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Numerical solution of third-order boundary value problems by using a two-step hybrid block method with a fourth derivative

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Abstract
This article proposes a two-step hybrid block method (TSHBM) with a fourth derivative for solving third-order boundary value problems in ordinary differential equations. The mathematical formulation of the proposed approach depends on interpolation and collocation techniques. The order of convergence of the TSHBM is showed to be seventh-order convergent and zero-stable. A few numerical examples are given to evaluate its performance. Numerical outcomes show that the TSHBM scheme is more efficient than some existing numerical techniques.

KEYWORDS
collocation and interpolation techniques, hybrid block method, linear and nonlinear problems, ordinary differential equations, third-order boundary value problems

1 | INTRODUCTION

This article considers the third-order boundary value problem (BVP) of the form

$$y'''(x) = f(x, y(x), y'(x), y''(x)), \quad x \in [x_0, x_M] \subset \mathbb{R} \tag{1}$$

with the boundary conditions of the form

$$y(x_0) = y_0,\ y'(x_0) = y'_0,\ y(x_M) = y_M, \tag{2}$$

although any of them can be replaced by any of the following

$$g_1(y(x_0), y'(x_0), y''(x_0)) = y_a, \quad g_2(y(x_M), y'(x_M), y''(x_M)) = y_b, \tag{3}$$

where y_0, y'_0, y_M, y_a, y_b are real constants. We assume that there exists a unique solution $y(x)$. Problem (1) emerges in the sandwich boundary layer and laminar flow beam, fluid mechanics and dynamics, the investigation of the obstacle, thin-film flow, the motion of a rocket, the study of stellar interiors and draining and coating flows. These problems additionally have critical applications in a variety of engineering and applied sciences (see, e.g., References 1-7 and references therein).

It is well known that not all BVP in (1) can be solved analytically. Due to this reason, numerical methods were introduced to provide an approximate solution to the BVP given in (1). Ramos and Rufai[8] have reported three main types of numerical methods for solving BVPs of ordinary differential equations (ODEs).

There are other strategies in the available literature for approximating the solution of the problem given in (1), for example, the modified Adomian decomposition technique, the automatic differentiation, spline technique, multi-step methods, block strategies (see References 8-19). In this article, we introduce a new two-step hybrid block method (TSHBM) that utilizes one-point fourth derivative to get the approximate solutions to the third-order BVP in (1)–(3) directly.

2 | CONSTRUCTION OF THE BLOCK METHOD

We are interested in obtaining approximations of the solution $y(x)$ at the grid points $x_0 < x_1 < \dots < x_M$ of the integration interval $[x_0, x_M]$, taking a constant step size $h = x_{j+1} - x_j$, $j = 0, 1, \dots, M-1$. To get the discrete formulas, we consider that $y(x)$ can be approximated on the interval $[x_n, x_{n+2}]$ by the following polynomial $q(x)$

$$y(x) \simeq q(x) = \sum_{n=0}^{8} a_n\, x^n, \tag{4}$$

where $a_n \in \mathbb{R}$ are real unknown coefficients that will be determined by imposing collocation conditions at selected points. Consider the intermediate points $x_{n+r} = x_n + (1/2)h, x_{n+s} = x_n + (3/2)h$ on $[x_n, x_{n+2}]$ and the approximation in (4), its first and second derivatives applied to the point x_n, its third derivative applied to the points $x_n, x_{n+r}, x_{n+1}, x_{n+s}, x_{n+2}$, and its fourth derivative applied to the points x_{n+2}. In this way, we get a system of nine equations with nine unknowns $a_n, n = 0(1)8$, given by

$$\begin{aligned} q(x_n) &= y_n, q'(x_n) = y'_n, q''(x_n) = y''_n, \\ q'''(x_n) &= f_n, q'''(x_{n+r}) = f_{n+r}, q'''(x_{n+1}) = f_{n+1}, q'''(x_{n+s}) = f_{n+s}, q'''(x_{n+2}) = f_{n+2}, \\ q''''(x_{n+2}) &= k_{n+2}, \end{aligned} \tag{5}$$

where $y_{n+i}, y'_{n+i}, y''_{n+i}, f_{n+i} = f(x_{n+i}, y_{n+i}, y'_{n+i}, y''_{n+i}), k_{n+i} = k(x_{n+i}, y_{n+i}, y'_{n+i}, y''_{n+i})$ denote approximations of $y(x_{n+i}), y'(x_{n+i})$, $y''(x_{n+i}), f(x_{n+i}, y(x_{n+i}), y'(x_{n+i}), y''(x_{n+i}))$ and $k(x_{n+i}, y(x_{n+i}), y'(x_{n+i}), y''(x_{n+i}))$ respectively, with

$$k(x, y, y', y'') = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' + \frac{\partial f}{\partial y''} f(x, y, y', y'').$$

The system in (5) may be written in matrix form as

$$\begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 & x_n^8 \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 & 6x_n^5 & 7x_n^6 & 8x_n^7 \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 & 42x_n^5 & 56x_n^6 \\ 0 & 0 & 0 & 6 & 24x_n & 60x_n^2 & 120x_n^3 & 210x_n^4 & 336x_n^5 \\ 0 & 0 & 0 & 6 & 24x_{n+r} & 60x_{n+r}^2 & 120x_{n+r}^3 & 210x_{n+r}^4 & 336x_{n+r}^5 \\ 0 & 0 & 0 & 6 & 24x_{n+1} & 60x_{n+1}^2 & 120x_{n+1}^3 & 210x_{n+1}^4 & 336x_{n+1}^5 \\ 0 & 0 & 0 & 6 & 24x_{n+s} & 60x_{n+s}^2 & 120x_{n+s}^3 & 210x_{n+s}^4 & 336x_{n+s}^5 \\ 0 & 0 & 0 & 6 & 24x_{n+2} & 60x_{n+2}^2 & 120x_{n+2}^3 & 210x_{n+2}^4 & 336x_{n+2}^5 \\ 0 & 0 & 0 & 0 & 24 & 120x_{n+2} & 360x_{n+2}^2 & 840x_{n+2}^3 & 1680x_{n+2}^4 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{pmatrix} = \begin{pmatrix} y_n \\ y'_n \\ y''_n \\ f_n \\ f_{n+r} \\ f_{n+1} \\ f_{n+s} \\ f_{n+2} \\ k_{n+2} \end{pmatrix}.$$

After obtaining the values of $a_n, n = 0(1)8$ and changing the variable, $x = x_n + zh$, the polynomial in (4) may be written as

$$\begin{aligned} q(x_n + zh) = {} & \alpha_0(z) y_n + h\alpha_1(z) y'_n + h^2 \alpha_2(z) y''_n \\ & + h^3(\beta_0(z) f_n + \beta_r(z) f_{n+r} + \beta_1(z) f_{n+1} + \beta_s(z) f_{n+s} + \beta_2(z) f_{n+2}) + \gamma_1(z) k_{n+2}), \end{aligned} \tag{6}$$

where h is the step size and $\alpha_0(z), \alpha_1(z), \alpha_2(z), \beta_0(z), \beta_r(z), \beta_1(z), \beta_s(z), \beta_2(z), \gamma_1(z)$ are continuous coefficients.

