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The Einstein-Maxwell-Particle System in the York Canonical Basis of ADM Tetrad Gravity: III) The Post-Minkowskian N-Body Problem, its Post-Newtonian Limit in Non-Harmonic 3-Orthogonal Gauges and Dark Matter as an Inertial Effect.

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Abstract

We conclude the study of the Post-Minkowskian linearization of ADM tetrad gravity in the York canonical basis for asymptotically Minkowskian space-times in the family of non-harmonic 3-orthogonal gauges parametrized by the York time ${}^3K(\tau, \vec{\sigma})$ (the inertial gauge variable, not existing in Newton gravity, describing the general relativistic remnant of the freedom in clock synchronization in the definition of the instantaneous 3-spaces). As matter we consider only N scalar point particles with a Grassmann regularization of the self-energies and with a ultraviolet cutoff making possible the PM linearization and the evaluation of the PM solution for the gravitational field.

We study in detail all the properties of these PM space-times emphasizing their dependence on the gauge variable ${}^3\mathcal{K}_{(1)} = \frac{1}{\triangle}\, {}^3K_{(1)}$ (the non-local York time): Riemann and Weyl tensors, 3-spaces, time-like and null geodesics, red-shift and luminosity distance. Then we study the Post-Newtonian (PN) expansion of the PM equations of motion of the particles. We find that in the two-body case at the 0.5PN order there is a damping (or anti-damping) term depending only on ${}^3\mathcal{K}_{(1)}$. This open the possibility to explain dark matter in Einstein theory as a relativistic inertial effect: the determination of ${}^3\mathcal{K}_{(1)}$ from the masses and rotation curves of galaxies would give information on how to find a PM extension of the existing PN Celestial frame (ICRS) used as observational convention in the 4-dimensional description of stars and galaxies. Dark matter would describe the difference between the inertial and gravitational masses seen in the non-Euclidean 3-spaces, without a violation of their equality in the 4-dimensional space-time as required by the equivalence principle.

I. INTRODUCTION

In Refs.[1, 2], quoted as papers I and II respectively, we studied Hamiltonian ADM tetrad gravity in asymptotically Minkowskian space-times in the York canonical basis defined in Ref.[3] and its Hamiltonian Post-Minkoskian (HPM) linearization in a family of non-harmonic 3-orthogonal gauges. Since in this formulation the instantaneous 3-spaces are well defined, we have control on the general relativistic remnant of the gauge freedom in clock synchronization, whose relevance for gravitational physics will be investigated in this paper, where the matter consists only of N scalar point particles (without the transverse electro-magnetic field present in papers I and II), in the Post-Minkowskian (PM) approximation.

The definition of 3-spaces, a pre-requisite for the formulation of the Cauchy problem for the field equations, is done by using radar 4-coordinates $\sigma^A = (\sigma^\tau = \tau; \sigma^r)$, $A = \tau, r$, adapted to the admissible 3+1 splitting of the space-time and centered on an arbitrary time-like observer $x^\mu(\tau)$ (origin of the 3-coordinates σ^r): they define a non-inertial frame centered on the observer, so that they are *observer and frame- dependent*. The time variable τ is an arbitrary monotonically increasing function of the proper time given by the atomic clock carried by the observer. The instantaneous 3-spaces identified by this convention for clock synchronization are denoted Σ_τ. The transformation $\sigma^A \mapsto x^\mu = z^\mu(\tau, \sigma^r)$ to world 4-coordinates defines the embedding $z^\mu(\tau, \vec{\sigma})$ of the Riemannian instantaneous 3-spaces Σ_τ into the space-time. By choosing world 4-coordinates centered on the time-like observer, whose world-line is the time axis, we have $x^\mu(\tau) = (x^o(\tau); 0)$: the condition $x^o(\tau) = const.$ is equivalent to $\tau = const.$ and identifies the instantaneous 3-space Σ_τ. If the time-like observer coincides with an asymptotic inertial observer $x^\mu(\tau) = x^\mu_o + \epsilon^\mu_\tau\, \tau$ with $\epsilon^\mu_\tau = (1; 0)$, $\epsilon^\mu_r = (0; \delta^i_r)$, $x^\mu_o = (x^o_o; 0)$, then the natural embedding describing the given 3+1 splitting of space-time is $z^\mu(\tau, \sigma^r) = x^\mu_o + \epsilon^\mu_A\, \sigma^A$ and the world 4-metric is ${}^4g_{\mu\nu} = \epsilon^A_\mu\, \epsilon^B_\nu\, {}^4g_{AB}$ (ϵ^A_μ are flat asymptotic cotetrads, $\epsilon^A_\mu\, \epsilon^\mu_B = \delta^A_B$, $\epsilon^A_\mu\, \epsilon^\nu_A = \delta^\mu_\nu$).

From now on we shall denote the curvilinear 3-coordinates σ^r with the notation $\vec{\sigma}$ for the sake of simplicity. Usually the convention of sum over repeated indices is used, except when there are too many summations.

The 4-metric ${}^4g_{AB}$ has signature $\epsilon\, (+ - - -)$ with $\epsilon = \pm$ (the particle physics, $\epsilon = +$, and general relativity, $\epsilon = -$, conventions). Flat indices (α), $\alpha = o, a$, are raised and lowered by the flat Minkowski metric ${}^4\eta_{(\alpha)(\beta)} = \epsilon\, (+ - - -)$. We define ${}^4\eta_{(a)(b)} = -\epsilon\, \delta_{(a)(b)}$ with a positive-definite Euclidean 3-metric. On each instantaneous 3-space Σ_τ we have that the 4-metric has a direction-independent limit to the flat Minkowski 4-metric (the asymptotic background) at spatial infinity ${}^4g_{AB}(\tau, \vec{\sigma}) \rightarrow {}^4\eta_{AB(asym)} = \epsilon\, (+ - - -)$.

After a review of the York canonical basis and of the HPM linearization in Subsections A and B respectively, we will outline the new results of this paper in Subsection C.

A. The York Canonical Basis

In the York canonical basis of ADM tetrad gravity of paper I

$$
\begin{array}{|ll|ll|l|l|l|}
\hline
\varphi_{(a)} & \alpha_{(a)} & n & \bar{n}_{(a)} & \theta^r & \tilde{\phi} & R_{\bar{a}} \\
\hline
\pi_{\varphi_{(a)}} \approx 0 & \pi^{(\alpha)}_{(a)} \approx 0 & \pi_n \approx 0 & \pi_{\bar{n}_{(a)}} \approx 0 & \pi^{(\theta)}_r & \pi_{\tilde{\phi}} & \Pi_{\bar{a}} \\
\hline
\end{array}
\qquad (1.1)
$$

the family of non-harmonic 3-orthogonal gauges is the family of Schwinger time gauges where we have

$$
\begin{aligned}
&\alpha_{(a)}(\tau,\vec{\sigma}) \approx 0, \qquad \varphi_{(a)}(\tau,\vec{\sigma}) \approx 0,\\
&\theta^i(\tau,\vec{\sigma}) \approx 0, \qquad \pi_{\tilde{\phi}}(\tau,\vec{\sigma}) = \frac{c^3}{12\pi\, G}\, {}^3K(\tau,\vec{\sigma}) \approx \frac{c^3}{12\pi\, G}\, F(\tau,\vec{\sigma}),
\end{aligned}
\tag{1.2}
$$

where $F(\tau,\vec{\sigma})$ is an arbitrary numerical function parametrizing the residual gauge freedom in clock synchronization, namely in the choice of the non-dynamical aspect of the instantaneous 3-spaces Σ_τ.

In the York canonical basis we have (${}^4E^A_{(\alpha)}$ are arbitrary tetrads; ${}^4\overset{\circ}{E}{}^A_{(\alpha)}$ and ${}^4\overset{\circ}{\bar{E}}{}^A_{(\alpha)}$, ${}^4\overset{\circ}{E}{}^A_{(a)} = R_{(a)(b)}(\alpha_{(e)})\, {}^4\overset{\circ}{\bar{E}}{}^A_{(b)}$, are tetrads adapted to the 3-spaces; ${}^3e^r_{(a)}$, ${}^3e_{(a)r}$ and ${}^3\bar{e}^r_{(a)}$, ${}^3\bar{e}_{(a)r}$, with ${}^3e^r_{(a)} = R_{(a)(b)}(\alpha_{(e)})\, {}^3\bar{e}^r_{(b)}$, ${}^3e_{(a)r} = R_{(a)(b)}(\alpha_{(e)})\, {}^3\bar{e}_{(b)r}$, are triads and cotriads on the 3-spaces Σ_τ; for the shift function we have $n_{(a)} = R_{(a)(b)}(\alpha_{(e)})\, \bar{n}_{(a)}$)

$$
\begin{aligned}
{}^4E^A_{(\alpha)} &= {}^4\overset{\circ}{\bar{E}}{}^A_{(o)}\, L^{(o)}{}_{(\alpha)}(\varphi_{(c)}) + {}^4\overset{\circ}{\bar{E}}{}^A_{(b)}\, R^T_{(b)(a)}(\alpha_{(c)})\, L^{(a)}{}_{(\alpha)}(\varphi_{(c)}) \approx {}^4\overset{\circ}{\bar{E}}{}^A_{(o)},\\
{}^4\overset{\circ}{\bar{E}}{}^A_{(o)} &= {}^4\overset{\circ}{E}{}^A_{(o)} = \frac{1}{1+n}\,(1; -\bar{n}_{(a)}\, {}^3\bar{e}^r_{(a)}) = l^A, \qquad {}^4\overset{\circ}{\bar{E}}{}^A_{(a)} = (0; {}^3\bar{e}^r_{(a)}) \approx {}^4\overset{\circ}{E}{}^A_{(a)},\\
{}^4\overset{\circ}{\bar{E}}{}^{(o)}_A &= {}^4\overset{\circ}{E}{}^{(o)}_A = (1+n)\,(1;\vec{0}) = \epsilon\, l_A, \qquad {}^4\overset{\circ}{\bar{E}}{}^{(a)}_A = (\bar{n}_{(a)}; {}^3\bar{e}_{(a)r}) \approx {}^4\overset{\circ}{E}{}^{(a)}_A,\\
{}^4g_{AB} &= {}^4E^{(\alpha)}_A\, {}^4\eta_{(\alpha)(\beta)}\, {}^4E^{(\beta)}_B = {}^4\overset{\circ}{E}{}^{(\alpha)}_A\, {}^4\eta_{(\alpha)(\beta)}\, {}^4\overset{\circ}{E}{}^{(\beta)}_B = {}^4\overset{\circ}{\bar{E}}{}^{(\alpha)}_A\, {}^4\eta_{(\alpha)(\beta)}\, {}^4\overset{\circ}{\bar{E}}{}^{(\beta)}_B,\\
{}^4g_{\tau\tau} &= \epsilon\, \Big[(1+n)^2 - \sum_a \bar{n}^2_{(a)}\Big],\\
{}^4g_{\tau r} &= -\epsilon\, \sum_a \bar{n}_{(a)}\, {}^3\bar{e}_{(a)r} = -\epsilon\, \tilde{\phi}^{1/3}\, Q_r,\\
{}^4g_{rs} &= -\epsilon\, {}^3g_{rs} = -\epsilon\, \sum_a {}^3\bar{e}_{(a)r}\, {}^3\bar{e}_{(a)s} = -\epsilon\, \phi^4\, {}^3\hat{g}_{rs} \,=\, -\epsilon\, \tilde{\phi}^{2/3}\, Q^2_r\, \delta_{rs},\\
Q_a &= e^{\Gamma^{(1)}_a} = e^{\sum_{\bar{a}}^{1,2} \gamma_{\bar{a}a}\, R_{\bar{a}}}, \qquad \tilde{\phi} = \phi^6 = \sqrt{\gamma} = \sqrt{det\, {}^3g} = {}^3\bar{e},\\
{}^3\bar{e}_{(a)r} &= \tilde{\phi}^{1/3}\, Q_a\, \delta_{ra}, \qquad {}^3\bar{e}^r_{(a)} = \tilde{\phi}^{-1/3}\, Q_a^{-1}\, \delta_{ra}.
\end{aligned}
\tag{1.3}
$$

The set of numerical parameters $\gamma_{\bar{a}a}$ satisfies [3] $\sum_u \gamma_{\bar{a}u} = 0$, $\sum_u \gamma_{\bar{a}u}\, \gamma_{\bar{b}u} = \delta_{\bar{a}\bar{b}}$, $\sum_{\bar{a}} \gamma_{\bar{a}u}\, \gamma_{\bar{a}v} = \delta_{uv} - \frac{1}{3}$. A different York canonical basis is associated to each solution of these equations. Let us remember [4] that to avoid coordinate singularities we must always have $N(\tau,\vec{\sigma}) = 1 + n(\tau,\vec{\sigma}) > 0$ (3-spaces at different times do not intersect each other), $\epsilon\, {}^4g_{\tau\tau}(\tau,\vec{\sigma}) > 0$ (no rotating disk pathology) and ${}^3g_{rs}(\tau,\vec{\sigma})$ with three distinct positive eigenvalues (Møller conditions).

B. The HPM Linearization

The standard decomposition used for the weak field approximation in the harmonic gauges is

$$
{}^4g_{\mu\nu} \,=\, {}^4\eta_{\mu\nu} + h_{\mu\nu}, \qquad |h_{\mu\nu}|, |\partial_\alpha\, h_{\mu\nu}|, |\partial_\alpha\, \partial_\beta\, h_{\mu\nu}| << 1, \tag{1.4}
$$

where ${}^4\eta_{\mu\nu}$ is the flat metric in an inertial frame of the background Minkowski space-time. This is equivalent to take a 3+1 splitting of our space-time with an inertial foliation, having Euclidean instantaneous 3-spaces, against the equivalence principle and against the fact (explicitly shown in paper I) that each solution of Einstein's equations has an associated dynamically selected preferred 3+1 splitting. The final $x^o = constant$ 3-spaces have a non-zero extrinsic curvature but are not well defined in this formalism and depend on the initial data at $\tau \rightarrow -\infty$ needed to solve the wave equations in the linearized harmonic gauges for the lapse and shift functions (see paper II).

Instead the HPM-linearization of paper II of the Hamilton-Dirac equations in the (non-harmonic) 3-orthogonal Schwinger time gauges (1.2) uses as background the asymptotic Minkowski 4-metric existing in our asymptotically Minkowskian space-times. By using radar 4-coordinates adapted to an admissible 3+1 splitting of space-time, we put

$$
{}^4g_{AB}(\tau, \sigma^r) \,=\, {}^4g_{(1)AB}(\tau, \sigma^r) + O(\zeta^2) \,\rightarrow {}^4\eta_{AB(asym)} \; at\, spatial\, infinity,
$$

$$
\begin{aligned}
&{}^4g_{(1)AB}(\tau, \sigma^r) = {}^4\eta_{AB(asym)} + {}^4h_{(1)AB}(\tau, \sigma^r),\\
&{}^4h_{(1)AB}(\tau, \sigma^r) = O(\zeta) \rightarrow 0 \; at\, spatial\, infinity,
\end{aligned} \tag{1.5}
$$

where $\zeta << 1$ is a small dimensionless parameter, the small perturbation ${}^4h_{(1)AB}$ has no intrinsic meaning in the bulk and ${}^3g_{(1)rs}(\tau, \sigma^r) \,=\, -\epsilon\, {}^4g_{(1)rs}(\tau, \sigma^r)$ is the positive-definite 3-metric on the instantaneous (non-Euclidean) 3-space Σ_τ.

The asymptotic 4-metric allows to define both a flat d'Alambertian $\Box = \partial_\tau^2 - \triangle$ and a flat Laplacian $\triangle = \sum_r \partial_r^2$ on Σ_τ ($\partial_A = \frac{\partial}{\partial\, \sigma^A}$). We will also need the flat distribution $c(\vec{\sigma}, \vec{\sigma}^{\,\prime}) = \frac{1}{\Delta}\, \delta^3(\vec{\sigma}, \vec{\sigma}^{\,\prime}) = -\frac{1}{4\pi}\, \frac{1}{|\vec{\sigma} - \vec{\sigma}^{\,\prime}|}$ with $|\vec{\sigma} - \vec{\sigma}^{\,\prime}| = \sqrt{\sum_u (\sigma^u - \sigma^{\prime\, u})^2}$, where $\delta^3(\vec{\sigma}, \vec{\sigma}^{\,\prime})$ is the Dirac delta function on the 3-manifold Σ_τ.

Therefore we will solve the constraints and the Hamilton-Dirac equations in a fictitious Euclidean inertial 3-space identified by the asymptotic Minkowski metric and the solution will describe the gravitational field in our well defined Riemannian 3-spaces modulo corrections of order $O(\zeta^2)$. The instantaneous 3-spaces will deviated from flat Euclidean 3-spaces by curvature effects of order $O(\zeta)$, in accord with the equivalence principle.

We assume that the dimensionless configurational tidal variables $R_{\bar{a}}$ in the York canonical basis satisfy the following conditions

$$|R_{\bar a}(\tau,\vec\sigma) = R_{(1)\bar a}(\tau,\vec\sigma)| = O(\zeta) << 1,$$

$$|\partial_u\, R_{\bar a}(\tau,\vec\sigma)| \sim \frac{1}{L} O(\zeta), \qquad |\partial_u\, \partial_v\, R_{\bar a}(\tau,\vec\sigma)| \sim \frac{1}{L^2} O(\zeta),$$
$$|\partial_\tau\, R_{\bar a}| = \frac{1}{L}\, O(\zeta), \qquad |\partial_\tau^2\, R_{\bar a}| = \frac{1}{L^2}\, O(\zeta), \qquad |\partial_\tau\, \partial_u\, R_{\bar a}| = \frac{1}{L^2}\, O(\zeta),$$

$$\Rightarrow\;\; Q_a(\tau,\vec\sigma) = e^{\sum_{\bar a}\, \gamma_{\bar a a}\, R_{\bar a}(\tau,\vec\sigma)} = 1 + \Gamma_a^{(1)}(\tau,\vec\sigma) + O(\zeta^2),$$
$$\Gamma_a^{(1)} = \sum_{\bar a} \gamma_{\bar a a}\, R_{\bar a}, \qquad \sum_a \Gamma_a^{(1)} = 0, \qquad R_{\bar a} = \sum_a \gamma_{\bar a a}\, \Gamma_a^{(1)}, \tag{1.6}$$

where L is a *big enough characteristic length interpretable as the reduced wavelength* $\lambda/2\pi$ *of the resulting gravitational waves (GW)*. Therefore the tidal variables $R_{\bar a}$ are slowly varying over the length L and times L/c. This also implies that the Riemann tensor ${}^4R_{ABCD}$, the Ricci tensor ${}^4R_{AB}$ and the scalar 4-curvature ${}^4R$ behave as $\frac{1}{L^2}\, O(\zeta)$. Also the intrinsic 3-curvature scalar of the instantaneous 3-spaces Σ_τ is of order $\frac{1}{L^2}\, O(\zeta)$. To simplify the notation we use $R_{\bar a}$ for $R_{(1)\bar a}$ in the rest of the paper. As shown in paper II, this condition defines a weak field approximation.

Eq.(1.5) can be implemented if we add the following assumptions

$$\begin{aligned}
\tilde\phi &= \phi^6 = \sqrt{det\, {}^3g_{rs}} = 1 + 6\, \phi_{(1)} + O(\zeta^2),\\
N &= 1 + n = 1 + n_{(1)} + O(\zeta^2), \qquad \bar n_{(a)} = \bar n_{(1)(a)} + O(\zeta^2),
\end{aligned}$$

$$\Downarrow$$

$$\begin{aligned}
{}^4g_{(1)\tau\tau} &= \epsilon + {}^4h_{(1)\tau\tau} = \epsilon\, (1 + 2\, n_{(1)}) = \epsilon + O(\zeta),\\
{}^4g_{(1)\tau r} &= {}^4h_{(1)\tau r} = - \epsilon\, \bar n_{(1)(r)} = O(\zeta),\\
{}^4g_{(1)rs} &= - \epsilon\, \delta_{rs} + {}^4h_{(1)rs} = - \epsilon\, [1 + 2\, (\Gamma_r^{(1)} + 2\, \phi_{(1)})]\, \delta_{rs} = - \epsilon\, \delta_{rs} + O(\zeta),
\end{aligned} \tag{1.7}$$

while the triads and cotriads become ${}^3\bar e^r_{(1)(a)} = \delta^r_a\, (1 - \Gamma_r^{(1)} - 2\, \phi_{(1)}) + O(\zeta^2)$ and ${}^3\bar e_{(1)(a)r} = \delta_{ra}\, (1 + \Gamma_r^{(1)} + 2\, \phi_{(1)}) + O(\zeta^2)$, respectively.

As shown in paper II, with these assumptions we have [1]

$$\begin{aligned}
\frac{8\pi\, G}{c^3}\, \Pi_{\bar a}(\tau,\vec\sigma) &= \frac{8\pi\, G}{c^3}\, \Pi_{(1)\bar a}(\tau,\vec\sigma) = \frac{1}{L}\, O(\zeta) \stackrel{\circ}{=} \Big[\partial_\tau\, R_{\bar a} - \sum_a \gamma_{\bar a a}\, \partial_a\, \bar n_{(1)(a)}\Big](\tau,\vec\sigma) + \frac{1}{L}\, O(\zeta^2),\\
&\sigma_{(a)(a)} = \sigma_{(1)(a)(a)} = - \frac{8\pi\, G}{c^3}\, \sum_{\bar a} \gamma_{\bar a a}\, \Pi_{(1)\bar a} + \frac{1}{L}\, O(\zeta^2).
\end{aligned} \tag{1.8}$$

[1] Let us remark that everywhere $\Pi_{(1)\bar a}$ appears in the combination $\frac{G}{c^3}\, \Pi_{(1)\bar a} = \frac{1}{L}\, O(\zeta)$, which behaves like $\partial_\tau\, R_{\bar a}$, i.e. it varies slowly over L.

where $\sigma_{(a)(a)}$ are the diagonal elements of the shear $\sigma_{(a)(b)}$ of the congruence of Eulerian observers, whose 4-velocity is the unit normal to the 3-spaces Σ_τ as 3-sub-manifolds of space-time (see Subsection IID of paper I). For the non-diagonal elements of the shear, for the momenta $\pi_i^{(\theta)}$ and for the extrinsic curvature the assumptions of paper II are

$$\sigma_{(a)(b)}|_{a\neq b} = \sigma_{(1)(a)(b)}|_{a\neq b} = \frac{1}{L}\, O(\zeta),$$
$$\Rightarrow \quad \frac{8\pi\, G}{c^3}\, \pi_i^{(\theta)} = \frac{1}{L}\, O(\zeta^2) = \sum_{a\neq b} (\Gamma_a^{(1)} - \Gamma_b^{(1)})\, \epsilon_{iab}\, \sigma_{(1)(a)(b)} + \frac{1}{L}\, O(\zeta^3),$$

$${}^3K = \frac{12\pi\, G}{c^3}\, \pi_{\tilde{\phi}} = {}^3K_{(1)} = \frac{12\pi\, G}{c^3}\, \pi_{(1)\tilde{\phi}} = \frac{1}{L}\, O(\zeta),$$

$$\Downarrow$$

$$\begin{aligned} {}^3K_{rs} &= {}^3K_{(1)rs} = \frac{1}{L}\, O(\zeta) = \\ &= (1-\delta_{rs})\, \sigma_{(1)(r)(s)} + \delta_{rs}\, \Big[\frac{1}{3}\, {}^3K_{(1)} - \partial_\tau\, \Gamma_r^{(1)} + \sum_a (\delta_{ra} - \frac{1}{3})\, \partial_a\, \bar{n}_{(1)(a)}\Big] + \frac{1}{L}\, O(\zeta^2). \end{aligned} \tag{1.9}$$

Let us now consider our matter, i.e. positive-energy scalar particles described by the 3-coordinates $\eta_i^r(\tau)$, $i = 1,..,N$, such that their world-lines are $x_i^\mu(\tau) = z^\mu(\tau, \vec{\eta}_i(\tau))$. $\kappa_{ir}(\tau)$ are the canonically conjugate 3-momenta. We have $\eta_i^r(\tau) = O(1)$ and $\dot{\eta}_i^r(\tau) = \frac{d\,\eta_i^r(\tau)}{d\tau} \stackrel{def}{=} \frac{v_i^r(t)}{c} = O(1)$ since $\tau = c\,t$ (in the non-relativistic limit we have $\dot{\vec{\eta}}_i = \vec{v}_i/c = O(1) \rightarrow_{c\rightarrow\infty} 0$).

As shown in paper II, to get a consistent approximation we must introduce a *ultraviolet cutoff* M on the masses and momenta of the particles so that the mass density and the mass current density (see the next Section for the energy-momentum of the particles) satisfy the following requirements

$$\begin{aligned} \mathcal{M}(\tau, \vec{\sigma}) &= \mathcal{M}^{(UV)}_{(1)}(\tau, \vec{\sigma}) + \mathcal{R}_{(2)}(\tau, \vec{\sigma}), \\ & m_i = M\, O(\zeta), \qquad \int d^3\sigma\, \mathcal{M}^{(UV)}_{(1)}(\tau, \vec{\sigma}) = M c\, O(\zeta), \qquad \int d^3\sigma\, \mathcal{R}_{(2)}(\tau, \vec{\sigma}) = M c\, O(\zeta^2), \\ \mathcal{M}_r(\tau, \vec{\sigma}) &= \mathcal{M}_{(1)r}(\tau, \vec{\sigma}), \qquad \int d^3\sigma\, \mathcal{M}^{(UV)}_{(1)r}(\tau, \vec{\sigma}) = M c\, O(\zeta). \end{aligned} \tag{1.10}$$

Here M is a finite mass defining the ultraviolet cutoff: $M\, c^2$ gives an estimate of the weak ADM energy of the 3-universe contained in the instantaneous 3-spaces Σ_τ. The associated length scale is the gravitational radius $R_M = 2M\, \frac{G}{c^2} \approx 10^{-29}\, M$.

