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# SIMULTANEOUS HEAT AND MASS TRANSFER ON OSCILLATORY FREE CONVECTION BOUNDARY LAYER FLOW

M. A. HOSSAIN*

*International Centre for Theoretical Physics, Trieste, Italy*

## SUMMARY

The problem of simultaneous heat and mass transfer in two-dimensional free convection from a semi-infinite vertical flat plate is investigated. An integral method is used to find a solution for zero wall velocity and for a mass transfer velocity at the wall with small-amplitude oscillatory wall temperature. Low- and high-frequency solutions are developed separately and are discussed graphically with the effects of the parameters *Gr* (the Grashof number for heat transfer), *Gc* (the Grashof number for mass transfer) and *Sc* (the Schmidt number) for $Pr = 0{\cdot}71$ representing aid at 20°C.

KEY WORDS Heat mass transfer Boundary-layer flow

## INTRODUCTION

The study of laminar boundary-layer in oscillatory flow with steady mean was first studied by Lighthill (1954) considering the effects of fluctuations in the free stream velocity on the skin-friction and heat transfer for plates and cylinder. Stuart (1955) in an attempt to verify certain results of Lighthill's analysis, discussed the problem of flow over an infinite flat plate with suction, when the main stream oscillates in time about a constant mean. The theory developed by Lighthill has later been extended for free convection boundary-layer flow along a semi-infinite vertical plate by Nanda and Sharma (1963), Eshghy *et al.* (1965), and Kelleher and Yang (1968). On the other hand, Muhuri and Maiti (1967) and Singh *et al.* (1978) have investigated the free-convection flow and heat transfer along a semi-infinite horizontal plate when the plate temperature oscillates about a constant and variable mean respectively.

But in many processes mass transfer and heat transfer occur simultaneously. In free convection these may either hinder or aid one another. The effects of mass transfer on free convection was studied by Somers (1956), Wilcox (1961), Gill *et al.* (1965), Adams and Lowell (1968), Gebhart and Pera (1971) and many others. All these studies were confined to steady flows only. The present paper is therefore devoted to a study of mass transfer effects on the unsteady free convection boundary-layer flow along a vertical flat plate, when the plate temperature oscillates in time about a constant non-zero mean, the free-stream is isothermal and the species concentration at the plate is constant. The treatment is restricted to small amplitude oscillations only. This enables us to effect linearization. Two different solutions for low- and high-frequency ranges are developed considering the fluid to be air only. In the present analysis we have adopted the method developed by Lighthill (1954) for both high- and low-frequency solutions. It is found that in the low-frequency range, the oscillating component of the skin friction always lags behind the plate temperature oscillations while the rate of heat transfer has a phase lead in the presence of foreign species. In the high-frequency range, the velocity and temperature in the boundary-layer are of 'shear-wave' type, predicting a phase lead of 45° in the rate of heat transfer fluctuation and an equivalent phase lag in the skin friction oscillations.

*On leave of absence from the Department of Mathematics, University of Dhaka, Bangladesh.

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## BASIC EQUATIONS

A two-dimensional unsteady free convection boundary-layer flow of a viscous incompressible fluid along a vertical flat plate in the presence of foreign species is considered, assuming that the plate temperature oscillates in time about a non-zero mean, the species concentration at the plate is constant, and the free stream is isothermal. The level of species concentration being very small, the Soret–Dufour effects can be neglected from the energy equation. Under the usual Boussinesq's approximation the flow, the heat transfer, and the mass transfer processes are governed by the following equations:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=g\beta(T-T_\infty)+g\beta^*(C-C_\infty)+\nu\frac{\partial^2 u}{\partial y^2} \tag{1}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{2}$$

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=k\frac{\partial^2 T}{\partial y^2} \tag{3}$$

$$\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}=D\frac{\partial^2 C}{\partial y^2} \tag{4}$$

where $u$ and $v$ are, respectively, the $x$- and $y$-components of the velocity field, $g$ the acceleration due to gravity, $\nu$ the kinematic viscosity, $T$ the temperature of the fluid, $T_\infty$ the temperature of the free stream, $C$ the concentration at a point, $C_\infty$ the species concentration at the free stream, $\beta$ the temperature densification coefficient, $\beta^*$ the concentration densification coefficient, $k$ the thermal conductivity and $D$ is the molecular diffusivity of the species. In equation (4) the chemical reaction term is neglected.