2.1 | Main formulas

Substituting the values of $\alpha_0(z)$, $\alpha_1(z)$, $\alpha_2(z)$, $\beta_0(z)$, $\beta_r(z)$, $\beta_1(z)$, $\beta_s(z)$, $\beta_2(z)$, $\gamma_1(z)$ into (6) and evaluating $q(x), q'(x), q''(x)$ at the point $x_{n+2} = x_n + 2h$, we obtain the following main two-step formulas of the proposed method

$$\begin{aligned}
y_{n+2} &= y_n + 2hy'_n + 2h^2y''_n + \frac{4}{945}h^3\left(224f_{n+r} + 96f_{n+s} + 66f_n - 36f_{n+1} - 35f_{n+2}\right) + \frac{2}{63}h^4k_{n+2},\\
y'_{n+2} &= y'_n + 2hy''_n + \frac{1}{945}h^2\left(1088f_{n+r} + 576f_{n+s} + 279f_n + 72f_{n+1} - 125f_{n+2}\right) + \frac{2}{63}h^3k_{n+2},\\
y''_{n+2} &= y''_n + \frac{1}{45}h\left(32\left(f_{n+r} + f_{n+s}\right) + 7f_n + 12f_{n+1} + 7f_{n+2}\right).
\end{aligned} \tag{7}$$

To form the TSHBM, we need to complete the above formulas. To get these formulas, we evaluate $q(x)$, $q'(x)$, and $q''(x)$ at the points $x_{n+r}, x_{n+1}, x_{n+s}$. The resulting formulas are given by

$$\begin{aligned}
y_{n+r} &= y_n + \frac{1}{8}h\left(4y'_n + hy''_n\right) + \frac{h^3\left(62480f_{n+r} + 45744f_{n+s} + 46731f_n - 54684f_{n+1} - 19631f_{n+2}\right)}{3870720} + \frac{139h^4k_{n+2}}{129024},\\
y_{n+1} &= y_n + hy'_n + h^2y''_n + \frac{h^3\left(4544f_{n+r} + 2304f_{n+s} + 1881f_n - 2700f_{n+1} - 989f_{n+2}\right)}{30240} + \frac{h^4k_{n+2}}{144},\\
y_{n+s} &= y_n + \frac{3}{8}h\left(4y'_n + 3hy''_n\right) + \frac{9h^3\left(7344f_{n+r} + 2960f_{n+s} + 2409f_n - 2484f_{n+1} - 1269f_{n+2}\right)}{143360} + \frac{243h^4k_{n+2}}{14336}.
\end{aligned} \tag{8}$$

$$\begin{aligned}
y'_{n+r} &= y'_n + \frac{hy''_n}{2} + \frac{h^2\left(53920f_{n+r} + 35424f_{n+s} + 29223f_n - 42948f_{n+1} - 15139f_{n+2}\right)}{483840} + \frac{107h^3k_{n+2}}{16128},\\
y'_{n+1} &= y'_n + hy''_n + \frac{h^2\left(3344f_{n+r} + 1296f_{n+s} + 1053f_n - 1350f_{n+1} - 563f_{n+2}\right)}{7560} + \frac{h^3k_{n+2}}{63},\\
y'_{n+s} &= y'_n + \frac{3hy''_n}{2} + \frac{3h^2\left(4768f_{n+r} + 1760f_{n+s} + 1297f_n - 396f_{n+1} - 709f_{n+2}\right)}{17920} + \frac{45h^3k_{n+2}}{1792}.
\end{aligned} \tag{9}$$

$$\begin{aligned}
y''_{n+r} &= y''_n + \frac{h\left(5888f_{n+r} + 3008f_{n+s} + 1873f_n - 3732f_{n+1} - 1277f_{n+2}\right)}{11520} + \frac{3h^2k_{n+2}}{128},\\
y''_{n+1} &= y''_n + \frac{h\left(1568f_{n+r} + 288f_{n+s} + 333f_n + 108f_{n+1} - 137f_{n+2}\right)}{2160} + \frac{h^2k_{n+2}}{72},\\
y''_{n+s} &= y''_n + \frac{h\left(896f_{n+r} + 576f_{n+s} + 201f_n + 396f_{n+1} - 149f_{n+2}\right)}{1280} + \frac{3h^2k_{n+2}}{128}.
\end{aligned} \tag{10}$$

The local truncation errors of the above formulas are obtained using the usual Taylor expansion tool, resulting in

$$\begin{aligned}
\mathcal{L}[y(x_{n+r}), h] &= -\frac{1931h^9y^{(9)}(x_n)}{464486400} + \mathcal{O}(h^{10})\\
\mathcal{L}[y(x_{n+1}), h] &= -\frac{193h^9y^{(9)}(x_n)}{7257600} + \mathcal{O}(h^{10})\\
\mathcal{L}[y(x_{n+s}), h] &= -\frac{27h^9u^{(9)}(x_n)}{409600} + \mathcal{O}(h^{10})\\
\mathcal{L}[y(x_{n+2}), h] &= -\frac{h^9y^{(9)}(x_n)}{8100} + \mathcal{O}(h^{10})\\
\mathcal{L}[y'(x_{n+r}), h] &= -\frac{781h^8y^{(9)}(x_n)}{30965760} + \mathcal{O}(h^9)\\
\mathcal{L}[y'(x_{n+1}), h] &= -\frac{h^8y^{(9)}(x_n)}{16128} + \mathcal{O}(h^9)\\
\mathcal{L}[y'(x_{n+s}), h] &= -\frac{111h^8y^{(9)}(x_n)}{1146880} + \mathcal{O}(h^9)\\
\mathcal{L}[y'(x_{n+2}), h] &= -\frac{h^8y^{(9)}(x_n)}{7560} + \mathcal{O}(h^9)
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}[y''(x_{n+r}), h] &= -\frac{337h^7y^{(9)}(x_n)}{3870720} + \mathcal{O}(h^8)\\
\mathcal{L}[y''(x_{n+1}), h] &= -\frac{h^7y^{(9)}(x_n)}{16128} + \mathcal{O}(h^8)\\
\mathcal{L}[y''(x_{n+s}), h] &= -\frac{11h^7y^{(9)}(x_n)}{143360} + \mathcal{O}(h^8)\\
\mathcal{L}[y''(x_{n+2}), h] &= -\frac{h^7y^{(9)}(x_n)}{15120} + \mathcal{O}(h^8),
\end{aligned} \tag{11}$$

from which we observe that each formula has a precision of at least sixth order.