The description of particles in our approximation will be reliable only if their masses and momenta are less of $M c\, O(\zeta)$ and at distances r from the particles satisfying $r = |\vec{\sigma} - \vec{\eta}_i(\tau)| >> R_M$ (that is at each instant we must enclose each particle in a sphere of radius R_M and our approximation is not reliable inside these spheres).

Therefore for the particles the validity of the weak field approximation requires

$$\vec{\eta}_i(\tau) = O(1), \qquad \frac{\vec{\kappa}_i(\tau)}{m_i c} = O(1), \qquad \frac{\vec{\kappa}_i(\tau)}{M c} = O(\zeta), \qquad \frac{m_i}{M} \leq O(\zeta). \tag{1.11}$$

Our results are equivalent to a re-summation of the post-Newtonian expansions valid for small rest masses still having relativistic velocities ($\frac{\vec{\kappa}_i^2}{m_i^2 c^2} = O(1)$, $\frac{\vec{v}_i}{c} = O(1)$).

Since, as said in Subsection IIE of paper I, we have that the matter energy-momentum tensor satisfies $\nabla_A T^{AB}(\tau, \vec{\sigma}) \equiv 0$ due to the Bianchi identities and since ${}^4g_{AB} = {}^4\eta_{AB(asym)} + O(\zeta)$, we must have $\partial_A T^{AB}_{(1)}(\tau, \vec{\sigma}) \equiv 0 + \partial_A \mathcal{R}^{AB}_{(2)}$. At the lowest order this implies

$$\begin{aligned}
&\partial_\tau \mathcal{M}^{(UV)}_{(1)} + \partial_r \mathcal{M}^{(UV)}_{(1)r} = 0 + \partial_A \mathcal{R}^{A\tau}_{(2)},\\
&\partial_\tau \mathcal{M}^{(UV)}_{(1)r} + \partial_s T^{rs}_{(1)} = 0 + \partial_A \mathcal{R}^{Ar}_{(2)},
\end{aligned} \tag{1.12}$$

as in inertial frames in Minkowski space-time. The equation $\partial_A T^{AB}_{(1)}(\tau, \vec{\sigma}) \equiv 0 + \partial_A \mathcal{R}^{AB}_{(2)}$ implies $\partial_A \Big(T^{AB}_{(1)}(\tau, \vec{\sigma})\, \sigma^C - T^{AC}_{(1)}(\tau, \vec{\sigma})\, \sigma^B\Big) \equiv 0 + \partial_A \mathcal{R}^{ABC}_{(2)}$ (angular momentum conservation).

In conclusion, since the weak field linearized solution can be trusted only at distances $d >> R_M$ from the particles, the GW's described by our linearization must have a wavelength satisfying $\lambda \approx L > d >> R_M$ (with the weak field approximation we have $\lambda << {}^4\mathcal{R}$ without the slow motion assumption).

If all the particles are contained in a compact set of radius l_c (the source), the frequency $\nu = \frac{c}{\lambda}$ of the emitted GW's will be of the order of the typical frequency ω_s of the motion inside the source, where the typical velocities are of the order $v \approx \omega_s\, l_c$. As a consequence we get $\nu = \frac{c}{\lambda} \approx \omega_s \approx v/l_c$ or $\lambda \approx \frac{c}{v}\, l_c >> R_M$, so that we get $\frac{v}{c} \approx \frac{l_c}{\lambda} << \frac{l_c}{R_M}$ and $l_c >> R_M$ if $\frac{v}{c} = O(1)$.

If the velocities of the particles become non-relativistic, i.e. in the slow motion regime with $v << c$ (for binary systems with total mass m and held together by weak gravitational forces we have also $\frac{v}{c} \approx \sqrt{\frac{R_m}{l_c}} << 1$), we have $\lambda >> l_c$ and we can have $l_c \approx R_M$.

As shown in paper II, this HPM linearization allows to get a consistent description of GW's in non-harmonic 3-orthogonal gauges reproducing their known properties in harmonic gauges.

C. Outline of the Paper

In this paper we look in detail at the properties of the PM space-times identified by our HPM solution and we will study the equations of motion of the particles. It will be shown how all the relevant gravitational quantities depend upon the York time, which is the general relativistic remnant of the special relativistic gauge freedom in clock synchronization. It will turn out that they (with the only exception of the ADM Lorentz generators) depend upon the gradients of the spatially non-local function ${}^3\mathcal{K}_{(1)}(\tau, \vec{\sigma}) = \frac{1}{\triangle}\, {}^3K_{(1)}(\tau, \vec{\sigma})$ (the non-local York time) of the lowest order component ${}^3K_{(1)}(\tau, \vec{\sigma})$ of the York time. This study will be done in our family of 3-orthogonal gauges, where the Riemannian instantaneous 3-spaces Σ_τ have a diagonal 3-metric but still depend on the arbitrary numerical function $F(\tau, \vec{\sigma})$ determining the inertial gauge variable ${}^3K_{(1)}(\tau, \vec{\sigma})$.

We will determine the explicit dependence of the proper time of time-like observers, of the time-like and null geodesics, of the redshift of light and of the luminosity distance upon the York time in these PM space-times.

Then we will study the consequences of the HPM linearization on the equations of motion for the particles and we will make their Post-Newtonian (PN) expansion at all $\frac{n}{2}PN$ orders. In particular we will show that at the astrophysical level there is a 0.5PN contribution to dark matter coming from the relativistic inertial effect connected to the choice of the function ${}^3\mathcal{K}_{(1)}$. In the non-Euclidean 3-space there is an effective inertial mass different from the gravitational mass, even if the equality on inertial and gravitational masses holds in the 4-dimensional space-time in accord with the equivalence principle.

In Section II we rewrite the solutions of the constraints and of the equations for the lapse and shift functions (all determined by elliptic equations inside the 3-spaces Σ_τ) in 3-orthogonal gauges, given in paper II, restricting them to the case in which the matter consists only of point particles. An equal time development of the retarded solution for the GW's is also given. Then, after the expression of the ADM Poincare' generators we give the Christoffel symbols and the Riemann and Weyl tensors of the PM space-times. Finally the expression of the 4-spin and 3-spin connections and the expression of the Ashtekar's variables in the York canonical basis are given.

In Section III we show properties of the PM space-times and of their Riemannian 3-spaces in 3-orthogonal gauges: the proper time of time-like observers; the 3-distance, the 3-curvature and the extrinsic curvature of 3-spaces; the PM time-like 4-geodesics. Also the comparison of the HPM solution for the 4-metric in 3-orthogonal gauges with the astronomical conventions for the 4-metric of the Solar System in a suitable harmonic gauge is given.

Section IV is devoted to the PM null geodesics, the red-shift, the geodesic deviation equation and the luminosity distance of PM space-times.

In Section V we give the PM equations of motion for the particles and discuss their qualitative properties and differences from the case of charged particles plus the electromagnetic field in Minkowski space-time. After a discussion of the problem of the center of mass and of the relative variables in the PM space-times (using the two-body problem as an example), we study the PN expansion of the equations of motion and we compare a 1PN binary system in 3-orthogonal gauges with the standard one in harmonic gauges. In

Appendix A there is the 1PN expression of the ADM Poincare' generators with terms till order $O(\zeta^2)$ included.

In Section VI we show that the non-local York time allows to describe the dark matter present in the masses of galaxies and their rotation curves as a relativistic inertial effect absent in Newton gravity.

In the Conclusions we comment on the gauge problem in GR, discussed in the Conclusions of paper II, and on how our explanation of dark matter may help in finding the PM extension of the Celestial reference frame (ICRS) [5] outside the Solar System. Then we show which lines of development are opened by our formulation, especially in cosmology and in particular for the possibility to reformulate the back-reaction approach for the elimination of dark energy, based on averages of scalar quantities in 3-volumes of 3-space, in the York canonical basis.

II. THE PM SOLUTION FOR THE GRAVITATIONAL FIELD

In this Section we review the results of paper II when the matter consists only of point particles. At this order the HPM linearized solution in the family of 3-orthogonal gauges depends on the York time ${}^3K$ only through the following function ${}^3\mathcal{K} = \frac{1}{\triangle}\,{}^3K$. Only the linearized ADM Lorentz generators have also a dependence on ${}^3K$. We give also the equal time development of the retarded solution for GW's. The metric, Christoffel symbols, spin connection, Riemann and Weyl tensors of PM space-times are given. Finally Ashtekar's variables are expressed in the York canonical basis and their PM limit is found.

A. The Energy-Momentum of the Particles

From Eqs.(3.9), (3.10) and (3.12) of paper II we get the following expression for the energy-momentum of the particles (in the following equations the notation $\frac{M}{L^3}\,O(\zeta^2)$ means $\sum_{i=1}^N\,\delta^3(\vec{\sigma},\vec{\eta}_i(\tau))\,M\,O(\zeta^2)$)

$$\begin{aligned}
\mathcal{M}^{(UV)}_{(1)}(\tau,\vec{\sigma}) \;&=\; T^{\tau\tau}_{(1)}(\tau,\vec{\sigma}) = \sum_{i=1}^N\,\delta^3(\vec{\sigma},\vec{\eta}_i(\tau))\,\eta_i\,\sqrt{m_i^2\,c^2+\vec{\kappa}_i^2(\tau)} =\\
&= \sum_{i=1}^N\,\delta^3(\vec{\sigma},\vec{\eta}_i(\tau))\,\eta_i\,\frac{m_i c}{\sqrt{1-\dot{\vec{\eta}}_i^2(\tau)}} + \frac{M}{L^3}\,O(\zeta^2),\\
&\quad M_{(1)}c = q^{|\tau\tau} = \sum_{i=1}^N\,\eta_i\,\sqrt{m_i^2c^2+\vec{\kappa}_i^2(\tau)} = \sum_{i=1}^N\,\eta_i\,\frac{m_i c}{\sqrt{1-\dot{\vec{\eta}}_i^2(\tau)}} + M\,O(\zeta^2),
\end{aligned}$$

$$\begin{aligned}
\mathcal{M}^{(UV)}_{(2)}(\tau,\vec{\sigma}) \;&=\; -\sum_{i=1}^N\,\delta^3(\vec{\sigma},\vec{\eta}_i(\tau))\,\eta_i\,\Big(\frac{2\,\phi_{(1)}\,\vec{\kappa}_i^2(\tau)+\sum_a\,\Gamma_a^{(1)}\,\kappa_{ia}^2(\tau)}{\sqrt{m_i^2\,c^2+\vec{\kappa}_i^2(\tau)}}\Big)(\tau,\vec{\sigma}) =\\
&= -\sum_{i=1}^N\,\delta^3(\vec{\sigma},\vec{\eta}_i(\tau))\,\eta_i\,m_i\,c\,\Big(\frac{2\,\phi_{(1)}\,\dot{\vec{\eta}}_i^2(\tau)+\sum_a\,\Gamma_a^{(1)}\,(\dot{\eta}_i^a(\tau))^2(\tau)}{\sqrt{1-\dot{\vec{\eta}}_i^2(\tau)}}\Big)(\tau,\vec{\sigma}),
\end{aligned}$$

$$\begin{aligned}
\mathcal{M}^{(UV)}_{(1)r}(\tau,\vec{\sigma}) \;&=\; T^{\tau r}_{(1)}(\tau,\vec{\sigma}) = \sum_{i=1}^N\,\delta^3(\vec{\sigma},\vec{\eta}_i(\tau))\,\eta_i\,\kappa_{ir}(\tau) = \sum_{i=1}^N\,\delta^3(\vec{\sigma},\vec{\eta}_i(\tau))\,\eta_i\,\frac{m_i c\,\dot{\eta}_i^r(\tau)}{\sqrt{1-\dot{\vec{\eta}}_i^2(\tau)}} + M\,O(\zeta^2),\\
&\quad q^{r|\tau s} = \sum_{i=1}^N\,\eta_i\,\eta_i^r(\tau)\,\kappa_{is}(\tau) = \sum_{i=1}^N\,\eta_i\,\frac{m_i c\,\eta_i^r(\tau)\,\dot{\eta}_i^s(\tau)}{\sqrt{1-\dot{\vec{\eta}}_i^2(\tau)}} + M\,O(\zeta^2),
\end{aligned}$$

$$
\begin{aligned}
T^{rs}_{(1)}(\tau,\vec{\sigma}) &= \sum_{i=1}^N \delta^3(\vec{\sigma},\vec{\eta}_i(\tau))\, \eta_i\, \frac{\kappa_{ir}(\tau)\,\kappa_{is}(\tau)}{\sqrt{m_i^2\, c^2 + \vec{\kappa}_i^2(\tau)}} = \\
&= \sum_{i=1}^N \delta^3(\vec{\sigma},\vec{\eta}_i(\tau))\, \eta_i\, \frac{m_i c\, \dot{\eta}^r_i(\tau)\, \dot{\eta}^s_i(\tau)}{\sqrt{1-\dot{\vec{\eta}}^2_i(\tau)}} + \frac{M}{L^3}\, O(\zeta^2), \\
& \qquad q^{|rs} = \sum_i \eta_i\, \frac{\kappa_{ir}(\tau)\,\kappa_{is}(\tau)}{\sqrt{m_i^2\, c^2 + \vec{\kappa}_i^2(\tau)}} = \sum_{i=1}^N \eta_i\, \frac{m_i c\, \dot{\eta}^r_i(\tau)\, \dot{\eta}^s_i(\tau)}{\sqrt{1-\dot{\vec{\eta}}^2_i(\tau)}} + M\, O(\zeta^2),
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{\triangle}\, \mathcal{M}^{(UV)}_{(1)}(\tau,\vec{\sigma}) = -\sum_{i=1}^N \eta_i\, \frac{\sqrt{m_i^2 c^2 + \vec{\kappa}_i^2(\tau)}}{4\pi\, |\vec{\sigma}-\vec{\eta}_i(\tau)|}, \qquad \frac{1}{\triangle} \sum_a T^{aa}_{(1)}(\tau,\vec{\sigma}) = -\sum_{i=1}^N \eta_i\, \frac{\frac{\vec{\kappa}_i^2(\tau)}{\sqrt{m_i^2 c^2+\vec{\kappa}_i^2(\tau)}}}{4\pi\, |\vec{\sigma}-\vec{\eta}_i(\tau)|}, \\
&\frac{1}{\triangle}\, \mathcal{M}^{(UV)}_{(1)r}(\tau,\vec{\sigma}) = -\sum_{i=1}^N \eta_i\, \frac{\kappa_{ir}(\tau)}{4\pi\, |\vec{\sigma}-\vec{\eta}_i(\tau)|}, \\
&\frac{\partial_a}{\triangle} \sum_c \partial_c\, \mathcal{M}^{(UV)}_{(1)c}(\tau,\vec{\sigma}) = -\sum_{i=1}^N \eta_i \sum_c \kappa_{ic}(\tau) \int \frac{d^3\sigma_1}{(4\pi)^2\, |\vec{\sigma}-\vec{\sigma}_1|\, |\vec{\sigma}_1-\vec{\eta}_i(\tau)|^3} \Big(\delta_{ac} - \\
&- \; 3\, \frac{(\sigma^a_1-\eta^a_i(\tau))\,(\sigma^c_1-\eta^c_i(\tau))}{|\vec{\sigma}_1-\vec{\eta}_i(\tau)|^2}\Big),
\end{aligned}
\tag{2.1}
$$

where we used $\vec{\kappa}_i = \frac{m_i c\, \dot{\vec{\eta}}_i}{\sqrt{1-\dot{\vec{\eta}}^2_i}} + Mc\, O(\zeta)$ and $\sqrt{m_i^2 c^2 + \vec{\kappa}_i^2} = \frac{m_i c}{\sqrt{1-\dot{\vec{\eta}}^2_i}} + Mc\, O(\zeta)$. We have also given the second order of the mass function. The quantities $q^{|\tau\tau}$, $q^{r|\tau s}$ and $q^{|rs}$ are the mass monopole, the momentum dipole and the stress tensor monopole respectively (see Appendix B of paper II).

B. The Solution of the Super-Hamiltonian and Super-Momentum Constraints and the Lapse and Shift Functions for the Family of 3-Orthogonal Gauges

From Eqs.(4.6), (4.7), (4.16) and (4.17) of paper II we get the following expressions for the solutions: a) $\tilde{\phi}_{(1)}(\tau,\vec{\sigma})$ of the super-Hamiltonian constraint; b) $N(\tau,\vec{\sigma}) = 1 + n_{(1)}(\tau,\vec{\sigma})$ and $\bar{n}_{(1)(a)}(\tau,\vec{\sigma})$ of the equations for the lapse and shift functions in the family of 3-orthogonal gauges; c) $\sigma_{(1)(a)(b)}|_{a\not= b}(\tau,\vec{\sigma})$ (the off-diagonal terms of the shear of the congruence of Eulerian observers) of the super-momentum constraints (see Eq.(1.9) for $\pi_i^{(\theta)}$):

$$
\begin{aligned}
\tilde{\phi}(\tau,\vec{\sigma}) &= 1 + 6\,\phi_{(1)}(\tau,\vec{\sigma}) + O(\zeta^2),\\
\phi_{(1)}(\tau,\vec{\sigma}) &\stackrel{\circ}{=} \Big[-\frac{2\pi\,G}{c^3}\,\frac{1}{\triangle}\,\mathcal{M}^{(UV)}_{(1)} + \frac{1}{4}\sum_c \frac{\partial_c^2}{\triangle}\,\Gamma_c^{(1)}\Big](\tau,\vec{\sigma})\stackrel{\circ}{=}\\
&\stackrel{\circ}{=} \frac{G}{2c^3}\sum_i \eta_i\,\frac{\sqrt{m_i^2\,c^2+\vec{\kappa}_i^2(\tau)}}{|\vec{\sigma}-\vec{\eta}_i(\tau)|} - \frac{1}{16\pi}\int d^3\sigma_1\,\frac{\sum_a \partial^2_{1a}\,\Gamma_a^{(1)}(\tau,\vec{\sigma}_1)}{|\vec{\sigma}-\vec{\sigma}_1|} =\\
&= \frac{G}{2c^2}\sum_i \eta_i\,\frac{\frac{m_i}{\sqrt{1-\dot{\vec{\eta}}_i^2(\tau)}}}{|\vec{\sigma}-\vec{\eta}_i(\tau)|} - \frac{1}{16\pi}\int d^3\sigma_1\,\frac{\sum_a \partial^2_{1a}\,\Gamma_a^{(1)}(\tau,\vec{\sigma}_1)}{|\vec{\sigma}-\vec{\sigma}_1|},
\end{aligned} \tag{2.2}
$$

$$
\begin{aligned}
n_{(1)}(\tau,\vec{\sigma}) &\stackrel{\circ}{=} \Big[\frac{4\pi\,G}{c^3}\,\frac{1}{\triangle}\Big(\mathcal{M}^{(UV)}_{(1)} + \sum_a T^{aa}_{(1)}\Big) - \partial_\tau\,{}^3\mathcal{K}_{(1)}\Big](\tau,\vec{\sigma})\stackrel{\circ}{=}\\
&\stackrel{\circ}{=} -\frac{G}{c^3}\sum_i \eta_i\,\frac{\sqrt{m_i^2\,c^2+\vec{\kappa}_i^2(\tau)}}{|\vec{\sigma}-\vec{\eta}_i(\tau)|}\Big(1+\frac{\vec{\kappa}_i^2}{m_i^2\,c^2+\vec{\kappa}_i^2}\Big) - \partial_\tau\,{}^3\mathcal{K}_{(1)}(\tau,\vec{\sigma}) =\\
&= -\frac{G}{c^2}\sum_i \eta_i\,\frac{\frac{m_i}{\sqrt{1-\dot{\vec{\eta}}_i^2(\tau)}}}{|\vec{\sigma}-\vec{\eta}_i(\tau)|}\,(1+\dot{\vec{\eta}}_i^2(\tau)) - \partial_\tau\,{}^3\mathcal{K}_{(1)}(\tau,\vec{\sigma}),
\end{aligned} \tag{2.3}
$$

$$
\begin{aligned}
\bar{n}_{(1)(a)}(\tau,\vec{\sigma}) &\stackrel{\circ}{=} \Big[\partial_a\,{}^3\mathcal{K}_{(1)} + \frac{4\pi\,G}{c^3}\,\frac{1}{\triangle}\Big(4\,\mathcal{M}^{(UV)}_{(1)a} - \frac{\partial_a}{\triangle}\sum_c \partial_c\,\mathcal{M}^{(UV)}_{(1)c}\Big) +\\
&+\ \frac{1}{2}\,\frac{\partial_a}{\triangle}\,\partial_\tau\Big(4\,\Gamma_a^{(1)} - \sum_c \frac{\partial_c^2}{\triangle}\,\Gamma_c^{(1)}\Big)\Big](\tau,\vec{\sigma})\stackrel{\circ}{=}\\
&\stackrel{\circ}{=} \partial_a\,{}^3\mathcal{K}_{(1)}(\tau,\vec{\sigma}) - \frac{G}{c^3}\sum_i \frac{\eta_i}{|\vec{\sigma}-\vec{\eta}_i(\tau)|}\Big(\frac{7}{2}\kappa_{ia}(\tau) +\\
&-\ \frac{1}{2}\frac{(\sigma^a-\eta_i^a(\tau))\,\vec{\kappa}_i(\tau)\cdot(\vec{\sigma}-\vec{\eta}_i(\tau))}{|\vec{\sigma}-\vec{\eta}_i(\tau)|^2}\Big) -\\
&-\ \int \frac{d^3\sigma_1}{4\pi\,|\vec{\sigma}-\vec{\sigma}_1|}\,\partial_{1a}\,\partial_\tau\Big[2\,\Gamma_a^{(1)}(\tau,\vec{\sigma}_1) - \int d^3\sigma_2\,\frac{\sum_c \partial^2_{2c}\,\Gamma_c^{(1)}(\tau,\vec{\sigma}_2)}{8\pi\,|\vec{\sigma}_1-\vec{\sigma}_2|}\Big] =
\end{aligned}
$$

$$
\begin{aligned}
&= \partial_a\, {}^3\mathcal{K}_{(1)}(\tau, \vec{\sigma}) - \frac{G}{c^2} \sum_i \frac{\eta_i}{|\vec{\sigma} - \vec{\eta}_i(\tau)|}\, \frac{m_i}{\sqrt{1 - \dot{\vec{\eta}}_i^2(\tau)}} \Big(\frac{7}{2} \dot{\eta}_i^a(\tau) + \\
&- \frac{1}{2} \frac{(\sigma^a - \eta_i^a(\tau))\, \dot{\vec{\eta}}_i(\tau) \cdot (\vec{\sigma} - \vec{\eta}_i(\tau))}{|\vec{\sigma} - \vec{\eta}_i(\tau)|^2}\Big) - \\
&- \int \frac{d^3\sigma_1}{4\pi\, |\vec{\sigma} - \vec{\sigma}_1|}\, \partial_{1a}\, \partial_\tau \Big[2\, \Gamma_a^{(1)}(\tau, \vec{\sigma}_1) - \int d^3\sigma_2\, \frac{\sum_c \partial^2_{2c}\, \Gamma_c^{(1)}(\tau, \vec{\sigma}_2)}{8\pi\, |\vec{\sigma}_1 - \vec{\sigma}_2|}\Big].
\end{aligned} \tag{2.4}
$$

$$
\begin{aligned}
\sigma_{(1)(a)(b)}|_{a\not= b}(\tau, \vec{\sigma}) &\,\circeq\, \frac{1}{2} \Big(\partial_a\, \bar{n}_{(1)(b)} + \partial_b\, \bar{n}_{(1)(a)}\Big)|_{a\not= b}(\tau, \vec{\sigma}) \circeq \\
&\circeq \Big[\partial_a\, \partial_b\, {}^3\mathcal{K}_{(1)} + \frac{8\pi\, G}{c^3} \Big[\frac{1}{\triangle} \Big(\partial_a\, \mathcal{M}^{(UV)}_{(1)b} + \partial_b\, \mathcal{M}^{(UV)}_{(1)a}\Big) - \frac{1}{2} \frac{\partial_a\, \partial_b}{\triangle} \sum_c \frac{\partial_c}{\triangle}\, \mathcal{M}^{(UV)}_{(1)c}\Big] + \\
&+ \partial_\tau\, \frac{\partial_a\, \partial_b}{\triangle} \Big(\Gamma_a^{(1)} + \Gamma_b^{(1)} - \frac{1}{2} \sum_c \frac{\partial_c^2}{\triangle}\, \Gamma_c^{(1)}\Big)\Big](\tau, \vec{\sigma}) \circeq
\end{aligned}
$$

$$
\begin{aligned}
&\circeq -\frac{1}{2} \sum_d (\delta_{ad}\, \partial_b + \delta_{bd}\, \partial_a) \quad \Big(\frac{G}{c^3} \sum_i \frac{\eta_i}{|\vec{\sigma} - \vec{\eta}_i(\tau)|} \Big(\frac{7}{2} \kappa_{id}(\tau) + \\
&- \frac{1}{2} \frac{(\sigma^d - \eta_i^d(\tau))\, \vec{\kappa}_i(\tau) \cdot (\vec{\sigma} - \vec{\eta}_i(\tau))}{|\vec{\sigma} - \vec{\eta}_i(\tau)|^2}\Big) + \\
&+ \int \frac{d^3\sigma_1}{4\pi\, |\vec{\sigma} - \vec{\sigma}_1|}\, \partial_{1d} \Big[2\, \partial_\tau\, \Gamma_d^{(1)}(\tau, \vec{\sigma}_1) + \int d^3\sigma_2\, \frac{\sum_c \partial_\tau\, \partial^2_{2c}\, \Gamma_c^{(1)}(\tau, \vec{\sigma}_2)}{8\pi\, |\vec{\sigma}_1 - \vec{\sigma}_2|}\Big]\,\Big) + \partial_a\, \partial_b\, {}^3\mathcal{K}_{(1)}(\tau, \vec{\sigma}).
\end{aligned} \tag{2.5}
$$

The action-at-a-distance part of the solution is explicitly shown. Only the PM volume element $\tilde{\phi}_{(1)} = 1 + 6\, \phi_{(1)}$ is independent from the York time. Eq.(2.4) describes gravito-magnetism in these PM space-times [2]: it has an inertial gauge part $\partial_a\, {}^3\mathcal{K}_{(1)}$.

