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A Class of Integral Operators that Fix Exponential Functions

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Abstract. In this paper we introduce a general class of integral operators that fix exponential functions, containing several recent modified operators of Gauss–Weierstrass, or Picard or moment type operators. Pointwise convergence theorems are studied, using a Korovkin-type theorem and a Voronovskaja-type formula is obtained.

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1. Introduction

The classical Bohman–Korovkin theorem (see [10,22,23]) is one of the pivotal results of approximation theory and several convergence theorems known in literature employ this basic tool. It states that a sequence of positive linear operators T_nf acting on the set of the continuous functions over a compact interval of the real line converges to the identity operator only if it converges on a finite number of test functions which form a so-called Chebyshev system. A complete treatment of the Korovkin theorem can be found in the monographies [2,3].

In this respect, if a sequence of operators T_nf is such that $T_n\varphi = \varphi$ for some continuous function φ, then to obtain the convergence appears very simple, if the functions φ belong to a Chebyshev system. Thus, it is of interest to define sequences of operators that have this property. In literature the so-called King type operators, have this property, especially in case of discrete operators (see e.g. [21]). In this paper we define an entire class of

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positive linear integral operators which fix exponential functions. This kind of results were obtained by Agratini, Aral and Deniz [1], by Aral [6] and recently by Yilmaz, Uysal and Aral [26], by considering modifications of specific operators (Picard, Gauss–Weiertsrass and moment-type operators). All the operators considered in these papers are special case of our present theory. Our approach includes also all the integral operators having a compactly supported kernel. Thus it applies for example to spline-type operators [11,24]. Related results can be found in [16,18,19] and [17].

For our operators, we apply the Gadjiev version of Bohman–Korovkin theorem (see [14,15]) in case of unbounded domains, for functions belonging to certain weighted spaces of continuous functions. Setting for every $n \in \mathbb{N}$ and $a > 0$,

$$A_{a,n} := \int_{-\infty}^{\infty} \exp(at) K_n(t) \mathrm{d}t, \quad \lambda_n(x) := x - \frac{1}{a} \log A_{a,n},$$

we define

$$(T_n f)(x) = A_{a,n} \int_{-\infty}^{\infty} \exp(-at) f\left(\lambda_n(x) + t\right) K_n(t) \mathrm{d}t,$$

where $\{K_n\}$ is a family of non-negative functions (kernel) belonging to a suitable function space. Setting for $a > 0$, $\exp_a(x) := e^{ax}$ for $x \in \mathbb{R}$, we show that $(T_n \exp_a)(x) = \exp(ax)$ and $(T_n \exp_{2a})(x) = \exp(2ax)$. Moreover we show that $T_n e_j \to e_j$ where $e_j(x) = x^j$, $j = 0, 1, 2$, so obtaining two uniform convergence theorems in weighted spaces of continuous functions. Then, using certain moduli of continuity we obtain certain quantitative estimates of the convergence and finally a Voronovskaja-type asymptotic formula. These kinds of asymptotic formulae are very useful also for applications, especially for discrete operators, like e.g. sampling-type operators (see, e.g., [11–13]. For integral operators of Mellin type see [7,8]). The last section is devoted to several examples.

We recall here, that general approaches to convergence of integral operators was recently given in [9], and more recently in [20] and [25] in the frame of nonlinear operators.

2. Basic Notations

Let us denote by $\mathbb{R}$ the set of real numbers, and by $\mathbb{N}$ the set of the positive integers. By $L^1(\mathbb{R})$ we denote the space comprising all the Lebesgue integrable functions on $\mathbb{R}$ with respect to the Lebesgue measure, and by $L^\infty(\mathbb{R})$ the space comprising all the essentially bounded functions on $\mathbb{R}$. By $C(\mathbb{R})$ we denote the space of all the continuous functions defined on $\mathbb{R}$. Finally, for $r \in \mathbb{N}$, we will say that a function $f : \mathbb{R} \to \mathbb{R}$ is locally of class C^r at a point $x \in \mathbb{R}$ if there is a neighbourhood U of x such that f is $(r-1)$-fold continuously differentiable in U and $f^{(r)}(x)$ exists.

3. The Class of Integral Operators

For any $a > 0$, we define the function $\exp_a(t) := e^{at}$, for $t \in \mathbb{R}$.

We define now the function space

$$L_{\exp_a}(\mathbb{R}) := \{g : \mathbb{R} \to \mathbb{R} : g(\cdot)e^{a|\cdot|} \in L^1(\mathbb{R})\}.$$

Let now $\{K_n\}$ be a non negative kernel, that is, for every $n \in \mathbb{N}$, $K_n(t) \geq 0$, for every $t \in \mathbb{R}$, $K_n \in L^1(\mathbb{R})$, and

$$\|K_n\|_1 = \int_{-\infty}^{\infty} K_n(t)\mathrm{d}t = 1.$$

In what follows we assume that $K_n \in L_{\exp_a}(\mathbb{R})$, for every sufficiently large $n \in \mathbb{N}$, namely for $n \geq n_0$, with $n_0 \in \mathbb{N}$. We set, for $n \geq n_0$,

$$A_{a,n} := \int_{-\infty}^{\infty} \exp_a(t)K_n(t)\mathrm{d}t, \qquad \lambda_n(x) := x - \frac{1}{a}\log A_{a,n}.$$

Note that from the assumption $K_n \in L_{\exp_a}(\mathbb{R})$, we have $A_{a,n} < +\infty$, $A_{-a,n} < +\infty$ and if moreover for every $n \in \mathbb{N}$, K_n is an even function, that is $K_n(t) = K_n(-t)$, for every $t \in \mathbb{R}$, one has easily $A_{a,n} = A_{-a,n}$. We introduce now the sequence of integral operators defined by

$$\begin{aligned}(T_n f)(x) &:= A_{a,n}\int_{-\infty}^{\infty} \exp_{-a}(t)f(\lambda_n(x)+t)K_n(t)\mathrm{d}t \\ &= A_{a,n}\int_{-\infty}^{\infty} e^{-at}f(\lambda_n(x)+t)K_n(t)\mathrm{d}t, \end{aligned} \tag{3.1}$$

for every function f belonging to the domain $D := \bigcap_{n \geq n_0} \operatorname{dom} T_n$, where $\operatorname{dom} T_n$ denotes the set of all the Lebesgue measurable functions f such that $(T_n|f|)(x)$ is convergent for almost all $x \in \mathbb{R}$.

