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Note

The list-chromatic index of K_6

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ABSTRACT

We prove that the list-chromatic index and paintability index of K_6 is 5. That indeed $\chi'_\ell(K_6) = 5$ was a still open special case of the List Coloring Conjecture. Our proof demonstrates how colorability problems can numerically be approached by the use of computer algebra systems and the Combinatorial Nullstellensatz.

1. Introduction

Given a graph G and an assignment of a family $\mathcal{L} = (L_e)_{e\in E(G)}$ of non-empty sets L_e to the edges e of G, we say that G is *$\mathcal{L}$-list-edge colorable* if it is possible to assign to each edge $e \in E(G)$ an element from the list L_e in such a way that adjacent edges of G are assigned distinct elements of $\mathcal{C} = \bigcup_{e\in E(G)} L_e$. The set $\mathcal{C}$ is conventionally called the set of *colors*. Given a positive integer k, we say that a graph G is k-list-edge colorable if, for any assignment $\mathcal{L} = (L_e)_{e\in E(G)}$, satisfying $|L_e| = k$ for every $e \in E(G)$, the graph G is $\mathcal{L}$-list-edge colorable. The *list-chromatic index* of G, denoted by $\chi'_\ell(G)$, is the minimum positive integer k such that G is k-list-edge colorable.

It is easy to see that $\chi'_\ell(G) \geq \chi'(G)$, where $\chi'(G)$ is the (ordinary) chromatic index of G, since, if $k = \chi'_\ell(G)$, and we set $\mathcal{L} = (L_e)_{e\in E(G)}$, where $L_e = \{1, 2, \dots, k\}$ for every $e \in E(G)$, then the fact that G is $\mathcal{L}$-list-edge colorable guarantees the existence of a k-edge coloring of G, and hence, by definition, $k \geq \chi'(G)$. The opposite was conjectured independently by several researchers; see [10, Section 12.20]:

Conjecture 1.1 (*List Coloring Conjecture*). *$\chi'_\ell(G) = \chi'(G)$ for every multigraph G.*

This conjecture appears to be very hard and it has been proven only for some special cases, most famously for bipartite graphs by Galvin in [6]. One indication of the difficulty of Conjecture 1.1 is provided by the fact that it is still open for some apparently trivial classes of graphs, such as the class of complete (simple) graphs. It is well known (see e.g. [5]) that the chromatic index of the complete graph K_n, with $n > 1$, is given by

$$\chi'(K_n) = \begin{cases} n-1 & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd, } n \geq 3. \end{cases}$$

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Thus, for complete graphs, Conjecture 1.1 reduces to proving that the list-chromatic index of K_n equals the quantity indicated above. The most successful attempt in this direction is given by the following theorem of Häggkvist and Janssen [7].

Theorem 1.2 (*Häggkvist and Janssen [7]*). *$\chi'_\ell(K_n) \leq n$ for every positive integer n.*

This result leaves the question open only for complete graphs of even order. For convenience, we state this case separately as a special case of the List Coloring Conjecture.

Conjecture 1.3. *$\chi'_\ell(K_{2m}) = 2m - 1$ for every positive integer m.*

Note that the Häggkvist–Janssen theorem can be derived as a consequence of Conjecture 1.3. To see this, we can clearly confine ourselves to the case of odd order. Let $m \in \mathbb{Z}^+$, and consider a list assignment $\mathcal{L} = (L_e)_{e\in E(K_{2m+1})}$ on the edges of K_{2m+1}, where $|L_e| = 2m + 1$ for every $e \in E(K_{2m+1})$. Let $v_0, v_1, \ldots, v_{2m}$ be the vertices of K_{2m+1}, and let $e_1 = v_0v_1$, $e_2 = v_0v_2, \ldots, e_{2m} = v_0v_{2m}$. Assign a distinct color to each of the edges $e_1, e_2, \ldots, e_{2m}$ from their respective lists (this is always possible since the size of each list is $2m+1$). Let $\varphi(e_1), \varphi(e_2), \ldots, \varphi(e_{2m})$ be the colors assigned to the edges $e_1, e_2, \ldots, e_{2m}$. Consider now the graph $K_{2m+1} - v_0$ and, for each edge $f = v_iv_j \in E(K_{2m+1} - v_0)$, consider the list

$$L'_f = L_f \setminus \{\varphi(e_i), \varphi(e_j)\},$$

where $L_f \in \mathcal{L}$. This defines a list assignment $\mathcal{L}'$ on the edges of $K_{2m+1} - v_0$ with

$$|L'_f| \geq 2m - 1 \quad \text{for every } f \in E(K_{2m+1} - v_0).$$

Assuming that Conjecture 1.3 holds, there exists an $\mathcal{L}'$-list edge coloring ψ of $K_{2m+1} - v_0$, as $K_{2m+1} - v_0 \cong K_{2m}$. Now, it is easy to see that, by construction, the function

$$\sigma(e) = \begin{cases} \varphi(e) & \text{if } e = e_1, e_2, \ldots, e_{2m}, \\ \psi(e) & \text{if } e \in E(K_{2m+1} - v_0) \end{cases}$$

is an $\mathcal{L}$-list-edge coloring of K_{2m+1}. This proves that $\chi'_\ell(K_{2m+1}) \leq 2m + 1$, and we obtain the Häggkvist–Janssen theorem.