C. The HPM Gravitational Waves

By using Eqs.(7.1) and (7.2) of paper II, the retarded solution for the tidal variables with the matter restricted to point particles is (see Eq. (3.12) of paper II for T^{rs}; the TT

[2] The gravito-magnetic potential $\vec{A}_G$ has the components $A_{G(r)} \sim c^2\, \bar{n}_{(1)(r)}$. The gravito-magnetic field $B_{G(r)} = c\, \Omega_{G(r)} = (\vec{\partial} \times \vec{A}_G)_r$ is proportional to the second term in the Christoffel symbol ${}^4\Gamma^u_{(1)\tau r}$ given in Eq.(2.15). Instead the gravito-electric potential is $\Phi_G = -\frac{c^2}{4}\, n_{(1)} = -\frac{8\pi\, G}{c}\, \frac{1}{\triangle}\, (\mathcal{M}^{(UV)}_{(1)} + \sum_a T^{aa}_{(1)}) + \frac{c^2}{4}\, \partial_\tau\, {}^3\mathcal{K}_{(1)})$. Both $A_{G(r)}$ and Φ_G depend on the non-local York time.

projector $\mathcal{P}_{rsuv}$ is defined in Eqs.(6.7) of paper II; the spatial operator $\tilde{M}^{-1}_{ab}$ is defined in Eqs. (6.24) and (6.25) of paper II) [3]

$$
\begin{aligned}
\Gamma^{(1)}_a(\tau,\vec{\sigma}) &= \sum_{\bar{a}} \gamma_{\bar{a}a}\, R_{\bar{a}}(\tau,\vec{\sigma}) \circeq \\
&\circeq -\frac{2\,G}{c^2} \sum_b \tilde{M}^{-1}_{ab}(\vec{\sigma}) \sum_i \eta_i\, m_i \sum_{uv} \mathcal{P}_{bbuv}(\vec{\sigma}) \\
&\int d^3\sigma_1\, \frac{\dot{\eta}^u_i(\tau-|\vec{\sigma}-\vec{\sigma}_1|)\,\dot{\eta}^v_i(\tau-|\vec{\sigma}-\vec{\sigma}_1|)}{\sqrt{1-\dot{\eta}^2_i(\tau-|\vec{\sigma}-\vec{\sigma}_1|)}}\, \frac{\delta^3(\vec{\sigma}_1-\vec{\eta}_i(\tau-|\vec{\sigma}-\vec{\sigma}_1|))}{|\vec{\sigma}-\vec{\sigma}_1|} + O(\zeta^2).
\end{aligned}
\tag{2.6}
$$

By making a equal time development of the retarded kernel like in Ref.[7] for the extraction of the Darwin potential from the Lienard-Wiechert solution (see Eqs. (5.1)-(5.21) of Ref.[7] with $\sum_s P^{rs}_{\perp}(\vec{\sigma})\,\dot{\eta}^s_i(\tau) \;\mapsto\; \sum_{uv} \mathcal{P}_{bbuv}(\vec{\sigma})\,\frac{\dot{\eta}^u_i(\tau)\,\dot{\eta}^v_i(\tau)}{\sqrt{1-\dot{\eta}^2_i(\tau)}}$) and by using the fact that $\ddot{\eta}^r_i(\tau) = O(\zeta)$ (see also Section V) we get the following expression of the HPM GW's from point masses

$$
\begin{aligned}
\Gamma^{(1)}_a(\tau,\vec{\sigma}) &\circeq -\frac{2\,G}{c^2} \sum_b \tilde{M}^{-1}_{ab}(\vec{\sigma}) \sum_i \eta_i\, m_i \sum_{uv} \mathcal{P}_{bbuv}(\vec{\sigma})\, \frac{\dot{\eta}^u_i(\tau)\,\dot{\eta}^v_i(\tau)}{\sqrt{1-\dot{\eta}^2_i(\tau)}} \\
&\Big[|\vec{\sigma}-\vec{\eta}_i(\tau)|^{-1} + \sum_{m=1}^{\infty} \frac{1}{(2m)!}\Big(\dot{\vec{\eta}}_i(\tau)\cdot\frac{\partial}{\partial\,\vec{\sigma}}\Big)^{2m} |\vec{\sigma}-\vec{\eta}_i(\tau)|^{2m-1}\Big] + O(\zeta^2).
\end{aligned}
\tag{2.7}
$$

As shown in paper II the multipolar expansion of the TT 3-metric and of the tidal variables reproduces the quadrupolar emission formula ($q^{uv|\tau\tau}$ is the mass quadrupole; the TT projector Λ_{abcd} is defined in Eqs.(7.17) of paper II)

---

[3] One could study the radiative fields $\Gamma^{(1)}_a(\tau,\vec{\sigma})$ at null infinity ($|\vec{\sigma}| \rightarrow \infty$ with the retarded time $\tau-|\vec{\sigma}|$ fixed) to see whether terms in $ln\,|\vec{\sigma}|$ appear like in the standard approach to GW's in harmonic gauges (see Section 5.3.4 of Ref.[6]), but this will done elsewhere.

$$
\begin{aligned}
{}^4h^{TT}_{(1)rs}(\tau,\vec{\sigma}) &\,\circeq\, -\epsilon\,\frac{2\,G}{c^3}\sum_{uv}\Lambda_{rsuv}(n)\,\frac{\partial^2_\tau\,q^{uv|\tau\tau}((\tau-|\vec{\sigma}|))}{|\vec{\sigma}|}+(higher\,multipoles)+O(1/r^2),\\
R_{\bar a}(\tau,\vec{\sigma}) &\,\circeq\, -\frac{G}{c^3}\sum_{ab}\gamma_{\bar a a}\,\tilde{M}^{-1}_{ab}(\vec{\sigma})\,\frac{\sum_{uv}\mathcal{P}_{bbuv}\,\partial^2_\tau\,q^{uv|\tau\tau}(\tau-|\vec{\sigma}|)}{|\vec{\sigma}|}+(higher\,multipoles)+O(1/r^2),\\
q^{uv|\tau\tau}(\tau-|\vec{\sigma}|) &= \int d^3\sigma_1\,\sigma_1^u\,\sigma_1^v\,\mathcal{M}^{(UV)}_{(1)}(\tau-|\vec{\sigma}|,\vec{\sigma}_1)=\\
&= \sum_{i=1}^N\eta_i\,\eta_i^u(\tau-|\vec{\sigma}|)\,\eta_i^v(\tau-|\vec{\sigma}|)\,\sqrt{m_i^2c^2+\vec{\kappa}_i^2(\tau-|\vec{\sigma}|)}=\\
&= \sum_{i=1}^N\eta_i\,\frac{m_ic\,\eta_i^u(\tau-|\vec{\sigma}|)\,\eta_i^v(\tau-|\vec{\sigma}|)}{\sqrt{1-\dot{\vec{\eta}}_i^2(\tau-|\vec{\sigma}|)}}.
\end{aligned}
\tag{2.8}
$$

Eq.(4.18) of paper II gives the following expression for the tidal momenta of Eqs.(1.8) (namely for diagonal elements $\sigma_{(1)(a)(a)}$ of the shear, see Eq.(1.8))

$$
\begin{aligned}
\frac{8\pi\,G}{c^3}\,\Pi_{\bar a}(\tau,\vec{\sigma}) &\,\circeq\, \partial_\tau\,R_{\bar a}(\tau,\vec{\sigma})-\sum_a\gamma_{\bar a a}\,\Big[\partial_\tau\,\frac{\partial_a^2}{2\,\triangle}\,(4\,\Gamma_a^{(1)}-\frac{1}{\triangle}\sum_c\partial_c^2\,\Gamma_c^{(1)})+\\
&+\ \frac{4\pi\,G}{c^3}\,\frac{1}{\triangle}\,(4\,\partial_a\,\mathcal{M}^{(UV)}_{(1)a}-\frac{\partial_a^2}{\triangle}\sum_c\partial_c\,\mathcal{M}^{(UV)}_{(1)c})+\partial_a^2\,{}^3\mathcal{K}_{(1)}\Big]=\\
&= \Big(\sum_{\bar b}M_{\bar a\bar b}\,\partial_\tau\,R_{\bar b}-\sum_a\gamma_{\bar a a}\,\Big[\frac{4\pi\,G}{c^3}\,\frac{1}{\triangle}\,(4\,\partial_a\,\mathcal{M}^{(UV)}_{(1)a}-\frac{\partial_a^2}{\triangle}\sum_c\partial_c\,\mathcal{M}^{(UV)}_{(1)c})+\\
&+\ \partial_a^2\,{}^3\mathcal{K}_{(1)}\Big]\Big)(\tau,\vec{\sigma}),\\
&\qquad M_{\bar a\bar b}=\delta_{\bar a\bar b}-\sum_a\gamma_{\bar a a}\,\frac{\partial_a^2}{\triangle}\,\Big(2\,\gamma_{\bar b a}-\frac{1}{2}\sum_b\gamma_{\bar b b}\,\frac{\partial_b^2}{\triangle}\Big),
\end{aligned}
\tag{2.9}
$$

While the tidal variables $R_{\bar a}$ do not depend on the York time, the tidal momenta $\Pi_{\bar a}$ depend upon it.

D. The Weak ADM Poincare' Generators

The HPM linearization of the ADM Poincare' generators is given in Eqs. (4-21)-(4.24) of paper II. The final expression of the generators is ($\sqrt{m_i^2\,c^2+\vec{\kappa}_i^2(\tau)}=m_i\,c/\sqrt{1-\dot{\vec{\eta}}_i^2(\tau)}$; the spatial operator $M_{\bar a\bar b}$ is given in Eq.(2.6); L is the GW wavelength of Section III of paper II)

$$
\begin{aligned}
\frac{1}{c}\,\hat{E}_{ADM} &= M_{(1)}\,c + \frac{1}{c}\,\hat{E}_{ADM(2)} + Mc\,O(\zeta^3) = \\
&= \sum_i \eta_i\,\sqrt{m_i^2\,c^2 + \vec{\kappa}_i^2(\tau)} - \\
&- \sum_i \eta_i\,\frac{\vec{\kappa}_i^2(\tau)\,\Big[\frac{G}{c^3}\,\sum_{j\neq i}\eta_j\,\frac{\sqrt{m_j^2\,c^2+\vec{\kappa}_j^2(\tau)}}{|\vec{\eta}_i(\tau)-\vec{\eta}_j(\tau)|} + \sum_c \frac{\partial_c^2}{2\,\triangle}\,\Gamma_c^{(1)}(\tau,\vec{\eta}_i(\tau))\Big] + \sum_c \kappa_{ic}^2(\tau)\,\Gamma_c^{(1)}(\tau,\vec{\eta}_i(\tau))}{\sqrt{m_i^2\,c^2+\vec{\kappa}_i^2(\tau)}} - \\
&- \frac{G}{c^3}\sum_{i>j}\eta_i\,\eta_j\,\frac{\sqrt{m_i^2\,c^2+\vec{\kappa}_i^2(\tau)}\,\sqrt{m_j^2\,c^2+\vec{\kappa}_j^2(\tau)}}{|\vec{\eta}_i(\tau)-\vec{\eta}_j(\tau)|} + \frac{G}{c^3}\sum_{i>j}\eta_i\,\eta_j\,\Big(\frac{4\,\vec{\kappa}_i(\tau)\cdot\vec{\kappa}_j(\tau)}{|\vec{\eta}_i(\tau)-\vec{\eta}_j(\tau)|} - \\
&- \frac{1}{4\,\pi}\sum_{rs}\kappa_{ir}(\tau)\,\kappa_{js}(\tau)\,\int d^3\sigma\,\frac{(\sigma^r-\eta_i^r(\tau))\,(\sigma^s-\eta_j^s(\tau))}{|\vec{\sigma}-\vec{\eta}_i(\tau)|^3\,|\vec{\sigma}-\vec{\eta}_j(\tau)|^3}\Big) + \\
&+ \frac{c^3}{16\pi G}\sum_{\bar{a}\bar{b}}\int d^3\sigma\,\Big[\partial_\tau\,R_{\bar{a}}\,M_{\bar{a}\bar{b}}\,\partial_\tau\,R_{\bar{b}} + \sum_a \partial_a\,R_{\bar{a}}\,M_{\bar{a}\bar{b}}\,\partial_a\,R_{\bar{b}}\Big]\Big)(\tau,\vec{\sigma}) - \\
&- \sum_i \eta_i\,\vec{\kappa}_i(\tau)\cdot\vec{\partial}^{\,3}\mathcal{K}_{(1)}(\tau,\vec{\eta}_i(\tau)) + Mc\,O(\zeta^3),
\end{aligned}
\tag{2.10}
$$

$$
\begin{aligned}
\hat{P}^r_{ADM} &= p^r_{(1)} + p^r_{(2)} + Mc\,O(\zeta^3) = \\
&= \sum_i \eta_i\,\kappa_{ir}(\tau) - \frac{c^3}{8\pi\,G}\int d^3\sigma\,\sum_{\bar{a}\bar{b}}\Big(\partial_r\,R_{\bar{a}}\,M_{\bar{a}\bar{b}}\,\partial_\tau\,R_{\bar{b}}\Big)(\tau,\vec{\sigma}) + \\
&+ \sum_i \eta_i\,\sum_a \kappa_{ia}(\tau)\,\frac{\partial_r\,\partial_a}{\triangle}\,\Big(\sum_c \frac{\partial_c^2}{2\,\triangle}\,\Gamma_c^{(1)} - 2\,\Gamma_a^{(1)}\Big)(\tau,\vec{\eta}_i(\tau)) - \\
&- \sum_i \eta_i\,\sqrt{m_i^2\,c^2+\vec{\kappa}_i^2(\tau)}\,\partial_r\,{}^3\mathcal{K}_{(1)}(\tau,\vec{\eta}_i(\tau)) + Mc\,O(\zeta^3) \approx 0,
\end{aligned}
\tag{2.11}
$$

$$
\begin{aligned}
\hat{J}^{rs}_{ADM} &= j^{rs}_{(1)} + j^{rs}_{(2)} + Mc\, O(\zeta^3) = \\
&= \sum_i \eta_i \Big(\eta^r_i(\tau)\, \kappa_{is}(\tau) - \eta^s_i(\tau)\, \kappa_{ir}(\tau)\Big) - \\
&- 2 \sum_i \eta_i \sum_u \kappa_{iu}(\tau) \Big(\eta^r_i(\tau)\, \frac{\partial}{\partial\, \eta^s_i} - \eta^s_i(\tau)\, \frac{\partial}{\partial\, \eta^r_i}\Big) \\
&\quad \frac{\partial_u}{\triangle} \Big(\Gamma^{(1)}_u(\tau, \vec{\eta}_i(\tau)) - \frac{1}{4} \sum_c \frac{\partial^2_c}{\triangle}\, \Gamma^{(1)}_c\Big) + \\
&+ 2 \sum_i \eta_i \Big[\kappa_{ir}(\tau)\, \frac{\partial_s}{\triangle} \Big(\Gamma^{(1)}_s - \frac{1}{4} \sum_c \frac{\partial^2_c}{\triangle}\, \Gamma^{(1)}_c\Big) - \\
&- \kappa_{is}(\tau)\, \frac{\partial_r}{\triangle} \Big(\Gamma^{(1)}_r - \frac{1}{4} \sum_c \frac{\partial^2_c}{\triangle}\, \Gamma^{(1)}_c\Big)\Big](\tau, \vec{\eta}_i(\tau)) - \\
&- \frac{c^3}{8\pi\, G} \int d^3\sigma\, \Big[\sum_{\bar{a}\bar{b}} (\sigma^r\, \partial_s - \sigma^s\, \partial_r)\, R_{\bar{a}}\, M_{\bar{a}\bar{b}}\, \partial_\tau\, R_{\bar{b}} + 2\, {}^3K_{(1)}\, \partial_r\, \partial_s\, (\Gamma^{(1)}_s - \Gamma^{(1)}_r) + \\
&+ 2\, (\partial_\tau\, \Gamma^{(1)}_r + \partial_\tau\, \Gamma^{(1)}_s - \frac{1}{2} \sum_c \frac{\partial^2_c}{\triangle}\, \partial_\tau\, \Gamma^{(1)}_c)\, \frac{\partial_r\, \partial_s}{\triangle}\, (\Gamma^{(1)}_s - \Gamma^{(1)}_r)\Big](\tau, \vec{\sigma}) + \\
&+ \sum_i \eta_i \sqrt{m^2_i\, c^2 + \vec{\kappa}^2_i(\tau)}\, \Big(\eta^r_i(\tau)\, \frac{\partial}{\partial \eta^s_i} - \eta^s_i(\tau)\, \frac{\partial}{\partial\, \eta^r_i}\Big)\, {}^3\mathcal{K}_{(1)}(\tau, \vec{\eta}_i(\tau)) + Mc\, L\, O(\zeta^3),
\end{aligned}
\tag{2.12}
$$

$$
\begin{aligned}
\hat{J}^{\tau r}_{ADM} &= -\hat{J}^{r\tau}_{ADM} = j^{\tau r}_{(1)} + j^{\tau r}_{(2)} + Mc\,L\,O(\zeta^3) = \\
&= -\sum_i \eta_i\, \eta^r_i(\tau)\, \sqrt{m_i^2 c^2 + \vec{\kappa}_i^2(\tau)} - \\
&- \frac{G}{c^3} \sum_{i\not= i} \eta_i \eta_j \frac{\vec{\kappa}_i^2(\tau)\, \sqrt{m_j^2 c^2 + \vec{\kappa}_j^2(\tau)}}{\sqrt{m_i^2 c^2 + \vec{\kappa}_i^2(\tau)}} \frac{\eta^r_i(\tau)}{\mid \vec{\eta}_i(\tau) - \vec{\eta}_j(\tau) \mid} + \\
&+ \sum_i \eta_i\, \eta^r_i(\tau)\, \sum_a \frac{\kappa^2_{ia}(\tau)\, \Big(\Gamma^{(1)}_a + \frac{1}{2}\, \sum_c \frac{\partial_c^2}{\Delta}\, \Gamma^{(1)}_c\Big)(\tau, \vec{\eta}_i(\tau))}{\sqrt{m_i^2 c^2 + \vec{\kappa}_i^2(\tau)}} - \\
&- \int d^3\sigma\, \sigma^r\, \Big[\frac{c^3}{16\pi G} \sum_{a,b} (\partial_a\, \Gamma^{(1)}_b(\tau, \vec{\sigma}))^2 - \\
&- \frac{c^3}{8\pi G} \sum_a \partial_a\, \Gamma^{(1)}_a(\tau, \vec{\sigma})\, \partial_a \Big(\Gamma^{(1)}_a - \frac{1}{2}\, \sum_c \frac{\partial_c^2}{\Delta}\, \Gamma^{(1)}_c\Big)(\tau, \vec{\sigma}) - \\
&- \frac{c^3}{32\pi G} \sum_a \partial_a \Big(\sum_c \frac{\partial_c^2}{\Delta}\, \Gamma^{(1)}_c(\tau, \vec{\sigma})\Big) \partial_a \Big(\sum_d \frac{\partial_d^2}{\Delta}\, \Gamma^{(1)}_d(\tau, \vec{\sigma})\Big) + \\
&+ \frac{1}{2} \sum_a \sum_i \eta_i\, \sqrt{m_i^2 c^2 + \vec{\kappa}_i^2(\tau)} \frac{\sigma^a - \eta^a_i(\tau)}{4\pi \mid \vec{\sigma} - \vec{\eta}_i(\tau) \mid^3}\, \partial_a \Big(\Gamma^{(1)}_a - \frac{1}{2} \sum_c \frac{\partial_c^2}{\Delta}\, \Gamma^{(1)}_c(\tau, \vec{\sigma})\Big) + \\
&+ \frac{2\, G}{\pi\, c^3} \sum_{i\not= j} \eta_i \eta_j\, \sqrt{m_i^2 c^2 + \vec{\kappa}_i^2(\tau)} \sqrt{m_j^2 c^2 + \vec{\kappa}_j^2(\tau)} \frac{(\vec{\sigma} - \vec{\eta}_i(\tau)) \cdot (\vec{\sigma} - \vec{\eta}_j(\tau))}{\mid \vec{\sigma} - \vec{\eta}_i(\tau) \mid^3 \mid \vec{\sigma} - \vec{\eta}_j(\tau) \mid^3} + \\
&+ \frac{c^3}{16\pi G} \sum_{a,b} \Big(\widetilde{M}_{ab}(\vec{\sigma})\, \partial_\tau\, \Gamma^{(1)}_b(\tau, \vec{\sigma})\Big)^2 + \\
&+ \frac{c^3}{16\pi G} \sum_{a\not= b} \Big[\frac{\partial_a \partial_b \partial_\tau}{\Delta} \Big(\Gamma^{(1)}_a + \Gamma^{(1)}_b - \frac{1}{2} \sum_c \frac{\partial_c^2}{\Delta}\, \Gamma^{(1)}_c\Big)(\tau, \vec{\sigma})\Big]^2 - \\
&- \frac{1}{2\pi} \sum_{a,b} \Big(\widetilde{M}_{ab}(\vec{\sigma})\, \partial_\tau\, \Gamma^{(1)}_b(\tau, \vec{\sigma})\Big) \sum_i \eta_i\, \frac{\kappa_{ia}(\tau)\, (\sigma^a - \eta^a_i(\tau))}{\mid \vec{\sigma} - \vec{\eta}_i(\tau) \mid^3} + \\
&+ \frac{1}{2\pi} \sum_{a\not= b} \frac{\partial_a \partial_b \partial_\tau}{\Delta} \Big(\Gamma^{(1)}_a + \Gamma^{(1)}_b - \frac{1}{2} \sum_c \frac{\partial_c^2 \Gamma^{(1)}_c}{\Delta}\Big)(\tau, \vec{\sigma})\, \sum_i \eta_i\, \frac{\kappa_{ia}(\tau)\, (\sigma^a - \eta^a_i(\tau))}{\mid \vec{\sigma} - \vec{\eta}_i(\tau) \mid^3} -
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{c^3}{8\pi G} \sum_{a,b} \Big(\widetilde{M}_{ab}(\vec{\sigma})\, \partial_\tau\, \Gamma_b^{(1)}(\tau,\vec{\sigma})\Big)\, \frac{\partial_a^2}{\Delta}\Big({}^3K_{(1)}(\tau,\vec{\sigma}) - \frac{G}{c^3} \sum_i \eta_i\, \frac{\vec{\kappa}_i(\tau)\cdot(\vec{\sigma}-\vec{\eta}_i(\tau))}{\mid \vec{\sigma}-\vec{\eta}_i(\tau)\mid^3}\Big) + \\
&+ \frac{c^3}{8\pi G} \sum_{a\neq b} \frac{\partial_a\partial_b\partial_\tau}{\Delta}\Big(\Gamma_a^{(1)} + \Gamma_b^{(1)} - \frac{1}{2}\sum_c \frac{\partial_c^2}{\Delta}\,\Gamma_c^{(1)}\Big)(\tau,\vec{\sigma}) \\
&\quad \frac{\partial_a\partial_b}{\Delta}\,\Big({}^3K_{(1)}(\tau,\vec{\sigma})) - \frac{G}{c^3}\sum_i \eta_i\, \frac{\vec{\kappa}_i(\tau)\cdot(\vec{\sigma}-\vec{\eta}_i(\tau))}{\mid \vec{\sigma}-\vec{\eta}_i(\tau)\mid^3}\Big) + \\
&+ \frac{c^3}{16\pi G} \sum_{a,b} \Big[\frac{\partial_a\partial_b}{\Delta}\Big({}^3K_{(1)}(\tau,\vec{\sigma}) - \frac{G}{c^3}\sum_i \eta_i\, \frac{\vec{\kappa}_i(\tau)\cdot(\vec{\sigma}-\vec{\eta}_i(\tau))}{\mid \vec{\sigma}-\vec{\eta}_i(\tau)\mid^3}\Big)\Big]^2 + \\
&+ \frac{1}{2\pi} \sum_{a,b}\sum_i \eta_i\, \frac{\kappa_{ib}(\tau)\,(\sigma^a - \eta_i^a(\tau))}{\mid \vec{\sigma}-\vec{\eta}_i(\tau)\mid^3}\, \frac{\partial_a\partial_b}{\Delta}\,\Big({}^3K_{(1)}(\tau,\vec{\sigma}) - \frac{G}{c^3}\sum_j \eta_j\, \frac{\vec{\kappa}_j(\tau)\cdot(\vec{\sigma}-\vec{\eta}_j(\tau))}{\mid \vec{\sigma}-\vec{\eta}_j(\tau)\mid^3}\Big) - \\
&- \frac{c^3}{48\pi G}\Big({}^3K_{(1)}(\tau,\vec{\sigma}) + \frac{3G}{c^3}\sum_i \eta_i\, \frac{\vec{\kappa}_i(\tau)\cdot(\vec{\sigma}-\vec{\eta}_i(\tau))}{\mid \vec{\sigma}-\vec{\eta}_i(\tau)\mid^3}\Big)^2 - \frac{c^3}{24\pi G}\Big({}^3K_{(1)}(\tau,\vec{\sigma})\Big)^2 + \\
&+ \frac{G}{2\pi\, c^3}\sum_{i\neq j}\eta_i\eta_j\, \frac{\vec{\kappa}_i(\tau)\cdot\vec{\kappa}_j(\tau)\,(\vec{\sigma}-\vec{\eta}_j(\tau))\cdot(\vec{\sigma}-\vec{\eta}_i(\tau)) + \vec{\kappa}_i(\tau)\cdot(\vec{\sigma}-\vec{\eta}_j(\tau))\,\vec{\kappa}_j(\tau)\cdot(\vec{\sigma}-\vec{\eta}_i(\tau))}{\mid \vec{\sigma}-\vec{\eta}_i(\tau)\mid^3 \mid \vec{\sigma}-\vec{\eta}_j(\tau)\mid^3}\ \Big] + \\
&\quad + \int d^3\sigma\, \Big[\frac{3}{8\pi}\sum_i \eta_i \frac{\sqrt{m_i^2c^2+\vec{\kappa}_i^2(\tau)}}{\mid \vec{\sigma}-\vec{\eta}_i(\tau)\mid}\,\partial_r\,\Gamma_r^{(1)} + \frac{3c^3}{16\pi G}\partial_r\,\Gamma_r^{(1)}\,\Big(\sum_c \frac{\partial_c^2}{\Delta}\,\Gamma_c^{(1)}\Big)\Big](\tau,\vec{\sigma}) + \\
&\quad - \int d^3\sigma\,\partial_r\,\Big[\frac{c^3}{16\pi G}\Big[2\Big(\Gamma_r^{(1)}(\tau,\vec{\sigma})\Big)^2 - \sum_s \Big(\Gamma_s^{(1)}(\tau,\vec{\sigma})\Big)^2 - \frac{1}{2}\Big(\sum_c \frac{\partial_c^2}{\Delta}\,\Gamma_c^{(1)}(\tau,\vec{\sigma})\Big)^2\Big] - \\
&\quad - \frac{G}{8\pi\, c^3}\sum_{i\neq j}\eta_i\eta_j\, \frac{\sqrt{m_i^2c^2+\vec{\kappa}_i^2(\tau)}\sqrt{m_j^2c^2+\vec{\kappa}_j^2(\tau)}}{\mid \vec{\sigma}-\vec{\eta}_i(\tau)\mid\mid \vec{\sigma}-\vec{\eta}_j(\tau)\mid}\ \Big] + Mc\,L\,O(\zeta^3) \approx 0.
\end{aligned}
\tag{2.13}
$$