Note that if $f \in L^\infty(\mathbb{R})$, then $f \in D$. Moreover, the functions $\exp_a$ and $\exp_{2a}$ both belong to D. We have the following

Proposition 3.1. *Under the established assumptions on the kernel* $\{K_n\}$, *we have*

$$(T_n \exp_a)(x) = \exp_a(x), \qquad (T_n \exp_{2a})(x) = \exp_{2a}(x). \tag{3.2}$$

Proof. We have

$$\begin{aligned}(T_n \exp_a)(x) &= A_{a,n}\int_{-\infty}^{\infty} e^{-at}e^{a(\lambda_n(x)+t)}K_n(t)\mathrm{d}t = A_{a,n}e^{a\lambda_n(x)}\int_{-\infty}^{\infty} K_n(t)\mathrm{d}t \\ &= A_{a,n}e^{a\lambda_n(x)} = \exp_a(x).\end{aligned}$$

Analogously, we have

$$\begin{aligned}(T_n \exp_{2a})(x) &= A_{a,n}\int_{-\infty}^{\infty} e^{-at}e^{2a(\lambda_n(x)+t)}K_n(t)\mathrm{d}t \\ &= A_{a,n}e^{2a\lambda_n(x)}\int_{-\infty}^{\infty} e^{at}K_n(t)\mathrm{d}t \\ &= A_{a,n}\exp_{2a}(x)\frac{A_{a,n}}{(A_{a,n})^2} = \exp_{2a}(x),\end{aligned}$$

that is the assertion □

Remark 3.2. In order to establish a connection with classical integral operators of convolution type of the form

$$(T_n^0 f)(x) := \int_{-\infty}^{\infty} f(x+t)K_n(t)\mathrm{d}t,$$

for continuous and bounded functions $f \in D$ we have

$$\lim_{a \to 0^+} (T_n f)(x) = (T_n^0 f)(x).$$

whenever

$$\lim_{a \to 0^+} \frac{1}{a} \log A_{a,n} = 0.$$

Indeed, under the above assumptions on the kernel $\{K_n\}$, it is easy to show that $\lim_{a\to 0^+} A_{a,n} = 1$, and $\lim_{a\to 0^+} \lambda_n(x) = x$. Thus under the assumptions on the function f the assertion follows by the Lebesgue theorem of dominated convergence.

4. Pointwise and Uniform Convergence

In this section we will study the pointwise convergence of $T_n f$ to f, where f belongs to a suitable weighted space of continuous functions, as introduced in [14], using a Korovkin-type theorem, established in [14] (see also [15]). In order to do that, we introduce the following constants

$$B_{-a,n} := \int_{-\infty}^{\infty} te^{-at} K_n(t)\mathrm{d}t$$
$$C_{-a,n} := \int_{-\infty}^{\infty} t^2 e^{-at} K_n(t)\mathrm{d}t.$$

We call these constants the *exponential moments* of orders 1 and 2 respectively. At the same way, we refer to $A_{-a,n}$ as the exponential moment of order 0. We introduce the following subspace of $L_{\exp_a}(\mathbb{R})$, in which the above moments are well-defined

$$L^*_{\exp_a}(\mathbb{R}) := \{g \in L_{\exp_a}(\mathbb{R}) : (c + d|\cdot|)^2 e^{a|\cdot|} g(\cdot) \in L^1(\mathbb{R})\},$$

with c, d positive constants. If $K_n \in L^*_{\exp_a}(\mathbb{R})$, then $B_{-a,n}$, $C_{-a,n}$ exist finite.

Let us introduce now the test functions $e_0(t) = 1$, $e_1(t) = t$, $e_2(t) = t^2$, $t \in \mathbb{R}$. Obviously, $e_j \in D$, for $j = 0, 1, 2$.

We have the following

Proposition 4.1. *Let $K_n \in L^*_{\exp_a}(\mathbb{R})$, for sufficiently large $n \in \mathbb{N}$. Then we have*

$$(T_n e_0)(x) = A_{a,n}\, A_{-a,n}$$
$$(T_n e_1)(x) = A_{a,n}\, A_{-a,n} x - \frac{1}{a} A_{a,n}\, A_{-a,n} \log A_{a,n} + A_{a,n}\, B_{-a,n}$$

and

$$(T_n e_2)(x) = A_{a,n}\, A_{-a,n}x^2 + \left(2A_{a,n}\, B_{-a,n} - 2A_{a,n}\, A_{-a,n}\frac{1}{a}\log A_{a,n}\right)x$$
$$+ A_{a,n}\, A_{-a,n}\frac{1}{a^2}\log^2 A_{a,n} + A_{a,n}\, C_{-a,n} - 2A_{a,n}\, B_{-a,n}\frac{1}{a}\log A_{a,n}.$$

Proof. We have

$$(T_n e_0)(x) = A_{a,n}\int_{-\infty}^{\infty} e^{-at}K_n(t)\mathrm{d}t = A_{a,n}\, A_{-a,n}.$$

Next,

$$\begin{aligned}(T_n e_1)(x) &= A_{a,n}\int_{-\infty}^{\infty} e^{-at}(\lambda_n(x)+t)K_n(t)\mathrm{d}t\\ &= A_{a,n}\,\lambda_n(x)\int_{-\infty}^{\infty} e^{-at}K_n(t)\mathrm{d}t + A_{a,n}\int_{-\infty}^{\infty} te^{-at}K_n(t)\mathrm{d}t\\ &= A_{a,n}\, A_{-a,n}x - A_{a,n}\, A_{-a,n}\frac{1}{a}\log A_{a,n} + A_{a,n}\, B_{-a,n}.\end{aligned}$$

Finally,

$$\begin{aligned}(T_n e_2)(x) &= A_{a,n}\int_{-\infty}^{\infty} e^{-at}(\lambda_n(x)+t)^2K_n(t)\mathrm{d}t\\ &= A_{a,n}\,(\lambda_n(x))^2A_{-a,n} + A_{a,n}\, C_{-a,n} + 2A_{a,n}\, B_{-a,n}\lambda_n(x)\\ &= A_{a,n}\, A_{-a,n}x^2 + 2A_{a,n}\left(B_{-a,n} - \frac{1}{a}A_{-a,n}\log A_{a,n}\right)x\\ &\quad + A_{a,n}\left(A_{-a,n}\frac{1}{a^2}\log^2 A_{a,n} + C_{-a,n} - \frac{2}{a}B_{-a,n}\log A_{a,n}\right).\end{aligned}$$

The proof is completed □

Corollary 4.2. *Under the assumptions of Proposition 4.1, if moreover*

$$\lim_{n\to+\infty} A_{a,n} = \lim_{n\to+\infty} A_{-a,n} = 1, \tag{4.1}$$

and

$$\lim_{n\to+\infty} B_{-a,n} = \lim_{n\to+\infty} C_{-a,n} = 0,$$

then

$$\lim_{n\to+\infty}(T_n e_j)(x) = e_j(x) \qquad (j = 0, 1, 2).$$

Proof. It is an immediate consequence of Proposition 4.1 □

Using Proposition 4.1 and Corollary 4.2, we now give a uniform convergence result for the operators $(T_n f)$ when f belongs to suitable weighted spaces of continuous functions. The key tool is a Korovkin-type theorem proved in [14] (see also [15]).

Given a continuous, strictly increasing function $\varphi : \mathbb{R} \to \mathbb{R}$, we define the function $\rho(x) := 1 + \varphi^2(x)$, and assume that $\lim_{x\to\pm\infty}\rho(x) = +\infty$.

We will consider two particular cases: $\varphi_1(x) = x$, and $\varphi_2(x) = \exp_a(x)$, and we set $\rho_1(x) := 1 + x^2$ and $\rho_2(x) := 1 + \exp_{2a}(x)$.

Let us consider the spaces, for $j = 1, 2$

$$C^0_{\rho_j}(\mathbb{R}) := \left\{ f \in C(\mathbb{R}) : \lim_{x \to \pm\infty} \frac{f(x)}{\rho_j(x)} = \ell_f \in \mathbb{R} \right\}.$$

Note that if $f \in C^0_{\rho_j}(\mathbb{R})$, then $|f(x)| \leq M_f \rho_j(x)$, for a suitable constant $M_f > 0$ and $x \in \mathbb{R}$. We define a norm on the space $C^0_{\rho_j}(\mathbb{R})$ on setting

$$\|f\|_{\rho_j} := \left\| \frac{f}{\rho_j} \right\|_\infty .$$

We are ready to prove the main theorem of this section.