Conjecture 1.3 clearly holds for $m = 1$ since K_2 is obviously 1-list-edge colorable. For $m = 2$, Conjecture 1.3 holds as a consequence of the fact that every 1-factorable planar graph satisfies Conjecture 1.1, a fact established by Ellingham and Goddyn [4]. A short direct and elementary proof of the case $m = 2$ of Conjecture 1.3 was given by Ko-Wei Lih and the first author in [2]. As far as we know Conjecture 1.3 is open for $m \geq 3$, and the purpose of this paper is to settle the case $m = 3$. Various attempts of the first and second authors to settle this case by an elementary, direct proof have not materialized.

2. The edge distance polynomial

To settle Conjecture 1.3 for K_6 ($m = 3$), we will examine the edge distance polynomial $P_{L(K_6)}$ of its line graph $L(K_6)$. Here, the edge distance polynomial P_G of a graph G on numbered vertices $v_1, v_2, \ldots, v_k$ is a polynomial in the variables $x_1, x_2, \ldots, x_k$, with one variable x_j for each vertex v_j. It is defined as the product over all differences $x_i - x_j$ with $v_iv_j \in E(G)$ and $i < j$, and we view it as a polynomial over $\mathbb{Q}$. The nonzeros in $\mathbb{Q}^k$ of this polynomial are precisely the vertex colorings of G with rational numbers as colors (and we always may view the set of colors $\mathcal{C}$ as contained in $\mathbb{Q}$). This is easy to see. The points $(x_1, x_2, \ldots, x_k) \in \mathbb{Q}^k$ are interpreted as vertex labelings of G via $v_j \longmapsto x_j$. If there is some difference $x_i - x_j$ that is zero, in such a point $(x_1, x_2, \ldots, x_k)$, this means that the vertices v_i and v_j receive the same color, $x_i = x_j$. Hence, indeed, the nonzeros of P_G are precisely those labelings that are correct colorings.

Since we are interested in edge colorings, but employ techniques for vertex colorings, first, we have to switch from K_6 to its line graph $L(K_6)$. Second, we consider the edge distance polynomial $P_{L(K_6)}$. Our process for generating $P_{L(K_6)}$ is presented in the first three parts of the algorithm in the next section. It also works for general complete graphs K_n. Note that our systematic approach requires double indexed variables $x_{i,j}$. This is because the vertices $v_{i,j}$ of the line graph are the edges v_iv_j of the underlying graph, and each edge v_iv_j has two ends v_i and v_j.

After the third part of the algorithm, the polynomial $P_{L(K_6)}$ is printed; it has $|E(K_6)| = 15$ variables and $|E(L(K_6))| = 60 = 15 \times 4$ factors. The expansion of this product is not advisable, as it would result in 2^{60} summands. We will use another method to determine the coefficient of the monomial

$$x_{1,2}^4 x_{1,3}^4 x_{1,4}^4 x_{1,5}^4 x_{1,6}^4 x_{2,3}^4 x_{2,4}^4 x_{2,5}^4 x_{2,6}^4 x_{3,4}^4 x_{3,5}^4 x_{3,6}^4 x_{4,5}^4 x_{4,6}^4 x_{5,6}^4 \tag{1}$$

as equal to -720. Since this is different from zero, the famous *Combinatorial Nullstellensatz* [1] will then guarantee a nonzero in any list assignment $\mathcal{L}$ with lists of size 5. In fact, edge distance polynomials are always homogeneous so that all monomials have maximal degree, and the following elegant formulation of the Combinatorial Nullstellensatz applies.

Theorem 2.1 (*Combinatorial Nullstellensatz*). *If $x_1^{\alpha_1}x_2^{\alpha_2}\ldots x_k^{\alpha_k}$ occurs as monomial of maximal degree in a polynomial $P(x_1, x_2, \ldots, x_k)$, then this polynomial has a nonzero in any domain $L_1 \times L_2 \times \cdots \times L_k$ with $|L_j| > \alpha_j$ for $j = 1, 2, \ldots, k$.*

As explained, the guaranteed nonzeros are vertex colorings of $L(K_6)$ and $\mathcal{L}$-list-edge colorings of K_6. Hence, our conjecture will then be verified for K_6. Moreover, the third author introduced in [11] (and already in [14]) the stronger concept of paintability. In this concept, the lists can still be modified during an interactive coloration process. He showed in [12] (and [14]) that the Combinatorial Nullstellensatz holds for paintability as well so that our approach will even prove that K_6 is edge 5-paintable.