Introducing the dimensionless quantities

$$x'=x/L,\ y'=y/L,\ t'=\nu t/L^2,\ u'=uL/\nu,\ v'=vL/\nu$$
$$T'=(T-T_\infty)/(T_w-T_\infty),\ C'=(C-C_\infty)/(C_w-C_\infty)$$

where $L$ is the characteristic length, $T_w$ the mean plate temperature and $C_w$ the species concentration at the plate, in equations (1)–(4) and dropping the primes for brevity we get

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=GrT+GcC+\frac{\partial^2 u}{\partial y^2} \tag{5}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{6}$$

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{1}{Pr}\frac{\partial^2 T}{\partial y^2} \tag{7}$$

$$\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}=\frac{1}{Sc}\frac{\partial^2 C}{\partial y^2} \tag{8}$$

where $Gr(=g\beta(T_w-T_\infty)L^3/\nu^2)$ and $Gc(=g\beta^*(C_w-C_\infty)L^3/\nu^2)$ are, respectively, the overall Grashof number for heat and mass transfer, $Pr(=\nu/k)$ the Prandtl number and $Sc(=\nu/D)$ the Schmidt number.

The boundary conditions to be satisfied by the equations (5)–(8) are

$$\left.\begin{array}{ll} u=0,\ v=0,\ T=1+\varepsilon\cos\omega t,\ C=1 & \text{at}\quad y=0 \\ u\to 0,\ T\to 0,\ C\to 0 & \text{as}\quad y\to\infty \end{array}\right\},\ \varepsilon\ll 1 \tag{9}$$

where $\omega$ is the dimensionless frequency $\omega L^2/\nu$.

## METHOD OF SOLUTIONS

In order to solve the above problem it is convenient to adopt the complex notations for harmonic functions. The solutions will be obtained in terms of complex functions, the real parts of which would be of our interest.

For small amplitude oscillations, we assume the solutions of the equations (5)–(8) in the following form:

$$\begin{aligned} u &= u_s + \varepsilon u_1 \exp^{i\omega t},\ v = v_s + \varepsilon v_1 \exp^{i\omega t} \\ T &= T_s + \varepsilon T_1 \exp^{i\omega t},\ C = C_s + \varepsilon C_1 \exp^{i\omega t} \end{aligned} \tag{10}$$

where $(u_s, v_s, T_s, C_s)$ is the steady mean flow and satisfies

$$\frac{\partial u_s}{\partial x} + \frac{\partial v_s}{\partial y} = 0 \tag{11}$$

$$u_s \frac{\partial u_s}{\partial x} + v_s \frac{\partial u_s}{\partial y} = \frac{\partial^2 u_s}{\partial y^2} + Gr\, T_s + Gc\, C_s \tag{12}$$

$$u_s \frac{\partial T_s}{\partial x} + v_s \frac{\partial T_s}{\partial y} = \frac{1}{Pr} \frac{\partial^2 T_s}{\partial y^2} \tag{13}$$

$$u_s \frac{\partial C_s}{\partial x} + v_s \frac{\partial C_s}{\partial y} = \frac{1}{Sc} \frac{\partial^2 C_s}{\partial y^2} \tag{14}$$

with the boundary conditions

$$\begin{aligned} &u_s = v_s = 0,\ T_s = 1,\ C_s = 1 \quad \text{at} \quad y = 0 \\ &u_s \to 0,\ T_s \to 0,\ C_s \to 0 \quad \text{as} \quad y \to \infty \end{aligned} \tag{15}$$

and $(u_1, v_1, T_1, C_1)$ is the unsteady part of the flow and satisfies

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} = 0 \tag{16}$$

$$i\omega u_1 + u_1 \frac{\partial u_s}{\partial x} + u_s \frac{\partial u_1}{\partial x} + v_1 \frac{\partial u_s}{\partial y} + v_s \frac{\partial u_1}{\partial y} = \frac{\partial u_1}{\partial y^2} + Gr\, T_1 + Gc C_1 \tag{17}$$

$$i\omega T_1 + u_1 \frac{\partial T_s}{\partial x} + u_s \frac{\partial T_1}{\partial x} + v_1 \frac{\partial T_s}{\partial y} + v_s \frac{\partial T_1}{\partial y} = \frac{1}{Pr} \frac{\partial^2 T_1}{\partial y^2} \tag{18}$$

$$i C_1 + u_1 \frac{\partial C_s}{\partial x} + u_s \frac{\partial C_1}{\partial x} + v_1 \frac{\partial C_s}{\partial y} + v_s \frac{\partial C_1}{\partial y} = \frac{1}{Sc} \frac{\partial^2 C_1}{\partial y^2} \tag{19}$$

which are obtained neglecting $\varepsilon^2$ and dividing by $\exp(i\omega t)$. The boundary conditions to be satisfied by the equations (16)–(19) are as follows:

$$\begin{aligned} &u_1 = 0,\ v_1 = 0,\ T_1 = 1,\ C_1 = 0 \quad \text{at} \quad y = 0 \\ &u_1 \to 0,\ T_1 \to 0,\ C_1 \to 0 \quad \text{as} \quad y \to \infty \end{aligned} \tag{20}$$