Theorem 4.3. *Let $j = 1, 2$ and let $f \in C^0_{\rho_j}(\mathbb{R})$. Then, under the assumptions of Proposition 4.1 and Corollary 4.2, we have*

$$\lim_{n \to +\infty} \|T_n f - f\|_{\rho_j} = 0.$$

Proof. First, consider the case $j = 1$. The test functions e_j obviously belong to $C^0_{\rho_1}(\mathbb{R})$ and moreover

$$\left\| \frac{T_n e_0 - e_0}{\rho_1} \right\|_\infty = \left\| \frac{A_{a,n}\, A_{-a,n} - 1}{\rho_1} \right\|_\infty \leq |A_{a,n}\, A_{-a,n} - 1|, \tag{4.2}$$

and the last term tends to 0 as $n \to +\infty$. Next,

$$\left\| \frac{T_n e_1 - e_1}{\rho_1} \right\|_\infty \leq |A_{a,n}\, A_{-a,n} - 1| \sup_{x \in \mathbb{R}} \frac{|x|}{x^2 + 1} + \frac{1}{a} A_{a,n}\, A_{-a,n} |\log A_{a,n}| + A_{a,n}\, |B_{-a,n}|,$$

and so

$$\lim_{n \to +\infty} \left\| \frac{T_n e_1 - e_1}{\rho_1} \right\|_\infty = 0.$$

Analogously, one can see that

$$\lim_{n \to +\infty} \left\| \frac{T_n e_2 - e_2}{\rho_1} \right\|_\infty = 0.$$

For $j = 2$, taking into account of (4.2) and Proposition 3.1, we obtain again the convergence on the test functions $\exp^k(ax)$, for $k = 0, 1, 2$. Applying Theorem 2 in [14] we obtain the assertion □

5. Quantitative Estimates

For $K_n \in L^*_{\exp_a}(\mathbb{R})$, we define the *absolute exponential moment of order* 1 of K_n as

$$\widetilde{B}_{-a,n} := \int_{-\infty}^{\infty} e^{-at} |t| K_n(t) \mathrm{d}t.$$

In the spaces $C^0_{\rho_j}(\mathbb{R})$, $j=1,2$ we can define various modulus of continuity. We begin with an estimate of the convergence expressed by Theorem 4.3, in terms of the classical modulus of continuity ω defined by

$$\omega(f,\delta) := \sup_{|h|\le\delta} |f(x+h)-f(x)| \qquad (\delta>0).$$

As it is well-known, for any uniformly continuous function f one has $\lim_{\delta\to 0}\omega(f,\delta)=0$. We have the following estimate

Theorem 5.1. *Let $K_n \in L^*_{\exp_a}(\mathbb{R})$, for sufficiently large values of $n\in\mathbb{N}$, and let $f\in C^0_{\rho_j}(\mathbb{R})$. Then for every $\delta>0$, we have*

$$\|T_n f - f\|_{\rho_j} \le \omega(f,\delta)\, A_{a,n}\left(A_{-a,n} + \frac{1}{\delta}\widetilde{B}_{-a,n} + \frac{1}{a\delta}A_{-a,n}\left|\log A_{a,n}\right|\right) + \|f\|_{\rho_j}\left|A_{a,n}A_{-a,n}-1\right|.$$

Proof. For every $x\in\mathbb{R}$ and $\delta>0$ we have

$$|(T_n f)(x) - f(x)| \le A_{a,n}\int_{-\infty}^{\infty} e^{-at}\omega\left(f, \left|t-\frac{1}{a}\log A_{a,n}\right|\right) K_n(t)\,\mathrm{d}t + |f(x)|\left|A_{a,n}A_{-a,n}-1\right| =: I_1 + I_2.$$

Thus we have to estimate only I_1. In order to do that, we use the following well-known property of ω (see e.g. [4])

$$\omega(f,\lambda\delta) \le (1+\lambda)\,\omega(f,\delta) \qquad (\lambda,\delta>0).$$

Setting $\lambda := \frac{\left|t-\frac{1}{a}\log A_{a,n}\right|}{\delta}$, we have

$$\begin{aligned} I_1 &\le A_{a,n}\omega(f,\delta)\int_{-\infty}^{\infty} e^{-at}\left(1+\frac{\left|t-\frac{1}{a}\log A_{a,n}\right|}{\delta}\right)K_n(t)\,\mathrm{d}t \\ &\le \omega(f,\delta)\,A_{a,n}\left(A_{-a,n}+\frac{1}{\delta}\int_{-\infty}^{\infty} e^{-at}\left(|t|+a^{-1}\left|\log A_{a,n}\right|\right)K_n(t)\,\mathrm{d}t\right) \\ &\le \omega(f,\delta)\,A_{a,n}\left(A_{-a,n}+\frac{1}{\delta}\widetilde{B}_{-a,n}+\frac{1}{a\delta}A_{-a,n}\left|\log A_{a,n}\right|\right). \end{aligned}$$

Passing to norm, we get the desired result

$$\|T_n f - f\|_{\rho_j} \le \omega(f,\delta)\, A_{a,n}\left(A_{-a,n} + \frac{1}{\delta}\widetilde{B}_{-a,n} + \frac{1}{a\delta}A_{-a,n}\left|\log A_{a,n}\right|\right) + \|f\|_{\rho_j}\left|A_{a,n}A_{-a,n}-1\right|$$

□

Corollary 5.2. *Let the assumptions of Theorem 5.1 be satisfied, and (4.1) holds. If moreover there is $\alpha>0$ such that $(n^\alpha\widetilde{B}_{-a,n})_n$, and $(n^\alpha|\log A_{a,n}|)_n$ are bounded sequences, then there exists an absolute constant $M>0$ such that*

$$\|T_n f - f\|_{\rho_j} \le M\omega\left(f,\frac{1}{n^\alpha}\right) + \left|A_{a,n}\,A_{-a,n}-1\right|\,\|f\|_{\rho_j},$$

for sufficiently large values of $n\in\mathbb{N}$.

Proof. It is a consequence of Theorem 5.1 on setting $\delta = n^{-\alpha}$ □

Remark 5.3. Note that, under the assumptions of Corollary 5.2, if the function f satisfies a Lipschitz condition of order 1, and $|A_{a,n}\, A_{-a,n} - 1| = \mathcal{O}(n^{-\alpha})$ as $n \to +\infty$, then

$$\|T_n f - f\|_{\rho_j} = \mathcal{O}(n^{-\alpha}) \qquad (n \to +\infty).$$

Now we state another estimate using a suitable weighted modulus of continuity. In order to do that, we introduce the exponential moment of order 4 of $K_n \in L^*_{\exp_a}(\mathbb{R})$ setting

$$E_{-a,n} := \int_{-\infty}^{\infty} e^{-at} t^4 K_n(t) \mathrm{d}t.$$

In order to establish rate of convergence, we will use special kind of modulus of continuity $\widetilde{\omega}$ which is compatible with the space $C^0_{\rho_1}(\mathbb{R})$. This weighted modulus of continuity was first introduced in [27] for $f \in C^0_{\rho_1}(\mathbb{R}^+_0)$, then considered in [5] for $f \in C^0_{\rho_1}(\mathbb{R})$ as follows:

$$\widetilde{\omega}(f, \delta) = \sup_{x \in \mathbb{R}, |h| \leq \delta} \frac{|f(x+h) - f(x)|}{1 + (h+x)^2}. \tag{5.1}$$

For any function $f \in C^0_{\rho_1}(\mathbb{R})$, $m \in \mathbb{N}$ and $\lambda, \delta \in \mathbb{R}^+$, $\widetilde{\omega}$ has the following properties (see [5,27]):

1. $\widetilde{\omega}(f, \delta)$ is a monotonically increasing function of δ,
2. $\lim_{\delta \to 0^+} \widetilde{\omega}(f, \delta) = 0$,
3. $\widetilde{\omega}(f, m\delta) \leq m\widetilde{\omega}(f, \delta)$, for every $m \in \mathbb{N}$,
4. $\widetilde{\omega}(f, \lambda\delta) \leq (1+\lambda)\widetilde{\omega}(f, \delta)$ for every $\lambda > 0$.