In order to apply the above formulas to solve the BVPs under consideration, we consider the formulas in (7)–(10) for $n = 0, 2, \ldots, M-2$ together with the given boundary conditions. In this way, we obtain a discretization of the given problem that allows us to approximate values at all the grid points. Considering the grid points $x_0 < x_1 < x_2 < \ldots < x_{M-1} < x_M$ with $M \in \mathbb{N}$, M an even positive integer, we obtain a system of $6M+3$ equations, including the boundary conditions, with $6M+3$ unknowns. The solution of this system provides the approximate solutions required.

3 | ANALYSIS OF CONVERGENCE

We begin by stating the definition of convergence of a numerical method for solving a BVP.

Definition 1. Let $u(x)$ be the exact solution of a BVP of the form in (1) with given boundary conditions as in (2), and let $\{y_j\}_{j=0}^{M}$ be the numerical approximations of $y(x)$ at the corresponding grid points, obtained by the proposed scheme. The method is said to have pth order of convergence if for sufficiently small step size h, there exists a constant K such that

$$\max_{0\le i\le M} \|y(x_j) - y_j\| \le Kh^p.$$

Theorem 1. *Let $y(x)$ be the exact solution of the BVP in (1)–(2), and $\{y_j\}_{j=0}^{M}$ the discrete solution provided by the proposed global method. Assuming that $y(x)$ is smooth enough, the proposed method has seventh order of convergence.*

Proof. Let A be the $6M \times 6M$ matrix defined as

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix},$$

where the $A_{j1}, j = 1, 2, 3$, are sub-matrices of dimension $2M \times (2M-1)$ as follows

$$A_{11} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & . & . & . & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & . & . & . & 0 & 0 & 0 & 0\\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & .\\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & .\\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & .\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & -1 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & -1 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & -1 & 0 & 0 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & -1 & 0 & 0 & 0
\end{pmatrix},$$

A_{21} and A_{31} are null sub-matrices. The sub-matrices $A_{j2}, j = 1, 2, 3$, have dimension $2M \times 2M$ and are given as follows

$$A_{12} = h\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_2 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_3 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_4 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & \alpha_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & \alpha_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & \alpha_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & \alpha_4 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where $\alpha_1 = -1/2, \alpha_2 = -1, \alpha_3 = -3/2, \alpha_4 = -2$;

$$A_{22} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & . & . & . & 0 & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & -1 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and A_{32} is a null sub-matrix. We note that removing the last column in the sub-matrix A_{22} we get the sub-matrix A_{11}.

The sub-matrices $A_{j3}, j = 1, 2, 3$ have dimension $2M \times (2M + 1)$ and are given as follows

$$A_{13} = h^2\begin{pmatrix} \beta_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\ \beta_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\ \beta_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\ \beta_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta_1 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta_2 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta_3 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta_4 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & \beta_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & \beta_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & \beta_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & \beta_4 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where $\beta_1 = -1/8, \beta_2 = -1/2, \beta_3 = -9/8, \beta_4 = -2,$

$$A_{23} = h\begin{pmatrix}
\alpha_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\
\alpha_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\
\alpha_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\
\alpha_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \alpha_2 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \alpha_3 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \alpha_4 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & \alpha_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & \alpha_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & \alpha_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & \alpha_4 & 0 & 0 & 0 & 0
\end{pmatrix},$$

with the α_i the same as in the sub-matrix A_{12}, and

$$A_{33} = \begin{pmatrix}
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & . & . & . & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & . & . & . & 0 & 0 & 0 & 0 & 0 \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & -1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & -1 & 0 & 0 & 0 & 1
\end{pmatrix}.$$

Note that if we remove the first column in the sub-matrix A_{33} we get the sub-matrix A_{22}.

Further, let V be the matrix of dimension $6M \times (4M+2)$ given by

$$V = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \\ v_{31} & v_{32} \end{pmatrix},$$

where the v_{ij} are sub-matrices of dimensions $2M \times (2M+1)$ given by

$$v_{11} = h^2 \begin{pmatrix} a_1^1 & a_2^1 & a_3^1 & a_4^1 & a_5^1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ b_1^1 & b_2^1 & b_3^1 & b_4^1 & b_5^1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ c_1^1 & c_2^1 & c_3^1 & c_4^1 & c_5^1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ d_1^1 & A_2^1 & d_3^1 & d_4^1 & d_5^1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_1^1 & a_2^1 & a_3^1 & a_4^1 & a_5^1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b_1^1 & b_2^1 & b_3^1 & b_4^1 & b_5^1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_1^1 & c_2^1 & c_3^1 & c_4^1 & c_5^1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & d_1^1 & d_2^1 & d_3^1 & d_4^1 & d_5^1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & \dots & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & \dots & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & \dots & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & a_1^1 & a_2^1 & a_3^1 & a_4^1 & a_5^1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & b_1^1 & b_2^1 & b_3^1 & b_4^1 & b_5^1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & c_1^1 & c_2^1 & c_3^1 & c_4^1 & c_5^1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & d_1^1 & d_2^1 & d_3^1 & d_4^1 & d_5^1 \end{pmatrix},$$

with $a_1^1 = -\frac{15577}{1290240}, a_2^1 = -\frac{781}{48384}, a_3^1 = \frac{217}{15360}, a_4^1 = -\frac{953}{80640}, a_5^1 = \frac{19631}{3870720}, b_1^1 = -\frac{209}{3360}, b_2^1 = -\frac{142}{945}, b_3^1 = \frac{5}{56}, b_4^1 = -\frac{8}{105}, b_5^1 = \frac{989}{30240}$, $c_1^1 = -\frac{21681}{143360}, c_2^1 = -\frac{4131}{8960}, c_3^1 = \frac{5589}{35840}, c_4^1 = -\frac{333}{1792}, c_5^1 = \frac{11421}{143360}, d_1^1 = -\frac{88}{315}, d_2^1 = -\frac{128}{135}, d_3^1 = \frac{16}{105}, d_4^1 = -\frac{128}{315}, d_5^1 = \frac{4}{27}$;

