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Complementary Riordan arrays

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ABSTRACT

Recently, the concept of the complementary array of a Riordan array (or recursive matrix) has been introduced. Here we generalize the concept and distinguish between dual and complementary arrays. We show a number of properties of these arrays, how they are computed and their relation with inversion. Finally, we use them to find explicit formulas for the elements of many recursive matrices.

1. Introduction

Let $\mathbb{K}$ be a field of characteristic 0; usually, we consider the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$. If t is an indeterminate over $\mathbb{K}$, a *formal power series* on $\mathbb{K}$ is a sum $f(t) = \sum_{n=0}^{\infty} f_n t^n$, where $f_n \in \mathbb{K}$, for all $n \in \mathbb{N}$; many times, $f(t)$ is called the *generating function* of the sequence $(f_n)_{n\in\mathbb{N}}$. The set of all formal power series over $\mathbb{K}$ is denoted by $\mathbb{K}[[t]]$ or simply $\mathcal{F}$ when no ambiguity exists about the field and the indeterminate.

Actually, $\mathcal{F}$ is an integral domain and can be extended to a field if we consider the set $\mathcal{F} \times \mathcal{F} \setminus \{0\}$, where the ordered pair $(f(t),\ g(t))$ with $f(t), g(t) \in \mathcal{F}$ and $g(t) \neq 0$ is interpreted as the ratio $f(t)/g(t)$ of the two formal power series. Every pair is called a *formal Laurent series* and can be represented by a sum $\ell(t) = \sum_{n=m}^{\infty} \ell_n t^n$, where $m \in \mathbb{Z}$. Obviously, the formal Laurent series with $m \geq 0$ do coincide with formal power series. Given a formal series[1] $f(t)$, the minimum index n for which $f_n \neq 0$ is called the *order* of $f(t)$. In general, $\mathcal{F}_m$ denotes the set of all formal series of order m. It is well-known that $\mathcal{F}_0$ is the set of *invertible* formal power series, that is, the formal power series $f(t)$ for which an inverse $g(t)$ exists such that $f(t)g(t) = 1$; $\mathcal{F}_0$ is also the set of formal series $f(t)$ having $f(0) \neq 0$; in this case, t is used as a variable rather than as an indeterminate. Finally, $\mathcal{F}_1$ is the set of *compositionally invertible* formal series, that is, the series $f(t)$ for which a formal series (usually denoted by $\overline{f}(t)$) exists such that $f(\overline{f}(t)) = \overline{f}(f(t)) = t$; they are characterized by the conditions: $f(0) = 0$ and $f'(0) \neq 0$.

The concept of a (proper) Riordan array was introduced in [18,19] as a generalization of the Pascal triangle. A Riordan array is defined as an ordered pair of formal power series $D = \mathcal{R}(d(t), h(t))$ with $d(0) \neq 0$, $h(0) = 0$, $h'(0) \neq 0$. The usual way to represent the Riordan array $\mathcal{R}(d(t), h(t))$ is by means of an infinite matrix $(d_{n,k})$, $n, k \in \mathbb{N}$, its generic element being:

$$d_{n,k} = [t^n]d(t)h(t)^k. \tag{1.1}$$

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[1] We say "formal series" whenever distinguishing between formal and Laurent is not relevant.

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The Pascal triangle is just the case $P = \mathcal{R}\left((1-t)^{-1},\ t(1-t)^{-1}\right)$. Many properties of Riordan arrays have been studied in the literature, in particular their connection with combinatorial sums. Actually, if $(s_n)_{n\in\mathbb{N}}$ is any sequence having $s(t) = \sum_{k=0}^{\infty} s_k t^k$ as its generating function, it is possible to prove that:

$$\sum_{k=0}^{n} d_{n,k}s_k = [t^n]d(t)s(h(t))$$

thus reducing the sum to the extraction of a coefficient. For alternative approaches see, e.g., [3,15].

For proper Riordan arrays, Rogers [16] has found an important characterization: every element $d_{n+1,k+1},\ n, k \in \mathbb{N}$, can be expressed as a linear combination of the elements in the preceding row, i.e.:

$$d_{n+1,k+1} = a_0 d_{n,k} + a_1 d_{n,k+1} + a_2 d_{n,k+2} + \cdots = \sum_{j=0}^{\infty} a_j d_{n,k+j}. \tag{1.2}$$

The sum is actually finite, the sequence $A = (a_k)_{k\in\mathbb{N}}$ is fixed and is called the *A-sequence* of the Riordan array. Finally, we recall that Riordan arrays are a formulation of the 1-*Umbral Calculus*, as defined by S. Roman in [17], although their name was only coined later in 1991 by Shapiro et al. [18]. The first appearance of the concept (in the form here considered) is due to Barnabei, Brini and Nicoletti [1]. For a different approach see [8]. Recently, Luzón and Morón [9], and then Sprugnoli [20] and Cheon and Jin [2] introduced the concept of *complementary* Riordan arrays, which is the main topic discussed in the present work. In particular, the paper is structured in the following way. In Section 2, we summarize the main properties of Riordan arrays, properties that will be used throughout these pages. Section 3 extends the concept of Riordan arrays to the equivalent but more comprehensive concept of recursive matrices, with which the following expansions are better explained; besides, we introduce the idea of a complementary and a dual array (of a given Riordan array) as specializations of the same concept; finally we give the algebraic characterization of these concepts. In Section 4, we prove a series of properties of complementary, dual and diagonal translation operators; in particular we introduce a subgroup of the Riordan group, which encompasses most of the subgroups studied in the literature; this subgroup allows us to connect algebraically all the quoted subgroups by means of the complementary and dual transformations, a result important in practice as well as in the theory. Finally, Section 5 gives formulas for the direct application of these transformations and proposes some examples relative to central binomial coefficients, trinomial and Motzkin triangles, showing how the last two cases are naturally connected.

2. Riordan arrays

We recall the main properties of Riordan arrays which will be used in this paper. According to the terminology introduced in [1], if $D = \mathcal{R}(d(t), h(t))$ is any Riordan array, the function $h(t)$ will be called the *recurrence rule* and the function $d(t)$ the *boundary value*. The product of two Riordan arrays is defined by:

$$D_1 * D_2 = \mathcal{R}(d_1(t), h_1(t)) * \mathcal{R}(d_2(t), h_2(t)) = \mathcal{R}(d_1(t)d_2(h_1(t)), h_2(h_1(t))); \tag{2.1}$$

it corresponds to the usual row-by-column product of two (infinite) matrices. The Riordan array $I = \mathcal{R}(1, t)$ acts as the identity and the inverse of $D = \mathcal{R}(d(t), h(t))$ is the Riordan array:

$$D^* = (d^*_{n,k}) = \mathcal{R}(d^*(t), h^*(t)) = \mathcal{R}\left(\frac{1}{d(\overline{h}(t))},\ \overline{h}(t)\right)$$

where $\overline{h}(t)$ is the compositional inverse of $h(t)$. Since the product $D * D^* = D^* * D$ equals the identity $I = \mathcal{R}(1, t)$, every Riordan array induces the two-parameters *basic identity*, where δ is the Kronecker delta:

$$\sum_{j=k}^{n} d_{n,j}d^*_{j,k} = \delta_{n,k}. \tag{2.2}$$

In general, a superscripted asterisk denotes quantities related to the inverse Riordan arrays; overlining denotes compositional inversion. We observe that $h^*(t) = \overline{h}(t)$, but $d^*(t) \neq \overline{d}(t)$. The following result concerns an important property of the compositional inverse, and will be used many times:

Theorem 2.1. *For any compositionally invertible formal power series $h(t)$, we have:*

$$h'(\overline{h}(t)) = \frac{1}{\overline{h}'(t)} \quad \textit{and} \quad \overline{h}'(h(t)) = \frac{1}{h'(t)}.$$

Proof. By definition we have $h(\overline{h}(t)) = t$, and we can differentiate by applying the chain rule:

$$1 = \frac{\mathrm{d}}{\mathrm{d}t}h(\overline{h}(t)) = h'(\overline{h}(t))\,\overline{h}'(t).$$

This is the first assertion, while the second is found starting with $\overline{h}(h(t)) = t$. ∎

The set $\mathcal{R}$ of all Riordan arrays is a group with the product defined above. Important subgroups are:

$\mathcal{R}_A$	Appell or Toeplitz subgroup	$\mathcal{R}(d(t), t)$
$\mathcal{R}_L$	Lagrange or associated subgroup	$\mathcal{R}(1, h(t))$
$\mathcal{R}_D$	Co-Lagrange or derivative subgroup	$\mathcal{R}(h'(t), h(t))$
$\mathcal{R}_N$	Renewal or bell subgroup	$\mathcal{R}(d(t), td(t))$
$\mathcal{R}_H$	Hitting-time subgroup	$\mathcal{R}(th'(t)/h(t), h(t))$.

Another important subgroup is the *checkerboard subgroup*, composed by the Riordan arrays $\mathcal{R}(d(t), h(t))$ in which $d(t)$ is odd $(d(-t) = -d(t))$ and $h(t)$ is even $(h(-t) = h(t))$. Other subgroups will be encountered in this paper, thus showing that Riordan arrays have a rich algebraic structure. An important observation is that every Riordan array can be seen as the product of an Appell by a Lagrange array:

$$\mathcal{R}(d(t), h(t)) = \mathcal{R}(d(t), t) * \mathcal{R}(1, h(t)). \tag{2.3}$$

Theorem 2.2 allows us to compute the generic element of the inverse array of $D = \mathcal{R}(d(t), h(t))$ by using the functions $d(t)$ and $h(t)$. The proof is based on the Lagrange Inversion Formula in the forms given in [5,21]. The LIF is important in the theory and applications of Riordan arrays (see, e.g., [11,12]), especially for the so-called *implicit Riordan arrays*, for which see [13].

Theorem 2.2. *Given the Riordan array $D = (d_{n,k}) = \mathcal{R}(d(t), h(t))$, the generic element of its inverse is given by:*

$$d^*_{n,k} = [t^{-k-1}]\frac{h'(t)}{d(t)h(t)^{n+1}} = [t^{n-k}]\frac{h'(t)}{d(t)(h(t)/t)^{n+1}} \tag{2.4}$$

or, equivalently:

$$d^*_{n,k} = \frac{1}{n}[t^{n-k}]\left(\frac{k}{d(t)} - \frac{td'(t)}{d(t)^2}\right)\left(\frac{t}{h(t)}\right)^n. \tag{2.5}$$

Several times, a sequence $(c_n)_{n\in\mathbb{N}}$ is defined as the diagonal elements of a set of functions:

$$c_n = [t^n]F(t)\phi(t)^n.$$

By the LIF, it is possible to prove the following useful *diagonalization rule*:

$$\sum_{n=0}^{\infty} c_n t^n = \left[\frac{F(w)}{1 - t\phi'(w)}\middle| w = t\phi(w)\right] \tag{2.6}$$

where $w(0) = 0$ and $[f(w) \mid w = g(t)] = f(g(t))$ is a linearization of $f(w)|_{w=g(t)}$. As a simple example, if we want to extract the following coefficient:

$$c_n = [t^n]\frac{1+t}{(1-t)^n}$$

by using formula (2.6) with $F(t) = 1 + t$ and $\phi(t) = 1/(1 - t)$, we have for $n \geq 1$:

$$c_n = [t^n]\frac{1}{2}\left(1 + \frac{1+2t}{\sqrt{1-4t}}\right) = \frac{1}{2}\binom{2n}{n} + \binom{2(n-1)}{n-1} = \frac{3n-1}{4n-2}\binom{2n}{n}.$$

For what concerns the A-sequence, we have the following results (see [4,10]):

Theorem 2.3. *An infinite lower triangular array $D = \left(d_{n,k}\right)_{n,k\in\mathbb{N}}$ is a Riordan array if and only if a sequence $A = (a_0 \neq 0, a_1, a_2, \ldots)$ exists such that for every $n, k \in \mathbb{N}$ relation (1.2) holds. Besides, the A-sequence is uniquely determined by the function $h(t)$, and vice versa, by the formulas:*

$$h(t) = tA(h(t)) \quad \text{and} \quad A(t) = \left[\frac{h(y)}{y}\middle| t = h(y)\right] = \left[\frac{t}{y}\middle| t = h(y)\right]. \tag{2.7}$$

The identity $h(t) = tA(h(t))$ will be called the *basic relation* (not to be confused with the basic identity (2.2)) and from it the following result follows immediately:

Theorem 2.4. *Let $D = \mathcal{R}(d(t), h(t))$ be any Riordan array, and D^* its inverse; then we have:*

$$A(t) = \frac{t}{\overline{h}(t)} \quad \text{and} \quad A^*(t) = \frac{t}{h(t)}.$$

Proof. The first relation is obtained by setting $t \mapsto \overline{h}(t)$ in the basic relation. The second relation follows from the fact that the recurrence rule of D^* is $\overline{h}(t)$. ■

Table 1
The trinomial extended triangle.

	−7	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6	7
−7	1														
−6	−6	1													
−5	10	−5	1												
−4	0	6	−4	1											
−3	−9	2	3	−3	1										
−2	2	−4	2	1	−2	1									
−1	1	0	−1	1	0	−1	1								
0	0	0	0	0	0	0	0	1							
1	0	0	0	0	0	0	1	1	1						
2	0	0	0	0	0	1	2	3	2	1					
3	0	0	0	0	1	3	6	7	6	3	1				
4	0	0	0	1	4	10	16	19	16	10	4	1			
5	0	0	1	5	15	30	45	51	45	30	15	5	1		
6	0	1	6	21	50	90	126	141	126	90	50	21	6	1	
7	1	7	28	77	161	266	357	393	357	266	161	77	28	7	1

3. Recursive matrices and their [m]-complementary

Let $D = \mathcal{R}(d(t), h(t))$ be any (proper) Riordan array; the corresponding *recursive matrix* is the bi-infinite triangle $(d_{n,k})_{k\le n\in\mathbb{Z}}$, denoted by $\mathcal{X}(d(t), h(t))$ and defined by:

$$d_{n,k} = [t^n]d(t)h(t)^k \quad \forall k, n \in \mathbb{Z},\ k \le n. \tag{3.1}$$