Theorem 5.4. *Let $K_n \in L^*_{\exp_a}$ be such that $E_{-a,n}$ exists finite. For $f \in C^0_{\rho_1}(\mathbb{R})$, there holds*

$$\begin{aligned}
&\|T_n f - f\|_{\rho_1} \\
&\quad \leq 16 A_{a,n} \widetilde{\omega}\left(f, \sqrt{C_{-a,n}}\right) \\
&\qquad \times \left\{ C_{-a,n} + (1 + C_{-a,n}) A_{-a,n} + \sqrt{E_{-a,n} + A_{-a,n}\left(1 + C^2_{-a,n}\right)} \right\} \\
&\qquad + \|f\|_{\rho_1} |A_{a,n} A_{-a,n} - 1|
\end{aligned}$$

provided that $\left|a^{-1} \log A_{a,n}\right| \leq \sqrt{C_{-a,n}}$ *for sufficiently large* $n \in \mathbb{N}$.

Proof. In view of Proposition 4.1, we have

$$\begin{aligned}
|(T_n f)(x) - f(x)| &\leq A_{a,n} \left| \int_{-\infty}^{\infty} e^{-at} f(\lambda_n(x) + t) K_n(t) \mathrm{d}t - f(x) \int_{-\infty}^{\infty} e^{-at} K_n(t) \mathrm{d}t \right| \\
&\quad + |f(x)| \left|(T_n e_0)(x) - 1\right| \\
&= A_{a,n} \left| \int_{-\infty}^{\infty} [f(\lambda_n(x) + t) - f(x)] e^{-at} K_n(t) \mathrm{d}t \right| \\
&\quad + |f(x)| \left|A_{a,n} A_{-a,n} - 1\right|.
\end{aligned}$$

For any $\delta > 0$, properties of $\widetilde{\omega}$ enable us to write

$$|f(\lambda_n(x) + t) - f(x)| = \left| f\left(x + t - a^{-1} \log A_{a,n}\right) - f(x) \right|$$

$$\leq \left(1+\left(x+t-a^{-1}\log A_{a,n}\right)^{2}\right)\widetilde{\omega}\left(f,\left|t-a^{-1}\log A_{a,n}\right|\right)$$
$$\leq \left(1+\left(x+t-a^{-1}\log A_{a,n}\right)^{2}\right)\left(1+\frac{\left|t-a^{-1}\log A_{a,n}\right|}{\delta}\right)\widetilde{\omega}\left(f,\delta\right).$$

In view of this observation, there holds

$$\begin{aligned}
&|(T_nf)(x)-f(x)|\leq \widetilde{\omega}(f,\delta)A_{a,n}\\
&\quad\times\int_{-\infty}^{\infty}\left(1+\left(x+t-a^{-1}\log A_{a,n}\right)^{2}\right)\left(1+\frac{\left|t-a^{-1}\log A_{a,n}\right|}{\delta}\right)e^{-at}K_n(t)\mathrm{d}t\\
&\quad+|f(x)|\left|A_{a,n}A_{-a,n}-1\right|\leq \widetilde{\omega}\left(f,\delta\right)A_{a,n}\\
&\quad\times\int_{-\infty}^{\infty}\left(1+\left(x+t-\frac{1}{a}\log A_{a,n}\right)^{2}\right)\left(1+\frac{\left|a^{-1}\log A_{a,n}\right|}{\delta}\right)e^{-at}K_n\left(t\right)\mathrm{d}t\\
&\quad+\widetilde{\omega}\left(f,\delta\right)A_{a,n}\int_{-\infty}^{\infty}\left(1+\left(x+t-\frac{1}{a}\log A_{a,n}\right)^{2}\right)\frac{|t|}{\delta}e^{-at}K_n\left(t\right)\mathrm{d}t\\
&\quad+|f(x)|\left|A_{a,n}A_{-a,n}-1\right|:=J_1+J_2+|f(x)|\left|A_{a,n}A_{-a,n}-1\right|.
\end{aligned}$$

For J_1, using the well-known inequality $|a-b|^p\leq 2^{p-1}\left(|a|^p+|b|^p\right)$, $a,b\in\mathbb{R}$, $p\geq 1$, we have

$$\begin{aligned}
&J_1\leq \widetilde{\omega}\left(f,\delta\right)A_{a,n}\\
&\quad\times\left(1+\frac{\left|a^{-1}\log A_{a,n}\right|}{\delta}\right)\left\{A_{-a,n}+2C_{-a,n}+2\left(x-a^{-1}\log A_{a,n}\right)^{2}A_{-a,n}\right\}\\
&\quad\leq 4A_{a,n}\widetilde{\omega}\left(f,\delta\right)\left(1+\frac{\left|a^{-1}\log A_{a,n}\right|}{\delta}\right)\left\{C_{-a,n}+\left(1+x^2+a^{-2}\log^2A_{a,n}\right)A_{-a,n}\right\}.
\end{aligned}$$

By the Cauchy–Schwarz inequality for J_2, we have

$$\begin{aligned}
J_2&=\widetilde{\omega}(f,\delta)A_{a,n}\int_{-\infty}^{\infty}\left(1+(x+t-a^{-1}\log A_{a,n})^2\right)\frac{|t|}{\delta}e^{-at}K_n(t)\mathrm{d}t\\
&\leq \widetilde{\omega}(f,\delta)A_{a,n}\int_{-\infty}^{\infty}(1+(x+t-a^{-1}\log A_{a,n})^2)^2e^{-at}K_n(t)\mathrm{d}t)^{1/2}\frac{(C_{-a,n})^{1/2}}{\delta}.
\end{aligned}$$

Repeated application of the inequality $|a-b|^p\leq 2^{p-1}\left(|a|^p+|b|^p\right)$, $a,b\in\mathbb{R}$, $p\geq 1$ yields

$$\begin{aligned}
\left(1+\left(x+t-a^{-1}\log A_{a,n}\right)^{2}\right)^{2}&\leq\left(1+2\left(t^2+\lambda_n^2(x)\right)\right)^{2}\\
&\leq 2\left(1+4\left(t^2+\lambda_n^2\left(x\right)\right)^{2}\right)\leq 2+16\left(t^4+\lambda_n^4\left(x\right)\right)\\
&=2+16t^4+16\left(x-a^{-1}\log A_{a,n}\right)^{4}\\
&\leq 2+16t^4+128\left(x^4+a^{-4}\log^4A_{a,n}\right).
\end{aligned}$$

Using above inequality, we have

$$J_2\leq 16\widetilde{\omega}\left(f,\delta\right)A_{a,n}\left(E_{-a,n}+A_{-a,n}\left(1+x^4+a^{-4}\log^4A_{a,n}\right)\right)^{1/2}\frac{(C_{-a,n})^{1/2}}{\delta}.$$