$$v_{21} = h \begin{pmatrix} a_1^2 & a_2^2 & a_3^2 & a_4^2 & a_5^2 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ b_1^2 & b_2^2 & b_3^2 & b_4^2 & b_5^2 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ c_1^2 & c_2^2 & c_3^2 & c_4^2 & c_5^2 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ d_1^2 & d_2^2 & d_3^2 & d_4^2 & d_5^2 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_1^2 & a_2^2 & a_3^2 & a_4^2 & a_5^2 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b_1^2 & b_2^2 & b_3^2 & b_4^2 & b_5^2 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_1^2 & c_2^2 & c_3^2 & c_4^2 & c_5^2 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & d_1^2 & d_2^2 & d_3^2 & d_4^2 & d_5^2 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & \dots & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & \dots & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & \dots & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & a_1^2 & a_2^2 & a_3^2 & a_4^2 & a_5^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & b_1^2 & b_2^2 & b_3^2 & b_4^2 & b_5^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & c_1^2 & c_2^2 & c_3^2 & c_4^2 & c_5^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & d_1^2 & d_2^2 & d_3^2 & d_4^2 & d_5^2 \end{pmatrix},$$

with $a_1^2 = -\frac{3247}{53760}, a_2^2 = -\frac{337}{3024}, a_3^2 = \frac{1193}{13440}, a_4^2 = -\frac{41}{560}, a_5^2 = \frac{15139}{483840}, b_1^2 = -\frac{39}{280}, b_2^2 = -\frac{418}{945}, b_3^2 = \frac{5}{28}, b_4^2 = -\frac{6}{35}, b_5^2 = \frac{563}{7560}, c_1^2 = -\frac{3891}{17920}, c_2^2 = -\frac{447}{560}, c_3^2 = \frac{297}{4480}, c_4^2 = -\frac{33}{112}, c_5^2 = \frac{2127}{17920}, d_1^2 = -\frac{31}{105}, d_2^2 = -\frac{1088}{945}, d_3^2 = -\frac{8}{105}, d_4^2 = -\frac{64}{105}, d_5^2 = \frac{25}{189}$;

$$v_{31} = \begin{pmatrix} a_1^3 & a_2^3 & a_3^3 & a_4^3 & a_5^3 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ b_1^3 & b_2^3 & b_3^3 & b_4^3 & b_5^3 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ c_1^3 & c_2^3 & c_3^3 & c_4^3 & c_5^3 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ d_1^3 & d_2^3 & d_3^3 & d_4^3 & d_5^3 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_1^3 & a_2^3 & a_3^3 & a_4^3 & a_5^3 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b_1^3 & b_2^3 & b_3^3 & b_4^3 & b_5^3 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_1^3 & c_2^3 & c_3^3 & c_4^3 & c_5^3 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & d_1^3 & d_2^3 & d_3^3 & d_4^3 & d_5^3 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & \dots & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & \dots & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & \dots & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & a_1^3 & a_2^3 & a_3^3 & a_4^3 & a_5^3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & b_1^3 & b_2^3 & b_3^3 & b_4^3 & b_5^3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & c_1^3 & c_2^3 & c_3^3 & c_4^3 & c_5^3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & d_1^3 & d_2^3 & d_3^3 & d_4^3 & d_5^3 \end{pmatrix},$$

with $a_1^3 = -\frac{1873}{11520}, a_2^3 = -\frac{23}{45}, a_3^3 = \frac{311}{960}, a_4^3 = -\frac{47}{180}, a_5^3 = \frac{1277}{11520}, b_1^3 = -\frac{37}{240}, b_2^3 = -\frac{98}{135}, b_3^3 = -\frac{1}{20}, b_4^3 = -\frac{2}{15}, b_5^3 = \frac{137}{2160}, c_1^3 = -\frac{201}{1280}, c_2^3 = -\frac{7}{10}, c_3^3 = -\frac{99}{320}, c_4^3 = -\frac{9}{20}, c_5^3 = \frac{149}{1280}, d_1^3 = -\frac{7}{45}, d_2^3 = -\frac{32}{45}, d_3^3 = -\frac{4}{15}, d_4^3 = -\frac{32}{45}, d_5^3 = -\frac{7}{45}$;

$$v_{12} = h^3 \begin{pmatrix} 0 & 0 & 0 & 0 & v_1^1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & v_2^1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & v_3^1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & v_4^1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v_1^1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v_2^1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v_3^1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v_4^1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & \dots & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & \dots & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & \dots & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & v_1^1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & v_2^1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & v_3^1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & v_4^1 \end{pmatrix},$$

with $v_1^1 = -\frac{139}{129024}, v_2^1 = -\frac{1}{144}, v_3^1 = -\frac{243}{14336}, v_4^1 = -\frac{2}{63}$;

$$v_{22} = h^2 \begin{pmatrix} 0 & 0 & 0 & 0 & v_1^2 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & v_2^2 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & v_3^2 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & v_4^2 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v_1^2 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v_2^2 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v_3^2 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v_4^2 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & \dots & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & \dots & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & \dots & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & v_1^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & v_2^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & v_3^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & v_4^2 \end{pmatrix},$$

with $v_1^2 = -\frac{107}{16128}, v_2^2 = -\frac{1}{63}, v_3^2 = -\frac{45}{1792}, v_4^2 = -\frac{2}{63}$;

$$v_{32} = h\begin{pmatrix} 0 & 0 & 0 & 0 & v_1^3 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & v_2^3 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & v_3^3 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & v_4^3 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v_1^3 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v_2^3 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v_3^3 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v_4^3 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & \dots & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & \dots & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & \dots & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & v_1^3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & v_2^3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & v_3^3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & v_4^3 \end{pmatrix},$$

with $v_1^3 = -\frac{3}{128}, v_2^3 = -\frac{1}{72}, v_3^3 = -\frac{3}{128}, v_4^3 = 0$.