In this way, the original Riordan array[2] is just the right lower part of the recursive matrix. For example, we can look at the trinomial recursive matrix in Table 1 (sequence A094531 in OEIS) defined as:

$$T = \mathcal{X}\left(\frac{1}{\sqrt{1-2t-3t^2}},\ \frac{1-t-\sqrt{1-2t-3t^2}}{2t}\right).$$

In fact, the trinomial coefficients are defined as:

$$T_{n,k} = [t^k]\left(\frac{1}{t}+1+t\right)^n = [t^{n+k}](1+t+t^2)^n;$$

this implies $T_{n+1,k+1} = T_{n,k} + T_{n,k+1} + T_{n,k+2}$ and the A-sequence is $A(t) = 1 + t + t^2$. The elements in column 0 are $T_n = [t^n](1+t+t^2)^n$, that is, the so-called *Central Trinomial Coefficients*. The existence of an A-sequence assures that the triangle is Riordan and its recurrence rule can be found by applying the relation $h(t) = tA(h(t))$. Besides, we can use the diagonalization rule (2.6) to find the generating function of column 0, the boundary values of the Riordan array:

$$\begin{aligned}[t^n](1+t+t^2)^n &= [t^n]\left[\frac{1}{1-t(1+2w)}\,\middle|\, w = t(1+w+w^2)\right]\\ &= [t^n]\left[\frac{1}{1-t(1+2w)}\,\middle|\, w = \frac{1-t-\sqrt{1-2t-3t^2}}{2t}\right] = [t^n]\frac{1}{\sqrt{1-2t-3t^2}}.\end{aligned}$$

It is immediate to prove that the triangle belongs to the hitting-time subgroup:

$$\frac{th'(t)}{h(t)} = t\cdot\frac{1-t-\sqrt{1-2t-3t^2}}{2t^2\sqrt{1-2t-3t^2}}\cdot\frac{2t}{1-t-\sqrt{1-2t-3t^2}} = \frac{1}{\sqrt{1-2t-3t^2}}.$$

By a well-known property of the arrays in this subgroup (see, e.g., [14]), if p is a prime number, all the elements in row p (except possibly the first and the last), are all divisible by p. For $p = 3,\ 5$ the property can be checked in the table.

The definition and properties of Riordan arrays are automatically valid for recursive matrices; in particular, the row-by-column product, its identity and the inverse for every element remain identical, although extended to positive and negative values. A final and important observation is that the A-sequence of the Riordan array becomes the A-sequence of the recursive matrix, and the relation $h(t) = tA(h(t))$ continues to hold (see [7]).

We now consider a transformation of the recursive matrix which has interesting properties. Table 2 illustrates the transformation for the trinomial recursive matrix and $m = -3$.

Definition 3.1. For any fixed $m \in \mathbb{Z}$, the $[m]$-*complementary* array of the recursive matrix $D = (d_{n,k})$ is defined as the array with elements $d^{[m]}_{k,n} = d_{m-n,m-k}$.

[2] By definition, $d(t)$ and $h(t)$ must satisfy $d(0) \neq 0$, $h(0) = 0$ and $h'(0) \neq 0$.

Table 2
The trinomial Riordan array and its $[-3]$-complementary matrix.

	0	1	2	3	4	5	6	7		0	1	2	3	4	5	6	7
0	1								0	1							
1	1	1							1	−3	1						
2	3	2	1						2	3	−4	1					
3	7	6	3	1					3	2	6	−5	1				
4	19	16	10	4	1				4	−9	0	10	−6	1			
5	51	45	30	15	5	1			5	9	−15	−5	15	−7	1		
6	141	126	90	50	21	6	1		6	3	24	−20	−14	21	−8	1	
7	393	357	266	161	77	28	7	1	7	−18	−6	49	−21	−28	28	−9	1

As we are now going to show, the bi-infinite triangle corresponding to Definition 3.1 is a recursive matrix with the corresponding Riordan array. These two objects will be called the $[m]$-*complementary* structures of the original ones, and noted as $D^{[m]} = \mathcal{R}(d^{[m]}(t),\ h^{[m]}(t))$ and $D^{[m]} = \mathcal{X}(d^{[m]}(t),\ h^{[m]}(t))$, respectively and without ambiguity.

Theorem 3.1. *Let $D = \mathcal{R}(d(t), h(t))$ be any Riordan array, then its $[m]$-complementary array is the Riordan array:*

$$D^{[m]} = \mathcal{R}\left(d(\overline{h}(t))\overline{h}'(t)\left(\frac{t}{\overline{h}(t)}\right)^{m+1},\ \overline{h}(t)\right).$$

Proof. If $d_{k,0}^{[m]}$ is the k-th element in column 0 of the $[m]$-complementary array, we have:

$$d_{k,0}^{[m]} = d_{m,m-k} = [t^m]d(t)h(t)^{m-k} = [t^k]\left(\frac{h(t)}{t}\right)^m d(t)\left(\frac{t}{h(t)}\right)^k.$$

The diagonalization rule (2.6) immediately applies:

$$d_{k,0}^{[m]} = [t^k]\left[\frac{h(w)^m d(w)}{w^m(1-t\phi'(w))}\bigg| w = t\phi(w)\right]$$

where $\phi(t) = t/h(t)$. The equation to be solved is:

$$w = t\frac{w}{h(w)}, \quad \text{that is,}\ w = \overline{h}(t).$$

Therefore, by applying Theorem 2.1:

$$\phi'(w) = \left[\frac{\mathrm{d}}{\mathrm{d}w}\frac{w}{h(w)}\bigg| w = \overline{h}(t)\right] = \left[\frac{h(w) - wh'(w)}{h(w)^2}\bigg| w = \overline{h}(t)\right] = \frac{t - \overline{h}(t)h'(\overline{h}(t))}{t^2} = \frac{1}{t} - \frac{\overline{h}(t)}{t^2\overline{h}'(t)}.$$

Consequently, we find:

$$1 - t\phi'(w) = 1 - t\left(\frac{1}{t} - \frac{\overline{h}(t)}{t^2\overline{h}'(t)}\right) = \frac{\overline{h}(t)}{t\overline{h}'(t)}.$$

By substituting $w \mapsto \overline{h}(t)$ we eventually have:

$$d_{k,0}^{[m]} = [t^k]\frac{t^m d(\overline{h}(t))t\overline{h}'(t)}{\overline{h}(t)^m\overline{h}(t)} = [t^k]\frac{t^{m+1}}{\overline{h}(t)^{m+1}}d(\overline{h}(t))\overline{h}'(t).$$

Once obtained the boundary value of the $[m]$-complementary array, it is sufficient to show that the recurrence rule exists and equals $\overline{h}(t)$. According to the first part of this proof, the k-th column generating function of the $[m]$-complementary array is:

$$d(\overline{h}(t))\overline{h}'(t)\left(\frac{t}{\overline{h}(t)}\right)^{m-k+1}t^k$$

where we took into account the elements in the column, shifted down by k positions. The ratio of two consecutive columns is therefore $t(\overline{h}(t)/t) = \overline{h}(t)$, and this completes the proof. ■

