frac{\omega'(\gamma(z))}{\omega(\gamma(z))}\right| \simeq \mathrm{e}^{\mu_{\mp}(s-z)},$$

then, for $s \to \infty$, we have

$$\big|u_1(s)\big| \int_0^s \big|\vartheta(z)^* \tilde{u}_i(z)\big|\,dz \simeq \int_0^s \mathrm{e}^{\mu_-(s-z)}\,dz \leqslant \frac{1}{|\mu_-|}.$$

A similar argument works when $s \to -\infty$. So we conclude that the dimension of the space of solutions of (1.8) that are bounded on $\mathbb{R}$ equals the dimension of solutions of (1.7) that are bounded on $\mathbb{R}$.

(d) Our approach can be modified to study coupled IODE such as

$$\begin{aligned} \omega(x_1)\dot{x}_1 &= F(x_1) + \varepsilon G_1(x_1, x_2, t, \varepsilon, \kappa), \\ \omega(x_2)\dot{x}_2 &= F(x_2) + \varepsilon G_2(x_1, x_2, t, \varepsilon, \kappa), \end{aligned} \tag{6.11}$$

where ω, F satisfy assumptions (C1)–(C5) and $G_i \in C^2(\mathbb{R}^{2n+m+2}, \mathbb{R}^n)$, $i = 1, 2$ are 1-periodic in t with $G_i(x_0, x_0, t, \varepsilon, \kappa) = 0$ for any $t \in \mathbb{R}$, $\kappa \in \mathbb{R}^m$ and ε sufficiently small. Assumptions (C1)–(C5) are assumed for (6.11), then we consider $\binom{\gamma(s)}{\gamma(s)}$ for (6.11). Observing this, we can consider a special case of (1.1) when $n = 2k$, $x = (x_1, \dots, x_k)$, $x_i \in \mathbb{R}^2$ and $A(x) = \operatorname{diag}\{A_0(x_1), \dots, A_0(x_k)\}$, $f(x) = (f_0(x_1), \dots, f_0(x_k))$ with A_0 is a two-dimensional matrix and f is a two-dimensional mapping. Then we perform in each x_i the adj-construction. To be more concrete, we may consider $n = 4$ and take

$$\begin{aligned} A_0(x_1)\dot{x}_1 &= f_0(x_1) + \varepsilon g_1(x_1, x_2, t, \varepsilon, \kappa), \\ A_0(x_2)\dot{x}_2 &= f_0(x_2) + \varepsilon g_2(x_1, x_2, t, \varepsilon, \kappa) \end{aligned} \tag{6.12}$$

with

$$x_i := \begin{pmatrix} u_i \\ v_i \end{pmatrix}, \qquad A_0(x) := \begin{pmatrix} \tilde{f}(u) & 0 \\ 0 & 1 \end{pmatrix},$$
$$f_0(x) := \begin{pmatrix} v \\ \bar{f}(u) \end{pmatrix},$$
$$g_i(x_1, x_2, t, \varepsilon, \kappa) := \begin{pmatrix} v \\ \tilde{g}_i(x_1, x_2, t, \varepsilon, \kappa) \end{pmatrix}.$$

Then (6.12) has a form of (6.11) with

$$\begin{aligned} \omega(x) &= \tilde{f}(u), \\ F(x) &= \begin{pmatrix} v \\ \tilde{f}(u)\bar{f}(u) \end{pmatrix}, \\ G_i(x_1, x_2, t, \varepsilon, \kappa) &= \begin{pmatrix} v \\ \tilde{f}(u_i)\tilde{g}_i(x_1, x_2, t, \varepsilon, \kappa) \end{pmatrix}. \end{aligned} \tag{6.13}$$

So we suppose $u^* \in \mathbb{R}$ exists such that $\tilde{f}(u^*) = 0$, $\tilde{f}'(u^*)\bar{f}(u^*) > 0$. Then $x_0 = \binom{u^*}{0}$, $F(x_0) = 0$, $DF(x_0)$ is hyperbolic and $G_i(x_0, x_0, t, \varepsilon, \kappa) = 0$. Next, (1.5) is a second order scalar ODE $\ddot{u} = \tilde{f}(u)\bar{f}(u)$ with a hyperbolic equilibrium u^*. We assume the existence of a homoclinic orbit $\gamma^*(s)$ to it. Then $\tilde{f}(\gamma^*(s)) \neq 0$ for any $s \in \mathbb{R}$. So taking $\gamma(s) := \binom{\gamma^*(s)}{\gamma^*(s)}$, conditions (C2)–(C5) are satisfied. We note that by [2], (6.12) is a codimension 4 problem, this fact being consistent with Remark 3.4.