At the lowest order they reduce to the special relativistic internal Poincare' generators in the rest-frame instant form of Ref.[4]. They are $p^o_{(1)} = M_{(1)}c = \sum_i \eta_i \sqrt{m_i^2c^2+\vec{\kappa}_i^2(\tau)}$, $p^r_{(1)} = \sum_i \eta_i\,\kappa_{ir}(\tau) \approx 0$, $j^{rs}_{(1)} = \sum_i \eta_i \Big(\eta_i^r(\tau)\,\kappa_{is}(\tau) - \eta_i^s(\tau)\,\kappa_{ir}(\tau)\Big)$, $j^{\tau r}_{(1)} = \sum_i \eta_i\,\eta_i^r(\tau)\,\sqrt{m_i^2c^2+\vec{\kappa}_i^2(\tau)} \approx 0$. The conditions $j^{\tau r}_{(1)} \approx 0$ and $p^r_{(1)} \approx 0$ are the rest-frame conditions eliminating the 3-center of mass and its conjugate 3-momentum inside the 3-spaces of the rest frame. As shown in Ref.[4] in special relativity (and also in PM canonical gravity) there is a decoupled external (canonical but not covariant) 4-center of mass to be used as collective variable.

Eqs.(2.12) and (2.13) show that the Lorentz generators depend both on the local and non-local York times at the second order.

For the effective Hamiltonian for 3-orthogonal gauges, not used in this paper, see Eq.(4.26) of paper II.

E. The 4-Metric, the Triads and Cotriads, the Σ_τ-Adapted Tetrads and Cotetrads

Eqs. (1.5) and (1.7) imply the following expression for triads, cotriads, tetrads, cotetrads and the 4-metric ($l^A_{(o)} = (1;0)$; $\epsilon\, l_{(o)A} = (1;0)$)

$$\begin{aligned}
{}^3\bar{e}^r_{(1)(a)} &= \delta^r_a\,(1 - \Gamma^{(1)}_r - 2\,\phi_{(1)}), \qquad {}^3\bar{e}_{(1)(a)r} = \delta_{ar}\,(1 + \Gamma^{(1)}_r + 2\,\phi_{(1)}),\\
{}^4\mathop{\bar{E}}\limits^{\circ}{}^A_{(1)(o)} &= l^A_{(o)} + l^A_{(1)} = \Big(1 - n_{(1)}; -\delta^r_a\,\bar{n}_{(1)(a)}\Big), \qquad {}^4\mathop{\bar{E}}\limits^{\circ}{}^A_{(1)(a)} = \Big(0; {}^3\bar{e}^r_{(1)(a)}\Big),\\
{}^4\mathop{\bar{E}}\limits^{\circ}{}^{(o)}_{(1)A} &= \epsilon\,(l_{(o)A} + l_{(1)A}) = (1 + n_{(1)})\,\Big(1;0\Big), \qquad {}^4\mathop{\bar{E}}\limits^{\circ}{}^{(a)}_{(1)A} = \Big(\bar{n}_{(1)(a)}; {}^3\bar{e}_{(a)r}\Big),\\
\epsilon\,{}^4g_{(1)\tau\tau} &= 1 + 2\,n_{(1)} = 1 + \frac{8\pi\,G}{c^3}\,\frac{1}{\triangle}\,\Big(\mathcal{M}^{(UV)}_{(1)} + \sum_a T^{aa}_{(1)}\Big) - 2\,\partial_\tau\,{}^3\mathcal{K}_{(1)},\\
\epsilon\,{}^4g_{(1)\tau r} &= -\bar{n}_{(1)(r)} = -\partial_r\,{}^3\mathcal{K}_{(1)} - \frac{4\pi\,G}{c^3}\,\frac{1}{\triangle}\,\Big(4\,\mathcal{M}^{(UV)}_r - \frac{\partial_r}{\triangle}\,\sum_c \partial_c\,\mathcal{M}^{(UV)}_c\Big) -\\
&- \frac{1}{2}\,\frac{\partial_r}{\triangle}\,\partial_\tau\,\Big(4\,\Gamma^{(1)}_r - \sum_c \frac{\partial^2_c}{\triangle}\,\Gamma^{(1)}_c\Big),\\
-\epsilon\,{}^4g_{(1)rs} &= {}^3g_{(1)rs} = \delta_{rs}\,[1 + 2\,(\Gamma^{(1)}_r + 2\,\phi_{(1)})] =\\
&= \delta_{rs}\,\Big[1 - \frac{8\pi\,G}{c^3}\,\frac{1}{\triangle}\,\mathcal{M}^{(UV)}_{(1)} + 2\,\Gamma^{(1)}_r + \sum_c \frac{\partial^2_c}{\triangle}\,\Gamma^{(1)}_c\Big].
\end{aligned} \tag{2.14}$$

The tetrads ${}^4\mathop{\bar{E}}\limits^{\circ}{}^A_{(1)(\alpha)}$ are adapted to the 3-spaces: ${}^4\mathop{\bar{E}}\limits^{\circ}{}^A_{(1)(o)} = l^A_{(o)} + l^A_{(1)}$ is the normal to Σ_τ. While the triads and the 3-metric in Σ_τ are independent from the non-local York time, the 4-metric components ${}^4g_{(1)\tau\tau}$, ${}^4g_{(1)\tau r}$ and the tetrads depend upon it.

F. The PM 4-Christoffel Symbols and the PM 4-Riemann and 4-Weyl Tensors

By using the PM linearized 4-metric given in Eq.(2.14) we can evaluate the Christoffel symbols and the Riemann and Weyl tensors of these PM space-times and study the properties of the Riemannian instantaneous 3-spaces. While the terms containing $\mathcal{M}^{(UV)}_{(1)}$, $\mathcal{M}^{(UV)}_{(1)r}$, $T^{rs}_{(1)}$, correspond to action-at-a-distance contributions, the terms containing $\Gamma^{(1)}_a = \sum_{\bar{a}} \gamma_{\bar{a}r}\,R_{\bar{a}}$ denote retarded GW contributions. The non-fixed gauge part is given by the terms depending upon ${}^3\mathcal{K}_{(1)} = \frac{1}{\triangle}\,{}^3K_{(1)}$.

1. The PM Christoffel Symbols

The Christoffel symbols and their linearization have the following expressions in our gauges [4] (${}^3\Gamma^u_{rs}$ is the Christoffel symbol built with the 3-metric ${}^3g_{rs}$ of the 3-space)

$$
\begin{aligned}
{}^4\Gamma^A_{BC} &= \frac{1}{2}\, {}^4g^{AE}\, (\partial_B\, {}^4g_{CE} + \partial_C\, {}^4g_{BE} - \partial_E\, {}^4g_{BC}) = \\
&= {}^4\Gamma^A_{(1)BC} + O(\zeta^2) = \frac{1}{2}\, {}^4\eta^{AE}\, (\partial_B\, {}^4g_{(1)CE} + \partial_C\, {}^4g_{(1)BE} - \partial_E\, {}^4g_{(1)BC}) + O(\zeta^2),
\end{aligned}
$$

$$
\begin{aligned}
{}^4\Gamma^\tau_{\tau\tau} &= \frac{1}{1+n}\Big(\partial_\tau\, n + \bar{n}_{(a)}\, {}^3\bar{e}^r_{(a)}\, \partial_r\, n - \bar{n}_{(a)}\, \bar{n}_{(b)}\, {}^3\bar{e}^s_{(a)}\, {}^3K_{rs}\, {}^3\bar{e}^r_{(b)}\Big) = \\
&= {}^4\Gamma^\tau_{(1)\tau\tau} + O(\zeta^2) = \partial_\tau\, n_{(1)} = \frac{4\pi\, G}{c^3}\, \frac{1}{\triangle}\, \partial_\tau\, \Big(\mathcal{M}^{(UV)}_{(1)} + \sum_a T^{aa}_{(1)}\Big) - \partial^2_\tau\, {}^3\mathcal{K}_{(1)} + O(\zeta^2), \\
{}^4\Gamma^\tau_{\tau r} &= \frac{1}{1+n}\Big(\partial_r\, n - {}^3K_{rs}\, {}^3e^s_{(a)}\, \bar{n}_{(a)}\Big) = \\
&= {}^4\Gamma^\tau_{(1)\tau r} + O(\zeta^2) = \partial_r\, n_{(1)} + \frac{4\pi\, G}{c^3}\, \frac{\partial_r}{\triangle}\, \Big(\mathcal{M}^{(UV)}_{(1)} + \sum_a T^{aa}_{(1)}\Big) - \partial_r\, \partial_\tau\, {}^3\mathcal{K}_{(1)} + O(\zeta^2), \\
{}^4\Gamma^\tau_{rs} &= -\frac{1}{1+n}\, {}^3K_{rs} = \\
&= {}^4\Gamma^\tau_{(1)rs} + O(\zeta^2) = -\frac{1}{2}\, (\partial_r\, \bar{n}_{(1)(s)} + \partial_s\, \bar{n}_{(1)(r)}) + \delta_{rs}\, \partial_\tau\, (\Gamma^{(1)}_r + 2\, \phi_{(1)}) + O(\zeta^2) = \\
&= -\frac{4\pi\, G}{c^3}\, \frac{1}{\triangle}\, \Big(2\, (\partial_r\, \mathcal{M}^{(UV)}_{(1)s} + \partial_s\, \mathcal{M}^{(UV)}_{(1)r}) - \frac{\partial_r\, \partial_s}{\triangle}\, \sum_c \partial_c\, \mathcal{M}^{(UV)}_{(1)c} + \delta_{rs}\, \partial_\tau\, \mathcal{M}^{(UV)}_{(1)}\Big) + \\
&+ \delta_{rs}\, \partial_\tau\, \Gamma^{(1)}_r - \frac{\partial_r\, \partial_s}{\triangle}\, \partial_\tau\, (\Gamma^{(1)}_r + \Gamma^{(1)}_s) + \frac{1}{2}\, (\delta_{rs} + \frac{\partial_r\, \partial_s}{\triangle})\, \partial_\tau\, \sum_c \frac{\partial^2_c}{\triangle}\, \Gamma^{(1)}_c - \\
&- \partial_r\, \partial_s\, {}^3\mathcal{K}_{(1)} + O(\zeta^2),
\end{aligned}
$$

---

[4] For the evaluation of ${}^4\Gamma^u_{\tau\tau}$ we need the kinematical Hamilton equations for the cotriads given in Ref.[8]. If we are in a Schwinger time-gauge with $\alpha_{(a)}(\tau, \vec{\sigma}) \neq 0$, we have to add to ${}^4\Gamma^u_{\tau\tau}$ the term $-\frac{1}{2}\sum_{abc}\, {}^3e^u_{(a)}\, \epsilon_{(a)(b)(c)}\, \hat{\mu}_{(b)}\, n_{(c)}$, where $\hat{\mu}_{(a)}(\tau, \vec{\sigma}) \circeq \partial_\tau\, \alpha_{(a)}(\tau, \vec{\sigma})$ is the Dirac multiplier in front of the rotation primary constraint.

$$
\begin{aligned}
{}^4\Gamma^u_{\tau\tau} &\stackrel{\circ}{=} \sum_a {}^3\bar{e}^u_{(a)} \Big[\partial_\tau \bar{n}_{(a)} - \frac{\partial_\tau n}{1+n} \bar{n}_{(a)} + \sum_{bcrs} \bar{n}_{(b)} \bar{n}_{(c)} {}^3\bar{e}^r_{(a)} {}^3\bar{e}^s_{(b)} (\partial_s {}^3\bar{e}_{(c)r} - \partial_r {}^3\bar{e}_{(c)s}) + \\
&+ \sum_{bcrs} {}^3e^r_{(b)} {}^3e^s_{(c)} n_{(b)} n_{(c)} \frac{n_{(a)}}{1+n} {}^3K_{rs} + (1+n) \sum_{rb} {}^3e^r_{(b)} \Big(\delta_{(a)(b)} - \frac{n_{(a)} n_{(b)}}{(1+n)^2}\Big) \partial_r n - \\
&- \frac{1}{2} \sum_{rb} n_{(b)} \Big({}^3e^r_{(a)} \partial_r n_{(b)} + {}^3e^r_{(b)} \partial_r n_{(a)}\Big)\Big] = \\
&= {}^4\Gamma^u_{(1)\tau\tau} + O(\zeta^2) = \partial_\tau \bar{n}_{(1)(u)} + \partial_u n_{(1)} + O(\zeta^2) = \\
&= \frac{4\pi G}{c^3} \Big[\frac{\partial_\tau}{\triangle} (4 \mathcal{M}^{(UV)}_{(1)u} - \frac{\partial_u}{\triangle} \sum_c \partial_c \mathcal{M}^{(UV)}_{(1)c}) + \frac{\partial_u}{\triangle} (\mathcal{M}^{(UV)}_{(1)} + \sum_a T^{aa}_{(1)})\Big] + \\
&+ \frac{1}{2} \frac{\partial_u}{\triangle} \partial^2_\tau (4 \Gamma^{(1)}_u - \sum_c \frac{\partial^2_c}{\triangle} \Gamma^{(1)}_c) + O(\zeta^2), \\
{}^4\Gamma^u_{\tau r} &= {}^3\bar{e}^u_{(a)} \Big[- \frac{\partial_r n}{1+n} \bar{n}_{(a)} - (1+n) (\delta_{(a)(b)} - \frac{\bar{n}_{(a)} \bar{n}_{(b)}}{(1+n)^2}) {}^3K_{rs} {}^3\bar{e}^s_{(b)} + {}^3e^s_{(a)} {}^3e_{(b)r} \partial_s n_{(b)} - \\
&- \frac{1}{2} \Big({}^3\bar{e}^v_{(a)} (\partial_r {}^3\bar{e}_{(b)v} - \partial_v {}^3\bar{e}_{(b)r}) - {}^3\bar{e}^v_{(b)} (\partial_r {}^3\bar{e}_{(a)v} - \partial_v {}^3\bar{e}_{(a)r}) + \\
&+ {}^3\bar{e}^v_{(a)} {}^3\bar{e}_{(c)r} {}^3\bar{e}^s_{(b)} (\partial_v {}^3\bar{e}_{(c)s} - \partial_s {}^3\bar{e}_{(c)v})\Big) \bar{n}_{(b)}\Big] = \\
&= {}^4\Gamma^u_{(1)\tau r} + O(\zeta^2) = \delta_{ur} \partial_\tau (\Gamma^{(1)}_r + 2 \phi_{(1)}) + \frac{1}{2} (\partial_r \bar{n}_{(1)(u)} - \partial_u \bar{n}_{(1)(r)}) + O(\zeta^2) = \\
&= \frac{8\pi G}{c^3} \frac{1}{\triangle} \Big(\partial_r \mathcal{M}^{(UV)}_{(1)(u)} - \partial_u \mathcal{M}^{(UV)}_{(1)(r)} - \frac{1}{2} \delta_{ur} \partial_\tau \mathcal{M}^{(UV)}_{(1)}\Big) + \\
&+ \delta_{ur} \partial_\tau \Big(\Gamma^{(1)}_r + \frac{1}{2} \sum_c \frac{\partial^2_c}{\triangle}\Big) - \frac{\partial_r \partial_u}{\triangle} \partial_\tau (\Gamma^{(1)}_r - \Gamma^{(1)}_u) + O(\zeta^2), \\
{}^4\Gamma^u_{rs} &= {}^3\Gamma^u_{rs} + \frac{\bar{n}_{(a)}}{1+n} {}^3\bar{e}^u_{(a)} {}^3K_{rs} = \\
&= {}^4\Gamma^u_{(1)rs} + O(\zeta^2) = {}^3\Gamma^u_{(1)rs} + O(\zeta^2) = \\
&= \delta_{ur} \partial_s (\Gamma^{(1)}_u + 2 \phi_{(1)}) + \delta_{us} \partial_r (\Gamma^{(1)}_u + 2 \phi_{(1)}) - \delta_{rs} \partial_u (\Gamma^{(1)}_r + 2 \phi_{(1)}) + O(\zeta^2) = \\
&= - \frac{4\pi G}{c^3} \frac{\delta_{ur} \partial_s + \delta_{us} \partial_r - \delta_{rs} \partial_u}{\triangle} \mathcal{M}^{(UV)}_{(1)} + \\
&+ (\delta_{ur} \partial_s + \delta_{us} \partial_r) \Gamma^{(1)}_u - \delta_{rs} \partial_u \Gamma^{(1)}_r + \frac{\delta_{ur} \partial_s + \delta_{us} \partial_r - \delta_{rs} \partial_u}{2 \triangle} \sum_c \partial^2_c \Gamma^{(1)}_c + O(\zeta^2).
\end{aligned}
\tag{2.15}
$$

2. The PM Riemann and Ricci Tensors

The 4-Riemann tensor and its linearization have the following expressions

$$
\begin{aligned}
{}^4R_{ABCD} &= {}^4g_{AE}\, {}^4R^E{}_{BCD} = \\
&= -\frac{1}{2}\Big(\partial_A\,\partial_c\, {}^4g_{BD} + \partial_B\,\partial_D\, {}^4g_{AC} - \partial_A\,\partial_D\, {}^4g_{BC} - \partial_B\,\partial_C\, {}^4g_{AD}\Big) + \\
&+ {}^4g_{EF}\Big({}^4\Gamma^E_{AD}\, {}^4\Gamma^F_{BC} - {}^4\Gamma^E_{AC}\, {}^4\Gamma^F_{BD}\Big) = \\
&= {}^4R_{(1)ABCD} + O(\zeta^2) = {}^4\eta_{AE}\, {}^4R^E_{(1)BCD} + O(\zeta^2) = \\
&= -\frac{1}{2}\,(\partial_A\,\partial_C\, {}^4g_{(1)BD} + \partial_B\,\partial_D\, {}^4g_{(1)AC} - \partial_A\,\partial_D\, {}^4g_{(1)BC} - \partial_B\,\partial_C\, {}^4g_{(1)AD}) + O(\zeta^2),
\end{aligned}
$$

$$
\begin{aligned}
{}^4R_{(1)rsuv} &= -\epsilon\, {}^3R_{(1)rsuv} = \\
&= -\epsilon\Big[\delta_{rv}\,\partial_s\,\partial_u\,(\Gamma_r^{(1)} + 2\,\phi_{(1)}) - \delta_{ru}\,\partial_s\,\partial_v\,(\Gamma_r^{(1)} + 2\,\phi_{(1)}) + \\
&+ \delta_{su}\,\partial_r\,\partial_v\,(\Gamma_s^{(1)} + 2\,\phi_{(1)}) - \delta_{sv}\,\partial_r\,\partial_u\,(\Gamma_s^{(1)} + 2\,\phi_{(1)})\Big] = \\
&= -\epsilon\Big[-\frac{4\pi\, G}{c^3}\,\frac{(\delta_{rv}\,\partial_u - \delta_{ru}\,\partial_v)\,\partial_s - (\delta_{sv}\,\partial_u - \delta_{su}\,\partial_v)\,\partial_r}{\triangle}\,\mathcal{M}^{(UV)}_{(1)} + \\
&+ (\delta_{rv}\,\partial_u - \delta_{ru}\,\partial_v)\,\partial_s\,\Gamma_r^{(1)} - (\delta_{sv}\,\partial_u - \delta_{su}\,\partial_v)\,\partial_r\,\Gamma_s^{(1)} + \\
&+ \frac{(\delta_{rv}\,\partial_u - \delta_{ru}\,\partial_v)\,\partial_s - (\delta_{sv}\,\partial_u - \delta_{su}\,\partial_v)\,\partial_r}{2\,\triangle}\,\sum_c \partial_c^2\,\Gamma_c^{(1)}\Big],
\end{aligned}
$$

$$
\begin{aligned}
{}^4R_{(1)\tau ruv} &= \epsilon\Big[(\delta_{rv}\,\partial_u - \delta_{ru}\,\partial_v)\,\partial_\tau\,(\Gamma_r^{(1)} + 2\,\phi_{(1)}) + \frac{1}{2}\,\partial_r\,(\partial_v\,\bar{n}_{(1)(u)} - \partial_u\,\bar{n}_{(1)(v)})\Big] = \\
&= \epsilon\Big[\frac{4\pi\, G}{c^3}\,\frac{1}{\triangle}\Big((\delta_{ru}\,\partial_v - \delta_{rv}\,\partial_u)\,\partial_\tau\,\mathcal{M}^{(UV)}_{(1)} - 2\,\partial_r\,(\partial_u\,\mathcal{M}^{(UV)}_{(1)v} - \partial_v\,\mathcal{M}^{(UV)}_{(1)u})\Big) + \\
&+ \partial_\tau\Big((\delta_{rv}\,\partial_u - \delta_{ru}\,\partial_v)\,(\Gamma_r^{(1)} + \frac{1}{2}\,\sum_c \frac{\partial_c^2}{\triangle}\,\Gamma_c^{(1)}) + \frac{\partial_r\,\partial_u\,\partial_v}{\triangle}\,(\Gamma_u^{(1)} - \Gamma_v^{(1)})\Big)\Big],
\end{aligned}
$$

$$
\begin{aligned}
{}^4R_{(1)\tau r\tau s} &= -\frac{\epsilon}{2}\Big(2\,\partial_r\,\partial_s\,n_{(1)} - 2\,\delta_{rs}\,\partial_\tau^2\,(\Gamma_r^{(1)} + 2\,\phi_{(1)}) + \partial_\tau\,(\partial_r\,\bar{n}_{(1)(s)} + \partial_s\,\bar{n}_{(1)(r)})\Big) = \\
&= \epsilon\Big[-\frac{4\pi\, G}{c^3}\,\frac{1}{\triangle}\Big(\partial_r\,\partial_s\,(\mathcal{M}^{(UV)}_{(1)} + \sum_a T^{aa}_{(1)}) + \delta_{rs}\,\partial_\tau^2\,\mathcal{M}^{(UV)}_{(1)} + \\
&+ \partial_\tau\Big[2\,(\partial_r\,\mathcal{M}^{(UV)}_{(1)s} + \partial_s\,\mathcal{M}^{(UV)}_{(1)r}) - \frac{\partial_r\,\partial_s}{\triangle}\sum_c \partial_c\,\mathcal{M}^{(UV)}_{(1)c}\Big]\Big) + \\
&+ \partial_\tau^2\Big(\delta_{rs}\,(\Gamma_r^{(1)} + \frac{1}{2}\,\sum_c \frac{\partial_c^2}{\triangle}\,\Gamma_c^{(1)}) - \frac{\partial_r\,\partial_s}{\triangle}\,(\Gamma_r^{(1)} + \Gamma_s^{(1)} - \frac{1}{2}\,\sum_c \frac{\partial_c^2}{\triangle}\,\Gamma_c^{(1)})\Big)\Big] = \\
&= \epsilon\Big[-\frac{4\pi\, G}{c^3}\,\frac{1}{\triangle}\Big(\partial_r\,\partial_s\,(\mathcal{M}^{(UV)}_{(1)} + \sum_a T^{aa}_{(1)}) + \delta_{rs}\,\partial_\tau^2\,\mathcal{M}^{(UV)}_{(1)} + \\
&+ \partial_\tau\Big[2\,(\partial_r\,\mathcal{M}^{(UV)}_{(1)s} + \partial_s\,\mathcal{M}^{(UV)}_{(1)r}) - \frac{\partial_r\,\partial_s}{\triangle}\sum_c \partial_c\,\mathcal{M}^{(UV)}_{(1)c}\Big]\Big) - \\
&- \frac{1}{2}\,\partial_\tau^2\, {}^4h^{TT}_{(1)rs}.
\end{aligned}
\tag{2.16}
$$

The final expression of ${}^4R_{(1)\tau r\tau s}$ has been obtained by using Eq.(6.12) of paper II and has been used in Eq.(7.39) of paper II. Let us remark that the Riemann tensor does not depend upon the York time ${}^3K$.