Now, let $y(x)$ be the true solution of the considered BVP, and define the $6M$-vector W as follows

$$W = \big(y(x_r), y(x_1), y(x_s), y(x_2), \dots, y(x_{N-2+s}), y'(x_r), y'(x_1), \dots, y'(x_M), y''(x_0), \dots, y''(x_M)\big)^T,$$

and the $(4M+2)$-vector F by

$$F = \big(f(x_0, y(x_0), y'(x_0), y''(x_0)), f(x_r, y(x_r), y'(x_r), y''(x_r)), \dots, f(x_M, y(x_M), y'(x_M), y''(x_M)),$$
$$k(x_0, y(x_0), y'(x_0), y''(x_0)), k(x_r, y(x_r), y'(x_r), y''(x_r)), \dots, k(x_M, y(x_M), y'(x_M), y''(x_M))\big)^T.$$

Using the above vector-matrix notation, the exact representation of the global system may be expressed as follows

$$A_{6M\times 6M}V_{6M} + hV_{6M\times(4N+2)}F_{4N+2} + K_{6M} = \mathcal{L}(h)_{6M}. \tag{12}$$

The subscripts in (12) denote the corresponding dimensions of vectors and matrices. The vector K_{6M} contains the known values, provided by the given boundary conditions, that is,

$$K_{6M} = (-y_0 - \frac{h}{2}y_0', -y_0 - hy_0', -y_0 - \frac{3h}{2}y_0', -y_0 - 2hy_0', 0, \dots, 0, y_M, -y_0', -y_0', -y_0', -y_0', 0, \dots, 0)^T,$$

and the vector $\mathcal{L}(h)_{6M}$ consists of the local truncation errors of the formulas, given by

$$\mathcal{L}(h) = \begin{pmatrix} \mathcal{L}[y(x_r), h] \\ \mathcal{L}[y(x_1), h] \\ \mathcal{L}[y(x_s), h] \\ \mathcal{L}[y(x_2), h] \\ \dots \\ \mathcal{L}[y(x_M), h] \\ \mathcal{L}[y'(x_r), h] \\ \mathcal{L}[y'(x_1), h] \\ \mathcal{L}[y'(x_s), h] \\ \mathcal{L}[y'(x_2), h] \\ \dots \\ \mathcal{L}[y'(x_M), h] \\ \mathcal{L}[y''(x_r), h] \\ \mathcal{L}[y''(x_1), h] \\ \mathcal{L}[y''(x_s), h] \\ \mathcal{L}[y''(x_2), h] \\ \dots \\ \mathcal{L}[y''(x_M), h] \end{pmatrix}.$$

Now, consider the system of approximate values of the problem expressed as follows

$$A_{6M\times 6M}\overline{W}_{6M} + hV_{6M\times(4N+2)}\overline{F}_{4N+2} + K_{6M} = 0, \tag{13}$$

where $\overline{W}_{6M}$ is used to denote the vector of approximate values of W_{6M}, that is,

$$\overline{W}_{6M} = (y_r, y_1, y_s, y_2, \dots, y_{M-2+s}, y'_r, y'_1, \dots, y'_M, y''_0, y''_r, \dots, y''_M)^T,$$

and $\overline{F}_{4M+2}$ is given by

$$\overline{F}_{4M+2} = (f_0, f_r, f_1, f_s, f_2, \dots, f_M, k_0, k_r, k_1, k_s, k_2, \dots, k_M)^T.$$

By subtracting (13) from (12), after some simplifications we get

$$D_{6M\times 6M}\mathcal{E}_{6M} + hV_{6M\times(4M+2)}(F - \overline{F})_{4M+2} = \mathcal{L}(h)_{6M}, \tag{14}$$

where

$$\mathcal{E}_{6M} = W_{6M} - \overline{W}_{6M} = (E_r, E_1, E_s, E_2, \dots, E_{M-2+s}, E'_r, \dots, E'_M, E''_0, E''_r, \dots, E''_M)^T$$

consists of the errors at off-steps and nodal points.

On the other hand, using the Mean Value Theorem, one can consider for $i = 0, r, 1, s, 2, 2+r, 3, 2+s, 4, \dots, M$ the identities

$$\begin{aligned} f(x_i, y(x_i), y'(x_i), y''(x_i)) - f(x_i, y_i, y'_i, y''_i) &= (y(x_i) - y_i)\frac{\partial f}{\partial y}(\xi_i) + (y'(x_i) - y'_i)\frac{\partial f}{\partial y'}(\xi_i) \\ &\quad + (y''(x_i) - y''_i)\frac{\partial f}{\partial y''}(\xi_i), \\ k(x_i, y(x_i), y'(x_i), y''(x_i)) - k(x_i, y_i, y'_i, y''_i) &= (y(x_i) - y_i)\frac{\partial k}{\partial y}(\eta_i) + (y'(x_i) - y'_i)\frac{\partial k}{\partial y'}(\eta_i) \\ &\quad + (y''(x_i) - y''_i)\frac{\partial k}{\partial y''}(\eta_i), \end{aligned} \tag{15}$$

where ξ_i and η_i stand for intermediate points on the line segment joining $(x_i, y(x_i), y'(x_i), y''(x_i))$ to (x_i, y_i, y_i', y_i''). Now, using the formulas in (15), we have

$$F - \overline{F} = \begin{pmatrix} \frac{\partial f}{\partial y}(\xi_0) & \dots & 0 & \frac{\partial f}{\partial y'}(\xi_0) & \dots & 0 & \frac{\partial f}{\partial y''}(\xi_0) & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ . & \dots & . & . & \dots & . & . & \dots & . \\ . & \dots & . & . & \dots & . & . & \dots & . \\ . & \dots & . & . & \dots & . & . & \dots & . \\ 0 & \dots & \frac{\partial f}{\partial y}(\xi_M) & 0 & \dots & \frac{\partial f}{\partial y'}(\xi_M) & 0 & \dots & \frac{\partial f}{\partial y''}(\xi_M) \\ \frac{\partial k}{\partial y}(\eta_0) & \dots & 0 & \frac{\partial k}{\partial y'}(\eta_0) & \dots & 0 & \frac{\partial k}{\partial y''}(\eta_0) & \dots & 0 \\ 0 & \dots & 0 & 0 \dots & 0 & 0 & 0 \dots & 0 & \\ . & \dots & . & . & \dots & . & . & \dots & . \\ . & \dots & . & . & \dots & . & . & \dots & . \\ . & \dots & . & . & \dots & . & . & \dots & . \\ 0 & \dots & \frac{\partial k}{\partial y}(\eta_M) & 0 & \dots & \frac{\partial k}{\partial y'}(\eta_M) & 0 & \dots & \frac{\partial k}{\partial y''}(\eta_M) \end{pmatrix} \begin{pmatrix} E_0 \\ E_r \\ . \\ E_M \\ E_0' \\ E_r' \\ . \\ E_M' \\ E_0'' \\ E_r'' \\ . \\ E_M'' \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \dots & 0 & 0 & \dots & 0 & \frac{\partial f}{\partial y''}(\xi_0) & \dots & 0 \\ \frac{\partial f}{\partial y}(\xi_r) & \dots & 0 & \frac{\partial f}{\partial y'}(\xi_r) & \dots & 0 & 0 & \dots & 0 \\ . & \dots & . & . & \dots & . & . & . & \\ . & \dots & . & . & \dots & . & . & . & \\ . & \dots & . & . & \dots & . & . & . & \\ 0 & \dots & \frac{\partial f}{\partial y}(\xi_{M-2+s}) & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & \frac{\partial f}{\partial y'}(\xi_M) & 0 & \dots & \frac{\partial f}{\partial y''}(\xi_M) \\ 0 & \dots & 0 & 0 & \dots & 0 & \frac{\partial g}{\partial y''}(\eta_0) & \dots & 0 \\ \frac{\partial k}{\partial y}(\eta_r) & \dots & 0 & \frac{\partial k}{\partial y'}(\eta_r) & \dots & 0 & 0 & \dots & 0 \\ . & \dots & . & . & \dots & . & . & \dots & . \\ . & \dots & . & . & \dots & . & . & \dots & . \\ . & \dots & . & . & \dots & . & . & \dots & . \\ 0 & \dots & \frac{\partial k}{\partial y}(\eta_{M-2+s}) & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & \frac{\partial k}{\partial y'}(\eta_M) & 0 & \dots & \frac{\partial k}{\partial y''}(\eta_M) \end{pmatrix} \begin{pmatrix} E_r \\ E_1 \\ . \\ E_{M-1} \\ E_{M-2+s} \\ E_r' \\ . \\ E_{M-2+s}' \\ E_M' \\ E_0'' \\ . \\ E_{M-2+s}'' \\ E_M'' \end{pmatrix}$$