We wish to observe explicitly that, in general, the inverse and the complementary matrices of a given Riordan array do not coincide, although occasionally this can happen, as for example in the Pascal triangle. Later on we will characterize the

Riordan arrays (or recursive matrices) D for which $D^* = D^{[m]}$, but for the moment let us emphasize the following simple, but important result:

Corollary 3.2. *Let D be any Riordan array (or recursive matrix). Then the A-sequence of the inverse and of the* $[m]$*-complementary arrays are the same.*

Proof. As we have seen, the inverse D^* and the $[m]$-complementary $D^{[m]}$ of the Riordan array $D = \mathcal{R}(d(t), h(t))$ have the same recurrence rule $\overline{h}(t)$ and the conclusion follows from Theorem 2.4. ■

Before proving other properties of $[m]$-complementary Riordan arrays, let us introduce the important concept of *diagonal translation operator*, here denoted by Φ. Following the procedure described in [6, p. 3614], if $D = \mathcal{R}(d(t), h(t))$ is any recursive matrix, ΦD is the same matrix D translated one position down and one position to the right. In other words, the former element $d_{n,k}$ becomes the element $d_{n-1,k-1}$, so that the new column 0 is what was column 1 in D, and the new row 0 is what was row 1 in D; therefore:

$$\Phi D = \Phi \mathcal{R}(d(t), h(t)) = \mathcal{R}\left(d(t)\frac{h(t)}{t}, h(t)\right).$$

The operator Φ corresponds to eliminate row 0 and column 0 from the original array. This operator was defined and used by two of the authors in their alternative approach to Riordan arrays (see [8]). Obviously, the operator Φ can be iterated and inverted; others of its relevant properties are collected in the following proposition, the proof of which is left to the reader.

Proposition 3.3. *The operator* Φ *defines a group isomorphism on the Riordan group and*

$$\Phi^m D = \Phi^m \mathcal{R}(d(t), h(t)) = \mathcal{R}\left(d(t)\left(\frac{h(t)}{t}\right)^m, h(t)\right) \quad \forall m \in \mathbb{Z}.$$

Moreover $D^{[m]} = \Phi^{-(m+1)}D^{[-1]}$ *and* $(\Phi^n D)^{[-1]} = \Phi^{-n}D^{[-1]}$.

The following theorem establishes an important connection between the diagonal translation operator and the $[m]$-complementary arrays.

Theorem 3.4. *Let* $D = \mathcal{R}(d(t), h(t))$ *be any Riordan array or, equivalently, any recursive matrix; then* $D^{[m][p]} = \Phi^{m-p}D$.

Proof. First of all, we observe that the recurrence rule of $D^{[m][p]}$ is just $h(t)$, that is, the compositional inverse of $\overline{h}(t)$. Furthermore, by applying Theorem 2.1, we have:

$$d^{[m][p]}(t) = \left(d(\overline{h}(t))\overline{h}'(t)\left(\frac{t}{\overline{h}(t)}\right)^{m+1}\right)^{[p]} = \left[d(\overline{h}(y))\overline{h}'(y)\left(\frac{y}{\overline{h}(y)}\right)^{m+1}\Big| y = h(t)\right] h'(t)\left(\frac{t}{h(t)}\right)^{p+1}$$
$$= d(t)\overline{h}'(h(t))\left(\frac{h(t)}{t}\right)^{m+1} h'(t)\left(\frac{t}{h(t)}\right)^{p+1} = d(t)\frac{1}{h'(t)}\left(\frac{h(t)}{t}\right)^{m-p} h'(t) = d(t)\left(\frac{h(t)}{t}\right)^{m-p}.$$

The last expression is just the boundary value of Φ^{m-p}. ■

We wish to point out that the $[m]$-complementary transformation is involutory. This is obvious from Definition 3.1 and the previous theorem:

Corollary 3.5. *The* $[m]$*-complementary transformation is involutory, that is,* $(D^{[m]})^{[m]} = D$.

Another important property of the $[m]$-complementary transformation is that it commutes with inversion:

Theorem 3.6. *Let* $D = (d(t), h(t))$ *be any Riordan array or, equivalently, any recursive matrix. Then* $D^{[m]*} = D^{*[m]}$.

Proof. The two arrays $D^{[m]*}$ and $D^{*[m]}$ have the same recurrence rule $h(t)$; for the boundary value:

$$d^{[m]*} = \left(d(\overline{h}(t))\overline{h}'(t)\left(\frac{t}{\overline{h}(t)}\right)^{m+1}\right)^* = \frac{1}{d(\overline{h}(h(t)))\overline{h}'(h(t))}\left(\frac{\overline{h}(h(t))}{h(t)}\right)^{m+1} = \frac{h'(t)}{d(t)}\,\frac{t^{m+1}}{h(t)^{m+1}},$$
$$d^{*[m]} = \left(\frac{1}{d(\overline{h}(t))}\right)^{[m]} = \frac{1}{d(\overline{h}(h(t)))}h'(t)\frac{t^{m+1}}{h(t)^{m+1}} = \frac{h'(t)}{d(t)}\,\frac{t^{m+1}}{h(t)^{m+1}}. \quad ■$$

4. Complementary and dual Riordan arrays

In this section, after some initial considerations, we only deal with Riordan arrays although what we are going to say applies also to recursive matrices. Riordan arrays are better suited for combinatorial applications and writing them as infinite, lower triangular arrays $(d_{n,k})_{0\le k\le n}$ allows us to grasp more easily their combinatorial meaning. Besides, comparing

Table 3
The dual (left) and the complementary (right) arrays of the trinomial Riordan array.

	0	1	2	3	4	5	6		0	1	2	3	4	5	6
0	1							0	1						
1	0	1						1	−1	1					
2	0	−1	1					2	0	−2	1				
3	0	0	−2	1				3	1	1	−3	1			
4	0	1	1	−3	1			4	−1	2	3	−4	1		
5	0	−1	2	3	−4	1		5	0	−4	2	6	−5	1	
6	0	0	−4	2	6	−5	1	6	1	2	−9	0	10	−6	1

two Riordan arrays is less difficult than to look at a recursive matrix, where the Riordan array and its $[m]$-complementary are displayed in different directions (see Table 1).

Actually, the concept of $[m]$-complementary Riordan array is very general and we think that it might be more instructive to restrict ourselves to some special cases, which can also be more interesting in applications. We focused our attention on the two special values $m = 0$ and $m = -1$, representing the two simplest instances of the concept.