Equations (11)–(14) are well known boundary-layer equations which describe the steady two-dimensional convection flow along a vertical flat plate with mass transfer. Approximate integral solutions of these equations are obtained by Wilcox (1961) and numerically by Callahan and Marner (1967). The approximate expressions for $u_s$, $T_s$ and $C_s$, as obtained by Wilcox (1961), are given as below:

$$\frac{u_s}{u_x} = \left(\frac{y}{\delta}\right) - 2\left(\frac{y}{\delta}\right)^2 + \left(\frac{y}{\delta}\right)^3 \tag{21}$$

$$T_s = 1 - \frac{3}{2}\left(\frac{y}{\delta'}\right) + \frac{1}{2}\left(\frac{y}{\delta'}\right)^3 \tag{22}$$

$$C_s = 1 - \frac{3}{2}\left(\frac{y}{\delta''}\right) + \frac{1}{2}\left(\frac{y}{\delta''}\right)^3 \tag{23}$$

where $\delta$, $\delta'$ and $\delta''$ are the boundary-layer thickness for the velocity, the temperature, and the concentration boundary-layers respectively, and

$$\begin{aligned} u_s &= 4{\cdot}42 \sqrt{\left(\frac{Gr + r\,Gr}{Sc + 0{\cdot}625}\right)} x \\ \delta = \delta'' &= 3{\cdot}41 \sqrt[4]{\left(\frac{x(1 + 0{\cdot}625/Sc)}{Sc(Gr + rGc)}\right)} \end{aligned} \tag{24}$$

where $r = \delta'/\delta''$, which is connected with the Prandtl number $Pr$ and the Schmidt number $Sc$ by the following relation:

$$\frac{Sc}{Pr} = r^3\left(\frac{21}{8} - \frac{105}{48}r + \frac{63}{112}r^2\right) \tag{25}$$

According to Wilcox (1961) the value of $r \simeq \sqrt{(Pr/Sc)}$ or $\sqrt{(Sc/Pr)}$ accordingly as $Pr > Sc$ or $Pr < Sc$. In this paper we shall always consider $r \simeq \sqrt{(Pr/Sc)}$, since for most gases the values of $Sc < Pr$ in air at 20°C and 1 atm (Gebhart, 1965).

*Low frequency fluctuation*

We now write the functions $u_1$, $v_1$, $T_1$ and $C_1$ as the sum of the in-phase and out-of-phase component as follows:

$$(u_1, v_1, T_1, C_1) = (u_r, v_r, T_r, C_r) + i(u_2, v_2, T_2, C_2) \tag{26}$$

and substituting in equations (16)–(19) and then separating the real and imaginary parts we get

$$\frac{\partial u_r}{\partial x} + \frac{\partial v_r}{\partial y} \tag{27}$$

$$-\omega u_2 + u_r\frac{\partial u_s}{\partial x} + u_s\frac{\partial u_r}{\partial x} + v_r\frac{\partial u_s}{\partial y} + v_s\frac{\partial u_r}{\partial y} = \frac{\partial^2 u_r}{\partial y^2} + GrT_r + Gc\,C_r \tag{28}$$

$$-\omega T_2 + u_r\frac{\partial T_s}{\partial x} + u_s\frac{\partial T_r}{\partial x} + v_r\frac{\partial T_s}{\partial y} + v_s\frac{\partial T_r}{\partial y} = \frac{1}{Pr}\frac{\partial^2 T_r}{\partial y^2} \tag{29}$$

$$-\omega C_2 + u_r\frac{\partial C_s}{\partial x} + u_s\frac{\partial C_r}{\partial x} + v_r\frac{\partial C_s}{\partial y} + v_s\frac{\partial C_r}{\partial y} = \frac{1}{Sc}\frac{\partial^2 C_r}{\partial y^2} \tag{30}$$

with the boundary conditions

$$\begin{aligned} u_r = v_r = 0,\ T_r = 1,\ C_r = 0 \quad &\text{at} \quad y = 0 \\ u_r \to 0,\ T_r \to 0,\ C_r \to 0 \quad &\text{as} \quad y \to \infty \end{aligned} \tag{31}$$