$$= J_{(4M+2)\times 6M}\mathcal{E}_{6M}.$$

In the above expressions, we have used the fact that the exact boundary conditions are known, that is, $E_0 = y(x_0) - y_0 = 0, E_0' = y'(x_0) - y_0' = 0$, and $E_M = y(x_M) - y_M = 0$. Finally, from Equation (14), we have that

$$A_{6M\times 6M}\mathcal{E}_{6M} + hV_{6M\times(4M+2)}J_{(4M+2)\times 6M}\mathcal{E}_{6M} = \mathcal{L}(h)_{6M}. \tag{16}$$

The formula in (16) may be rewritten as

$$\left(A_{6M\times 6M} + hV_{6M\times(4M+2)}J_{(4M+2)\times 6M}\right)\mathcal{E}_{6M} = \mathcal{L}(h)_{6M}, \tag{17}$$

and putting $\mathcal{M} = A + hVJ$ we have

$$\mathcal{M}_{6M\times 6M}\mathcal{E}_{6M} = \mathcal{L}(h)_{6M}. \tag{18}$$

Hence, the equation in (18) is rewritten as

$$\mathcal{E}_{6M} = \mathcal{M}^{-1}_{6M\times 6M}\mathcal{L}(h)_{6M}. \tag{19}$$

We consider the maximum norm in $\mathbb{R}^{6M}$, $\|\mathcal{E}\| = \max_i |E_i|$, and the corresponding induced matrix norm in $\mathbb{R}^{6M\times 6M}$. Then, expanding each term of $\mathcal{M}^{-1}_{6M\times 6M}$ in a series about h, and after some simplification we have $\|\mathcal{M}^{-1}_{6M\times 6M}\| = \mathcal{O}(h^{-2})$. Finally, from Equation (19) and assuming that $y(x)$ has enough bounded derivatives, we get

$$\|\mathcal{E}_{6M}\| \le \|\mathcal{M}^{-1}_{6M\times 6M}\|\ \|\mathcal{L}(h)_{6M}\| = |\mathcal{O}(h^{-2})|\ |\mathcal{O}(h^7)| \le Kh^5.$$

As we are interested only in the grid points, looking at the vector $L(h)$, we see that

$$|e_i| = |y(x_i) - y_i| \le |\mathcal{O}(h^{-2})|\ |\mathcal{O}(h^9)| \le Kh^7, \quad i = 1, 2, \dots, M,$$

which proves that the proposed method exhibits a seventh-order convergence. ■

4 | IMPLEMENTATION ISSUES

The proposed method is implemented in a block mode. We rewrite the systems in (12) as $\boldsymbol{W}(\boldsymbol{u}) = \mathbf{0}$, and the unknowns as

$$\left\{y_j, y_j', y_j''\right\}_{j=0,\dots,M} \bigcup \left\{y_{j+r}, y_{j+s}, y_{j+r}', y_{j+s}', y_{j+r}'', y_{j+s}''\right\}_{j=0,\dots,M-2}.$$

Then, we use Newton's method to solve nonlinear equations since the TSHBM is an implicit scheme. The i-step iteration of the Newton method is given by

$$\tilde{\mathbf{U}}^{\mathbf{i}+1} = \tilde{\mathbf{U}}^{\mathbf{i}} - \left(\mathbf{J}^{\mathbf{i}}\right)^{-1}\mathbf{F}^{\mathbf{i}},$$

where $\mathbf{J}$ represents the Jacobian matrix of $\mathbf{F}$. The starting values for solving the systems on each iteration are taken as those provided by the linear interpolation obtained throughout the boundary values, while the stopping criterion considers a maximum number of 100 iterations and an error between two successive approximations less than 10^{-16}.

We summarize how the TSHBM is used to give numerical solutions to third-order BVPs:

1. Let us take $M > 0 \in \mathbb{N}$, and define $h = \frac{x_M - x_0}{M}$ to generate the partition

$$P_M = \{x_0 + jh\}_{j=0,\dots,M} \bigcup \{x_0 + (c+j)h\}_{c=r,s;\ j=0,\dots,M-2}.$$

2. Using equations in (8)–(10), form the system of equations with variables

$$\left\{y_j, y_j', y_j''\right\}_{j=0,\dots,M} \bigcup \left\{y_{j+r}, y_{j+s}, y_{j+r}', y_{j+s}', y_{j+r}'', y_{j+s}''\right\}_{j=0,\dots,M-2}.$$

3. Make just one block matrix equation by joining all the equations generated in the previous step on the partition P_M with the given boundary conditions.
4. Solve the single block matrix equation to get the approximate solutions for the BVP on the whole interval $[x_0, x_M]$.

5 | NUMERICAL EXAMPLES

This section will give four numerical experiments to confirm and verify the proposed method's accuracy and efficiency. The codes considered for comparisons are:

- TSHBM: The two-step hybrid block method developed in this article.