- $m = 0$ represents the elementary transformation of the array, better explained in terms of recursive matrices. The element $d_{n,k}$ of the matrix becomes the element $d^{[0]}_{-k,-n}$ in the [0]-complementary recursive matrix, as explained in the previous section. The new elements $d^{[0]}_{n,k}$, with $0 \leq k \leq n$, constitute the *dual Riordan array* of the original array $D = \mathcal{R}(d(t), h(t))$, embedded in the corresponding recursive matrix. This dual array will be denoted $D^{\diamond} = \mathcal{R}\left(d^{\diamond}(t),\ h^{\diamond}(t)\right)$ and Theorem 3.1 gives us the formula:

$$D^{\diamond} = \mathcal{R}\left(d(\overline{h}(t))\overline{h}'(t)\frac{t}{\overline{h}(t)},\ \overline{h}(t)\right).$$

- $m = -1$ represents the simplest algebraic transformation, since it cancels the factor $(t/\overline{h}(t))^{m+1}$. The element $d_{n,k}$ of the matrix becomes the element $d^{[-1]}_{-k-1,-n-1}$ in the $[-1]$-complementary recursive matrix. The resulting array will be called the *complementary Riordan array* and denoted by a superscript $\perp$, so that $D^{\perp} = \mathcal{R}\left(d^{\perp}(t),\ h^{\perp}(t)\right)$ and Theorem 3.1 gives us the appropriate formula:

$$D^{\perp} = \mathcal{R}\left(d(\overline{h}(t))\overline{h}'(t),\ \overline{h}(t)\right).$$

  We also have the obvious relation $\Phi D^{\diamond} = D^{\perp}$.

Given a Riordan array D, its dual and complementary arrays only differ by one column; the reader can see Table 3 relative to our example on trinomial coefficients; the original array appears in Tables 1 and 2. For the sake of completeness, we remember that the linearized dual array $T^{\diamond}$ is sequence A198295 of the OEIS, while the linearized complementary $T^{\perp}$ is sequence A104562.

By Corollary 3.5, the complementary and dual transformations are involutory. However, in general, their properties are quite different, as we are now going to illustrate. For example, in the case of trinomial numbers, we can observe that $T^{\diamond}$ is a Lagrange array while $T^{\perp}$ is a renewal array.

The first general question is: in which case the dual and complementary transformations coincide? The answer is not a surprise:

Theorem 4.1. *Let $D = \mathcal{R}(d(t), h(t))$ be any Riordan array; $D^{\diamond} = D^{\perp}$ if and only if $D \in \mathcal{R}_A$.*

Proof. By using the formulas above, we have $D^{\diamond} = D^{\perp}$ if and only if $t/h(t) = 1$, that is, $h(t) = t$, and this characterizes Appell Riordan arrays. ∎

The dual and complementary arrays of a given Riordan array $D = \mathcal{R}(d(t), h(t))$ have the same recurrence rule, that is the compositional inverse of the recurrence rule in D. Therefore it can be difficult to find all the Riordan arrays D for which $D = D^{\diamond}$ or $D = D^{\perp}$. In this latter case, for example, we should solve the functional equation $d(t) = d(\overline{h}(t))\overline{h}'(t)$, a rather complicated task. Clearly, all Appell arrays enjoy the property ($h(t) = t$) and in the last section we will see another example, but the general problem remains open.

Instead, the problem of finding all the Riordan arrays $D = \mathcal{R}(d(t), h(t))$ for which $D^{\perp} = D^*$ or $D^{\diamond} = D^*$ can be easily solved.

Theorem 4.2. *If $D = \mathcal{R}(d(t), h(t))$ is any Riordan array, then $D^{\perp} = D^*$ if and only if $d(t)^2 = h'(t)$. Furthermore, $D^{\diamond} = D^*$ if and only if $d(t)^2 = th'(t)/h(t)$.*

Proof. Since the recurrence rules of D^*, $D^{\perp}$ and $D^{\diamond}$ are the same, it is sufficient to prove under which conditions the boundary values coincide. If $D^{\perp} = D^*$, by performing the substitution $t \mapsto h(t)$, we have:

$$\frac{1}{d(\overline{h}(t))} = d(\overline{h}(t))\overline{h}'(t) \quad \text{or} \quad \frac{1}{d(\overline{h}(h(t)))} = d(\overline{h}(h(t)))\overline{h}'(h(t)).$$

By applying Theorem 2.1 we find:

$$\frac{1}{d(t)} = \frac{d(t)}{h'(t)} \quad \text{or} \quad d(t)^2 = h'(t).$$

The proof is complete when we observe that these computations can be reverted. For the dual Riordan arrays the starting point is:

$$\frac{1}{d(\overline{h}(t))} = d(\overline{h}(t))\overline{h}'(t)\frac{t}{\overline{h}(t)}$$

but the rest of the derivation is just as before. ■

The Pascal triangle is the classical example of a Riordan array D such that $D^* = D^\perp$; in fact we have:

$$d(t)^2 = \frac{1}{(1-t)^2} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{t}{1-t} = h'(t).$$

By starting with $h(t) = t/(1-4t)$, we find the appropriate boundary value so that $D^\diamond = D^*$:

$$\mathcal{R}\left(\frac{1}{\sqrt{1-4t}}, \frac{t}{1-4t}\right).$$

Another interesting case is $D = D^\perp$ or $D = D^\diamond$. Both imply that $h(t) = \overline{h}(t)$, a non-obvious problem. Particular examples are not difficult to find, as the following:

$$V = \mathcal{R}\left(\frac{1}{\sqrt{1-4t}}, \frac{-t}{1-4t}\right).$$

The reader is invited to perform the necessary computations (which are rather simple), but we can proceed by finding the general formula for the elements of the array:

$$V_{n,k} = [t^n]\frac{1}{\sqrt{1-4t}}\frac{(-t)^k}{(1-4t)^k} = (-1)^k\binom{-k-1/2}{n-k}(-4)^{n-k}.$$

Let us now perform the index transformation corresponding to the dual array:

$$\begin{aligned} V^\diamond_{n,k} &= (-1)^{-n}\binom{n-1/2}{n-k}(-4)^{n-k} = (-1)^n\binom{-n+1/2+n-k-1}{n-k}(-1)^{n-k}(-4)^{n-k} \\ &= (-1)^k\binom{-k-1/2}{n-k}(-4)^{n-k}, \end{aligned}$$

that is, the Riordan array is self-dual.

Let us now return to the subgroups listed in Section 2; the last four subgroups $\mathcal{R}_L$, $\mathcal{R}_D$, $\mathcal{R}_R$, $\mathcal{R}_H$ have a common flavor, and in fact the following results prove that they belong to the same family.

Theorem 4.3. *Every hitting-time Riordan array is the complementary of a renewal array, and vice versa. Furthermore, every derivative Riordan array is the complementary of a Lagrange array, and vice versa.*

Proof. Let us consider the renewal array $D = \mathcal{R}(h(t)/t, h(t))$ and apply the complementary transform:

$$D^\perp = \mathcal{R}\left(\frac{h(\overline{h}(t))}{\overline{h}(t)}\overline{h}'(t), \overline{h}(t)\right) = \mathcal{R}\left(\frac{t\overline{h}'(t)}{\overline{h}(t)}, \overline{h}(t)\right)$$

that is a hitting-time Riordan array. The converse is obvious since $D^{\perp\perp} = D$. The second part of the theorem is proved analogously, starting with $\mathcal{R}(h'(t), h(t))$. ■

These results can be summarized in the following diagrams:

$$\begin{tikzcd} & \boxed{\mathcal{R}_N} \arrow[dl, dash, "*"'] \arrow[dr, dash, "\perp"] & \\ \boxed{\mathcal{R}_N} \arrow[dr, dash, "\perp"'] & & \boxed{\mathcal{R}_H} \arrow[dl, dash, "*"] \\ & \boxed{\mathcal{R}_H} & \end{tikzcd} \qquad \begin{tikzcd} & \boxed{\mathcal{R}_L} \arrow[dl, dash, "*"'] \arrow[dr, dash, "\perp"] & \\ \boxed{\mathcal{R}_L} \arrow[dr, dash, "\perp"'] & & \boxed{\mathcal{R}_D} \arrow[dl, dash, "*"] \\ & \boxed{\mathcal{R}_D} & \end{tikzcd}$$

In order to unify all these results (and many others) let us begin by introducing a family of infinite subgroups of $\mathcal{R}$, among which there are $\mathcal{R}_L$, $\mathcal{R}_D$, $\mathcal{R}_R$, $\mathcal{R}_H$.