and

$$\frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y} = 0 \tag{32}$$

$$u_r + u_2\frac{\partial u_s}{\partial x} + u_s\frac{\partial u_2}{\partial x} + v_2\frac{\partial u_s}{\partial y} + v_s\frac{\partial u_2}{\partial y} = \frac{\partial^2 u_2}{\partial y^2} + Gr\,T_2 + GcC_2 \tag{33}$$

$$T_r + u_2\frac{\partial T_s}{\partial x} + u_s\frac{\partial T_2}{\partial x} + v_2\frac{\partial T_s}{\partial y} + v_s\frac{\partial T_2}{\partial y} = \frac{1}{Pr}\frac{\partial^2 T_2}{\partial y^2} \tag{34}$$

$$C_r + u_2\frac{\partial C_s}{\partial x} + u_s\frac{\partial C_2}{\partial x} + v_2\frac{\partial C_s}{\partial y} + v_s\frac{\partial C_2}{\partial y} = \frac{1}{Sc}\frac{\partial^2 C_2}{\partial y^2} \tag{35}$$

with the boundary conditions

$$\begin{aligned} &u_2 = v_2 = 0,\ T_2 = C_2 = 0 \quad \text{at} \quad y = 0 \\ &u_2 \to 0,\ T_2 \to 0,\ C_2 \to 0 \quad \text{as} \quad y \to \infty \end{aligned} \tag{36}$$

The difference in phase between the velocity, the temperature and the concentration fluctuation at a point within the boundary layer and in the plate temperature fluctuations is

$$\tan^{-1}\left(\frac{u_2}{u_r}\right),\ \tan^{-1}\left(\frac{T_2}{T_r}\right),\ \text{and}\ \tan^{-1}\left(\frac{C_2}{C_r}\right)$$

When the frequency of oscillation is small, it is to be expected that the phase shift will be small. Therefore, $u_2$, $T_2$ and $C_2$ would be small in comparison with $u_r$, $T_r$ and $C_r$. Thus, when $\omega$ is small, the terms $-\omega u_2$, $-\omega T_2$ and $-\omega C_2$ can be neglected in equations (27)–(30). $u_r$, $T_r$ and $C_r$ will then be quasi-steady solutions corresponding to $\omega = 0$. This can be seen from the fact that the same equations can be obtained by substituting $u = u_s + \varepsilon u_r$, $v = v_s + \varepsilon v_r$, $T = T_s + \varepsilon T_r$ and $C = C_s + \varepsilon C_r$ in the steady-flow boundary layer equations. Following Lighthill (1954), $u_r$, $v_r$, $T_r$ and $C_r$ can be easily obtained as

$$\begin{aligned} u_r &= u_s + \frac{1}{2}y\frac{\partial u_s}{\partial y}, \quad v_r = \frac{1}{2}\left(v_s + y\frac{\partial v_s}{\partial y}\right) \\ T_r &= T_s + \frac{1}{2}y\frac{\partial T_s}{\partial y}, \quad C_r = \frac{1}{2}\left(C_s + y\frac{\partial C_s}{\partial y}\right) \end{aligned} \tag{37}$$

We now use Karman–Pohlhausen approximate integral method to solve the equations (32)–(35). Accordingly, we assume the following expression for $u_2$, $T_2$ and $C_2$:

$$u_2 = A_1\left[\frac{y}{\delta} - 3\left(\frac{y}{\delta}\right)^3 + 2\left(\frac{y}{\delta}\right)^4\right] \tag{38}$$

$$T_2 = A_2\left[\frac{y}{\delta'} - 3\left(\frac{y}{\delta'}\right)^3 + 2\left(\frac{y}{\delta'}\right)^4\right] + \frac{1}{2}\omega Pr\, y^2\left[1 - 2\left(\frac{y}{\delta'}\right) + \left(\frac{y}{\delta'}\right)^2\right] \tag{39}$$

$$C_2 = A_3\left[\frac{y}{\delta''} - 3\left(\frac{y}{\delta''}\right)^3 + 2\left(\frac{y}{\delta''}\right)^4\right] \tag{40}$$

which should satisfy the following averaging conditions

$$\omega\int_0^{\delta} u_r\,dy + 2\frac{d}{dx}\int_0^{\delta} u_2 u_s\,dy = Gr\int_0^{\delta'} T_2\,dy + Gc\int_0^{\delta''} C_2\,dy - \left(\frac{\partial u_2}{\partial y}\right)_{y=0} \tag{41}$$

$$\omega\int_0^{\delta'} T_r\,dy + \frac{d}{dx}\int_0^{\delta'}\{u_2 T_s + u_s T_2\}\,dy = -\frac{1}{Pr}\left(\frac{\partial T_2}{\partial y}\right)_{y=0} \tag{42}$$

$$\omega\int_0^{\delta''} C_r\,dy + \frac{d}{dx}\int_0^{\delta''}\{u_2 C_s + u_s C_2\}\,dy = -\frac{1}{Sc}\left(\frac{\partial C_2}{\partial y}\right)_{y=0} \tag{43}$$