Collecting all inequalities we get

$$|(T_nf)\left(x\right)-f(x)|$$

$$\leq 4A_{a,n}\widetilde{\omega}(f,\delta)\left(1+\frac{\left|a^{-1}\log A_{a,n}\right|}{\delta}\right)\left\{C_{-a,n}+\left(1+x^2+a^{-2}\log^2 A_{a,n}\right)A_{-a,n}\right\}$$
$$+16\widetilde{\omega}(f,\delta)A_{a,n}\left(E_{-a,n}+A_{-a,n}\left(1+x^4+a^{-4}\log^4 A_{a,n}\right)\right)^{1/2}\frac{(C_{-a,n})^{1/2}}{\delta}$$
$$+|f(x)|\left|A_{a,n}A_{-a,n}-1\right|.$$

Choosing $\delta=(C_{-a,n})^{1/2}$ with $\left|a^{-1}\log A_{a,n}\right|\leq\delta$ for sufficiently large n, dividing both sides by $1+x^2$ and taking supremum over all x, we obtain

$$\begin{aligned}\|T_nf-f\|_{\rho_1}&\\ &\leq 16A_{a,n}\widetilde{\omega}\left(f,\sqrt{C_{-a,n}}\right)\Big\{C_{-a,n}+\left(1+a^{-2}\log^2 A_{a,n}\right)A_{-a,n}\\ &\quad+\sqrt{E_{-a,n}+A_{-a,n}\left(1+a^{-4}\log^4 A_{a,n}\right)}\Big\}+|f(x)|\left|A_{a,n}A_{-a,n}-1\right|\\ &\leq 16A_{a,n}\widetilde{\omega}\left(f,\sqrt{C_{-a,n}}\right)\Big\{C_{-a,n}+(1+C_{-a,n})A_{-a,n}\\ &\quad+\sqrt{E_{-a,n}+A_{-a,n}\left(1+(C_{-a,n})^2\right)}\Big\}\\ &\quad+\|f\|_{\rho_1}\left|A_{a,n}A_{-a,n}-1\right|\end{aligned}$$

that is the assertion □

Remark 5.5. The estimate of Theorem 5.4 gives an evaluation of the convergence in terms of the modulus of continuity with parameter $\sqrt{C_{-a,n}}$, under the further assumptions that $C_{-a,n}\to 0$ and $|A_{a,n}A_{-a,n}-1|\to 0$ as $n\to+\infty$. Thus one can obtain a corresponding corollary, as for Theorem 5.1 (see Corollary 5.2 and Remark 5.3).

6. An Asymptotic Formula

Here we establish two asymptotic formulae of Voronovskaya type for functions $f\in C^0_{\rho_j}(\mathbb{R})$, $j=1,2$, which are locally of class C^2 at a point x. These kinds of formulae give an exact evaluation of the order of pointwise convergence. We examine the case $j=1$.

Theorem 6.1. *Let $f\in C^0_{\rho_1}(\mathbb{R})$ be locally of class C^2 at a point $x\in\mathbb{R}$. Let $K_n\in L^*_{\exp_a}(\mathbb{R})$, for sufficiently large values of $n\in\mathbb{N}$, be a kernel satisfying the assumptions of Corollary 4.2. Assume that there exist $\alpha>0$ such that the kernel $\{K_n\}$ satisfies further the following conditions:*

(i) $\lim_{n\to+\infty}n^\alpha(A_{a,n}A_{-a,n}-1)=\ell_0$, $\lim_{n\to+\infty}n^\alpha B_{-a,n}=\ell_1$, $\lim_{n\to+\infty}n^\alpha C_{-a,n}=\ell_2$ *and* $\lim_{n\to+\infty}n^\alpha\frac{1}{a}\log A_{a,n}=\ell_3$, *with* $\ell_j\in\mathbb{R}$, *for* $j=0,1,2,3$.
(ii) *For every* $\eta>0$,

$$\lim_{n\to+\infty}n^\alpha\int_{|t|\geq\eta}e^{-at}\left(t-\frac{1}{a}\log A_{a,n}\right)^2K_n(t)\mathrm{d}t=0.$$

Then,

$$\lim_{n\to+\infty} n^{\alpha}[(T_nf)(x)-f(x)] = \ell_0 f(x) + (\ell_1-\ell_3)f'(x) + \frac{\ell_2}{2}f''(x). \quad (6.1)$$

Proof. Since f has a polynomial growth of order 2, and locally of class C^2 at the point x, using a local Taylor formula of the second order, we can write

$$\begin{aligned} f(\lambda_n(x)+t) &= f\left(x-\frac{1}{a}\log A_{a,n}+t\right) = f(x)+f'(x)\left(t-\frac{1}{a}\log A_{a,n}\right) \\ &\quad +\frac{f''(x)}{2}\left(t-\frac{1}{a}\log A_{a,n}\right)^2 + r_x\left(t-\frac{1}{a}\log A_{a,n}\right) \\ &\quad \times\left(t-\frac{1}{a}\log A_{a,n}\right)^2, \end{aligned} \quad (6.2)$$

where $r_x(y)$ is a bounded function such that $\lim_{y\to 0} r_x(y)=0$.

We write

$$\begin{aligned} (T_nf)(x)-f(x) &= A_{a,n}\int_{-\infty}^{\infty} e^{-at}[f(\lambda_n(x)+t)-f(x)]K_n(t)\mathrm{d}t \\ &\quad + (A_{a,n}\,A_{-a,n}-1)f(x) =: I_1+I_2. \end{aligned}$$

As to I_2 we have by assumptions that $n^{\alpha}I_2 \to \ell_0 f(x)$ as $n\to+\infty$. Therefore we evaluate now the term I_1. Inserting (6.2) in I_1, we can write

$$\begin{aligned} I_1 &= A_{a,n}\left(B_{-a,n}-A_{-a,n}\frac{1}{a}\log A_{a,n}\right)f'(x) \\ &\quad +A_{a,n}\left(C_{-a,n}+\frac{A_{-a,n}}{a^2}\log^2 A_{a,n}-\frac{2}{a}B_{-a,n}\log A_{a,n}\right)\frac{f''(x)}{2}+R_x \\ &=: I_1^1+I_1^2+R_x, \end{aligned} \quad (6.3)$$

where

$$R_x := A_{a,n}\int_{-\infty}^{\infty} e^{-at} r_x\left(t-\frac{1}{a}\log A_{a,n}\right)\left(t-\frac{1}{a}\log A_{a,n}\right)^2 K_n(t)\mathrm{d}t.$$

Now,

$$\lim_{n\to+\infty} n^{\alpha}I_1^1 = (\ell_1-\ell_3)f'(x), \qquad \lim_{n\to+\infty} n^{\alpha}I_1^2 = \frac{\ell_2}{2}f''(x).$$

Thus we have to estimate the remainder term R_x.