Theorem 4.4. *Let r, s be two fixed real (or complex) numbers; then the set of the Riordan arrays:*

$$\mathcal{H}[r,s] = \left\{ \mathcal{R}\left(\left(\frac{h(t)}{t}\right)^r h'(t)^s, h(t)\right) \middle| h(t) \in \mathcal{F}_1 \right\}$$

with the usual row-by-column product, is a subgroup of $\mathcal{R}$.

Proof. The proof consists in two straight-forward computations relative to the product closure and the inverse. For what concerns the closure:

$$\begin{aligned}
&\mathcal{R}\left(\left(\frac{h_1(t)}{t}\right)^r h_1'(t)^s,\ h_1(t)\right) * \mathcal{R}\left(\left(\frac{h_2(t)}{t}\right)^r h_2'(t)^s,\ h_2(t)\right)\\
&= \mathcal{R}\left(\left(\frac{h_1(t)}{t}\right)^r h_1'(t)^s \left(\frac{h_2(h_1(t))}{h_1(t)}\right)^r h_2'(h_1(t))^s,\ h_2(h_1(t))\right)\\
&= \mathcal{R}\left(\left(\frac{h_2(h_1(t))}{t}\right)^r \left(\frac{\mathrm{d}}{\mathrm{d}t} h_2(h_1(t))\right)^s,\ h_2(h_1(t))\right).
\end{aligned}$$

We conclude with the inverse:

$$\mathcal{R}\left(\left(\frac{h(t)}{t}\right)^r h'(t)^s, h(t)\right)^* = \mathcal{R}\left(\left(\frac{\overline{h}(t)}{h(\overline{h}(t))}\right)^r \frac{1}{h'(\overline{h}(t))'s},\ \overline{h}(t)\right) = \mathcal{R}\left(\left(\frac{\overline{h}(t)}{t}\right)^r \overline{h}'(t)^s,\ \overline{h}(t)\right). \quad \blacksquare$$

The following correspondence is immediate:

Lagrange or associated subgroup	$\mathcal{R}(1, h(t))$	$\mathcal{R}_L = \mathcal{H}[0,0]$
Co-Lagrange or derivative subgroup	$\mathcal{R}(h'(t), h(t))$	$\mathcal{R}_D = \mathcal{H}[0,1]$
Renewal or Rogers subgroup	$\mathcal{R}(h(t)/t, h(t))$	$\mathcal{R}_N = \mathcal{H}[1,0]$
Hitting-time subgroup	$\mathcal{R}(th'(t)/h(t), h(t))$	$\mathcal{R}_H = \mathcal{H}[-1,1]$.

From our point of view, the important fact is that the $[m]$-complementary of a Riordan array in $\mathcal{H}[r,s]$ is a Riordan array of the same type $\mathcal{H}[r',s']$, usually with $r \neq r'$ and/or $s \neq s'$. Here is the exact transformation:

$$\begin{aligned}
\mathcal{R}\left(\left(\frac{h(t)}{t}\right)^r h'(t)^s, h(t)\right) &\overset{[m]}{\to} \mathcal{R}\left(\left(\frac{h(\overline{h}(t))}{\overline{h}(t)}\right)^r h'(\overline{h}(t))^s h'(t)\left(\frac{t}{\overline{h}(t)}\right)^{m+1},\ \overline{h}(t)\right)\\
&= \mathcal{R}\left(\left(\frac{\overline{h}(t)}{t}\right)^{-r-m-1} (\overline{h}'(t))^{1-s},\ \overline{h}(t)\right).
\end{aligned}$$

The important cases are $m = 0$ and $m = -1$, which can be summarized as follows:

$$\text{dual transform: } \mathcal{H}[r,s] \overset{\diamond}{\to} \mathcal{H}[-r-1, 1-s], \qquad \text{complementary transform: } \mathcal{H}[r,s] \overset{\perp}{\to} \mathcal{H}[-r, 1-s].$$

Putting together the previous observation and theorem, we can prove that the $[m]$-complementary (in particular, the dual and complementary) transformations are group anti-isomorphisms between subgroups of the type described. Recall that an anti-isomorphism φ is a bijective map between groups transforming the product of two elements D_1 and D_2 in the first group into the product, in the opposite order, of the transformed elements: $\varphi(D_1 * D_2) = \varphi(D_2) * \varphi(D_1)$. As for isomorphisms, an anti-isomorphism transforms subgroups into subgroups. One can easily prove that two anti-isomorphic groups are in fact isomorphic groups. By using the definition of product (2.1) and Theorem 3.1, we now prove that $[-1]$-complementarity enjoys this property.

Theorem 4.5. *Let $D_1 = \mathcal{R}(d_1(t),\ h_1(t))$ and $D_2 = \mathcal{R}(d_2(t),\ h_2(t))$ be two proper Riordan arrays; then we have $D_1^{\perp} * D_2^{\perp} = (D_2 * D_1)^{\perp}$, with the two factors exchanged.*

Proof. First of all we have:

$$D_2 * D_1 = \mathcal{R}\left(d_2(t) \cdot d_1(h_2(t)),\ h_1(h_2(t))\right);$$

the complementary is:

$$(D_2 * D_1)^{\perp} = \mathcal{R}\left(d_2(\overline{h_1(h_2}(t))) \cdot d_1(h_2(\overline{h_1(h_2}(t)))) \cdot D(\overline{h_1(h_2}(t))),\ \overline{h_1(h_2}(t))\right),$$

where D denotes differentiation with respect to t. We observe that $\overline{h_1(h_2(t))} = \bar{h}_2(\bar{h}_1(t))$ and by the chain rule: $D(\bar{h}_2(\bar{h}_1(t))) = \bar{h}_2'(\bar{h}_1(t)) \cdot \bar{h}_1'(t)$, so that we find:

$$(D_2 * D_1)^{\perp} = \mathcal{R}\left(d_2(\bar{h}_2(\bar{h}_1(t))) \cdot d_1(\bar{h}_1(t)) \cdot \bar{h}_2'(\bar{h}_1(t)) \cdot \bar{h}_1(t),\ \bar{h}_2(\bar{h}_1(t))\right).$$

Let us now pass to the right side of the formula:

$$D_1^{\perp} = \mathcal{R}\left(d_1(\bar{h}_1(t)) \cdot \bar{h}_1'(t),\ \bar{h}_1(t)\right) \qquad D_2^{\perp} = \mathcal{R}\left(d_2(\bar{h}_2(t)) \cdot \bar{h}_2'(t),\ \bar{h}_2(t)\right)$$

and their product:

$$D_1^{\perp} * D_2^{\perp} = \mathcal{R}\left(d_1(\bar{h}_1(t)) \cdot \bar{h}_1'(t) \cdot d_2(\bar{h}_2(\bar{h}_1(t))) \cdot \bar{h}_2'(\bar{h}_1(t)),\ \bar{h}_2(\bar{h}_1(t))\right);$$

apart from the order of the factors, this is the same expression as the one for the left side. ∎

We wish to point out that this result could also be proved, in an elegant way, by using the canonical decomposition of Riordan arrays given by formula (2.3). For the general case, following Proposition 3.3 we have that $D^{[m]}$ is the composition of the complementary transformation with an isomorphism, $D^{[m]} = \Phi^{-(m+1)}D^{[-1]}$, and the composition of an anti-isomorphism with an isomorphism is again an anti-isomorphism.