Substitution of the equations (37)–(39) into (40)–(42) yields

$$A_1 = -\frac{c_2(b_1d_2+b_2d_1)-126b_2c_1}{c(a_1b_2+a_2b_1)+51c_1b_2}\,\omega x$$

$$A_2 = -\frac{c_2(a_1d_2-a_2d_1)+c_1(126a_2+51d_2)}{c_2(a_1b_2+a_2b_1)+51c_1b_2}\,\omega x^{\frac{1}{2}}$$

$$A_3 = \frac{126(a_1b_2+a_2b_1)+51(b_1d_2+b_2d_1)}{c_2(a_1b_2+a_2b_1)+51c_1b_2}\,\omega x^{\frac{1}{2}} \qquad (44)$$

where

$$a_1 = (31{\cdot}01+10{\cdot}26\,Gc)\left(\frac{Gc+rGr}{Sc+0{\cdot}625}\right)^{\frac{1}{2}}$$

$$a_2 = 20r\left(\frac{1}{10}-\frac{9}{140}r^2+\frac{1}{40}r^3\right),\quad b_1 = 18rGr$$

$$b_2 = \left[1{\cdot}36+88{\cdot}0r\left(\frac{1}{15}-\frac{1}{14}r+\frac{3}{140}r^3\right)\right]\left(\frac{Gc+rGr}{Sc+0{\cdot}625}\right)^{\frac{1}{2}}$$

$$c_1 = 18{\cdot}0Gc,\quad c_2 = 119.47\left(\frac{Gc+rGr}{Sc+0{\cdot}625}\right)^{\frac{1}{2}}$$

$$d_1 = 22{\cdot}15\left(\frac{Gc+rGr}{Sc+0{\cdot}625}\right)^{\frac{1}{2}}-23{\cdot}39Gr\left(\frac{Sc+0{\cdot}625}{Gc+rGr}\right)^{\frac{1}{2}}$$

$$d_2 = 3+518{\cdot}15r\left(\frac{1}{60}-\frac{2}{105}r+\frac{1}{168}r^2\right)$$

Values of $A_1$ and $A_2$ are entered in Table I for different values of the parameters. In the present analysis, the values of the Grashof numbers $Gr$ are taken considerably large and positive. The value of the Grashof number $Gc$ for mass transfer is taken arbitrarily. In order to be realistic, the value of the Prandtl number $Pr$ is chosen in such a way that it represents air ($Pr = 0{\cdot}71$). The values of the Schmidt number $Sc$ are also chosen in such a way that they represent the diffusing chemical species of most common interest in air and less than the values of $Pr$. From Table I it can be seen that the values of $A_1$ and $A_2$ are always negative. Finally, the equations (37) and (38) together with (43) give the expressions for $u_2$ and $T_2$.

### *High frequency fluctuations*

Lighthill (1954) has shown that, for high frequency, the oscillatory flow is to a close approximation to an ordinary 'shear-wave' unaffected by the mean flow. The flow field can be described as a superposition of the steady mean flow and a shear-wave flow corresponding to the oscillating component of the plate temperature.

Table I. Values of $A_1$ and $A_2$ for $Pr = 0{\cdot}71$

| $Gr$ | $Gc$ | $Sc$ | $A_1$ | $A_2$ |
|---|---|---|---|---|
| 2 | 0 | — | −4·22 | −4·90 |
| 2 | 1 | 0·22 | −3·47 | −4·66 |
| 4 | 1 | 0·22 | −3·80 | −3·38 |
| 2 | 2 | 0·22 | −2·95 | −4·43 |
| 2 | 1 | 0·30 | −1·82 | −2·75 |
| 2 | 1 | 0·60 | −0·58 | −1·23 |