Since $\lim_{y\to 0} r_x(y) = 0$, given an arbitrary $\varepsilon>0$, there is $\delta\in]0,1[$ such that $|r_x(y)|<\varepsilon$ whenever $|y|\le\delta$. We take now an index $\overline{n}$ such that for $n\ge\overline{n}$, one has $a^{-1}|\log A_{a,n}|<\delta/2$. Thus for $|t|<\delta/2$, we have also $|t-a^{-1}\log A_{a,n}|<\delta$. Thus

$$\left|r_x\left(t-\frac{1}{a}\log A_{a,n}\right)\right| < \varepsilon \qquad (|t|<\delta/2). \quad (6.4)$$

Writing

$$\begin{aligned} R_x &= A_{a,n}\left\{\int_{-\delta/2}^{\delta/2}+\int_{|t|\ge\delta/2}\right\} e^{-at} r_x\left(t-\frac{1}{a}\log A_{a,n}\right)\left((t-\frac{1}{a}\log A_{a,n}\right)^2 K_n(t)\mathrm{d}t \\ &=: R_1+R_2, \end{aligned}$$

we have, by (6.4),

$$|R_1| \leq A_{a,n}\varepsilon \left(C_{-a,n} + A_{-a,n}\frac{1}{a^2}\log^2 A_{a,n} - \frac{2B_{-a,n}}{a}\log A_{a,n} \right).$$

Therefore using the assumptions (i) we have, for a suitable absolute positive constant M,

$$\limsup_{n\to+\infty} n^\alpha |R_1| \leq M\varepsilon.$$

As to the term R_2, using the boundedness of r_x we can write

$$|R_2| \leq A_{a,n}\|r_x\|_\infty \int_{|t|\geq\delta/2} e^{-at}\left(t - \frac{1}{a}\log A_{a,n}\right)^2 K_n(t)\mathrm{d}t,$$

and by (ii) we obtain

$$\lim_{n\to+\infty} n^\alpha |R_2| = 0.$$

This implies

$$\limsup_{n\to+\infty} n^\alpha |R_x| \leq M\varepsilon,$$

that is $|R_x| \to 0$ as $n \to +\infty$. Thus the theorem is completely proved □

Theorem 5.4 works perfectly in several particular examples, as we will see in the next section. But in certain situations, as for example, the moment-type operators, formula (6.1) becomes

$$\lim_{n\to+\infty} n^\alpha[(T_n f)(x) - f(x)] = 0,$$

for a suitable constant α for which all the assumptions are satisfied. In case of the moment-type operator, we have $\alpha = 1$. If we try to take $\alpha = 2$ some of the assumptions (i) of Theorem 5.4 are not satisfied. This result is however interesting, but it gives no exact information about the pointwise order of approximation at a point x. Therefore, we will formulate a slight generalization of the above theorem, in order to include also the case of moment-type operators, by changing assumptions (i). We have the following

Theorem 6.2. *Let $f \in C^0_{\rho_1}(\mathbb{R})$ be locally of class C^2 at a point $x \in \mathbb{R}$. Let $K_n \in L^*_{\exp_a}(\mathbb{R})$, for sufficiently large values of $n \in \mathbb{N}$, be a kernel satisfying the assumptions of Corollary 4.2. Assume that there exist $\alpha > 0$ such that the kernel $\{K_n\}$ satisfies assumption* (ii) *of Theorem 5.4 and the following conditions:*

(j) $\lim_{n\to+\infty} n^\alpha (A_{a,n}\, A_{-a,n} - 1) = \ell_0,$
$\lim_{n\to+\infty} n^\alpha \left(B_{-a,n} - A_{-a,n}\frac{1}{a}\log A_{a,n}\right) = \lambda_1,$

(jj) $\lim_{n\to+\infty} n^\alpha \left(C_{-a,n} + \frac{A_{-a,n}}{a^2}\log^2 A_{a,n} - \frac{2}{a}B_{-a,n}\log A_{a,n}\right) = \lambda_2$

Then,

$$\lim_{n\to+\infty} n^\alpha[(T_n f)(x) - f(x)] = \ell_0 f(x) + \lambda_1 f'(x) + \frac{\lambda_2}{2}f''(x). \tag{6.5}$$

Proof. The proof is clearly exactly the same. □

7. Examples

In this section we discuss some examples of kernels $\{K_n\}$ for which the theory developed can be applied.

(1) *The Gauss–Weierstrass kernel*

For $n \in \mathbb{N}$, let us consider the kernel $\{K_n\}$ with

$$K_n(t) = \sqrt{\frac{n}{\pi}} e^{-nt^2} \qquad (t \in \mathbb{R}).$$

This kernel is defined by even functions. First we evaluate the exponential moments of orders $0, 1, 2$. In order to do that, we first calculate the coefficients $A_{a,n}$ using a differentiation under the integral. By solving a simple first order linear differential equation, we obtain

$$A_{a,n} = A_{-a,n} = \sqrt{\frac{n}{\pi}} \int_{-\infty}^{\infty} e^{at} e^{-nt^2} \mathrm{d}t = e^{a^2/(4n)}.$$

Hence $A_{a,n}\, A_{-a,n} = e^{a^2/(2n)}$. Taking $\alpha = 1$, we obtain

$$\lim_{n\to+\infty} n(A_{a,n}\, A_{-a,n} - 1) = \lim_{n\to+\infty} n(e^{a^2/(2n)} - 1) = \frac{a^2}{2}.$$

Moreover, using partial integrations, one has

$$B_{-a,n} = -\frac{a}{2n} e^{a^2/4n},\ C_{-a,n} = \frac{1}{2n} A_{a,n} + \frac{a^2}{4n^2} A_{a,n},$$

and so

$$\lim_{n\to+\infty} nB_{-a,n} = \frac{1}{2},\ \lim_{n\to+\infty} nC_{-a,n} = \frac{1}{2}.$$

Next, we have

$$\lim_{n\to+\infty} n \log A_{a,n} = \frac{a^2}{4},$$

therefore we have also

$$\lim_{n\to+\infty} n \log^2 A_{a,n} = 0.$$

Therefore, $\ell_1 - \ell_3 = -3a/4,\ \ell_2 = 1/2$. Now we prove that the Gauss–Weierstrass kernel satisfies also assumption (ii) of Theorem 5.4. Let $\eta > 0$ be fixed. Let us set

$$J := n \int_{|t|\geq\eta} e^{-at} \left(t - \frac{1}{a} \log A_{a,n}\right)^2 K_n(t)\mathrm{d}t.$$

Then

$$J \leq n \int_{|t|\geq\eta} e^{-at} t^2 K_n(t)\mathrm{d}t + n\left(\frac{1}{a^2}\log^2 A_{a,n}\right) \int_{|t|\geq\eta} e^{-at} K_n(t)\mathrm{d}t$$
$$+ 2n\frac{1}{a} |\log A_{a,n}| \int_{|t|\geq\eta} e^{-at} |t| K_n(t)\mathrm{d}t =: J_1 + J_2 + J_3.$$

As to J_1 we can write

$$\begin{aligned} J_1 &= \frac{n\sqrt{n}}{\sqrt{\pi}} \int_{|t|\geq \eta} e^{-at} t^2 e^{-nt^2} \mathrm{d}t \\ &= \frac{n\sqrt{n}}{\sqrt{\pi}} \left\{ \int_\eta^\infty e^{-at} t^2 e^{-nt^2} \mathrm{d}t + \int_\eta^\infty e^{at} t^2 e^{-nt^2} \mathrm{d}t \right\} \\ &=: J_1^1 + J_1^2. \end{aligned}$$