Using the same language as for homomorphisms, for the dual and complementary transformations we have:

Theorem 4.6. *The dual transformation is an anti-endomorphism for the only subgroup* $\mathcal{H}[-1/2, 1/2]$. *The complementary transformation is an anti-endomorphism for the unique subgroup* $\mathcal{H}[0, 1/2]$.

Proof. If $\mathcal{H}[r, s] \overset{\diamond}{\to} \mathcal{H}[r', s']$, this transformation is an endomorphism if and only if $r = r'$ and $s = s'$, that is $r = -r - 1$ and $s = 1 - s$. The only solution is $r = -1/2$ and $s = 1/2$ corresponding to the subgroup:

$$\mathcal{H}\left[-\frac{1}{2}, \frac{1}{2}\right] = \left\{\mathcal{R}\left(\sqrt{\frac{th'(t)}{h(t)}}, h(t)\right) \middle| h(t) \in \mathcal{F}_1\right\}.$$

To show that the endomorphism is not the identity, it is enough to consider $h(t) = t/(1 - 4t)$:

$$\mathcal{R}\left(\frac{1}{\sqrt{1-4t}},\ \frac{t}{1-4t}\right) \overset{\diamond}{\to} \mathcal{R}\left(\frac{1}{\sqrt{1+4t}},\ \frac{t}{1+4t}\right).$$

For the complementary transformation we proceed in a similar way. ∎

The transformations can be displayed in chains, where a subgroup is followed by the corresponding one, according to the dual or the complementary transformation used. Since the two transformations are involutory, they are applied alternatively. The most important chain is:

$$\cdots \mathcal{H}[2,0] \overset{\perp}{\to} \mathcal{H}[-2,1] \overset{\diamond}{\to} \mathcal{H}[1,0] \overset{\perp}{\to} \mathcal{H}[-1,1] \overset{\diamond}{\to} \mathcal{H}[0,0] \overset{\perp}{\to} \mathcal{H}[0,1]$$
$$\overset{\diamond}{\to} \mathcal{H}[-1,0] \overset{\perp}{\to} \mathcal{H}[1,1] \overset{\diamond}{\to} \mathcal{H}[-2,0] \cdots.$$

In fact, by the correspondence observed above, this is equivalent to:

$$\cdots \mathcal{H}[2,0] \overset{\perp}{\to} \mathcal{H}[-2,1] \overset{\diamond}{\to} \mathcal{R}_N \overset{\perp}{\to} \mathcal{R}_H \overset{\diamond}{\to} \mathcal{R}_L \overset{\perp}{\to} \mathcal{R}_D \overset{\diamond}{\to} \mathcal{H}[-1,0] \overset{\perp}{\to} \mathcal{H}[1,1] \overset{\diamond}{\to} \mathcal{H}[-2,0] \cdots.$$

We summarize these facts as follows:

1. every renewal array is the complementary of a hitting-time array, and vice versa;
2. every derivative array is the complementary of a Lagrange array, and vice versa;
3. every Lagrange array is the dual of a hitting-time array, and vice versa.

5. Formulas

An important application of complementary Riordan arrays is that, in several cases, they can be used to compute array items when it is too difficult to evaluate them directly. In fact, given a Riordan array $D = \mathcal{R}(d(t), h(t))$, we actually have four matrices: the basic D, the inverse D^* (formula at the beginning of Section 2), the dual $D^{\diamond}$ and the complementary $D^{\perp}$ (formulas at the beginning of Section 4). Besides, we also have specific formulas for the single elements: Theorem 2.2 for passing from the basic formulation to the inverse array, and vice versa; the transformations:

$$\begin{array}{lll} k \mapsto -n-1 & n \mapsto -k-1 & \text{For the complementary} \\ k \mapsto -n & n \mapsto -k & \text{For the dual array.} \end{array}$$

The strategy is therefore: (i) obtain the generating functions of the array of interest, together with the inverse, the dual and the complementary arrays; (ii) choose the simplest among the four formulations (“simplest” is purposely rather vague;

it is intended as the formulation the user can handle in the best way); (iii) use theorems or index transformation for returning to the base case. Our first example concerns the central binomial coefficient (sequence A094527 in OEIS), defined as:

$$\mathbf{B} = \mathcal{R}\left(\frac{1}{\sqrt{1-4t}},\ \frac{1-2t-\sqrt{1-4t}}{2t}\right);$$

the name derives from column 0. In order to find the general formula for its elements, we should extract a coefficient, which requires the application of the Lagrange Inversion Formula. Alternatively, we can proceed in the following way. First we observe that **B** belongs to the hitting-time subgroup, so, according to fact 3 at the end of the previous section, its dual array belongs to the Lagrange subgroup. Therefore, the boundary value is 1 and we only have to find the compositional inverse of the recurrence rule. So

$$\mathbf{B}^{\diamond} = \mathcal{R}\left(1,\ \frac{t}{(1+t)^2}\right)$$

and we can easily extract the coefficient:

$$\mathbf{B}^{\diamond}_{n,k}[t^n]\frac{t^k}{(1+t)^{2k}} = [t^{n-k}](1+t)^{-2k} = \binom{-2k}{n-k} = (-1)^{n-k}\binom{n+k-1}{n-k}.$$

Finally, we perform the change of indices relative to the dual transformation:

$$\mathbf{B}_{n,k} = (-1)^{n-k}\binom{-k-n-1}{n-k} = \binom{2n}{n-k},$$

valid for every $k, n \in \mathbb{Z}$.