For high-frequency the differential set equations (17)–(19) reduces to

$$\frac{\partial^2 u_1}{\partial y^2} - i\omega u_1 = -GrT_1 - GcC_1$$

$$\frac{\partial^2 T_1}{\partial y^2} - i\omega PrT_1 = 0, \frac{\partial^2 C_1}{\partial y^2} - i\omega ScC_1 = 0$$

from which we easily obtain

$$u_1(y) = \frac{Gr}{i\omega(Pr-1)}[\exp(-\sqrt{(i\omega)}y) - \exp(-\sqrt{(i\omega Pr)}y)] \tag{45}$$

$$T_1(y) = \exp(-\sqrt{(i\omega Pr)}y) \tag{46}$$

$$C_1(y) = 0 \tag{47}$$

## DISCUSSIONS AND CONCLUSIONS

For small-frequency, the longitudinal component of the velocity and temperature may be written in the form

$$\begin{aligned} u &= u_s + \varepsilon R_1 \cos(\omega t + \beta_1) \\ T &= T_s + \varepsilon R_2 \cos(\omega t + \beta_2) \end{aligned} \tag{48}$$

where

$$R_1 = (u_r^2 + u_2^2)^{\frac{1}{2}}, R_2 = (T_r^2 + T_2^2)^{\frac{1}{2}} \tag{49}$$

and

$$\beta_1 = \tan^{-1}(u_2/u_r), \quad \beta_2 = \tan^{-1}(T_2/T_r) \tag{50}$$

The velocity and the temperature in the 'shear-wave' flow are as follows:

$$u = u_s + \varepsilon R_3 \cos(\omega t - \beta_3) \tag{51}$$

$$T = T_s + \varepsilon R_4 \cos(\omega t - \beta_4) \tag{52}$$

where

$$R_3 = (P^2 + Q^2)^{\frac{1}{2}}, R_4 = \exp(-\sqrt{(\tfrac{1}{2}\omega Pr)}y)$$

$$\beta_3 = \tan^{-1}(Q/P), \beta_4 = \sqrt{(\tfrac{1}{2}\omega Pr)}y$$

$$P = \frac{Gr}{\omega(1-Pr)}[\exp(-\sqrt{(\tfrac{1}{2}\omega)}y)\sin(\sqrt{(\tfrac{1}{2}\omega)}y) - \exp(-\sqrt{(\tfrac{1}{2}\omega Pr)}y)\sin(\sqrt{(\tfrac{1}{2}\omega Pr)}y)]$$

and

$$Q = \frac{Gr}{\omega(1-Pr)}[\exp(-\sqrt{(\tfrac{1}{2}\omega Pr)}y)\cos(\sqrt{(\tfrac{1}{2}\omega)}y) - \exp(-\sqrt{(\tfrac{1}{2}\omega)}y)\cos(\sqrt{(\tfrac{1}{2}\omega)}y)]$$

Now we shall compare the frequency response to the heat transfer and the friction factor. The local heat transfer from the surface to the fluid may be calculated from the following non-dimensional relation:

$$q = -\left[\left(\frac{\partial T_s}{\partial y}\right)_{y=0} + \varepsilon \exp^{i\omega t}\left(\frac{\partial T_1}{\partial y}\right)_{y=0}\right]$$

The temperature gradient in 'shear-wave' flow is given by

$$\mathrm{Re}\left[\exp^{i\omega t}\left(\frac{\partial T_1}{\partial y}\right)_{y=0}\right] = -(\omega Pr)^{\frac{1}{2}}\cos(\omega t + \tfrac{1}{4}\pi)$$

Its amplitude increases with frequency and its phase is ahead of that of the fluctuation in the surface temperature by 45°. For the low-frequency we have

$$\mathrm{Re}\left[\exp^{i\omega t}\left(\frac{\partial T_1}{\partial y}\right)_{y=0}\right] = -\left[0{\cdot}29(\omega Pr)^{\frac{1}{2}}\frac{Sc + 0{\cdot}625^{\frac{1}{4}}}{(Gc + rGr)x_1}\frac{81}{16} + A_2^{2\,\frac{1}{2}}\right]\cos(\omega t + \phi) \tag{53}$$

where

$$\phi = \tan^{-1}\left(-\frac{4}{9}A_2\sqrt{x_1}\right) \quad \text{and} \quad x_1 = \omega^2 x$$

The variations of phase angle and amplitude of the wall temperature gradient as a function of $x_1$ are shown in Figures 1 and 2, respectively. The corresponding asymptotic values of $x_1$ are obtained from

$$x_1 = \frac{81}{16}A_2^{-2} \tag{54}$$

and entered in Table II. The high- and low-frequency solutions may be matched on the basis of heat-transfer oscillation, taking the matching point as the frequency at which the low-frequency solution predicts a phase advance equal to the shear-wave solution. Thus values of $x_1$ may be taken as the boundary between the regions of applicability of high- and low-frequency solutions. The temperature profiles obtained on the basis of high- and low-frequency are compared in Figure 3.