For the 4-Ricci tensor and the 4-curvature scalar we have ($\Box = \partial_\tau^2 - \triangle$)

$$
\begin{aligned}
{}^4R_{(1)AB} &= {}^4\eta^{EF}\,{}^4R_{(1)EAFB} = \epsilon\,\Big({}^4R_{(1)\tau A\tau B} - \sum_r {}^4R_{(1)rArB}\Big),\\
{}^4R_{(1)} &= {}^4\eta^{AB}\,{}^4R_{(1)AB} = \epsilon\,\Big({}^4R_{(1)\tau\tau} - \sum_r {}^4R_{(1)rr}\Big),
\end{aligned}
$$

$$
\begin{aligned}
{}^4R_{(1)\tau\tau} &= -6\,\partial_\tau^2\,\phi_{(1)} + \triangle\,n_{(1)} + \sum_r \partial_\tau\,\partial_r\,\bar{n}_{(1)(r)} =\\
&= \frac{4\pi\,G}{c^3}\,\Big((1 + 3\,\frac{\partial_\tau^2}{\triangle})\,\mathcal{M}^{(UV)}_{(1)} + \sum_a T^{aa}_{(1)} + 3\,\frac{\partial_\tau}{\triangle}\,\sum_c \partial_c\,\mathcal{M}^{(UV)}_{(1)c}\Big),\\
{}^4R_{(1)\tau r} &= \partial_\tau\,\partial_r\,(\Gamma_r^{(1)} - 4\,\phi_{(1)}) + \frac{1}{2}\,\sum_s \partial_s\,(\partial_r\,\bar{n}_{(1)(s)} - \partial_s\,\bar{n}_{(1)(r)}) =\\
&= \frac{8\pi\,G}{c^3}\,\Big(\frac{\partial_\tau\,\partial_r}{\triangle}\,\mathcal{M}^{(UV)}_{(1)} + \sum_s \frac{\partial_s}{\triangle}\,(\partial_r\,\mathcal{M}^{(UV)}_{(1)s} - \partial_s\,\mathcal{M}^{(UV)}_{(1)r})\Big),\\
{}^4R_{(1)rs} &= \partial_r\,\partial_s\,(-n_{(1)} + \Gamma_r^{(1)} + \Gamma_s^{(1)} - 2\,\phi_{(1)}) + \delta_{rs}\,(\partial_\tau^2 - \triangle)\,(\Gamma_r^{(1)} + 2\,\phi_{(1)}) -\\
&- \frac{1}{2}\,\partial_\tau\,(\partial_r\,\bar{n}_{(1)(s)} + \partial_s\,\bar{n}_{(1)(r)}) =\\
&= -\frac{1}{2}\,\Box\,{}^4h^{TT}_{(1)rs} + \frac{4\pi\,G}{c^3}\,\Big(-\,\delta_{rs}\,\frac{\Box}{\triangle}\,\mathcal{M}^{(UV)}_{(1)} - \frac{\partial_r\,\partial_s}{\triangle}\,\sum_a T^{aa}_{(1)} -\\
&- 2\,\frac{\partial_\tau}{\triangle}\,(\partial_r\,\mathcal{M}^{(UV)}_{(1)s} + \partial_s\,\mathcal{M}^{(UV)}_{(1)r}) + \frac{\partial_r\,\partial_s\,\partial_\tau}{\triangle^2}\,\sum_c \partial_c\,\mathcal{M}^{(UV)}_{(1)c}\Big),
\end{aligned}
$$

$$
\begin{aligned}
{}^4R_{(1)} &= 2\,\Big(-\sum_r \partial_r^2\,\Gamma_r^{(1)} + \triangle\,n_{(1)} + \sum_r \partial_\tau\,\partial_r\,\bar{n}_{(1)(r)} + 8\,\triangle\,\phi_{(1)} - 12\,\partial_\tau^2\,\phi_{(1)}\Big) =\\
&= -\frac{8\pi\,G}{c^3}\,\Big((1 - 3\,\frac{\partial_\tau^2}{\triangle})\,\mathcal{M}^{(UV)}_{(1)} - \sum_a T^{aa}_{(1)} - 3\,\frac{\partial_\tau}{\triangle}\,\sum_c \partial_c\,\mathcal{M}^{(UV)}_{(1)c}\Big).
\end{aligned}
\tag{2.17}
$$

By using Eqs.(1.12), it can be checked that Einstein equations ${}^4R_{AB} - \frac{1}{2}\,{}^4g_{AB}\,{}^4R \overset{\circ}{=} \frac{8\pi\,G}{c^3}\,T_{AB}$ are verified, namely we have ${}^4R_{(1)AB} - \frac{1}{2}\,{}^4\eta_{AB}\,{}^4R_{(1)} \overset{\circ}{=} \frac{8\pi\,G}{c^3}\,T_{(1)AB} + O(\zeta^2)$.

3. *The PM Weyl Tensor*

For the Weyl tensor and its electric and magnetic components with respect to the Eulerian observers, whose unit 4-velocity l^A is the normal to the 3-space Σ_τ with the zeroth order expression $l^A_{(o)} = (l^\tau_{(o)} 1; l^r_{(o)} = 0)$ [see Eqs.(2.14)], we have

$$
\begin{aligned}
{}^4C_{ABCD} &= {}^4R_{ABCD} - \frac{1}{2}\,({}^4g_{AC}\,{}^4R_{BD} + {}^4g_{BD}\,{}^4R_{AC} - {}^4g_{AD}\,{}^4R_{BC} - {}^4g_{BC}\,{}^4R_{AD}) + \\
&+ \frac{1}{6}\,({}^4g_{AC}\,{}^4g_{BD} - {}^4g_{AD}\,{}^4g_{BC})\,{}^4R = {}^4C_{(1)ABCD} + O(\zeta^2),
\end{aligned}
$$

$$
\begin{aligned}
&{}^4C_{ABCD} = {}^4C_{CDAB} = -{}^4C_{BACD} = -{}^4C_{ABDC}, \\
&{}^4C_{ABCD} + {}^4C_{ADBC} + {}^4C_{ACDB} = 0,
\end{aligned}
$$

$$
\begin{aligned}
{}^4C_{(1)\tau r\tau s} &= {}^4R_{(1)\tau r\tau s} - \frac{\epsilon}{2}\Big({}^4R_{(1)rs} - \delta_{rs}\,{}^4R_{(1)\tau\tau}\Big) - \frac{1}{6}\,\delta_{rs}\,{}^4R_{(1)} = \\
&= -\frac{1}{4}\,(\Box - \triangle)\,{}^4h^{TT}_{(1)rs} + \frac{4\pi\,G}{c^3}\Big[(\frac{1}{3}\,\delta_{rs} - \frac{\partial_r\,\partial_s}{\triangle})\,(\mathcal{M}^{(UV)}_{(1)} + \frac{1}{2}\,\sum_a T^{aa}_{(1)}) + \\
&+ \frac{1}{2}\,(\delta_{rs} + \frac{\partial_r\,\partial_s}{\triangle})\,\sum_c \frac{\partial_c\,\partial_\tau}{\triangle}\,\mathcal{M}^{(UV)}_{(1)c} - \frac{\partial_\tau}{\triangle}\,(\partial_r\,\mathcal{M}^{(UV)}_{(1)s} + \partial_s\,\mathcal{M}^{(UV)}_{(1)r})\Big], \\
{}^4C_{(1)\tau ruv} &= {}^4R_{(1)\tau ruv} + \frac{\epsilon}{2}\Big(\delta_{rv}\,{}^4R_{(1)\tau u} - \delta_{ru}\,{}^4R_{(1)\tau v}\Big) = \\
&= \partial_\tau\,\Big[(\delta_{rv}\,\partial_u - \delta_{ru}\,\partial_v)\,(\Gamma^{(1)}_r + \frac{1}{2}\,\sum_c \frac{\partial_c^2}{\triangle}\,\Gamma^{(1)}_c) + \frac{\partial_r\,\partial_u\,\partial_v}{\triangle}\,(\Gamma^{(1)}_u - \Gamma^{(1)}_v)\Big] + \\
&+ \frac{4\pi\,G}{c^3}\,\frac{1}{\triangle}\,\Big[\sum_c\,[\delta_{ru}\,(\delta_{vc} - \partial_v\,\partial_c) - \delta_{rv}\,(\delta_{uc} - \partial_u\,\partial_c)]\,\mathcal{M}^{(UV)}_{(1)c} - 2\,\partial_r\,(\partial_u\,\mathcal{M}^{(UV)}_{(1)v} - \partial_v\,\mathcal{M}^{(UV)}_{(1)u})\Big],
\end{aligned}
$$

$$
\begin{aligned}
{}^4C_{(1)rsuv} &= {}^4R_{(1)rsuv} + \frac{\epsilon}{2}\Big(\delta_{ru}\,{}^4R_{(1)sv} + \delta_{sv}\,{}^4R_{(1)ru} - \\
&- \delta_{rv}\,{}^4R_{(1)su} - \delta_{su}\,{}^4R_{(1)rv}\Big) + \frac{1}{6}\Big(\delta_{ru}\,\delta_{sv} - \delta_{rv}\,\delta_{su}\Big)\,{}^4R_{(1)} = \\
&= -\frac{1}{4}\,\Box\,(\delta_{ru}\,{}^4h^{TT}_{(1)rv} + \delta_{sv}\,{}^4h^{TT}_{(1)ru} - \delta_{rv}\,{}^4h^{TT}_{(1)su} - \delta_{su}\,{}^4h^{TT}_{(1)rv}) - \\
&- (\delta_{rv}\,\partial_u\,\partial_s + \delta_{su}\,\partial_v\,\partial_r - \delta_{ru}\,\partial_v\,\partial_s - \delta_{sv}\,\partial_u\,\partial_r)\,(\Gamma^{(1)}_s + \frac{1}{2}\,\sum_c \frac{\partial_c^2}{\triangle}\,\Gamma^{(1)}_c) + \\
&+ \frac{4\pi\,G}{c^3}\Big[(\frac{2}{3}\,(\delta_{ru}\,\delta_{sv} - \delta_{rv}\,\delta_{su}) + \delta_{rv}\,\frac{\partial_u\,\partial_s}{\triangle} + \delta_{su}\,\frac{\partial_v\,\partial_r}{\triangle} - \delta_{ru}\,\frac{\partial_v\,\partial_s}{\triangle} - \delta_{sv}\,\frac{\partial_u\,\partial_r}{\triangle})\,\mathcal{M}^{(UV)}_{(1)} + \\
&+ (\frac{1}{3}\,(\delta_{ru}\,\delta_{sv} - \delta_{rv}\,\delta_{su}) + 2\,\delta_{rv}\,\frac{\partial_u\,\partial_s}{\triangle} + 2\,\delta_{su}\,\frac{\partial_v\,\partial_r}{\triangle} - 2\,\delta_{ru}\,\frac{\partial_v\,\partial_s}{\triangle} - 2\,\delta_{sv}\,\frac{\partial_u\,\partial_r}{\triangle})\,\sum_a T^{aa}_{(1)} + \\
&+ (\delta_{ru}\,\delta_{sv} - \delta_{rv}\,\delta_{su} - 2\,\delta_{rv}\,\frac{\partial_u\,\partial_s}{\triangle} - 2\,\delta_{su}\,\frac{\partial_v\,\partial_r}{\triangle} + 2\,\delta_{ru}\,\frac{\partial_v\,\partial_s}{\triangle} + 2\,\delta_{sv}\,\frac{\partial_u\,\partial_r}{\triangle})\,\sum_c \frac{\partial_\tau\,\partial_c}{\triangle}\,\mathcal{M}^{(UV)}_{(1)c} + \\
&+ \frac{\partial_\tau}{\triangle}\Big(\delta_{ru}\,(\partial_s\,\mathcal{M}^{(UV)}_{(1)v} + \partial_v\,\mathcal{M}^{(UV)}_{(1)s}) + \delta_{sv}\,(\partial_r\,\mathcal{M}^{(UV)}_{(1)u} + \partial_u\,\mathcal{M}^{(UV)}_{(1)r}) - \\
&- \delta_{rv}\,(\partial_s\,\mathcal{M}^{(UV)}_{(1)u} + \partial_u\,\mathcal{M}^{(UV)}_{(1)s}) - \delta_{su}\,(\partial_r\,\mathcal{M}^{(UV)}_{(1)v} + \partial_v\,\mathcal{M}^{(UV)}_{(1)r})\Big)\Big],
\end{aligned}
$$

$$
\begin{aligned}
E^A_{(1)B} &= {}^4\eta^{AC}\, {}^4C_{(1)CEBF}\, l^E_{(o)}\, l^F_{(o)} = {}^4\eta^{AC}\, {}^4C_{(1)C\tau B\tau} = -\epsilon \sum_{rs} \delta^{Ar}\, \delta_{Bs}\, {}^4C_{(1)\tau r\tau s},\\
H_{(1)AB} &= \frac{1}{2}\, \epsilon_{AMCD}\, l^M_{(o)}\, {}^4\eta^{DE}\, {}^4\eta^{CF}\, {}^4C_{(1)EFBG}\, l^G_{(o)} = \frac{1}{2} \sum_{rsuv} \delta_{As}\, \delta_{Br}\, \epsilon_{suv}\, C_{(1)\tau nrs} =\\
&= \sum_{rsuv} \delta_{As}\, \delta_{Br} \Big(\partial_\tau \Big[\epsilon_{rsu}\, \partial_u\, (\Gamma^{(1)}_r + \frac{1}{2} \sum_c \frac{\partial^2_c}{\triangle}\, \Gamma^{(1)}_c) + \epsilon_{suv}\, \frac{\partial_r\, \partial_u\, \partial_v}{\triangle}\, \Gamma^{(1)}_u\Big] -\\
&- \frac{4\pi\, G}{c^3}\, \frac{1}{\triangle} \Big[\epsilon_{rsu} \sum_c (\delta_{uc} - \partial_u\, \partial_c)\, \mathcal{M}^{(UV)}_{(1)c} + \epsilon_{suv}\, \partial_r\, (\partial_u\, \mathcal{M}^{(UV)}_{(1)v} - \partial_v\, \mathcal{M}^{(UV)}_{(1)u})\Big]\Big).
\end{aligned}
\tag{2.18}
$$

Their Newtonian limit, in particular the vanishing of $H_{(1)AB}$, is consistent with Ref.[9].

G. The 4-Spin and 3-Spin Connections

The 4-spin connection ${}^4\omega_{A\,(\beta)}^{\;(\alpha)}$ associated with the general tetrads ${}^4E^A_{(\alpha)}$ is connected with the Σ_τ-adapted one ${}^4\overset{\circ}{\omega}_{A\,(\beta)}^{\;(\alpha)}$ by means of the Lorentz boosts with parameters $\varphi_{(a)}$ [8]. The expressions of these 4-spin connections are ($\hat{\mu}_{(a)}(\tau,\vec{\sigma}) \overset{\circ}{=} \partial_\tau\, \alpha_{(a)}(\tau,\vec{\sigma})$ are the Dirac multipliers in front of the primary first class rotation constraints; they vanish in the gauges $\alpha_{(a)}(\tau,\vec{\sigma}) \approx 0$; ${}^3K_{rs}$ is given in Eqs.(2.3) of paper II)

$$
\begin{aligned}
{}^4\omega_A{}^{(\alpha)}{}_{(\beta)} &= {}^4E^{(\alpha)}_B \Big(\partial_A\, {}^4E^B_{(\beta)} + {}^4\Gamma^B_{AC}\, {}^4E^C_{(\beta)}\Big) =\\
&= \Big[L(\varphi_{(a)})\, {}^4\overset{o}{\omega}_A\, L^{-1}(\varphi_{(a)}) + \partial_A\, L(\varphi_{(a)})\, L^{-1}(\varphi_{(a)})\Big]^{(\alpha)}{}_{(\beta)},\\
{}^4\overset{o}{\omega}_A{}^{(\alpha)}{}_{(\beta)} &= {}^4\overset{\circ}{E}{}^{(\alpha)}_B \Big(\partial_A\, {}^4\overset{\circ}{E}{}^B_{(\beta)} + \Gamma^B_{AC}\, {}^4\overset{\circ}{E}{}^C_{(\beta)}\Big),
\end{aligned}
$$

$$
\begin{aligned}
{}^4\overset{o}{\omega}_\tau{}^{(o)}{}_{(a)} &= \epsilon\, {}^4\overset{o}{\omega}_{\tau(o)(a)} = {}^3e^r_{(a)}\, \partial_r\, n - {}^3e^r_{(a)}\, {}^3K_{rs}\, {}^3e^s_{(b)}\, n_{(b)},\\
{}^4\overset{o}{\omega}_\tau{}^{(a)}{}_{(b)} &= -\epsilon\, {}^4\overset{o}{\omega}_{\tau(a)(b)} = -\epsilon_{(a)(b)(c)}\, \hat{\mu}_{(c)} +\\
&+ \sum_u \Big({}^3e^u_{(b)}\, \partial_u\, n_{(a)} - {}^3e^u_{(a)}\, \partial_u\, n_{(b)} + \sum_{sc} n_{(c)}\, {}^3e^s_{(c)}\, {}^3e^u_{(b)}\, \partial_s\, {}^3e_{(a)u}\Big) -\\
&- \frac{1}{2} \sum_c n_{(c)} \sum_{su} \Big[{}^3e^s_{(c)} \Big({}^3e^u_{(b)}\, \partial_u\, {}^3e_{(a)s} - {}^3e^u_{(a)}\, \partial_u\, {}^3e_{(b)s}\Big) - {}^3e^u_{(a)}\, {}^3e^s_{(b)} \Big(\partial_u\, {}^3e_{(c)s} - \partial_s\, {}^3e_{(c)u}\Big)\Big],\\
{}^4\overset{o}{\omega}_r{}^{(o)}{}_{(a)} &= \epsilon\, {}^4\overset{o}{\omega}_{r(o)(a)} = -\epsilon\, {}^4\overset{o}{\omega}_{r(a)(o)} = -{}^3K_{rs}\, {}^3e^s_{(a)},\\
{}^4\overset{o}{\omega}_r{}^{(a)}{}_{(b)} &= -\epsilon\, {}^4\overset{o}{\omega}_{r(a)(b)} = {}^3\omega_{r(a)(b)}.
\end{aligned}
\tag{2.19}
$$

The 3-spin connection [5] ${}^3\omega_{r(a)(b)} = {}^4\overset{\circ}{\omega}_{r(a)(b)}$ with ${}^3\omega_{r(a)(b)}$ depending on triads and cotriads and also on the angles $\alpha_{(a)}$ is connected to the 3-spin connection ${}^3\bar{\omega}_{r(a)(b)}$ in Schwinger time gauges by local SO(3) rotations $R(\alpha_{(a)})$ ($R^T = R^{-1}$) and has the expression in gauges near the 3-orthogonal ones having $\theta^i(\tau,\vec{\sigma}) = \theta^i_{(1)}(\tau,\vec{\sigma}) = O(\zeta) \neq 0$ (we use $V_{ra}(\theta^i) = \delta_{ra} - \sum_i \epsilon_{rai}\, \theta^i_{(1)} + O(\zeta^2)$, see before Eqs.(2,9) of paper I)

$$
{}^3\omega_{r(a)(b)} \;=\; \epsilon_{(a)(b)(c)}\, {}^3\omega_{r(c)} = \Big[R(\alpha_{(e)})\, {}^3\bar{\omega}_r\, R^T(\alpha_{(e)}) + R(\alpha_{(e)})\, \partial_r\, R^T(\alpha_{(e)}) \Big]_{(a)(b)},
$$

$$
\begin{aligned}
{}^3\bar{\omega}_{r(a)} \;&=\; \frac{1}{2} \sum_{bc} \epsilon_{(a)(b)(c)}\, {}^3\bar{\omega}_{r(b)(c)} = \\
&=\; \frac{1}{2} \sum_{bcu} \epsilon_{(a)(b)(c)}\, {}^3\bar{e}^u_{(b)} \Big[\partial_r\, {}^3\bar{e}_{(c)u} - \partial_u\, {}^3\bar{e}_{(c)r} + \sum_{dv} {}^3\bar{e}^v_{(c)}\, {}^3\bar{e}_{(d)r}\, \partial_v\, {}^3\bar{e}_{(d)u} \Big] = \\
&=\; {}^3\bar{\omega}_{(1)r(a)} + O(\zeta^2) = \\
&=\; \frac{1}{2} \sum_{bc} \epsilon_{(a)(b)(c)} \Big[(\delta_{rb}\, \partial_c - \delta_{rc}\, \partial_b)\, (\Gamma^{(1)}_r + 2\, \phi_{(1)}) - \sum_i \epsilon_{(b)(c)(i)}\, \theta^i_{(1)} \Big] + O(\zeta^2),\\
{}^3\bar{\omega}_{(1)r(a)(b)} \;&=\; \epsilon_{(a)(b)(c)}\, {}^3\bar{\omega}_{(1)r(c)}. \qquad (2.20)
\end{aligned}
$$

Once we know the PM 3-spin connection ${}^3\bar{\omega}_{(1)r(a)(b)}$ in the 3-orthogonal Schwinger time gauges, we can go to non-Schwinger gauges near the 3-orthogonal ones (with $\alpha_{(a)}(\tau,\vec{\sigma}) \neq 0$, $\varphi_{(a)}(\tau,\vec{\sigma}) \neq 0$, $\theta^i(\tau,\vec{\sigma}) = \theta^i_{(1)}(\tau,\vec{\sigma}) = O(\zeta) \neq 0$) and find (we use Eqs. (1.9), (2.5) and (2.14) and $n_{(a)} = \sum_b R_{(a)(b)}(\alpha_{(c)})\, \bar{n}_{(b)}$, ${}^3e^r_{(a)} = \sum_b R_{(a)(b)}(\alpha_{(c)})\, {}^3\bar{e}^r_{(b)}$)

$$
\begin{aligned}
{}^4\overset{o}{\omega}_r{}^{(a)}{}_{(b)} \;&=\; {}^3\omega_{r(a)(b)} = \epsilon_{(a)(b)(c)}\, {}^3\omega_{r(c)} = \Big[R(\alpha_{(e)})\, \partial_r\, R^T(\alpha_{(e)}) + R(\alpha_{(e)})\, {}^3\bar{\omega}_r\, R^T(\alpha_{(e)}) \Big]_{(a)(b)} = \\
&=\; \Big[R(\alpha_{(e)})\, \partial_r\, R^T(\alpha_{(e)}) + R(\alpha_{(e)})\, {}^3\bar{\omega}_{(1)r}\, R^T(\alpha_{(e)}) \Big]_{(a)(b)} + O(\zeta^2),\\
{}^4\overset{o}{\omega}_r{}^{(o)}{}_{(a)} \;&=\; \sum_b R_{(a)(b)}(\alpha_{(e)})\, {}^3K_{(1)rs} + O(\zeta^2),\\
{}^4\overset{o}{\omega}_\tau{}^{(a)}{}_{(b)} \;&=\; -\sum_c \epsilon_{(a)(b)(c)}\, \partial_\tau\, \alpha_{(c)} + \sum_{cd} R_{(a)(d)}(\alpha_{(e)})\, R_{(b)(c)}(\alpha_{(e)})\, (\partial_c\, \bar{n}_{(1)(d)} - \partial_d\, \bar{n}_{(1)(c)}) + \\
&+\; \sum_{cd} \Big(R_{(b)(c)}(\alpha_{(e)})\, \partial_c\, R_{(a)(d)} - R_{(a)(c)}\, \partial_c\, R_{(b)(d)} \Big)\, \bar{n}_{(1)(d)} + O(\zeta^2),\\
{}^4\overset{o}{\omega}_\tau{}^{(o)}{}_{(a)} \;&=\; \sum_b R_{(a)(b)}(\alpha_{(e)})\, \partial_b\, n_{(1)} + O(\zeta^2). \qquad (2.21)
\end{aligned}
$$

---

[5] It is defined by the vanishing of the generalized covariant derivative acting on both types of indices of the triad: $\partial_r\, {}^3e^u_{(a)} + {}^3\Gamma^u_{rs}\, {}^3e^s_{(a)} + {}^3\omega_{r(a)(b)}\, {}^3e^u_{(b)} = {}^3\nabla_r\, {}^3e^u_{(a)} + {}^3\omega_{r(a)(b)}\, {}^3e^u_{(b)} = 0$, so that the 3-Christoffel symbols (see also the last of Eqs.(2.15)) have the expression ${}^3\Gamma^u_{rs} = \frac{1}{2} \Big[{}^3\bar{e}^u_{(a)} \Big({}^3\bar{e}_{(b)r}\, {}^3\bar{\omega}_{s(a)(b)} + {}^3\bar{e}_{(b)s}\, {}^3\bar{\omega}_{r(a)(b)} \Big) - \Big({}^3\bar{e}_{(a)r}\, \partial_s\, {}^3\bar{e}^u_{(a)} + {}^3\bar{e}_{(a)s}\, \partial_r\, {}^3\bar{e}^u_{(a)} \Big) \Big]$.