Acknowledgment

We thank the referee for valuable comments and suggestions to improve our paper.

References

[1] J. Andres, L. Górniewicz, Topological Principles for Boundary Value Problems, Kluwer, Dordrecht, 2003.

[2] F. Battelli, C. Fečkan, Nonlinear RLC Circuits and Implicit ODEs, submitted for publication.

[3] F. Battelli, C. Lazzari, Exponential dichotomies, heteroclinic orbits, and Melnikov functions, J. Differential Equations 86 (1990) 342–366.

[4] A. Calamai, M. Franca, Mel'nikov methods and homoclinic orbits in discontinuous systems, J. Dynam. Differential Equations 25 (2013) 733–764.

[5] C. Chicone, Ordinary Differential Equations with Applications, second ed., Springer-Verlag, New York, 2006.

[6] E.A. Coddington, N. Levinson, Theory of Ordinary Differential Equations, Tata McGraw–Hill Publ. Co., New Delhi, 1972 (reprint 1987).

[7] M. Fečkan, Existence results for implicit differential equations, Math. Slovaca 48 (1998) 35–42.

[8] M. Frigon, T. Kaczynski, Boundary value problems for systems of implicit differential equations, J. Math. Anal. Appl. 179 (1993) 317–326.

[9] L. Górniewicz, Topological Fixed Point Theory of Multivalued Mappings, Springer, Berlin, 2009.

[10] S. Heikkilä, M. Kumpulainen, S. Seikkala, Uniqueness and comparison results for implicit differential equations, Dynam. Systems Appl. 7 (1998) 237–244.

[11] P. Kunkel, V. Mehrmann, Differential-Algebraic Equations, Analysis and Numerical Solution, European Math. Soc., 2006.

[12] N. Lazarides, M. Eleftheriou, G.P. Tsironis, Discrete breathers in nonlinear magnetic metamaterials, Phys. Rew. Lett. 97 (2006) 157406.

[13] D. Li, Peano's theorem for implicit differential equations, J. Math. Anal. Appl. 258 (2001) 591–616.

[14] M. Medved', Normal forms of implicit and observed implicit differential equations, Riv. Mat. Pura Appl. 10 (1991) 95–107.

[15] M. Medved', Qualitative properties of generalized vector fields, Riv. Mat. Pura Appl. 15 (1994) 7–31.

[16] P.J. Rabier, Implicit differential equations near a singular point, J. Math. Anal. Appl. 144 (1989) 425–449582.

[17] P.J. Rabier, W.C. Rheinboldt, A general existence and uniqueness theorem for implicit differential algebraic equations, Differential Integral Equations 4 (1991) 563–582.

[18] K.J. Palmer, Exponential dichotomies and transversal homoclinic points, J. Differential Equations 55 (1984) 225–256.

[19] P.J. Rabier, W.C. Rheinbold, A geometric treatment of implicit differential-algebraic equations, J. Differential Equations 109 (1994) 110–146.

[20] P.J. Rabier, W.C. Rheinbold, On impasse points of quasilinear differential algebraic equations, J. Math. Anal. Appl. 181 (1994) 429–454.

[21] P.J. Rabier, W.C. Rheinbold, On the computation of impasse points of quasilinear differential algebraic equations, Math. Comp. 62 (1994) 133–154.

[22] R. Riaza, Differential-Algebraic Systems, Analytical Aspects and Circuit Applications, World Sci. Publ. Co. Pte. Ltd., 2008.

[23] G.P. Veldes, J. Cuevas, P.G. Kevrekidis, D.J. Frantzeskakis, Quasidiscrete microwave solitons in a split-ring-resonator-based left-handed coplanar waveguide, Phys. Rev. E 83 (2011) 046608.