We now obtain the expression for skin-friction from the following non-dimensional relation:

$$\tau = \left(\frac{\partial u_s}{\partial y}\right)_{y=0} + \varepsilon\exp^{i\omega t}\left(\frac{\partial u_1}{\partial y}\right)_{y=0} \tag{55}$$

The velocity gradient in the 'shear-wave' flow is given by

$$\mathrm{Re}\left[\exp^{i\omega t}\left(\frac{\partial u_1}{\partial y}\right)_{y=0}\right] = \frac{Gr}{\sqrt{(\omega(1 + Pr))}}\cos(\omega t - \tfrac{1}{4}\pi)$$

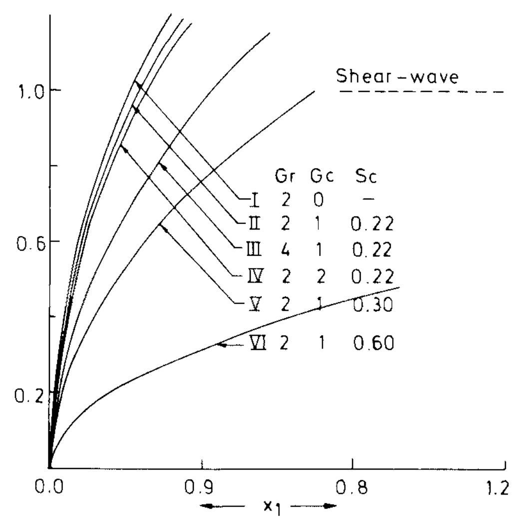


Figure 1. Phase angle of temperature gradient for $Pr = 0{\cdot}71$. Curve I is for $Gr = 2$, $Gc = 0$ (Nanda and Sharma, 1963)

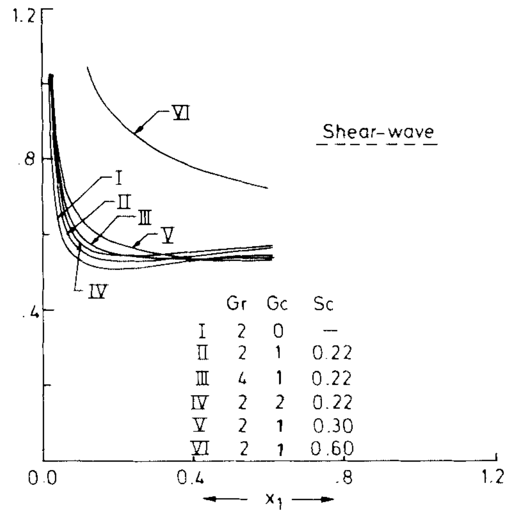


Figure 2. Amplitude of temperature gradient for $Pr = 0{\cdot}71$. Curve I is for $Gr = 2$, $Gc = 0$ (Nanda and Sharma, 1963)

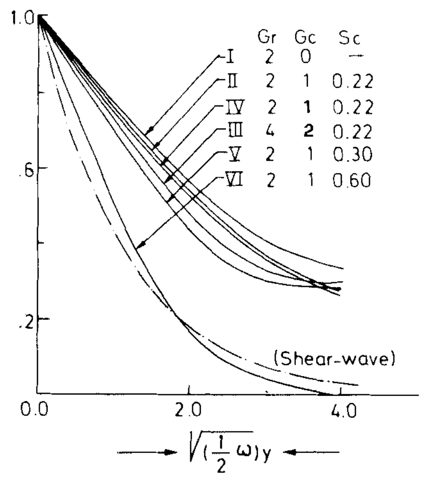


Figure 3. Amplitude of the temperature profiles for $Pr = 0{\cdot}71$. Curve I is for $Gr = 2$, $Gc = 0$ (Nanda and Sharma, 1963)

Table II. Values of critical $x_1$ for $Pr = 0{\cdot}71$

| *Gr* | *Gc* | *Sc* | Temperature field | Velocity field |
|---|---|---|---|---|
| 2 | 0 | — | 0·21 | 10·54 |
| 2 | 1 | 0·22 | 0·23 | 19·97 |
| 4 | 1 | 0·22 | 0·44 | 29·55 |
| 2 | 2 | 0·22 | 0·25 | 33·61 |
| 2 | 1 | 0·30 | 0·67 | 58·76 |
| 2 | 1 | 0·60 | 0·37 | 340·11 |

the amplitude of which decreases with frequency and its phase lags behind the plate temperature oscillation by 45°. For low-frequency we have

$$\mathrm{Re}\left[\exp^{i\omega t}\left(\frac{\partial u_1}{\partial y}\right)_{y=0}\right] = \left[0{\cdot}29 Sc^{\frac{1}{2}}\left(\frac{Sc+0{\cdot}625}{Gc+rGr}x\right)^{\frac{1}{4}}\left(A_1^2 x_1 + 44{\cdot}15\frac{Gc+rGr}{Sc+0{\cdot}65}\right)\right]\cos(\omega t+\psi) \tag{56}$$

where

$$\psi = \tan^{-1}\left(\tfrac{2}{3}A_1\sqrt{x_1}\right)$$

As before the variations of amplitude and phase angle of the friction factor at the surface of the plate as a function of $x_1$ are shown in Figures 4 and 5, respectively. The corresponding asymptotic values of $x_1$ may be