- NPST: The non-polynomial spline technique in Reference 20.
- CSM: The cubic spline method proposed in Reference 21.
- PAM: The Padé approximation method presented in Reference 22.
- QSM: The quartic spline method in Reference 23.
- FDM: The finite difference method of algebraic order-six in Reference 24.

5.1 | Example 1

Consider the following singularly perturbed model problem in fluid mechanics and engineering

$$\begin{cases} -\epsilon y'''(x) + y(x) = 81\epsilon^2 \cos(3x) + 3\epsilon \sin(3x), \\ y(0) = 0, y(1) = 3\epsilon \sin(3), y''(0) = 0, 0 \le x \le 1. \end{cases} \tag{20}$$

The exact solution is $y(x) = 3\epsilon \sin(3x)$.

5.2 | Example 2

Consider a model BVP in References 20 and 21

$$\begin{cases} y'''(x) = xy(x) + \left(x^3 - 2x^2 - 5x - 3\right) \exp(x), \\ y(0) = 0, y'(0) = 1, y(1) = 0, 0 \le x \le 1. \end{cases} \tag{21}$$

The exact solution is given as $y(x) = x(1 - x) \exp(x)$.

5.3 | Example 3

Consider the following sandwich beam BVP given in Tirmizi et al.[4]

$$\begin{cases} y'''(x) - c^2 y'(x) + m = 0, \\ y'(0) = 0,\ y'(1) = 0,\ y\left(\frac{1}{2}\right) = 0,\ 0 \le x \le 1, \end{cases} \tag{22}$$

where $c^2 = \frac{(G_u L^2)}{(D_u A_e)(C_2 A_e - C_1^2)}$, $m = \frac{(C_1 L^3)}{(D_u A_e)}$, L is the span of the beam, u represents shear wrapping, A_e represents the effective area of cross-section of the beam, C_1 and C_2 are shear parameters, D_u is shear rigidity, and G_u is face shear moduli. The exact solution is $y(x) = \left(\frac{m}{k^3}\right)\left[\left(\sinh\left(\frac{c}{2}\right) - \sinh(cx)\right) + c\left(x - \frac{1}{2}\right) + \tanh\left(\left(\cosh(cx) - \cosh\left(\frac{c}{2}\right)\right)\right)\right]$.

5.4 | Example 4

Consider the following general third-order BVP with mixed boundary conditions

$$\begin{cases} y'''(x) = -\frac{y''(x)}{x} + \frac{y'(x)}{x^2} + \frac{1}{x}, \\ y(2) = 0, y''(1) + 0.3y'(1) = 0, y''(2) + 0.15y'(2) = 0, 1 \le x \le 2. \end{cases} \tag{23}$$

The exact solution is given as $y(x) = c_1 + c_2 \log(x) + c_3 x^2 - \frac{x^2}{4} + \frac{1}{4}x^2 \log(x)$, where $c_1 = \frac{33}{26} + \frac{\log(2)(7+26\log(2))}{21}$, $c_2 = -\frac{(26\log(2))}{21}$, $c_3 = -\frac{7}{104} - \frac{\log(2)}{3}$.

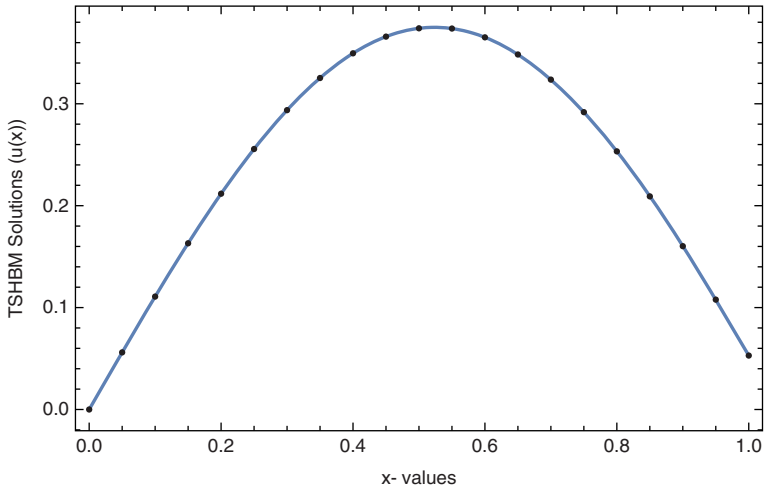

FIGURE 1 Numerical results of the TSHBM and exact solution for problem (20) taking $M = 20$

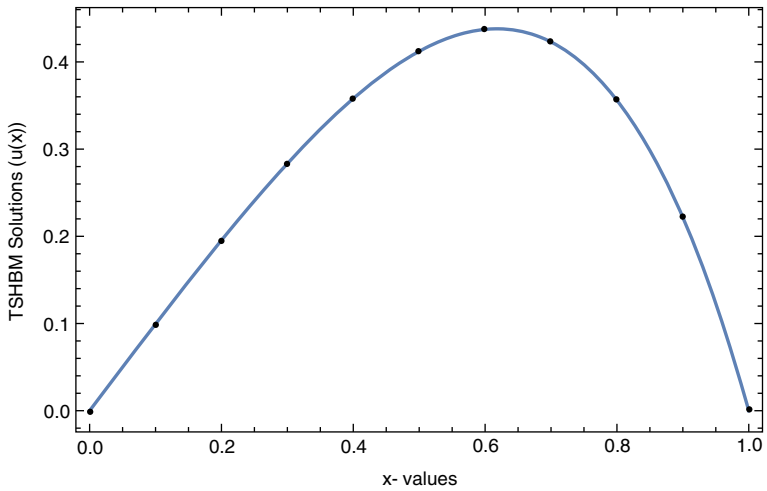

FIGURE 2 Numerical results of the TSHBM and exact solution for problem (21) taking $M = 10$

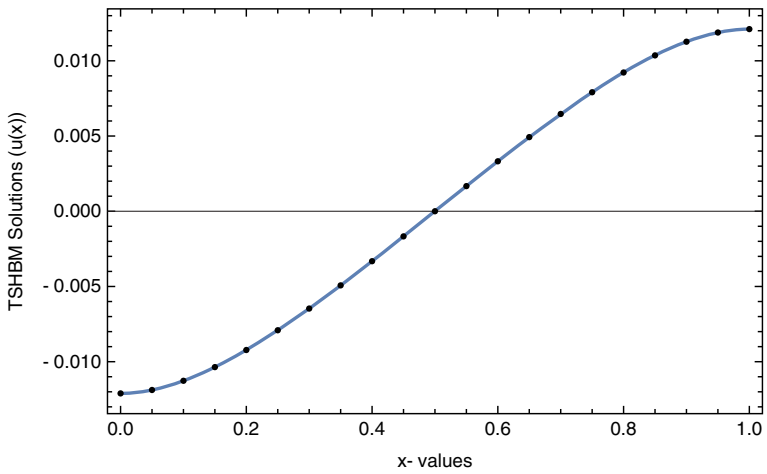

FIGURE 3 Numerical results of the TSHBM and exact solution with $m = 1, c = 5$ for problem (21) taking $M = 20$