From Eqs. (3.5) we have ${}^3K_{(1)rs} = \sigma_{(1)(r)(s)}|_{r\neq s} + \delta_{rs}\Big(\frac{1}{3}\,{}^3K - \partial_\tau\,\Gamma_r^{(1)} + \partial_r\,\bar{n}_{(1)(r)} - \sum_a\,\partial_a\,\bar{n}_{(1)(a)}\Big)$ with $\sigma_{(1)(r)(s)}|_{r\neq s}$ given in Eq. (2.5) after the solution of the constraints.

For the relation between the 4-field strength and the 4-curvature tensors and between their 3-dimensional analogues we have

$$\begin{aligned}
{}^4\Omega_{AB}{}^{(\alpha)}{}_{(\beta)} &= {}^4E^{(\gamma)}_A\,{}^4E^{(\delta)}_B\,{}^4\Omega^{(\alpha)}{}_{(\beta)(\gamma)(\delta)} = {}^4R^C{}_{DAB}\,{}^4E^{(\alpha)}_C\,{}^4E^D_{(\beta)} = \\
&= \partial_A{}^4\omega_B{}^{(\alpha)}{}_{(\beta)} - \partial_B\,{}^4\omega_A{}^{(\alpha)}{}_{(\beta)} + {}^4\omega_A{}^{(\alpha)}{}_{(\gamma)}\,{}^4\omega_B{}^{(\gamma)}{}_{(\beta)} - {}^4\omega_B{}^{(\alpha)}{}_{(\gamma)}\,{}^4\omega_A{}^{(\gamma)}{}_{(\beta)},\\
{}^4R^A{}_{BCD} &= {}^4E^A_{(\gamma)}\,{}^4E^{(\delta)}_B\,{}^4\Omega_{CD}{}^{(\gamma)}{}_{(\delta)},\\
{}^3\Omega_{rs(a)} &= \frac{1}{2}\epsilon_{(a)(b)(c)}\,{}^3\Omega_{rs(b)(c)} = \frac{1}{2}\,\epsilon_{(a)(b)(c)}\,{}^3e_{(b)t}\,{}^3e^w_{(c)}\,{}^3R^t{}_{wrs} = \\
&= \partial_r\,{}^3\omega_{s(a)} - \partial_s\,{}^3\omega_{r(a)} - \epsilon_{(a)(b)(c)}\,{}^3\omega_{r(b)}\,{}^3\omega_{s(c)},\\
{}^3R^r{}_{stw} &= \epsilon_{(a)(b)(c)}\,{}^3e^r_{(a)}\,\delta_{(b)(n)}\,{}^3e^{(n)}_s\,{}^3\Omega_{tw(c)}, \qquad {}^3R_{rsuv} = \epsilon_{(a)(b)(c)}\,{}^3e_{(a)r}\,{}^3e_{(b)s}\,{}^3\Omega_{uv(c)}.
\end{aligned} \tag{2.22}$$

The first Bianchi identity ${}^3R^t{}_{rsu} + {}^3R^t{}_{sur} + {}^3R^t{}_{urs} \equiv 0$ implies the cyclic identity ${}^3\Omega_{rs(a)}\,{}^3e^s_{(a)} \equiv 0$.

H. The Ashtekar Variables in the York Canonical Basis and their PM Limit

As shown in Ref. [8, 10], the canonical basis ${}^3e_{(a)r}$, $\pi^r_{(a)}$, formed by the cotriads on Σ_τ and by their conjugate momenta (see Eq.(2.8) of paper I) can be replaced by the following canonical basis of Ashtekar's variables (γ is the Immirzi parameter; we use the conventions of Ref.[11]; $Q_a = e^{\sum_{\bar{a}}\,\gamma_{\bar{a}a}\,R_{\bar{a}}}$)

$$\begin{aligned}
{}^3\mathcal{E}^r_{(a)} &= {}^3e\,{}^3e^r_{(a)} = \tilde{\phi}^{2/3}\sum_b R_{(a)(b)}(\alpha_{(e)})\,V_{ra}(\theta^i)\,e^{-\sum_{\bar{a}}\,\gamma_{\bar{a}a}\,R_{\bar{a}}},\\
{}^3A_{(\gamma)(a)r} &= {}^3\omega_{r(a)} + \gamma\,{}^3e^s_{(a)}\,{}^3K_{rs},
\end{aligned} \tag{2.23}$$

with ${}^3\omega_{r(a)}$ of Eqs.(2.20) and with ${}^3K_{rs}$ of Eq.(2.3) of paper II. This formalism is usually defined in the Schwinger time gauges $\varphi_{(a)}(\tau,\vec{\sigma}) \approx 0$ of ADM tetrad gravity.

In Ref.[11] it is shown that we have $\{A_{(\gamma)(a)r}(\tau,\vec{\sigma}), A_{(\gamma)(b)s}(\tau,\vec{\sigma}_1)\} = 0$ due to the results $\{{}^3K_{(a)r}(\tau,\vec{\sigma}), {}^3\omega_{s(b)}(\tau,\vec{\sigma}_1)\} = \{{}^3K_{(b)s}(\tau,\vec{\sigma}), {}^3\omega_{r(a)}(\tau,\vec{\sigma}_1)\} = 0$ (${}^3K_{(a)r} = {}^3K_{rs}\,{}^3e^s_{(a)}$), which are a consequence of the fact that ${}^3e^r_{(a)}\,\delta\,{}^3\omega_{r(a)}$ is a pure divergence (this implies ${}^3\omega_{r(a)}(\tau,\vec{\sigma}) = [{}^3e_{(b)r}\,{}^3\omega_{(b)(a)}](\tau,\vec{\sigma}) = \frac{\delta}{\delta\,{}^3e^r_{(a)}(\tau,\vec{\sigma})}\int d^3\sigma_1\,\sum_b\,{}^3\omega_{(b)(b)}(\tau,\vec{\sigma}_1))$. We also have $\{{}^3A_{(\gamma)(a)r}(\tau,\vec{\sigma}), {}^3\mathcal{E}^s_{(b)}(\tau,\vec{\sigma}_1)\} = \gamma\,\delta^s_r\,\delta_{(a)(b)}\,\delta^3(\vec{\sigma}-\vec{\sigma}_1)$.

The SO(3) connection $A_{(\gamma)(a)r}$ is considered as a SU(2) connection with field strength ${}^3F_{(\gamma)(a)rs} = \partial_r\,{}^3A_{(\gamma)(a)s} - \partial_s\,{}^3A_{(\gamma)(a)r} + \epsilon_{(a)(b)(c)}\,{}^3A_{(\gamma)(b)r}\,{}^3A_{(\gamma)(c)s}$. Instead the true

SO(3) connection, associated with the O(3) subgroup of the Lorentz group O(3,1), is $\frac{1}{2}\,\epsilon_{(a)(b)(c)}\,\Big[R(\alpha_{(e)})\,\partial_r\,R^T(\alpha_{(e)})\Big]_{(b)(c)} + R_{(b)(m)}(\alpha_{(e)})\,{}^3\bar{\omega}_{r(m)(n)}\,R^T_{(m)(c)}(\alpha_{(e)})\Big)$.

Instead the densitized triad ${}^3\mathcal{E}^r_{(a)}$ is considered an analogue of an electric field.

In the Ashstekar formalism the non-abelianized rotation constraint ${}^3M_{(a)}(\tau,\vec{\sigma}) \approx 0$ of Ref.[3], whose Abelianized form in the York canonical basis is $\pi^{(\alpha)}_{(a)} = -\sum_b {}^3M_{(b)}\,A_{(b)(a)}(\alpha_{(e)}) \approx 0$, is replaced by the Gauss law constraint $G_{(a)} = \sum_r \partial_r\,{}^3\mathcal{E}^r_{(a)} + \epsilon_{(a)(b)(c)}\,{}^3A_{(\gamma)(b)r}\,{}^3\mathcal{E}^r_{(c)} \approx 0$. The super-Hamiltonian constraint $\mathcal{H}(\tau,\vec{\sigma}) \approx 0$ takes the form $({}^3e)^{-2}\,\sum_{ars}\,\Big[{}^3F_{(\gamma)(a)rs} - (1+\gamma^2)\,\sum_{bc}\,\epsilon_{(a)(b)(c)}\,{}^3K_{r(b)}\,{}^3K_{s(c)}\Big]\,\sum_{mn}\,\epsilon_{(a)(m)(n)}\,{}^3\mathcal{E}^r_{(m)}\,{}^3\mathcal{E}^s_{(n)} + \gamma^{-1}\,(1+\gamma^2)\,\sum_{ar}\,G_{(a)}\,\partial_r\,\Big[({}^3e)^{-2}\,{}^3\mathcal{E}^r_{(a)}\Big] \approx 0$, while the super-momentum constraints become $\gamma^{-1}\,\sum_a\,\Big[\sum_s\,{}^3F_{(\gamma)(a)rs}\,{}^3\mathcal{E}^s_{(a)} - (1+\gamma^2)\,{}^3K_{r(a)}\,G_{(a)}\Big] \approx 0$.

The linearized Ashtekar variables in gauges near the 3-orthogonal ones (with $\alpha_{(a)}(\tau,\vec{\sigma}) \neq 0$, $\varphi_{(a)}(\tau,\vec{\sigma}) \neq 0$, $\theta^i(\tau,\vec{\sigma}) = \theta^i_{(1)}(\tau,\vec{\sigma}) = O(\zeta) \neq 0$) are

$$
\begin{aligned}
{}^3\mathcal{E}^r_{(a)} &= \sum_b R_{(a)(b)}(\alpha_{(e)})\,\Big[(1 - \Gamma_r^{(1)} + 4\,\phi_{(1)})\,\delta_{rb} - \sum_i \epsilon_{rbi}\,\theta^i_{(1)}\Big] + O(\zeta^2),\\
{}^3A_{(\gamma)(a)r} &= \frac{1}{2}\sum_{bc}\epsilon_{(a)(b)(c)}\,\Big(\Big[R(\alpha_{(e)})\,\partial_r\,R^T(\alpha_{(e)})\Big]_{(b)(c)} + \\
&+ \sum_{mn} R_{(b)(m)}(\alpha_{(e)})\,R_{(c)(n)}(\alpha_{(e)})\,\Big[(\delta_{rm}\,\partial_n - \delta_{rn}\,\partial_m)\,(\Gamma_r^{(1)} + 2\,\phi_{(1)}) - \sum_i \epsilon_{mni}\,\partial_r\,\theta^i_{(1)}\Big] + \\
&+ \gamma\sum_b R_{(a)(b)}(\alpha_{(e)})\,\Big[\sigma_{(1)(r)(b)}|_{r\neq b} + \delta_{rb}\,\Big(\frac{1}{3}\,{}^3K - \partial_\tau\,\Gamma_r^{(1)} + \partial_r\,\bar{n}_{(1)(r)} - \sum_a \partial_a\,\bar{n}_{(1)(a)}\Big)\Big]\Big).
\end{aligned}
\tag{2.24}
$$

After having solved the super-Hamiltonian and super-momentum constraints with no fixation of the gauge one has to replace $\phi_{(1)}$ and $\sigma_{(1)(r)(s)}|_{r\neq s}$ with their expressions given in Eqs.(2.2) and (2.5). The expression in the 3-orthogonal Schwinger time-gauges is obtained by putting $\alpha_{(a)}(\tau,\vec{\sigma}) = 0$, $\varphi_{(a)}(\tau,\vec{\sigma}) = 0$, $\theta^i(\tau,\vec{\sigma}) = \theta^i_{(1)}(\tau,\vec{\sigma}) = 0$.

With these results we can find the PM expression of

1) the holonomy along a closed loop Γ, i.e. $P\,e^{\int_\Gamma A} = \sum_{n=0}^\infty\,{}_{1>s_n>..>s_1>0}\,\int ... \int\, A(\Gamma(s_1))..A(\Gamma(s_n))\,ds_1\,..ds_n$, where $A[\Gamma] = \int_\Gamma A = \int_o^1\,ds\,A_{(\gamma)(c)a}(x(s))\,\frac{dx^a(s)}{ds}\,\tau_{(c)}$ ($\tau_{(c)}$ are Pauli matrices);

2) the flux of the electric field across a surface S, i.e. $\int_S\,d^2\sigma\,n_r\,{}^3e^r_{(a)} = E_{(a)}(S)$.

III. THE PM SPACE-TIME AND ITS INSTANTANEOUS 3-SPACES

In this Section we illustrate the general properties of PM Einstein space-times and how their properties depend on the choice of the York time (selecting a member of our family of 3-orthogonal gauges) through the non-local function ${}^3\mathcal{K}_{(1)} = \frac{1}{\triangle}\, {}^3K_{(1)}$. After describing the proper time of a time-like observer and the properties of the non-Euclidean 3-spaces, we compare the results in 3-orthogonal gauges with the IAU conventions in harmonic gauges for the Solar System. Then we study the PM time-like geodesics.

A. The PM Proper Time of a Time-like Observer

Given a time-like observer located in (τ, σ^r) (not too near to the particles), the evaluation of the observer proper time is done with the line element $\epsilon\, ds^2|_{(\tau,\sigma^r)} = \epsilon\, {}^4g_{\tau\tau}(\tau, \sigma^r)\, d\tau^2 = d\,\mathcal{T}^2_{(\tau,\sigma^r)}$. Therefore from Eqs.(2.14) we get

$$\begin{aligned} d\,\mathcal{T}_{(\tau,\sigma^r)} &= \sqrt{\epsilon\, {}^4g_{\tau\tau}(\tau, \sigma^r)}\, d\tau = \sqrt{1 + 2\, n_{(1)}(\tau, \sigma^r)}\, d\tau = \\ &= \Big[1 - \frac{G}{c^3} \sum_i \frac{\eta_i\, \sqrt{m_i^2c^2 + \vec{\kappa}_i^2(\tau)}}{|\vec{\sigma} - \vec{\eta}_i(\tau)|}\, (1 + \frac{\vec{\kappa}_i^2(\tau)}{m_i^2c^2 + \vec{\kappa}_i^2(\tau)}) - \\ &- \partial_\tau\, {}^3\mathcal{K}_{(1)}(\tau, \vec{\sigma})\Big]\, d\tau. \end{aligned} \tag{3.1}$$

As a consequence, the proper time depends on the τ-derivative of the non-local York time ${}^3\mathcal{K}_{(1)}$ at the position of the observer in the 3-space Σ_τ.

B. The Instantaneous PM 3-Spaces Σ_τ

1. *The Spatial 3-Distance on the Instantaneous 3-Space Σ_τ*

Let us consider two points on the instantaneous 3-space Σ_τ (whose intrinsic 3-curvature will be given in Eq.(3.6)) with radar 3-coordinates σ^r_o and σ^r_1. They will be joined by a unique 3-geodesic $\xi^r(\tau, s) = \sigma^r_0 + (\sigma^r_1 - \sigma^r_o)\, s + \xi^r_{(1)}(\tau, s)$, $\xi^r(0) = \sigma^r_o$, $\xi^r(1) = \sigma^r_1$, $\xi^r_{(1)}(0) = \xi^r_{(1)}(1) = 0$, solution of the geodesic equation $\frac{d^2\, \xi^r(\tau,s)}{ds^2} = - \sum_{uv}\, {}^3\Gamma^r_{(1)uv}(\tau, \vec{\xi}(\tau, s))\, \frac{d\xi^u(\tau,s)}{ds}\, \frac{d\xi^v(\tau,s)}{ds}$ with the 3-Christoffel symbol given in Eq.(2.15).

At order $O(\zeta)$ we get the following solution for the 3-geodesic

$$\begin{aligned} \xi^r(\tau, s) &= \sigma^r_o + (\sigma^r_1 - \sigma^r_o)\, s + \\ &+ \sum_{uv} (\sigma^u_1 - \sigma^u_o)\, (\sigma^v_1 - \sigma^v_o)\, \Big(\int_o^1 - \int_o^s\Big)\, ds_1 \int_o^{s_1} ds_2\, {}^3\Gamma^r_{(1)uv}(\tau, \vec{\sigma}_o + (\vec{\sigma}_1 - \vec{\sigma}_o)\, s_2). \end{aligned} \tag{3.2}$$

Since Eqs.(2.14) implies that at the first order the line 3-element joining the two points is

$$
\begin{aligned}
d\mathcal{S}(\tau) &= \sqrt{-\epsilon \sum_{rs} {}^4g_{(1)rs}(\tau, \vec{\xi}(\tau, s))\, \frac{d\xi^r(\tau, s)}{ds}\, \frac{d\xi^s(\tau, s)}{ds}}\, ds = \\
&= \sqrt{\Big(\frac{d\vec{\xi}(\tau, s)}{ds}\Big)^2 + 2 \sum_r (\sigma_1^r - \sigma_o^r)^2\, (2\, \phi_{(1)} + \Gamma_r^{(1)})(\tau, \vec{\xi}(\tau, s))}\, ds,
\end{aligned} \tag{3.3}
$$

the geodesic 3-distance between the two points is ($d_{Euclidean}(\vec{\sigma}_0, \vec{\sigma}_1) = |\vec{\sigma}_1 - \vec{\sigma}_o| = \sqrt{\sum_r (\sigma_1^r - \sigma_o^r)^2}$ is the Euclidean distance with respect to the flat asymptotic 3-metric)

$$
\begin{aligned}
d(\vec{\sigma}_o, \vec{\sigma}_1)(\tau) &= \int_o^1 d\mathcal{S}(\tau) = d_{Euclidean}(\vec{\sigma}_0, \vec{\sigma}_1) + \\
&+ \sum_r \frac{\sigma_1^r - \sigma_o^r}{|\vec{\sigma}_1 - \vec{\sigma}_o|} \int_o^1 ds\, \Big((\sigma_1^r - \sigma_o^r)\, (2\, \phi_{(1)} + \Gamma_r^{(1)})\, (\tau, \vec{\sigma}_o + (\vec{\sigma}_1 - \vec{\sigma}_o)\, s) - \\
&- \sum_s (\sigma_1^s - \sigma_o^s) \int_0^s ds_1\, \Big[2\, (\sigma_1^r - \sigma_o^r)\, \partial_s\, (2\, \phi_{(1)} + \Gamma_r^{(1)}) - \\
&- (\sigma_1^s - \sigma_o^s)\, \partial_r\, (2\, \phi_{(1)} + \Gamma_s^{(1)}) \Big] (\tau, \vec{\sigma}_o + (\vec{\sigma}_1 - \vec{\sigma}_o)\, s_1) \Big).
\end{aligned} \tag{3.4}
$$

As expected it does not depend upon the inertial gauge variable ${}^3\mathcal{K}_{(1)}$ [6].

Let us remark that in general a 3-geodesic of the 3-metric ${}^3g_{(1)rs} = -\epsilon\, {}^4g_{(1)rs}$ on the 3-space Σ_τ is not a space-like geodesics of the 4-metric ${}^4g_{(1)AB}$.

2. The Extrinsic 3-Curvature Tensor

From Eqs.(1.8) and by using $\sum_{\bar{a}} \gamma_{\bar{a}a}\, \gamma_{\bar{a}b} = \delta_{ab} - \frac{1}{3}$, we get that the extrinsic curvature tensor of our 3-spaces in our family of 3-orthogonal gauges is the following first order quantity

$$
{}^3K_{(1)rs}(\tau, \vec{\sigma}) = \sigma_{(1)(r)(s)}|_{r \neq s}(\tau, \vec{\sigma}) + \delta_{rs}\, \Big(\frac{1}{3}\, {}^3K_{(1)} - \partial_\tau\, \Gamma_r^{(1)} + \partial_r\, \bar{n}_{(1)(r)} - \sum_a \partial_a\, \bar{n}_{(1)(a)} \Big)(\tau, \vec{\sigma}), \tag{3.5}
$$

with $\bar{n}_{(1)(r)}$ and $\sigma_{(1)(r)(s)}|_{r \neq s}$ given in Eqs.(2.4) and (2.5), respectively, and with $\Gamma_r^{(1)}$ given by Eq.(2.6). Therefore, our (dynamically determined) 3-spaces have a first order deviation from Euclidean 3-spaces, embedded in the asymptotically flat space-time, determined by both instantaneous inertial matter effects and retarded tidal ones. Moreover the inertial gauge variable ${}^3K_{(1)}$ (non existing in Newtonian gravity) is the free numerical function labeling the members of the family of 3-orthogonal gauges. The extrinsic curvature tensor depends on the local (${}^3K_{(1)}$) and also on the non-local one (${}^3\mathcal{K}_{(1)}$) through the shift function.

---

[6] Instead a space-like 4-geodesic depends on it. Indeed the extrinsic curvature tensor ${}^3K_{rs}$ is a measure, at a point in the 3-space Σ_τ, of the curvature of a space-time geodesic tangent to the 3-geodesic (3.2) at that point, see Refs.[12]

3. The Intrinsic 3-Curvature Tensor

The 3-Riemann tensor is given in Eq.(2.16). The 3-Ricci tensor and the 3-curvature scalar are

$$\begin{aligned}
{}^3R_{(1)rs} &= \sum_u {}^3R_{(1)urus} = -\delta_{rs}\,\triangle\,(\Gamma_r^{(1)} + 2\,\phi_{(1)}) + \partial_r\,\partial_s\,(\Gamma_r^{(1)} + \Gamma_s^{(1)} - 2\,\phi_{(1)}) = \\
&= \frac{4\pi\,G}{c^3}\,(\delta_{rs} + \frac{\partial_r\,\partial_s}{\triangle})\,\mathcal{M}^{(UV)}_{(1)} - \\
&- \delta_{rs}\,\triangle\,(\Gamma_r^{(1)} + \frac{1}{2}\sum_c \frac{\partial_c^2}{\triangle}\,\Gamma_c^{(1)}) + \partial_r\,\partial_s\,(\Gamma_r^{(1)} + \Gamma_s^{(1)} - \frac{1}{2}\sum_c \frac{\partial_c^2}{\triangle}\,\Gamma_c^{(1)}) = \\
&= \frac{4\pi\,G}{c^3}\,(\delta_{rs} + \frac{\partial_r\,\partial_s}{\triangle})\,\mathcal{M}^{(UV)}_{(1)} + \frac{1}{2}\,\triangle\,{}^4h^{TT}_{(1)rs}, \\
{}^3R_{(1)} &= \sum_r {}^3R_{(1)rr} = -8\,\triangle\,\phi_{(1)} + 2\sum_a \partial_a^2\,\Gamma_a^{(1)} = \frac{16\pi\,G}{c^3}\,\mathcal{M}^{(UV)}_{(1)}.
\end{aligned} \tag{3.6}$$

We see that, apart from distributional contributions from the particles, the intrinsic 3-curvature ${}^3R_{(1)}$ of these non-Euclidean 3-spaces is determined only by the tidal variables, i.e. by the PM GW's propagating inside these 3-spaces.

C. Comparison with the Barycentric Celestial Reference System (BCRS) of IAU2000 in the Harmonic Gauge used for the Solar System.