For J_1^1 we have by a suitable substitution,

$$J_1^1 \leq \frac{e^{-a\eta}}{\sqrt{\pi}} \int_{\sqrt{n}\eta}^\infty v^2 e^{-v^2} dv,$$

and so by the absolute continuity of the Lebesgue integral, we obtain

$$\lim_{n\to+\infty} J_1^1 = 0.$$

Analogously,

$$J_1^2 \leq \frac{1}{\sqrt{\pi}} \int_{\sqrt{n}\eta}^\infty e^{av} v^2 e^{-v^2} dv,$$

and again, since the integrand in the right-hand side is Lebesgue integrable, we obtain $J_1^2 \to 0$ as $n \to +\infty$. Next, we evaluate J_2. We have immediately

$$J_2 \leq \frac{n}{a^2} \log^2 A_{a,n} A_{-a,n} \to 0, \quad (n \to +\infty).$$

Finally, we evaluate J_3. Using the same reasonings as for the estimate of J_1, we have

$$\begin{aligned} J_3 &\leq \frac{a\sqrt{n}}{2\sqrt{\pi}} \left\{ e^{-a\eta} \int_\eta^\infty t e^{-nt^2} \mathrm{d}t + \int_\eta^\infty e^{at} t e^{-nt^2} \mathrm{d}t \right\} \\ &=: J_3^1 + J_3^2. \end{aligned}$$

As for J_1 we obtain easily that $J_3^1 \to 0$, $J_3^2 \to 0$ as $n \to +\infty$. Concluding, we obtain assumption (ii) of Theorem 5.4. Therefore all the assumptions introduced are satisfied with $\alpha = 1$. The asymptotic formula of Theorem 5.4 reads

$$\lim_{n\to+\infty} n((T_n f)(x) - f(x)) = \frac{a^2}{2} f(x) - \frac{3a}{4} f'(x) + \frac{1}{4} f''(x),$$

which is a result of [26].

(2) *The Picard Kernel*

For $n \in \mathbb{N}$, let us consider the kernel $\{K_n\}$ with

$$K_n(t) = \frac{\sqrt{n}}{2} e^{-\sqrt{n}|t|} \quad (t \in \mathbb{R}).$$

Also this kernel is defined by even functions. Let us evaluate the exponential moments of order $0, 1, 2$. First we have

$$A_{a,n} = A_{-a,n} = \frac{n}{n-a^2}, \text{ and } A_{a,n}^2 = \left(\frac{n}{n-a^2}\right)^2$$

and from sufficiently large values of n, $A_{a,n}$ is positive. Moreover

$$\lim_{n\to+\infty} n(A_{a,n}A_{-a,n} - 1) = 2a^2, \quad \lim_{n\to+\infty} n \log A_{a,n} = a^2.$$

Now, we evaluate the moment $B_{-a,n}$. Using elementary calculation, based on partial integration, we can see that

$$B_{-a,n} = -\frac{2an}{(n-a^2)^2}, \; C_{-a,n} = \sqrt{n}\left(\frac{1}{(a+\sqrt{n})^3} + \frac{1}{(\sqrt{n}-a)^3}\right).$$

Therefore,

$$\lim_{n\to+\infty} nB_{-a,n} = -2a, \; \lim_{n\to+\infty} nC_{-a,n} = 2.$$

Finally we check assumption (ii) of Theorem 5.4. We proceed as in the previous example. Let $\eta > 0$ be fixed and set

$$J := n\int_{|t|\geq\eta} e^{-at}\left(t - \frac{1}{a}\log A_{a,n}\right)^2 K_n(t)\mathrm{d}t.$$

We have

$$J \leq \frac{n\sqrt{n}}{2}\left\{\int_{|t|\geq\eta} e^{-at}t^2 e^{-\sqrt{n}|t|}\mathrm{d}t + \frac{1}{a^2}\log^2\frac{n}{n-a^2}\int_{|t|\geq\eta} e^{-at}e^{-\sqrt{n}|t|}\mathrm{d}t \right.$$
$$\left. +\frac{2}{a}\log\frac{n}{n-a^2}\int_{|t|\geq\eta} e^{-at}|t|e^{-\sqrt{n}|t|}\mathrm{d}t\right\} =: J_1 + J_2 + J_3.$$

Using now elementary calculations, based on suitable substitutions, it is easy to see that

$$J_1 \leq \frac{e^{-a\eta}}{2}\int_{\sqrt{n}\eta}^{\infty} u^2 e^{-u/2}du + \frac{1}{2}\int_{\sqrt{n}\eta}^{\infty} u^2 e^{-u/2}du,$$

for every n such that $1 - a/\sqrt{n} > 1/2$. Thus, $J_1 \to 0$ as $n \to +\infty$.

Next, let us consider J_2. We have easily

$$J_2 \leq \frac{n}{a^2}\log^2\frac{n}{n-a^2}A_{-a,n},$$

and so $J_2 \to 0$ as $n \to +\infty$. Finally, as to J_3 we have

$$J_3 \leq \frac{e^{-a\eta}\sqrt{n}}{a}\log\frac{n}{n-a^2}\int_{\sqrt{n}\eta}^{\infty} ve^{-v}dv + \frac{\sqrt{n}}{a}\log\frac{n}{n-a^2}\int_{\sqrt{n}\eta}^{\infty} ue^{-u/2}du,$$

for every n such that $1 - a/\sqrt{n} > 1/2$. Therefore, $J_3 \to 0$ as $n \to +\infty$, and assumption (ii) is satisfied. Therefore all the assumptions introduced are satisfied with $\alpha = 1$. The asymptotic formula of Theorem 5.4 reads

$$\lim_{n\to+\infty} n((T_n f)(x) - f(x)) = 2a^2 f(x) - 3af'(x) + f''(x)$$

which is a result of [6].

(3) *The moment kernel*

For $n \in \mathbb{N}$ let us consider the kernel $\{K_n\}$ with

$$K_n(t) = n\chi_{[0,1/n]}(t) \qquad (t \in \mathbb{R}).$$

This kernel is not even, and the functions K_n have compact support. We calculate now the coefficients $A_{a,n}$, $A_{-a,n}$, $B_{-a,n}$ and $C_{-a,n}$ and some their properties. We have

$$A_{a,n} = n\int_0^{1/n} e^{at}\mathrm{d}t = \frac{n}{a}(e^{a/n} - 1),\ A_{-a,n} = \frac{n}{a}(1 - e^{-a/n}),$$

and so

$$\lim_{n\to+\infty} A_{a,n} = \lim_{n\to+\infty} A_{-a,n} = 1, \quad \lim_{n\to+\infty} \frac{1}{a} n \log A_{a,n} = \frac{1}{2}.$$

Moreover,

$$\lim_{n\to+\infty} n(A_{a,n}A_{-a,n} - 1) = 0.$$

Next,

$$B_{-a,n} = -\frac{e^{-a/n}}{a} + \frac{n}{a^2}(1 - e^{-a/n}),\ C_{-a,n} = -\frac{e^{-a/n}}{an} + \frac{2}{a}B_{-a,n},$$

and both tend to zero as $n \to +\infty$. Now,

$$\lim_{n\to+\infty} nB_{-a,n} = 1/2,\ \lim_{n\to+\infty} nC_{a,n} = 0.$$

Concerning assumption (ii) of Theorem 5.4, since the functions K_n have supports $[0, 1/n]$ for any given $\eta > 0$, one can find an integer n_0 such that for any $n \geq n_0$ the set $\{t : |t| \geq \eta\}$ does not intersect the interval $[0, 1/n]$, and so denoting $H_{n,\eta} := \{t : |t| \geq \eta\} \cap [0, 1/n]$, we have

$$\int_{H_n} e^{-at}\left(t - \frac{1}{a}\log A_{a,n}\right)^2 \mathrm{d}t = 0$$

This implies that assumption (ii) is trivially satisfied. Therefore all the assumptions introduced are satisfied with $\alpha = 1$. The asymptotic formula of Theorem 5.4 reads

$$\lim_{n\to+\infty} n((T_n f)(x) - f(x)) = 0.$$

This result is not fully satisfactory, due to the fact that we have no a precise order of pointwise approximation. Therefore we now employ Theorem 6.2 with $\alpha = 2$. In order to do that, we have to check the assumptions (j) and (jj). As to (j), we first calculate the limit

$$\lim_{n\to+\infty} n^2\left(B_{-a,n} - \frac{1}{a}\log A_{a,n}\right),$$

which can be written as

$$\lim_{n\to+\infty} \frac{n^2}{a^2}\left(-ae^{-a/n} + a\frac{n}{a}(1 - e^{-a/n}) - a\log\left(\frac{n}{a}(e^{a/n} - 1)\right)\right).$$