Our second example concerns the Riordan array:

$$V = \mathcal{R}\left(\frac{1}{\sqrt{-1+2\sqrt{1-4t}}\sqrt{1-4t}},\ \frac{1-\sqrt{-1+2\sqrt{1-4t}}}{2}\right),$$

which seems hard for coefficient extraction and, on the other hand, does not give any hint for belonging to any specific subgroup. Fortunately, according to the proposed strategy, we have:

$$V^* = \mathcal{R}\left(1-4t+6t^2-4t^3,\ t\left(1-2t+2t^2-t^3\right)\right);$$
$$V^{\diamond} = \mathcal{R}\left(\left(1-2t+2t^2-t^3\right)^{-1},\ t\left(1-2t+2t^2-t^3\right)\right);$$
$$V^{\perp} = \mathcal{R}\left(1,\ t(1-2t+2t^2-t^3)\right),$$

and we can take the complementary array $V^{\perp}$ to perform the computations relative to a single (but generic) element. So, routinely we have:

$$\begin{aligned}
V^{\perp}_{n,k} &= [t^{n-k}](1-2t+2t^2-t^3)^k = [t^{n-k}]\sum_{j=0}^{k}\binom{k}{j}(-2t)^j\left(1-t+\frac{t^2}{2}\right)^j \\
&= \sum_{j=0}^{k}\binom{k}{j}(-2)^j[t^{n-k-j}]\sum_{r=0}^{j}\binom{j}{r}(-t)^r\left(1-\frac{t}{2}\right)^r \\
&= \sum_{j=0}^{k}\binom{k}{j}(-2)^j\sum_{r=0}^{j}\binom{j}{r}(-1)^r[t^{n-k-j-r}]\left(1-\frac{t}{2}\right)^r
\end{aligned}$$

and conclude:

$$V^{\perp}_{n,k} = \sum_{j=0}^{k}\sum_{r=0}^{j}\binom{k}{j}\binom{j}{r}\binom{r}{n-k-j-r}\frac{(-1)^{n-k}}{2^{n-k-2j-r}}.$$

We can consider $n-k$ instead of k in the upper limit of the first sum since when $n-k \geq j \geq k$ the first binomial coefficient is equal to zero while when $n-k \leq j \leq k$ the third binomial coefficient is zero. Finally, by applying the transformations $k \mapsto -n-1$ and $n \mapsto -k-1$ we get the desired formula:

$$V_{n,k} = \sum_{j=0}^{n-k}\sum_{r=0}^{j}\binom{-n-1}{j}\binom{j}{r}\binom{r}{n-k-j-r}\frac{(-1)^{n-k}}{2^{n-k-2j-r}}.$$

These formulas can be verified by means of the upper part of V and its dual array shown in Table 4. The interested reader can perform the obvious last computations to obtain the formulas for the inverse and dual arrays.

Table 4
The V example with its complementary array.

	0	1	2	3	4	5	6		0	1	2	3	4	5	6
0	1							0	1						
1	4	1						1	0	1					
2	18	6	1					2	0	−2	1				
3	84	32	8	1				3	0	2	−4	1			
4	400	165	50	10	1			4	0	−1	8	−6	1		
5	1932	840	286	72	12	1		5	0	0	−10	18	−8	1	
6	9436	4256	1568	455	98	14	1	6	0	0	8	−35	32	−10	1

Let us now return to our main example of trinomial coefficients, for which we want to find a general formula. Traditionally, their definition is directly applied:

$$T_{n,k} = [t^k]\left(\frac{1}{t}+1+t\right)^n = [t^{n+k}](1+t+t^2)^n = [t^{n+k}]\sum_{j=0}^{n}\binom{n}{j}t^j(1+t)^j$$
$$= \sum_{j=0}^{n}\binom{n}{j}[t^{n+k-j}](1+t)^j = \sum_{j=0}^{n}\binom{n}{j}\binom{j}{n+k-j}.$$

This formula computes correctly all the trinomial coefficients with $n \geq 0$, but produces erroneous results for $n < 0$, when values reflected on column 0 are generated. At this point, the simple trick is to change $k \mapsto -k$ to obtain the desired formula.

A neater procedure uses the approach outlined above. As noted earlier, T is a hitting-time array, so its dual belongs to the Lagrange subgroup and column 0 is just 1. The compositional inverse of the recursion rule is found by solving in y the functional equation:

$$\frac{1-y-\sqrt{1-2y-3y^2}}{2y} = t, \quad \text{i.e., } y = \overline{h}_T(t) = \frac{t}{1+t+t^2}.$$

This implies

$$T^{\diamond} = \mathcal{R}\left(1,\ \frac{t}{1+t+t^2}\right) \tag{5.1}$$

and we have a formula for the generic element of the dual recursive matrix:

$$T^{\diamond}_{n,k} = [t^n]\frac{t^k}{(1+t+t^2)^k} = [t^{n-k}](1+t+t^2)^{-k} = \sum_{j=0}^{\infty}\binom{-k}{j}[t^{n-k-j}](1+t)^j$$
$$= \sum_{j=0}^{n-k}\binom{-k}{j}\binom{j}{n-k-j} = \sum_{j=0}^{n-k}(-1)^j\binom{k+j-1}{j}\binom{j}{n-k-j}.$$

The upper limit of the sum is determined by observing that the first binomial coefficient is defined for every value of j, while the second is different from 0 only when $0 \leq j \leq n-k$. Finally, we perform the change of indexes relative to the dual Riordan array and find:

$$T_{n,k} = \sum_{j=0}^{n-k}\binom{n}{j}\binom{j}{n-k-j} \qquad \forall k \leq n \in \mathbb{Z}.$$

An interesting result is obtained by the appropriate transformation formulas:

$$T = \mathcal{X}\left(\frac{1}{\sqrt{1-2t-3t^2}},\ \frac{1-t-\sqrt{1-2t-3t^2}}{2t}\right)$$
$$T^* = \mathcal{X}\left(\frac{1-t^2}{1+t+t^2},\ \frac{t}{1+t+t^2}\right) \qquad T^{\perp} = \mathcal{X}\left(\frac{t}{1+t+t^2},\ \frac{t}{1+t+t^2}\right)$$
$$T^{*\perp} = T^{\perp *} = \mathcal{X}\left(\frac{1-t-\sqrt{1-2t-3t^2}}{2t^2},\ \frac{1-t-\sqrt{1-2t-3t^2}}{2t}\right).$$

According to Theorem 4.3, T and T^* belong to the hitting-time subgroup, while $T^{\perp}$ and $T^{*\perp}$ are renewal arrays. This last recursive matrix is the *Motzkin triangle*, shown in Table 5 (sequence A026300 in OEIS).

Therefore, we have the following theorem:

Theorem 5.1. *The Motzkin triangle M is the complementary of the inverse of the trinomial triangle T, that is, $M = T^{*\perp}$.*

Table 5
Motzkin recursive matrix.

	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6
−6	1												
−5	−5	1											
−4	5	−4	1										
−3	5	2	−3	1									
−2	−5	4	0	−2	1								
−1	−1	−1	2	−1	−1	1							
0	0	0	0	0	−1	0	1						
1	0	0	0	−1	−1	0	1	1					
2	0	0	−1	−2	−2	0	2	2	1				
3	0	−1	−3	−5	−4	0	4	5	3	1			
4	−1	−4	−9	−12	−9	0	9	12	9	4	1		
5	−5	−14	−25	−30	−21	0	21	30	25	14	5	1	
6	−20	−44	−69	−76	−51	0	51	76	69	44	20	6	1

This theorem implies $M^{\perp} = T^{*}$ and $M^{*} = T^{\perp}$, an important fact. As a final example, let us determine the dual array of the Motzkin triangle; we easily find:

$$M^{\diamond} = \mathcal{X}\left(1 - t^2,\ \frac{t}{1 + t + t^2}\right).$$

Comparing this formula with (5.1) we have immediately the well-known identity:

$$M_{n,k} = T_{n,k} - T_{n,k-2}.$$

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