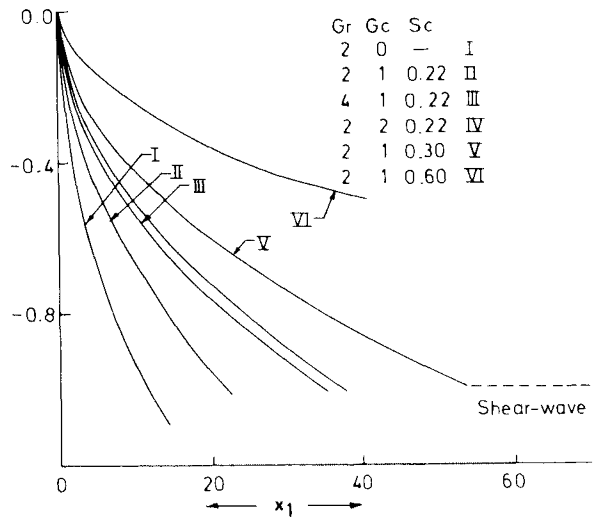


Figure 4. Phase angle of velocity gradient for $Pr = 0{\cdot}71$. Curve I is for $Gr = 2$, $Gc = 0$ (Nanda and Sharma, 1963)

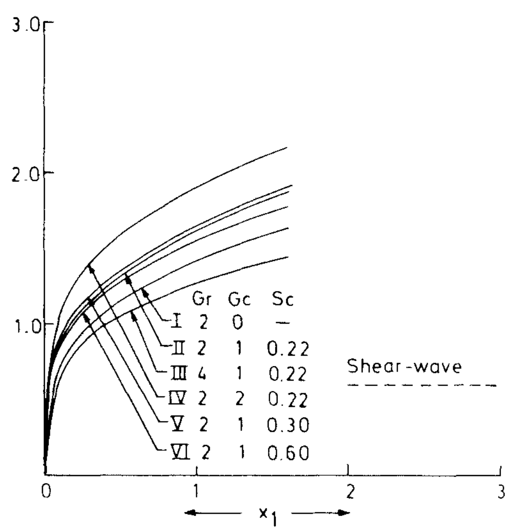


Figure 5. Amplitude of velocity gradient for $Pr = 0{\cdot}71$. Curve I is for $Gr = 2$, $Gc = 0$ (Nanda and Sharma, 1963)

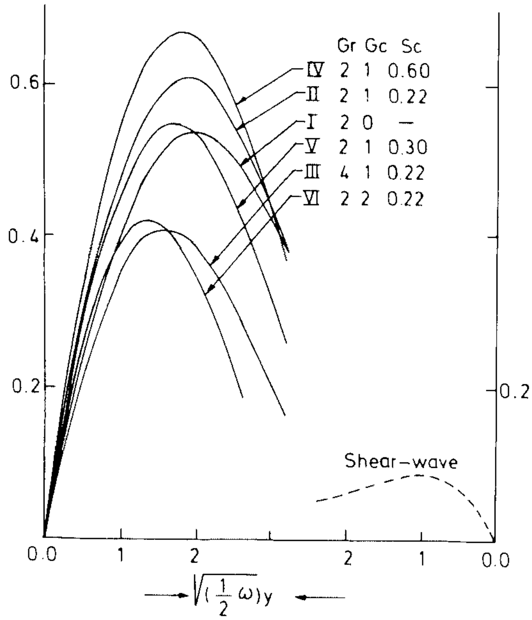


Figure 6. Amplitude of the velocity profiles for $Pr = 0{\cdot}71$. Curve I is for $Gr = 2$, $Gc = 0$ (Nanda and Sharma, 1963)

obtained from the following relation:

$$x_1 = 32{\cdot}55\left(\frac{Gc + rGr}{Sc + 0{\cdot}625}\right)A_1^{-2} \tag{57}$$

The values of which for different values of the parameters are given in Table II. It is also of interest to compare the velocity profiles obtained on the basis of low- and high-frequencies. The comparison of the velocity profiles is made in Figure 6 with different asymptotic values of $x_1$.

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