This can be interpreted as a restriction of the limit

$$\lim_{x\to 0}\frac{a}{x^2}\left(-e^{-x}+\frac{1-e^{-x}}{x}-\log\left(\frac{e^x-1}{x}\right)\right).$$

The above limit can be solved using elementary techniques, based on the L'Hospital rule and its value is given by $-3a/8$. Now, in order to calculate λ_1 we write

$$\begin{aligned} &n^2\left(B_{-a,n}-A_{-a,n}\frac{1}{a}\log A_{a,n}\right)\\ &\quad= n^2\left(B_{-a,n}-\frac{1}{a}\log A_{a,n}\right)+n^2\frac{1}{a}\log A_{a,n}\,(1-A_{-a,n}), \end{aligned}$$

thus we consider only the limit

$$\lim_{n\to+\infty} n^2\frac{1}{a}\log A_{a,n}\,(1-A_{-a,n}).$$

Since as we have seen before $\lim_{n\to+\infty}(n/a)\log A_{a,n}=1/2$, we have to calculate only $\lim_{n\to+\infty} n(1-A_{-a,n})$ and it is not difficult to see that its value is given by $a/2$. Thus, finally we obtain $\lambda_1=-a/8$.

Now we proceed to the calculation of λ_2. At first we calculate the limit

$$\lim_{n\to+\infty} n^2 C_{-a,n}.$$

Again this limit can be considered as a restriction of the limit

$$\lim_{x\to 0}\frac{1}{x^2}\left(-xe^{-x}-2e^{-x}+2\frac{1-e^{-x}}{x}\right),$$

and using an analogous elementary approach, its value is given by $1/3$.

Next, using the previous results, we have immediately

$$\lim_{n\to+\infty} n^2\frac{A_{-a,n}}{a^2}\log^2 A_{a,n}=\frac{1}{4},$$

and

$$\lim_{n\to+\infty}\frac{2}{a}n^2 B_{-a,n}\log A_{a,n}=1/2.$$

Thus, we have $\lambda_2=1/12$.

The last limit $\ell_0=\lim_{n\to+\infty} n^2(A_{a,n}\,A_{-a,n}-1)$ can be obtained with the same reasonings as a restriction of the limit

$$\lim_{x\to 0} a^2\frac{e^x+e^{-x}-2-x^2}{x^4}$$

which is given by $a^2/12$. Concluding, the Voronovskaya formula (6.5) with $\alpha=2$ for the moment type operator is given by (see also [26])

$$\lim_{n\to+\infty} n^2[(T_nf)(x)-f(x)]=\frac{a^2}{12}f(x)-\frac{a}{8}f'(x)+\frac{1}{24}f''(x).$$

(4) *A spline kernel*

For $n \in \mathbb{N}$ let us consider the kernel $\{K_n\}$ with

$$K_n(t) = n(1 - |nt|)\chi_{[-1/n,1/n]}(t) \qquad (t \in \mathbb{R}).$$

For this kernel we have, by elementary calculations

$$A_{a,n} = A_{-a,n} = n\int_{-1/n}^{1/n} e^{at}(1 - n|t|)\mathrm{d}t = \frac{2n^2}{a^2}(\cosh(a/n) - 1),$$

and so

$$\lim_{n\to+\infty} A_{\pm a,n} = 1, \qquad \lim_{n\to+\infty} n^2(A_{a,n}A_{-a,n} - 1) = \frac{a^2}{6},$$

and $\lim_{n\to+\infty} n^2 \log A_{a,n} = a^2/12$. Next,

$$B_{-a,n} = n\int_{-1/n}^{1/n} te^{-at}(1 - n|t|)\mathrm{d}t = -\frac{2n}{a^2}\sinh(a/n) + \frac{4n^2}{a^3}(\cosh(a/n) - 1),$$

and

$$\lim_{n\to+\infty} n^2 B_{-a,n} = -\frac{a}{6}.$$

As to $C_{-a,n}$ we have

$$C_{-a,n} = n\int_{-1/n}^{1/n} t^2(1 - n|t|)e^{-at}\mathrm{d}t$$
$$= \frac{2}{a^2}\cosh(a/n) - \frac{8n}{a^3}\sinh(a/n) + \frac{12n^2}{a^4}(\cosh(a/n) - 1),$$

and so

$$\lim_{n\to+\infty} n^2 C_{-a,n} = \frac{1}{6}.$$

Finally, since the functions K_n have compact supports $[-1/n, 1/n]$, we easily see, as in the previous example, that (ii) of Theorem 5.4 is satisfied. Thus, all the assumptions used in the previous theory are satisfied with $\alpha = 2$. The corresponding asymptotic formula is given by

$$\lim_{n\to+\infty} n^2((T_n f)(x) - f(x)) = \frac{a^2}{6}f(x) - \frac{a}{4}f'(x) + \frac{1}{12}f''(x).$$

Remark 7.1. The kernel of Example (4) is generated by the spline function of second order, defined by

$$\beta_2(x) = (1 - |x|)\chi_{[-1,1]}(x) \qquad (x \in \mathbb{R}).$$

The functions K_n are given by $K_n(x) = n\beta_2(nx)$, for $x \in \mathbb{R}$. The spline functions of order k are given by the formula (see e.g. [11,24])

$$\beta_k(x) = \frac{1}{(k-1)!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}\left(\frac{k}{2} + x - j\right)_+^{k-1},$$

where for any real number r, r_+ denotes its positive part. The theory may be applied also to any kernel generated by the spline β_k.

Remark 7.2. Note that the kernels of any of the previous examples satisfy the assumptions of Theorem 5.4, in particular the inequality

$$\frac{1}{a}|\log A_{a,n}| \leq \sqrt{C_{-a,n}},$$

for every $a > 0$ and sufficiently large $n \in \mathbb{N}$. First, note that since in any of the above examples $A_{a,n} \geq 1$, we have $\log A_{a,n} \geq 0$. For the modified Gauss–Weierstrass kernel, one has easily

$$\sqrt{C_{-a,n}} > \frac{a}{2n} > \frac{1}{a}\log A_{a,n}.$$

For the modified Picard kernel, one can employ the calculations of the limits in Example (2): since $nC_{-a,n} \to 2$ as $n \to +\infty$, and

$$\lim_{n\to+\infty} \frac{\sqrt{n}}{a}\log A_{a,n} = 0,$$

for sufficiently large values of n we obtain the assertion. Similar arguments can be used for the remaining examples.

Remark 7.3. Further examples can be obtained using Phillips type operators and Post-Widder type operators which act on functions defined over the positive real axis, (see [16,18,19] in which some modified version are introduced).

Remark 7.4. Our approach may be applied also in case of general exponential functions of the form a^{-x} for $a > 1$ as studied in [17] for Baskakov-type operators that act on functions defined over the positive real axis.

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