In Refs.[13] there is the 4-metric chosen in the astronomical conventions IAU2000 to describe the Solar System in the Barycentric Celestial Reference System (BCRS) centered in its barycenter by using a PN approximation of Einstein's equations in a special system of harmonic 4-coordinates x^μ_B. The barycenter world-line (a time-like geodesic of the PN 4-metric ${}^4g_{B\mu\nu}(x_B)$) is the time axis $x^\mu_{B(B)}(\tau_B) = \Big(x^o_B(\tau_B); 0^i\Big)$, where τ_B is the proper time of a standard clock in the solar system barycenter, $((d\tau_B)^2 = \epsilon\,g_{Boo}(x_{B(B)})\,(dx^o_B)^2)$. It is approximately a straight line if we neglect galactic and extra-galactic influences. Through each point of this world-line we consider the hyper-surfaces $x^o_B = c\,t_B = const.$ as instantaneous 3-spaces $\Sigma_{x^o_B}$ with *rectangular* 3-coordinates (practically they are the quasi-Euclidean 3-spaces of a quasi-inertial frame of Minkowski space-time, even if they do not correspond to Einstein's 1/2 clock synchronization convention). In each point of the barycenter world-line there is a *tetrad* with the time-like 4-vector given by the barycenter 4-velocity and with the 3 mutually orthogonal *kinematically non-rotating* spatial axes (no systematic rotation with respect to certain fixed stars (radio sources) in the instantaneous 3-spaces $t_B = const.$). This is a *global* reference system, with the following PN solution of Einstein's equations for the 4-metric ${}^4g_{B\mu\nu}(x_B)$ (the potentials w_B and w_{BI} are static and of order G, so that $w_B^2 = O(G^2)$)

$$
\begin{aligned}
{}^4g_{Boo}(x_B) &= \epsilon\,\Big[N_B^2 - {}^3g_B^{ij}\,N_{Bi}\,N_{Bj}\Big](x_B) == \epsilon\,\Big[1 - \frac{2\,w_B}{c^2} - \frac{2\,w_B^2}{c^4} + O(c^{-5})\Big](x_B),\\
{}^4g_{Boi}(x_B) &= -\epsilon\,N_{Bi}(x_B) = -\epsilon\,\Big[\frac{4\,w_{Bi}}{c^3} + O(c^{-5})\Big](x_B),\\
{}^4g_{Bij}(x_B) &= -\epsilon\,{}^3g_{Bij} = -\epsilon\,\Big[(1 + \frac{2\,w_B}{c^2})\,\delta_{ij} + O(c^{-4})\Big](x_B).
\end{aligned}
\qquad (3.7)
$$

Eqs.(3.7) imply an extrinsic curvature tensor ${}^3K_{Bij} = \frac{1}{2N_B}\,(N_{Bi|j} + N_{Bj|i} - \partial_o\,{}^3g_{Bij})$ of order $O(c^{-2})$, but the 3-sub-manifolds $x^o_B = const.$ of space-time (the harmonic 3-spaces) are not specified: one has to solve the inverse problem of finding the 3-sub-manifolds with the given extrinsic curvature tensor.

By comparison let us consider the N particles in non-harmonic 3-orthogonal gauges as the Sun and the planets of the Solar System. Let us neglect gravitational waves (so that the 3-spaces have negligible intrinsic 3-curvature except for a distributional singularity at the particle locations, [see Eqs.(3.6)], where our approximation breaks down). Then by using Eqs.(2.2), (2.3), (2.4), the non-relativistic limit of the 4-metric (2.14) in radar 4-coordinates (see the embedding in the Introduction to get world 4-coordinates like the ones of BCRS) has the following form

$$
\begin{aligned}
{}^4g_{(1)\tau\tau}(\tau,\vec{\sigma}) &= \epsilon\,\Big[1 - \frac{2\,w}{c^2} - \frac{2\,\tilde{w}}{c^4} - 2\,\partial_\tau\,{}^3\mathcal{K}_{(1)} + O(c^{-5})\Big](\tau,\vec{\sigma}),\\
{}^4g_{(1)\tau r}(\tau,\vec{\sigma}) &= -\epsilon\,\Big(\frac{4\,w_r}{c^3} + \partial_r\,{}^3\mathcal{K}_{(1)} + O(c^{-5})\Big)(\tau,\vec{\sigma}),\\
{}^4g_{(1)rs}(\tau,\vec{\sigma}) &= -\epsilon\,\delta_{rs}\,\Big[1 + \frac{2\,w}{c^2} + O(c^{-4})\Big](\tau,\vec{\sigma}),
\end{aligned}
$$

$$
\begin{aligned}
&w(\tau,\vec{\sigma}) = \sum_i w_i(\tau,\vec{\sigma}), \qquad w_i(\tau,\vec{\sigma}) = \eta_i\,\frac{G\,m_i}{|\vec{\sigma} - \vec{\eta}_i(\tau)|}, \qquad \tilde{w}(\tau,\vec{\sigma}) = \sum_i \frac{3\,\vec{\kappa}_i^2(\tau)}{2\,m_i^2\,c^2}\,w_i(\tau,\vec{\sigma}),\\
&w_r(\tau,\vec{\sigma}) = -\frac{G}{2}\,\sum_i \frac{\eta_i}{|\vec{\sigma} - \vec{\eta}_i(\tau)|}\,\Big(\kappa_{ir}(\tau) + \frac{(\sigma^r - \eta_i^r(\tau))\,\vec{\kappa}_i(\tau)\cdot(\vec{\sigma} - \vec{\eta}_i(\tau))}{|\vec{\sigma} - \vec{\eta}_i(\tau)|^2}\Big).
\end{aligned}
\qquad (3.8)
$$

Also in this 3-orthogonal gauge we can get quasi-static potentials (ignoring the motion of the sources and assuming that $\partial_\tau\,{}^3\mathcal{K}_{(1)}$ and $\partial_r\,{}^3\mathcal{K}_{(1)}$ are slowing varying functions of τ) and the same pattern as in Eq.(3.7) till the order $1/c^3$ included. The main difference is that $\tilde{w} \neq w^2 = O(G^2)$. Here w is the Newton potential and w_r the gravito-magnetic one.

If we choose the special 3-orthogonal gauge ${}^3\mathcal{K}_{(1)}(\tau,\vec{\sigma}) = 0$ we recover agreement with the Solar System conventions. Let us remark that the instantaneous 3-spaces are not hyperplanes due to Eq.(3.5), giving the non-vanishing extrinsic curvature tensor ${}^3K_{(1)rs} = O(c^{-3})$ even if ${}^3K_{(1)} = 0$.

See Ref.[14] for the status of knowledge on the possibility of the presence of dark matter or of modifications of gravity in the Solar System for explaining effects like the Pioneer anomaly (to be mimicked by means of ${}^3\mathcal{K}_{(1)}(\tau, \vec{\sigma})$ if needed). Further restrictions on ${}^3\mathcal{K}_{(1)}$ near the Earth will come from the gravito-magnetic Lense-Thirring (or frame-dragging) effect (see Refs.[15], Ref.[16] for Lageos and Ref.[17] for Gravity Probe B) when the experimental errors will become acceptable.

Like in the case of the IAU 4-metric and by assuming that the dependence on the inertial gauge variable ${}^3\mathcal{K}_{(1)}$ is negligible inside the Solar System, by using the 4-metric (3.8) one could reproduce the standard general relativistic effects like the perihelion precession and the deflection of light rays by the Sun [7] also in 3-orthogonal gauges. See Ref.[19] for the derivation of the Shapiro time delay and for the gravitational redshift induced by the geo-potential (by using its multipolar description). With only one body (the Sun) in the limit of spherical symmetry one can find the perihelion advance of planets with the standard method of using the geodesic equation for test particles (see Refs.[15, 18, 20]).

D. PM Time-like Geodesics

Let now us consider a time-like geodesic $y^\mu(s) = z^\mu(\sigma^A(s)) = x^\mu_o + \epsilon^\mu_A\, \sigma^A(s)$ (we use the natural adapted embedding of the Introduction) with affine parameter s and with radar 4-coordinates $\sigma^A(s) = \Big(\tau(s); \sigma^u(s)\Big)$ to be used as the trajectory of a planet or of a star. The tangent to the geodesic is $u^\mu(s) = \frac{dy^\mu(s)}{ds} = \epsilon^\mu_A\, p^A(s)$ with $p^A(s) = \frac{d\sigma^A(s)}{ds}$.

At the first order the parametrization of the geodesic (with 4-velocity $p^A(s)$) and the geodesic equation are

$$
\begin{aligned}
\sigma^A(s) &= \sigma^A_o(s) + \sigma^A_{(1)}(s) + O(\zeta^2), \qquad \sigma^A_o(s) = a^A + b^A\, s,\\
p^A(s) &= \frac{d\,\sigma^A(s)}{ds} = b^A + \frac{\sigma^A_{(1)}(\sigma_o(s))}{ds},\\
\frac{d^2\sigma^A(s)}{ds^2} &= -{}^4\Gamma^A_{(1)BC}(\sigma(s))\, \frac{d\sigma^B(s)}{ds}\, \frac{d\sigma^C(s)}{ds} = -{}^4\Gamma^A_{(1)BC}(\sigma_o(s))\, b^B\, b^C,
\end{aligned}
\tag{3.9}
$$

where $\sigma^\alpha_o(s) = a^\alpha + b^\alpha\, s$ is the flat Minkowski geodesic (with respect to the asymptotic flat 4-metric). The Christoffel symbols are given in Eq.(2.15).

The solution of the geodetic equation is

$$
\sigma^A(s) = a^A + b^A\, s - b^B\, b^C \int_0^s ds_1 \int_0^{s_1} ds_2\, {}^4\Gamma^A_{(1)BC}(a + b\, s_2). \tag{3.10}
$$

As Cauchy data at $s = 0$ we take the position $y^\mu(0) = x^\mu_o + \epsilon^\mu_A\, a^A$ with $a^A = \sigma^A(0) = \sigma^A_o$ and the tangent $u^\mu(0) = \epsilon^\mu_A\, p^A(0)$.

[7] For them a 4-metric approximating the static spherically symmetric Schwartzschild solution is enough: see for instance Ref.[18].

For a time-like geodesics the tangent in the origin satisfies $\epsilon\, u^2(0) = 1$, i.e. $\epsilon\, {}^4g_{(1)AB}(\sigma(0))\, p^A(0)\, p^B(0) = 1$, if the parameter s is the proper time. If $u^i(0) = \mathcal{U}^i$, then we have $u^\mu(0) = (\sqrt{1+\vec{\mathcal{U}}^2}; \mathcal{U}^i)$, $\vec{\mathcal{U}}^2 = \sum_r (\mathcal{U}^r)^2$. Therefore, with $b^r = \mathcal{U}^r$ and with the 4-metric of Eq.(2.14), for future-oriented geodesics the condition $\epsilon\, u^2(0) = 1$ leads to the following result for b^A

$$\begin{aligned}
b^\tau &= \sqrt{1+\vec{\mathcal{U}}^2} + d_{(1)}(\sigma_o),\\
d_{(1)}(\sigma_o) &= -\sqrt{1+\vec{\mathcal{U}}^2}\, \Big[2\, n_{(1)}(\sigma_o) - \frac{1}{2} \sum_r \mathcal{U}^r\, \bar{n}_{(1)(r)}(\sigma_o) +\\
&+ \sum_r (\mathcal{U}^r)^2\, (\Gamma_r^{(1)} + 2\, \phi_{(1)})(\sigma_o)\Big].\\
&\Rightarrow \qquad b^A = b^A_{(o)} + \delta^{A\tau}\, d_{(1)}(\sigma_o), \qquad b^A_{(o)} = (\sqrt{1+\vec{\mathcal{U}}^2}; \mathcal{U}^r).
\end{aligned} \tag{3.11}$$

Therefore, with these Cauchy data and by using Eqs.(2.15), the geodesic and its tangent take the form

$$\begin{aligned}
\tau(s) &= \sigma^\tau(s) = \tau_o + \Big(\sqrt{1+\vec{\mathcal{U}}^2} + d_{(1)}(\sigma_o)\Big)\, s -\\
&- \int_0^s ds_1 \int_0^{s_1} ds_2\, \Big((1+\vec{\mathcal{U}}^2)\, \partial_\tau\, n_{(1)} + 2\, \sqrt{1+\vec{\mathcal{U}}^2}\, \sum_u \mathcal{U}^u\, \partial_u\, n_{(1)} +\\
&+ \sum_{uv} \mathcal{U}^u\, \mathcal{U}^v\, \Big[-\frac{1}{2}\, (\partial_u\, \bar{n}_{(1)(v)} + \partial_v\, \bar{n}_{(1)(u)}) + \delta_{uv}\, \partial_\tau\, (\Gamma_r^{(1)} + 2\, \phi_{(1)})\Big]\Big)(\sigma_o + \mathcal{U}\, s_2) =\\
&\stackrel{def}{=} \tau_{({}^3K=0)}(s) + \tau_{({}^3K)}(s),\\
&\quad \tau_{({}^3K)}(s) = 2\, \sqrt{1+\vec{\mathcal{U}}^2}\, \partial_\tau\, {}^3\mathcal{K}_{(1)}(\sigma_o) - \frac{1}{2} \sum_r \mathcal{U}^r\, \partial_r\, {}^3\mathcal{K}_{(1)}(\sigma_o) -\\
&- \int_0^s ds_1 \int_0^{s_1} ds_2\, \Big(- (1+\vec{\mathcal{U}}^2)\, \partial_\tau^2\, {}^3\mathcal{K}_{(1)} -\\
&- 2\, \sqrt{1+\vec{\mathcal{U}}^2}\, \sum_u \mathcal{U}^u\, \partial_u\, \partial_\tau\, {}^3\mathcal{K}_{(1)} - \sum_{uv} \mathcal{U}^u\, \mathcal{U}^v\, \partial_u\, \partial_v\, {}^3\mathcal{K}_{(1)}\Big)(\sigma_o + \mathcal{U}\, s_2),
\end{aligned}$$

$$
\begin{aligned}
\sigma^r(s) &= \sigma^r(0) + \mathcal{U}^r\, s - \int_0^s ds_1 \int_0^{s_1} ds_2 \Big((1 + \vec{\mathcal{U}}^2)\, (\partial_r\, n_{(1)} + \partial_\tau\, \bar{n}_{(1)(r)}) + \\
&+ 2\sqrt{1 + \vec{\mathcal{U}}^2} \sum_u \mathcal{U}^u \Big[\delta_{ur}\, \partial_\tau\, (\Gamma_r^{(1)} + 2\,\phi_{(1)}) - \frac{1}{2}\, (\partial_r\, \bar{n}_{(1)(u)} - \partial_u\, \bar{n}_{(1)(r)}) \Big] + \\
&+ \sum_{uv} \mathcal{U}^u \mathcal{U}^v \Big[2\,\delta_{ru}\, \partial_v\, (\Gamma_r^{(1)} + 2\,\phi_{(1)}) - \delta_{uv}\, \partial_r\, (\Gamma_u^{(1)} + 2\,\phi_{(1)}) \Big] \Big) (\sigma(0) + \mathcal{U}\, s_2) = \\
&\stackrel{def}{=} \sigma^r_{({}^3K=0)}(s) + \sigma^r_{({}^3K)}(s) = \sigma_{({}^3K=0)}(s), \\
&\quad \sigma^r_{({}^3K)}(s) = 0, \\
&\\
p^A(s) &= b^A_{(o)} + p^A_{(1)}(s), \\
&\\
p^\tau(s) &= \sqrt{1 + \vec{\mathcal{U}}^2} + d_{(1)}(\sigma_o) - \\
&- \int_0^s ds_2 \Big((1 + \vec{\mathcal{U}}^2)\, \partial_\tau\, n_{(1)} + 2\sqrt{1 + \vec{\mathcal{U}}^2} \sum_u \mathcal{U}^u\, \partial_u\, n_{(1)} + \\
&+ \sum_{uv} \mathcal{U}^u \mathcal{U}^v \Big[- \frac{1}{2}\, (\partial_u\, \bar{n}_{(1)(v)} + \partial_v\, \bar{n}_{(1)(u)}) + \delta_{uv}\, \partial_\tau\, (\Gamma_r^{(1)} + 2\,\phi_{(1)}) \Big] \Big) (\sigma_o + \mathcal{U}\, s_2), \\
p^r(s) &= \mathcal{U}^r - \int_0^s ds_2 \Big((1 + \vec{\mathcal{U}}^2)\, (\partial_r\, n_{(1)} + \partial_\tau\, \bar{n}_{(1)(r)}) + \\
&+ 2\sqrt{1 + \vec{\mathcal{U}}^2} \sum_u \mathcal{U}^u \Big[\delta_{ur}\, \partial_\tau\, (\Gamma_r^{(1)} + 2\,\phi_{(1)}) - \frac{1}{2}\, (\partial_r\, \bar{n}_{(1)(u)} - \partial_u\, \bar{n}_{(1)(r)}) \Big] + \\
&+ \sum_{uv} \mathcal{U}^u \mathcal{U}^v \Big[2\,\delta_{ru}\, \partial_v\, (\Gamma_r^{(1)} + 2\,\phi_{(1)}) - \delta_{uv}\, \partial_r\, (\Gamma_u^{(1)} + 2\,\phi_{(1)}) \Big] \Big) (\sigma(0) + \mathcal{U}\, s_2).
\end{aligned}
\tag{3.12}
$$

This is the *trajectory of a massive test particle.*

By using Eqs.(2.2) - (2.4), it turns out that all the dependence of the geodesic upon the non-local York time is contained in the function $\tau_{({}^3K)}(s)$, which contributes with $\frac{d\,\tau_{({}^3K)}(s)}{ds}$ to the component $p^\tau(s)$ of the tangent.

Once the time-like geodesic $\sigma^A(s)$ starting at $\sigma^A_o = \sigma^A(s=0)$ and arriving at $\sigma^A_1 = \sigma^A(s=1)$ is known in terms the 4-metric of Eq.(2.14) and denoted γ_{01}, we can evaluate the HPM expression of the Synge world function (see Refs. [21–23]), i.e. of the two-point function (for a space-like 4-geodesics it has the opposite sign)

$$
\begin{aligned}
\Omega(\sigma_o, \sigma_1) &= \frac{1}{2} \int_{(\Gamma_{01})\, 0}^{1} ds\, \epsilon\, {}^4g_{AB}(\sigma^D(s))\, \frac{d\, \sigma^A(s)}{ds}\, \frac{d\, \sigma^B(s)}{ds} = \\
&= \frac{1}{2} \int_{(\Gamma_{01})\, 0}^{1} ds\, \Big[\sqrt{1 + \vec{\mathcal{U}}^2}\, (\sqrt{1 + \vec{\mathcal{U}}^2} + 2\, p^{\tau}_{(1)}(\sigma(s))) - \\
&- \sum_r \mathcal{U}^r\, (\mathcal{U}^r + 2\, p^r_{(1)}(\sigma(s))) + (1 + \vec{\mathcal{U}}^2)\, (1 + 2\, n_{(1)}(\sigma(s))) - \\
&- 2\sqrt{1 + \vec{\mathcal{U}}^2} \sum_r \mathcal{U}^r\, \bar{n}_{(1)(r)} - 2\, (\mathcal{U}^r)^2\, (\Gamma^{(1)}_r + 2\, \phi_{(1)})(\sigma(s))\Big].
\end{aligned} \tag{3.13}
$$

This is a 4-scalar in both points (the simplest case of bi-tensors [22]) defined in terms of the 4-geodesic distance between them. Its gradients with respect to the end points give the vectors tangent to the 4-geodesic at the end points.

IV. PM NULL GEODESICS, THE RED-SHIFT, THE GEODESIC DEVIATION EQUATION AND THE PM LUMINOSITY DISTANCE

In this Section we study the null geodesics, the red-shift, the geodesic deviation equation and the luminosity distance in PM space-times.

A. The PM Null Geodesics and the Red-Shift

Let us now consider a null geodesic $y^\mu(s) = z^\mu(\sigma^A(s)) = x^\mu_o + \epsilon^\mu_A\, \sigma^A(s)$ through the point $y^\mu_o = y^\mu(0) = x^\mu_o + \epsilon^\mu_A\, \sigma^A(0)$ with $\sigma^A(0) = \sigma^A_o = (\tau_o; \vec{\sigma}_o)$. It will have the form (4.2) with $a^A = \sigma^A_o$.

However now the tangent vector $u^\mu(s) = \epsilon^\mu_A\, p^A(s)$, with $p^A(s) = \frac{d\sigma^A(s)}{ds} = b^A - b^B\, b^C\, \int_0^s ds_2\, {}^4\Gamma^A_{(1)BC}(\sigma_o + b\, s_2)$, is a null vector, $\epsilon\, {}^4g_{(1)AB}(\sigma(s))\, p^A(s)\, p^B(s) = 0$. Therefore we must require the initial condition $\epsilon\, {}^4g_{(1)AB}(\sigma_o)\, p^A(0)\, p^B(0) = \epsilon\, {}^4g_{(1)AB}(\sigma_o)\, b^A\, b^B + O(\zeta^2) = 0$ on $b^A = (b^\tau; b^r)$.

By using Eq.(2.14) we get that to each given value of b^r there are two values of b^τ determined by the following equation

$$[1 + 2\, n_{(1)}(\sigma_o)]\, (b^\tau)^2 - 2\, b^\tau \sum_r b^r\, \bar{n}_{(1)(r)}(\sigma_o) - [\vec{b}^2 + 2 \sum_r (b^r)^2\, (\Gamma^{(1)}_r + 2\, \phi_{(1)})(\sigma_o)] = 0,$$

$$\Downarrow$$

$$b^\tau \;=\; \pm\sqrt{\vec{b}^2} + c_{(1)\pm}(\sigma_o),$$

$$c_{(1)\pm}(\sigma_o) = \mp\sqrt{\vec{b}^2}\, [2\, n_{(1)}(\sigma_o) + \sum_r (b^r)^2\, (\Gamma^{(1)}_r + 2\, \phi_{(1)})(\sigma_o)] + \frac{1}{2} \sum_r b^r\, \bar{n}_{(1)(r)}(\sigma_o),$$

$$b^A = b^A_{(o)\pm} + \delta^{A\tau}\, c_{(1)\pm}(\sigma_o), \qquad b^A_{(o)\pm} = (\pm\sqrt{\vec{b}^2}; b^r). \tag{4.1}$$

Therefore we get the following form of a future-oriented null geodesic emanating from σ^A_o with tangent $b^A = b^A_{(o)+} + \delta^{A\tau}\, c_{(1)+}(\sigma_o)$

$$\sigma^A(s) \;=\; \sigma^A_o + (b^A_{(o)+} + \delta^{A\tau}\, c_{(1)+})\, s - b^B_{(o)+}\, b^C_{(o)+} \int_0^s ds_1 \int_0^{s_1} ds_2\, {}^4\Gamma^A_{(1)BC}(\sigma_o + b_{(o)+}\, s_2),$$

$$
\begin{aligned}
\tau(s) &= \tau_o + (\sqrt{\vec{b}^2} + c_{(1)+}(\sigma_o))\, s - \\
&- \int_0^s ds_1 \int_0^{s_1} ds_2 \Big(\vec{b}^2\, \partial_\tau\, n_{(1)} + 2\sqrt{\vec{b}^2} \sum_u \mathcal{U}^u\, \partial_u\, n_{(1)} + \\
&+ \sum_{uv} b^u\, b^v \Big[-\frac{1}{2}\, (\partial_u\, \bar{n}_{(1)(v)} + \partial_v\, \bar{n}_{(1)(u)}) + \delta_{uv}\, \partial_\tau\, (\Gamma_r^{(1)} + 2\,\phi_{(1)}) \Big] \Big) (\sigma(0) + b_{(o)+}\, s_2) = \\
&\stackrel{def}{=} \tau_{({}^3K=0)}(s) + \tau_{({}^3K)}(s), \\
&\qquad \tau_{({}^3K)}(s) = - \int_0^s ds_1 \int_0^{s_1} ds_2 \Big(-\vec{b}^2\, \partial_\tau^2\, {}^3\mathcal{K}_{(1)} - \\
&- 2\sqrt{\vec{b}^2} \sum_u b^u\, \partial_u\, \partial_\tau\, {}^3\mathcal{K}_{(1)} - \sum_{uv} b^u\, b^v\, \partial_u\, \partial_v\, {}^3\mathcal{K}_{(1)} \Big) (\sigma_o + b_{(o)+}\, s_2),
\end{aligned}
$$

$$
\begin{aligned}
\sigma^r(s) &= \sigma_o^r + b^r\, s - \int_0^s ds_1 \int_0^{s_1} ds_2 \Big(\vec{b}^2\, (\partial_r\, n_{(1)} + \partial_\tau\, \bar{n}_{(1)(r)}) + \\
&+ 2\sqrt{\vec{b}^2} \sum_u b^u \Big[\delta_{ur}\, \partial_\tau\, (\Gamma_r^{(1)} + 2\,\phi_{(1)}) - \frac{1}{2}\, (\partial_r\, \bar{n}_{(1)(u)} - \partial_u\, \bar{n}_{(1)(r)}) \Big] + \\
&+ \sum_{uv} b^u\, b^v \Big[2\,\delta_{ru}\, \partial_v\, (\Gamma_r^{(1)} + 2\,\phi_{(1)}) - \delta_{uv}\, \partial_r\, (\Gamma_u^{(1)} + 2\,\phi_{(1)}) \Big] \Big) (\sigma(0) + b_{(o)+}\, s_2) = \\
&\stackrel{def}{=} \sigma^r_{({}^3K=0)}(s) + \sigma^r_{({}^3K)}(s) = \sigma^r_{({}^3K=0)}(s), \\
&\qquad \sigma^r_{({}^3K)}(s) = 0.
\end{aligned}
\tag{4.2}
$$

This is the trajectory of a *ray of light*.