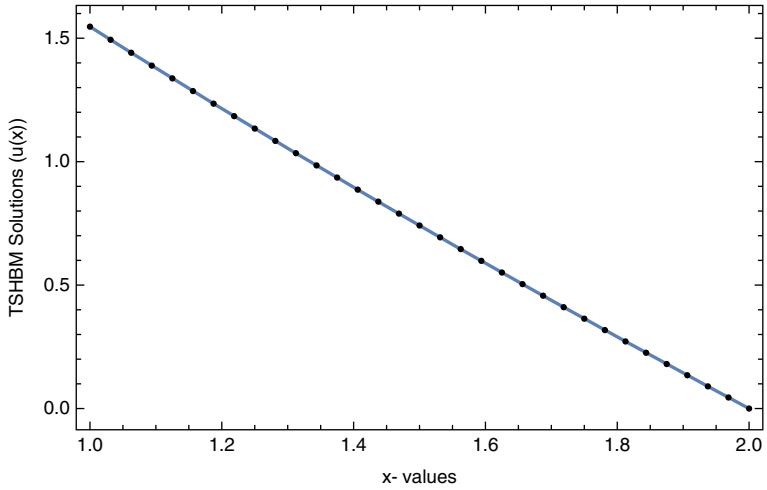

FIGURE 4 Numerical results of the TSHBM and exact solution for problem (23) taking $M = 32$

TABLE 1 Maximum absolute errors (MAXAE) for problem (20)

M	ϵ	Methods	MAXAE
10	$\frac{1}{16}$	TSHBM	2.50472×10^{-9}
20	$\frac{1}{16}$	TSHBM	3.97180×10^{-11}
10	$\frac{1}{16}$	QSM	2.50000×10^{-3}
20	$\frac{1}{16}$	QSM	1.90000×10^{-4}
10	$\frac{1}{32}$	TSHBM	9.02593×10^{-10}
20	$\frac{1}{32}$	TSHBM	1.43122×10^{-11}
10	$\frac{1}{32}$	QSM	6.80000×10^{-4}
20	$\frac{1}{32}$	QSM	5.70000×10^{-5}
10	$\frac{1}{64}$	TSHBM	3.00527×10^{-10}
20	$\frac{1}{64}$	TSHBM	4.67482×10^{-12}
10	$\frac{1}{64}$	QSM	1.20000×10^{-4}
20	$\frac{1}{64}$	QSM	1.30000×10^{-5}

5.5 | Explanation of results

This section reports the numerical results. The data obtained with the proposed TSHBM are presented in Tables 1–4. The new method has been utilized to solve various third-order problems. The numerical results are compared with the results of some existing techniques, such as the non-polynomial spline technique in Reference 20, the cubic spline method presented by Al-Said and Noor,[21] the Padé approximation method introduced by Tirmizi et al.,[4] the quartic spline method proposed by Akram,[23] and the finite difference method in Reference 24. Comparison of the theoretical versus numerical solutions obtained with the proposed TSHBM for problems (20)–(23) are displayed in Figures 1–4. The numerical results confirm that the proposed TSHBM is an efficient scheme for solving the problems considered.

6 | CONCLUDING REMARKS

This article has applied the collocation and interpolation techniques to derive a uniform seventh-order TSHBM with a fourth derivative for solving third-order BVPs of ODEs directly. The basic properties, including the convergence and stability analysis of the technique, have been well studied. Comparisons of the absolute and maximum errors show that the new approach performs better than some existing methods. The TSHBM introduced in this article has been used to solve some model third-order BVPs, and it presents good performance. Hence, it might be considered for solving third-order BVPs in ODEs.

TABLE 2 Maximum absolute error (MAXAE) for problem (21)

h	Methods	MAXAE
$\frac{1}{4}$	TSHBM	6.14072×10^{-11}
$\frac{1}{16}$	NPST	5.29920×10^{-7}
$\frac{1}{16}$	CSM	1.68610×10^{-3}
$\frac{1}{8}$	TSHBM	2.58404×10^{-13}
$\frac{1}{32}$	NPST	2.61270×10^{-8}
$\frac{1}{32}$	CSM	4.45100×10^{-4}
$\frac{1}{16}$	TSHBM	1.11022×10^{-16}
$\frac{1}{64}$	NPST	1.49990×10^{-9}
$\frac{1}{64}$	CSM	1.12930×10^{-4}

TABLE 3 Absolute errors (AE) for $m = 1, c = 5$ for problem (22)

x-value	AE with TSHBM	AE with PAM
0.0000	1.37308×10^{-11}	6.65300×10^{-5}
0.1000	1.29817×10^{-11}	6.50000×10^{-5}
0.2000	1.02789×10^{-11}	5.25400×10^{-5}
0.3000	6.81773×10^{-12}	3.63000×10^{-5}
0.4000	3.39005×10^{-12}	1.87500×10^{-5}
0.6000	3.39005×10^{-12}	1.73400×10^{-5}
0.7000	6.81773×10^{-12}	3.40500×10^{-5}
0.8000	1.02789×10^{-11}	4.98000×10^{-5}
0.9000	1.29817×10^{-11}	6.20100×10^{-5}
1.0000	1.37308×10^{-11}	6.34700×10^{-5}

TABLE 4 Maximum absolute error (MAXAE) for problem (23)

h	Methods	MAXAE
$\frac{1}{8}$	TSHBM	8.17298×10^{-7}
$\frac{1}{8}$	FDM	1.40000×10^{-5}
$\frac{1}{16}$	TSHBM	1.44941×10^{-8}
$\frac{1}{16}$	FDM	2.91000×10^{-7}
$\frac{1}{32}$	TSHBM	2.37099×10^{-10}
$\frac{1}{32}$	FDM	5.00000×10^{-9}
$\frac{1}{40}$	TSHBM	6.26244×10^{-11}
$\frac{1}{40}$	FDM	1.33000×10^{-9}

CONFLICT OF INTEREST

The authors declare no potential conflict of interests.

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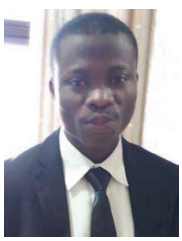

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