Following the suggestion of the referee, we explain the concept of paintability and the mentioned version of the Combinatorial Nullstellensatz here briefly. The idea is that, if the set of all colors $\mathcal{C}$ is contained in $\mathbb{Z}^+$, one may try to use color 1 at first, of course, only for vertices v whose lists L_v contain color 1. Afterwards, one may allow the change of the remaining lists $L_v \setminus \{1\}$, without changing their cardinalities, and then extend the partial coloring with color 2. This extension process is then repeated with color 3, color 4 and so forth, where, in between, the remaining tails of the color lists may be altered. This more flexible on-line list coloring can also be formulated for polynomials. The partial colorings just correspond to partial substitutions that do not make the polynomial zero. However, on the more general level of polynomials, there is the additional difficulty that the assumption $\mathcal{C} \subseteq \mathbb{Z}^+$ is not sufficient, as the Paintability Nullstellensatz requires algebraically independent colors. Without lose of generality, here, one may assume $\mathcal{C} \subseteq \{T_1, T_2, \ldots\}$, with symbolic variables T_j. Substantially new techniques are also required in the proof of the Paintability Nullstellensatz, but the result itself is not surprising. In fact, the great majority of all list coloring theorems in graph theory could already be generalized to paintability; see e.g. [11,13,12,8,15,9,3]. In these graph-theoretic cases, an elegant recursive way to define k-paintability may even have added some clarity to the proofs of these theorems.

All that is left in our approach to K_6, whether its edge 5-paintability or just its edge 5-choosability, is to calculate the coefficient of monomial (1). To do so, we use the simple fact that the coefficient $P_{\alpha_1,\alpha_2,\ldots,\alpha_k}$ of $x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_k^{\alpha_k}$ in P is given by

$$\alpha_1!\,\alpha_2!\ldots\alpha_k!\,P_{\alpha_1,\alpha_2,\ldots,\alpha_k} = \left.\left(\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}}\frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}}\cdots\frac{\partial^{\alpha_k}}{\partial x_k^{\alpha_k}}P\right)\right|_{x_1=x_2=\cdots=x_k=0}. \tag{2}$$

Actually, this calculation is still too heavy for a normal personal computer, as the product rule for differentiation brings us many summands. However, we can accelerate it by plugging in the zeros as early as possible. In this way, we kill many terms and save memory and time. If, for example, we want to calculate the coefficient $P_{2,3}$ of $x_1^2 x_2^3$ in a polynomial $P(x_1, x_2)$, we can use

$$2!\,3!\,P_{2,3} = \left.\left(\frac{\partial^2}{\partial x_1^2}\frac{\partial^3}{\partial x_2^3}P\right)\right|_{x_1=x_2=0} = \left.\left(\frac{\partial^2}{\partial x_1^2}\left[\left.\left(\frac{\partial^3}{\partial x_2^3}P\right)\right|_{x_2=0}\right]\right)\right|_{x_1=0}. \tag{3}$$

This trick is incorporated into the last part of our algorithm, where, for each variable at a time, we differentiate $n-2=4$ many times, substitute zero and then divide by $(n-2)! = 4!$. In the main line of this algorithm, the variable x[e[1], e[2]] belongs to the edge e $\in$ Egs $:= E(K_n)$, e.g. $x_{3,5}$ = x[3, 5] belongs to the edge $\{3, 5\}$. The expression `diff(P,x[e[1],e[2]]$n-2)` denotes the $(n-2)$nd derivative of P in that variable. Finally, to substitute 0 for x[e[1], e[2]], this command has been extended to

subs(diff(P, x[e[1], e[2]]$n − 2), x[e[1], e[2]] = 0)/(n − 2)!,

which includes the division by $(n-2)!$ already. As explained, this command has to be repeated for each edge e $\in$ Egs $:= E(K_n)$, and eventually results in the value of our coefficient.

As mentioned, the initial three parts of the algorithm calculate the edge distance polynomial P of $L(K_6)$, which is printed right before the fourth and last part. Its 60 factors could have been entered manually, but we wanted to provide a generic solution that works for arbitrary n. The edge distance polynomial is just the following product of edge distances, where the edges le of the line graph are listed in LEgs as ordered pairs $[\{i,j\},\{k,\ell\}]$ of adjacent edges $\{i,j\}$ and $\{k,\ell\}$ of the underlying complete graph:

$$\mathrm{P} := \prod_{\substack{[\{i,j\},\{k,\ell\}]\in \mathrm{LEgs}\\ i<j,k<\ell}} (x_{k,\ell} - x_{i,j}) = \prod_{\mathrm{le}\in\mathrm{LEgs}} \big(\mathtt{x[le[2][1],le[2][2]]} - \mathtt{x[le[1][1],le[1][2]]}\big).$$

3. The algorithm

Our algorithm is written in the easily understandable MuPAD language, in red. The output of the different steps is printed in blue. The running time on a usual personal computer, using MATLAB with Symbolic Math Toolbox, was about five minutes. More than 450 megabytes memory was allocated by the process. On future faster high performance computers it might become possible to do the next step, by replacing the very first command “n := 6” with “n := 8”:

```
n:=6: Egs:=[]: print(Unquoted,NoNL,"Edges of K_".n.": ");
for i from 1 to n-1 do for j from i+1 to n do Egs:=Egs.[{i,j}]
end_for end_for //output Egs
```

Edges of K_6: [{1, 2}, {1, 3}, {1, 4}, {1, 5}, {1, 6}, {2, 3}, {2, 4}, {2, 5}, {2, 6}, {3, 4}, {3, 5}, {3, 6}, {4, 5}, {4, 6}, {5, 6}]

```
LEgs:=[]: print(Unquoted,NoNL,"Arcs of L(K_".n."): ");
for i from 1 to nops(Egs)-1 do for j from i+1 to nops(Egs) do
    if card(Egs[i] intersect Egs[j])=1 then LEgs:=LEgs.[[Egs[i],Egs[j]]]
    end_if
end_for end_for //output LEgs
```

Arcs of $L(K\_6)$: $[[\{1,2\},\{1,3\}],[\{1,2\},\{1,4\}],[\{1,2\},\{1,5\}],\ldots,[\{4,6\},\{5,6\}]]$

```
P:=1: print(Unquoted,NoNL,"Edge distance polynomial of L(K_".n."):   ");
for le in LEgs do P:=P*(x[le[2][1],le[2][2]]-x[le[1][1],le[1][2]])
end_for //output P
```

Edge distance polynomial of $L(K_6)$: $(x_{1,2}-x_{1,3})(x_{1,2}-x_{1,4})(x_{1,2}-x_{2,3})(x_{1,2}-x_{1,5})$
$(x_{1,2}-x_{2,4})(x_{1,3}-x_{1,4})(x_{1,3}-x_{2,3})(x_{1,2}-x_{1,6})(x_{1,2}-x_{2,5})(x_{1,3}-x_{1,5})(x_{1,2}-x_{2,6})(x_{1,3}-x_{1,6})$
$(x_{1,3}-x_{3,4})(x_{1,4}-x_{1,5})(x_{1,4}-x_{2,4})(x_{2,3}-x_{2,4})(x_{1,3}-x_{3,5})(x_{1,4}-x_{1,6})(x_{1,4}-x_{3,4})(x_{2,3}-x_{2,5})$
$(x_{2,3}-x_{3,4})(x_{1,3}-x_{3,6})(x_{2,3}-x_{2,6})(x_{2,3}-x_{3,5})(x_{1,5}-x_{1,6})(x_{1,5}-x_{2,5})(x_{2,4}-x_{2,5})(x_{2,4}-x_{3,4})$
$(x_{1,4}-x_{4,5})(x_{2,3}-x_{3,6})(x_{1,5}-x_{3,5})(x_{2,4}-x_{2,6})(x_{1,4}-x_{4,6})(x_{1,5}-x_{4,5})(x_{2,4}-x_{4,5})(x_{1,6}-x_{2,6})$
$(x_{2,5}-x_{2,6})(x_{2,5}-x_{3,5})(x_{3,4}-x_{3,5})(x_{2,4}-x_{4,6})(x_{1,6}-x_{3,6})(x_{2,5}-x_{4,5})(x_{3,4}-x_{3,6})(x_{3,4}-x_{4,5})$
$(x_{1,5}-x_{5,6})(x_{1,6}-x_{4,6})(x_{3,4}-x_{4,6})(x_{2,6}-x_{3,6})(x_{3,5}-x_{3,6})(x_{3,5}-x_{4,5})(x_{1,6}-x_{5,6})(x_{2,5}-x_{5,6})$
$(x_{2,6}-x_{4,6})(x_{2,6}-x_{5,6})(x_{3,5}-x_{5,6})(x_{3,6}-x_{4,6})(x_{4,5}-x_{4,6})(x_{3,6}-x_{5,6})(x_{4,5}-x_{5,6})(x_{4,6}-x_{5,6})$

```
print(Unquoted,NoNL,"'Leading' coefficient: ");
for e in Egs do P:=subs(diff(P,x[e[1],e[2]]$n-2),x[e[1],e[2]]=0)/(n-2)!
end_for //output P
```

'Leading' coefficient: -720

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