The tangent to the null geodesic is

$$
\begin{aligned}
p^A(s) &= b^A_{(o)} + p^A_{(1)}(s) = \\
&= b^A_{(o)+} + \delta^{A\tau}\, c_{(1)+}(\sigma_o) - b^B_{(o)+}\, b^C_{(o)+} \int_0^s ds_2\, {}^4\Gamma^A_{(1)BC}(\sigma_o + b_{(o)+}\, s_2), \\
p^A(0) &= b^A_{(o)+} + \delta^{A\tau}\, c_{(1)+}(\sigma_o),
\end{aligned}
$$

$$
\begin{aligned}
p^\tau(s) &= \sqrt{\vec{b}^2} + c_{(1)+}(\sigma_o) - \int_0^s ds_2 \Big(\vec{b}^2\, \partial_\tau\, n_{(1)} + 2\, \sqrt{\vec{b}^2} \sum_u \mathcal{U}^u\, \partial_u\, n_{(1)} + \\
&+ \sum_{uv} b^u\, b^v \Big[-\frac{1}{2} \left(\partial_u\, \bar{n}_{(1)(v)} + \partial_v\, \bar{n}_{(1)(u)}\right) + \delta_{uv}\, \partial_\tau \left(\Gamma_r^{(1)} + 2\, \phi_{(1)}\right)\Big]\Big)(\sigma(0) + b_{(o)+}\, s_2) = \\
&\stackrel{def}{=} \sqrt{\vec{b}^2} + c_{(1)+}(\sigma_o) + p^\tau_{(1)(^3K=0)}(s) + p^\tau_{(1)(^3K)}(s), \\
&\quad p^\tau_{(1)(^3K)}(s) = \int_0^s ds_2 \Big(\vec{b}^2\, \partial^2_\tau\, {}^3\mathcal{K}_{(1)} + \\
&+ 2\, \sqrt{\vec{b}^2} \sum_u b^u\, \partial_u\, \partial_\tau\, {}^3\mathcal{K}_{(1)} + \sum_{uv} b^u\, b^v\, \partial_u\, \partial_v\, {}^3\mathcal{K}_{(1)} \Big)(\sigma_o + b_{(o)+}\, s_2),
\end{aligned}
$$

$$
\begin{aligned}
p^r(s) &= b^r - \int_0^s ds_2 \Big(\vec{b}^2 \left(\partial_r\, n_{(1)} + \partial_\tau\, \bar{n}_{(1)(r)}\right) + \\
&+ 2\, \sqrt{\vec{b}^2} \sum_u b^u \Big[\delta_{ur}\, \partial_\tau \left(\Gamma_r^{(1)} + 2\, \phi_{(1)}\right) - \frac{1}{2} \left(\partial_r\, \bar{n}_{(1)(u)} - \partial_u\, \bar{n}_{(1)(r)}\right)\Big] + \\
&+ \sum_{uv} b^u\, b^v \Big[2\, \delta_{ru}\, \partial_v \left(\Gamma_r^{(1)} + 2\, \phi_{(1)}\right) - \delta_{uv}\, \partial_r \left(\Gamma_u^{(1)} + 2\, \phi_{(1)}\right)\Big]\Big)(\sigma(0) + b_{(o)+}\, s_2) = \\
&\stackrel{def}{=} b^r + p^r_{(1)(^3K=0)}(s),
\end{aligned}
\qquad (4.3)
$$

with $\epsilon\, {}^4g_{(1)AB}(\sigma(s))\, p^A(s)\, p^B(s) = 0 + O(\zeta^2)$.

The point $\sigma^A_1 = \sigma^A(s=1)$ satisfies the equation

$$
\begin{aligned}
(\tau_1 - \tau_o)^2 &- (\vec{\sigma}_1 - \vec{\sigma}_2)^2 = 2\, \sqrt{\vec{b}^2}\, c_{(1)+}(\sigma_o) - \\
&- 2\, b^B_{(o)+}\, b^C_{(o)+} \int_0^1 ds_1 \int_0^{s_1} ds_2 \Big[\sqrt{\vec{b}^2}\, {}^4\Gamma^\tau_{(1)BC} - \sum_r b^r\, {}^4\Gamma^r_{(1)BC}\Big](\sigma_o + b_{(o)+}\, s_2),
\end{aligned}
\qquad (4.4)
$$

which gives an idea of the first order deviation of the null geodesic from the flat one joining the same two points σ^A_o and σ^A_1 on the Minkowski light-cone $\epsilon\, {}^4\eta_{AB}\, (\sigma^A_1 - \sigma^A_o)\, (\sigma^B_1 - \sigma^B_o) = (\sigma_1 - \sigma_o)^2 = (\tau_1 - \tau_o)^2 - \sum_r (\sigma^r_1 - \sigma^r_o)^2 = 0$. See Ref. [24] for the use of a similar equations in the IAU conventions for the definition of the radial velocity of stars.

Let us remark that the already introduced Synge world function $\Omega(\sigma_o, \sigma_1)$ of Eq.(4.5) vanishes when evaluated along a null 4-geodesics joining the two points: therefore $\Omega(\sigma_o, \sigma) = 0$ is the equation of the null cone at the point σ^A_o. If one solves the equation $\Omega(\sigma_o, \sigma_1) = 0$ in τ_1, one can find the emission time transfer function for an electromagnetic signal emitted at τ_o in σ^r_o and absorbed in σ^r_1 and then study *time delays* [23] and their dependence upon the York time.

By using the embedding given in the Introduction we get the following expressions for the end points and the tangent vector

$$y^\mu(s) = x^\mu_o + \epsilon^\mu_A\, \sigma^A(s) = x^\mu_{2(\vec\sigma(s))}(\tau_s = \tau(s)), \qquad y^\mu_o = y^\mu(0) = x^\mu_o + \epsilon^\mu_A\, \sigma^A_o = x^\mu_{1(\vec\sigma_o)}(\tau_o),$$

$$k^\mu(s) = \frac{d\, y^\mu(s)}{ds} = \epsilon^\mu_A\, p^A(s). \tag{4.5}$$

With the PM null geodesics one can study the light deflection from a massive body and the Shapiro time delay (see for instance Ref.[23, 25]): in both cases the main ${}^3\mathcal{K}_{(1)}$-dependence comes from the lapse function $n_{(1)}$.

1. The PM Red-Shift

If $v^\mu_1(0) = \frac{\dot x^\mu_1(\tau_o)}{\sqrt{\epsilon\, \dot x^2_1(\tau_o)}}$ is the unit 4-velocity of the object emitting the ray of light at τ_o and $v^\mu_2(s) = \frac{\dot x^\mu_2(\tau_s)}{\sqrt{\epsilon\, \dot x^2_2(\tau_s)}}$ of the observer detecting it at $\tau_s = \sigma^\tau(s)$, the emitted frequency $\omega(0)$, the absorbed frequency $\omega(s)$ and the red-shift $z(s)$ (see Ref.[25]) have the following PM expressions

$$\begin{aligned}
\omega(0) &= c\, k^\mu(0)\, v_{1\mu}(0) = c\, v_{1\mu}(0)\, \epsilon^\mu_A\, p^A(0) = c\, v_{1\mu}(0)\, \epsilon^\mu_A\, (b^A_{(o)+} + \delta^{A\tau}\, c_{(1)+}(\sigma_o)),\\
\omega(s) &= c\, k^\mu(s)\, v_{2\mu}(s) = c\, v_{2\mu}(s)\, \epsilon^\mu_A\, p^A(s),
\end{aligned}$$

$$\frac{1}{1+z(s)} = \frac{\omega(s)}{\omega(0)} = \frac{v_{2\mu}(s)\, \epsilon^\mu_A\, p^A(s)}{v_{1\mu}(0)\, \epsilon^\mu_A\, p^A(0)},$$

$$\begin{aligned}
z(s) &= 1 - \frac{v_{1\mu}(0)\, \Big(\epsilon^\mu_\tau\, \sqrt{\vec b^2} + \epsilon^\mu_r\, b^r\Big)}{v_{2\mu}(s)\, \Big(\epsilon^\mu_\tau\, \sqrt{\vec b^2} + \epsilon^\mu_r\, b^r\Big)} \times \Big[1 + \frac{v_{1\mu}(o)\, \epsilon^\mu_\tau\, c_{(1)+}(\sigma_o)}{v_{1\mu}(0)\, \Big(\epsilon^\mu_\tau\, \sqrt{\vec b^2} + \epsilon^\mu_r\, b^r\Big)} - \\
&- \frac{v_{2\mu}(s)\, \Big(\epsilon^\mu_\tau\, \Big[c_{(1)+}(\sigma_o) + p^\tau_{(1)({}^3K=0)}(s) + p^\tau_{(1)({}^3K)}(s)\Big] + \epsilon^\mu_r\, p^r_{(1)({}^3K=0)}(s)\Big)}{v_{2\mu}(s)\, \Big(\epsilon^\mu_\tau\, \sqrt{\vec b^2} + \epsilon^\mu_r\, b^r\Big)}\Big].
\end{aligned} \tag{4.6}$$

This equation allows to find the dependence of the red-shift $z(s)$ upon the non-local York time ${}^3\mathcal{K}(\sigma(s))$.

B. The PM Geodesic Deviation Equation along a PM Null Geodesic and the PM Luminosity Distance

In the inertial frames of Minkowski space-time the flat null geodesics joining x^μ_1 to x^μ_2 with $(x_1 - x_2)^2 = 0$ is $x^\mu(p) = x^\mu_1 + (x^\mu_2 - x^\mu_1)\, p$, $k^\mu = \frac{dx^\mu(p)}{dp} = x^\mu_2 - x^\mu_1$: this implies

$|x^o_1 - x^o_2| = \sqrt{(\vec{x}_1 - \vec{x}_2)^2} = d_{Euclidean}(1, 2)$, where $d_{Euclidean}$ is the Euclidean spatial distance between the two points in the instantaneous inertial 3-spaces.

In curved space-time we have to solve the equation for the null geodesics (see the previous Subsection and the Appendix of the first paper in Refs.[19]). However in astrophysics one uses the *luminosity distance* [25] between the emission point on a star and the absorption point on the Earth. We have to find the relation of the luminosity distance with the dynamical spatial distance between the star and the Earth in the dynamical instantaneous PM 3-spaces.

1. The PM Geodesic Deviation Equation

As shown in Ref.[25], to find the luminosity distance between a point (a star) emitting a ray of light (eikonal approximation) and a point (the Earth) where the ray of light (propagating along a null geodesics) is absorbed, we must solve the geodesics deviation equation for nearby null geodesics with the same emission point and propagate the resulting deviation vector to the absorption point.

Let the emitting star S have the world-line $y^\mu_S(\tau(s_S)) = x^\mu_o + \epsilon^\mu_A\, \sigma^A_S(s_S)$ (a time-like geodesic with parameter s_S if the star is considered a test particle) with the unit time-like 4-velocity $v^\mu_S(\tau(s_S)) = \epsilon^\mu_A\, u^A_S(s_S) = \epsilon^\mu_A\, \frac{\sigma^A_S(s_S)}{ds_S}$ (s_S is the proper time). Let $s_S = 0$, with $\sigma^A_S(0) = \sigma^A_o$, be the proper time of the emission point.

Let $y^\mu(s) = \epsilon^\mu_A\, \sigma^A(s)$, $\sigma^A(s) = \sigma^A_o + b^A\, s + \sigma^A_{(1)}(s)$ be the null geodesic (4.2) followed by the emitted ray of light, whose tangent vector $k^\mu(s) = \epsilon^\mu_A\, p^A(s)$, $p^A(s) = b^A + p^A_{(1)}(s)$ ($b^A = (\sqrt{\vec{b}^2}; b^r)$), is given in Eq.(4.3).

At the emission point we have the unit time-like vector $u^A_S(\sigma_o)$ and the null vector $p^A(0) = b^A + \delta^{A\tau}\, c_{(1)}(\sigma_o)$ satisfying $\epsilon\, {}^4g_{(1)AB}(\sigma_o)\, u^A_S(\sigma_o)\, u^B_S(\sigma_o) = 1$ and $\epsilon\, {}^4g_{(1)AB}(\sigma_o)\, p^A(0)\, p^B(0) = 0$ [8], respectively. To form a (non-orthogonal) frame at σ^A_o we must add two space-like vectors $E^A_{S(\lambda)}(\sigma_o)$, $\lambda = 1, 2$ satisfying $\epsilon\, {}^4g_{(1)AB}(\sigma_o)\, u^A_S(\sigma_o)\, E^B_{S(\lambda)} = \epsilon\, {}^4g_{(1)AB}(\sigma_o)\, p^A(0)\, E^B_{S(\lambda)} = 0$ and $\epsilon\, {}^4g_{(1)AB}(\sigma_o)\, E^A_{S(\lambda)}\, E^B_{S(\lambda_1)} = -\delta_{\lambda\lambda_1}$ (they span a 2-plane orthogonal the star velocity and to the tangent to the ray of light at the emission point).

A set of four vectors satisfying these conditions is (${}^4\eta_{AB}\, b^A\, b^B = 0$, ${}^4\eta_{AB}\, b^A\, E^B_{(o)S(\lambda)} = 0$, $\epsilon\, {}^4\eta_{AB}\, b^A\, u^B_{(o)S} = 1$, $\epsilon\, {}^4g_{(1)AB}(\sigma_o)\, p^A(0)\, u^B_S(\sigma_o) = 1 + (c_{(1)} + n_{(1)} - \bar{n}_{(1)(3)})(\sigma_o) = \frac{\omega_S(\sigma_o)}{c}$ with $\omega_S(\sigma_o)$ the emission frequency)

$$
\begin{aligned}
u^A_S(\sigma_o) &= u^A_{(o)S} - \delta^{A\tau}\, n_{(1)}(\sigma_o), \qquad u^A_{(o)S} = (1; 0, 0, 0),\\
p^A(0) &= b^A + \delta^{A\tau}\, c_{(1)}(\sigma_o), \qquad b^A = (1; 0, 0, 1),\\
E^A_{S(\lambda)}(\sigma_o) &= E^A_{(o)S(\lambda)} + E^A_{(1)S(\lambda)}(\sigma_o),
\end{aligned}
$$

[8] With the 4-metric (2.14), we have $\epsilon\, {}^4g_{(1)AB}\, (A^A_{(o)} + A^A_{(1)})\, (B^B_{(o)} + B^B_{(1)}) = A^\tau_{(o)}\, B^\tau_{(o)} - \sum_r A^r_{(o)}\, B^r_{(o)} + 2\, A^\tau_{(o)}\, B^\tau_{(o)}\, n_{(1)} + A^\tau_{(o)}\, B^\tau_{(1)} + A^\tau_{(1)}\, B^\tau_{(o)} - \sum_r (A^\tau_{(o)}\, B^r_{(o)} + A^r_{(o)}\, B^\tau_{(o)})\, \bar{n}_{(1)(r)} - \sum_r (A^r_{(o)}\, B^r_{(1)} + A^r_{(1)}\, B^r_{(o)}) - 2 \sum_r A^r_{(o)}\, B^r_{(o)} (\Gamma^{(1)}_r + 2\, \phi_{(1)})$.

$$
\begin{aligned}
&E^A_{(o)S(\lambda)} = \Big(0; e^1_{(o)S(\lambda)}, e^2_{(o)S(\lambda)}, 0\Big),\\
&E^A_{(1)S(\lambda)}(\sigma_o) = \Big(\sum_{s\neq 3} \bar{n}_{(1)(s)}(\sigma_o)\, e^s_{(o)S(\lambda)}; -(\Gamma^{(1)}_1 + 2\,\phi_{(1)})(\sigma_o)\, e^1_{(o)S(\lambda)},\\
&\quad -(\Gamma^{(1)}_2 + 2\,\phi_{(1)})(\sigma_o)\, e^2_{(o)S(\lambda)}, 0\Big),\\
&\sum_{r\neq 3} e^r_{(o)S(\lambda)}\, e^r_{(o)S(\lambda_1)} = \delta_{\lambda\lambda_1}, \qquad e^3_{(o)S(\lambda)} = 0.
\end{aligned}
\tag{4.7}
$$

Let the absorbing Earth E have the world-line $y^\mu_E(\tau(s_E)) = x^\mu_o + \epsilon^\mu_A\, \sigma^A_E(s_E)$ (a time-like geodesic with parameter s_E if the Earth is considered a test particle) with the unit time-like 4-velocity $v^\mu_E(\tau(s_E)) = \epsilon^\mu_A\, u^A_E(s_E) = \epsilon^\mu_A\, \frac{\sigma^A_E(s_E)}{ds_E}$ (s_E is the proper time). Let $s_E = s_1$, with $\sigma^A_S(s_1) = \sigma^A_1$, be the proper time of the absorption point.

At the absorption point $s = s_1$ we have the unit time-like vector $u^A_E(\sigma_1)$ and the null vector $p^A(s_1) = b^A + p^A_{(1)}(s_1)$, with $p^A_{(1)}(s_1)$ given in Eq.(4.1), satisfying $\epsilon\, {}^4g_{(1)AB}(\sigma_1)\, u^A_E(\sigma_1)\, u^B_E(\sigma_1) = 1$ and $\epsilon\, {}^4g_{(1)AB}(\sigma_1)\, p^A(s_1)\, p^B(s_1) = 0$, respectively.

To form a (non-orthogonal) frame at σ^A_1 we must add two space-like vectors $F^A_{E(\lambda)}(\sigma_1)$, $\lambda = 1,2$ satisfying $\epsilon\, {}^4g_{(1)AB}(\sigma_1)\, u^A_E(\sigma_1)\, F^B_{E(\lambda)} = \epsilon\, {}^4g_{(1)AB}(\sigma_1)\, p^A(s_1)\, F^B_{E(\lambda)} = 0$ and $\epsilon\, {}^4g_{(1)AB}(\sigma_1)\, F^A_{E(\lambda)}\, F^B_{E(\lambda_1)} = -\delta_{\lambda\lambda_1}$ (they span a 2-plane orthogonal the Earth velocity and to the tangent to the ray of light at the absorption point).

A set of four vectors satisfying these conditions is (${}^4\eta_{AB}\, b^A\, b^B = 0$, ${}^4\eta_{AB}\, b^A\, F^B_{(o)E(\lambda)} = 0$, $\epsilon\, {}^4g_{(1)AB}(\sigma_1)\, p^A(s_1)\, u^B_E(\sigma_1) = 1 + (p^\tau_{(1)} + n_{(1)} - \bar{n}_{(1)(3)})(\sigma_1) = \frac{\omega_E(\sigma_1)}{c}$ with $\omega_E(\sigma_1)$ the absorption frequency)

$$
\begin{aligned}
u^A_E(\sigma_1) &= u^A_{(o)E} - \delta^{A\tau}\, n_{(1)}(\sigma_1), \qquad u^A_{(o)E} = (1;0,0,0),\\
p^A(s_1) &= b^A + p^A_{(1)}(s_1), \qquad b^A = (1;0,0,1),\\
F^A_{E(\lambda)}(\sigma_1) &= F^A_{(o)E(\lambda)} + F^A_{(1)E(\lambda)}(\sigma_1),
\end{aligned}
$$

$$
\begin{aligned}
&F^A_{(o)E(\lambda)} = \Big(0; f^1_{(o)E(\lambda)}, f^2_{(o)E(\lambda)}, 0\Big),\\
&F^A_{(1)E(\lambda)}(\sigma_1) = \Big(\sum_{s\neq 3} \bar{n}_{(1)(s)}(\sigma_1)\, f^s_{(o)E(\lambda)}; -(\Gamma^{(1)}_1 + 2\,\phi_{(1)})(\sigma_1)\, f^1_{(o)E(\lambda)},\\
&-(\Gamma^{(1)}_2 + 2\,\phi_{(1)})(\sigma_1)\, f^2_{(o)E(\lambda)}, -\sum_{s\neq 3} p^s_{(1)}(s_1)\, f^s_{(o)E(\lambda)}\Big),\\
&\sum_{r\neq 3} f^r_{(o)E(\lambda)}\, f^r_{(o)E(\lambda_1)} = \delta_{\lambda\lambda_1}, \qquad f^3_{(o)E(\lambda)} = 0.
\end{aligned}
\tag{4.8}
$$

We can choose $e^r_{(o)S(\lambda)} = f^r_{(o)E(\lambda)} = g^r_{(o)(\lambda)}$ with $\sum_{r=1,2} g^r_{(o)(\lambda)}\, g^r_{(o)(\lambda_1)} = \delta_{\lambda\lambda_1}$ and $g^3_{(o)(\lambda} = 0$.

As shown in Ref.[25] the deviation vector $Y^\mu(y(s)) = \epsilon^\mu_A\, Y^A(\sigma(s))$, with $Y^A(\sigma_o) = 0$, along the null geodesic connecting σ^A_o and σ^A_1 has the following properties:

A) it vanishes at σ^A_o;

B) its covariant derivative along the tangent to the null geodesic

$$\frac{D\, Y^A(\sigma(s))}{ds} = p^B(s)\, \Big[\partial_B\, Y^A(\sigma(s)) + {}^4\Gamma^A_{BC}(\sigma(s))\, Y^C(\sigma(s))\Big], \tag{4.9}$$

is orthogonal to the star velocity $u^A_S(\sigma_o)$ and to the tangent $p^a(0)$ to the ray of light at the emission point σ^A_o;

C) its covariant differential along the tangent to the null geodesic is also orthogonal to the Earth velocity $u^A_E(\sigma_1)$ and to the tangent $p^A(s_1)$ to the ray of light at the absorption point σ^A_1.

Therefore we have

$$\begin{aligned} Y^A(\sigma_o) \;&=\; 0,\\ \frac{D\, Y^A(\sigma(s))}{ds}|_{\sigma_o} \;&=\; \sum_{\lambda=1,2} A_{(\lambda)}\, E^A_{S(\lambda)}(\sigma_o),\\ \frac{D\, Y^A(\sigma(s))}{ds}|_{\sigma_1} \;&=\; \sum_{\lambda=1,2} B_{(\lambda)}\, F^A_{E(\lambda)}(\sigma_1), \end{aligned} \tag{4.10}$$

The deviation vector is solution of the geodesic deviation equation

$$\begin{aligned} \frac{D^2\, Y^A(\sigma(s))}{ds^2} \;&=\; p^B(s)\, \Big(\partial_B\, \Big[p^C(s)\, \Big(\partial_C\, Y^A(\sigma(s)) + {}^4\Gamma^A_{CD}(\sigma(s))\, Y^D(\sigma(s))\Big)\Big] +\\ &+\; {}^4\Gamma^A_{BE}(\sigma(s))\, \Big[p^C(s)\, \Big(\partial_C\, Y^E(\sigma(s)) + {}^4\Gamma^E_{CD}(\sigma(s))\, Y^D(\sigma(s))\Big)\Big]\Big) =\\ &=\; {}^4g^{AB}(\sigma(s))\, {}^4R_{BCDE}(\sigma(s))\, p^C(s)\, p^D(s)\, Y^E(\sigma(s)), \end{aligned} \tag{4.11}$$

with the initial data $Y^A(\sigma_o) = 0$ and $\frac{D\, Y^A(\sigma(s))}{ds}|_{\sigma_o} = \sum_{\lambda=1,2} A_{(\lambda)}\, E^A_{S(\lambda)}(\sigma_o)$.

Its solution, evaluated at the absorption point σ^A_1, can be put in the form [25]

$$
\begin{aligned}
Y^A(\sigma_1) &= J^A{}_B(E,S)\,\frac{c}{\omega_S(\sigma_o)}\,\frac{D\,Y^B(\sigma(s))}{ds}|_{s=0},\\
&= \sum_{\lambda\lambda_1} F^A_{E(\lambda_1)}(\sigma_1)\,\mathcal{J}_{\lambda_1\lambda}(E,S)\,E_{S(\lambda)B}(\sigma_o)\,\frac{A_{(\lambda)}}{\omega_S(\sigma_o)/c},\\
\textit{with} \qquad & J^A{}_B(E,S) = \sum_{\lambda_1\lambda} F^A_{E(\lambda_1)}(\sigma_1)\,\mathcal{J}_{\lambda_1\lambda}\,E_{S(\lambda)B}(\sigma_o),\\
& \epsilon^{\,4} g_{(1)AC}(\sigma_1)\,Y^A(\sigma_1)\,F^C_{E(\lambda_1)}(\sigma_1) = -\sum_{\lambda}\mathcal{J}_{\lambda_1\lambda}(E,S)\,\frac{A_{(\lambda)}}{\omega_S(\sigma_o)/c},
\end{aligned}
\tag{4.12}
$$

where $\omega_S(\sigma_o)$ is the emission circular frequency of the light-ray. The *Jacobi map* $J^\mu{}_\nu(E,S)$ maps vectors at S into vectors at E

2. The PM Luminosity Distance

The *luminosity distance* is [25]

$$
d_{lum}(S,E) = (1+z)\,\sqrt{|det\,\mathcal{J}|} = \frac{\omega_S(\sigma_o)}{\omega_E(\sigma_1)}\,\sqrt{|det\,\mathcal{J}|}, \tag{4.13}
$$

where z is the *red-shift* of the source as seen by the observer: $1+z = \omega_S(\sigma_o)/\omega_E(\sigma_1)$, with $\omega_E(\sigma_1)$ the absorption frequency. The *corrected luminosity distance* is $D_{lum}(S,E) = \sqrt{|det\,\mathcal{J}|}$.

In the inertial frames of Minkowski space-time one gets $d_{lum}(S,E) = (1+z)\,d_{Euclidean}(S,E)$, namely the corrected luminosity distance is the Euclidean spatial distance.

In the weak field approximation, by using $\sigma^A(s) = \sigma_{(o)}(s) + \sigma^A_{(1)}(s)$ we get

$$\begin{aligned}
Y^A(\sigma(s)) &= Y^A(\sigma_{(o)}(s)+\sigma_{(1)}(s)) = \\
&= Y^A(\sigma_{(o)}(s)) + \frac{\partial\, Y^A(\sigma_{(o)}(s))}{\partial\, \sigma^E}\, \sigma^E_{(1)}(s) + O(\zeta^2) = \\
&= Y^A_{(o)}(\sigma_{(o)}(s)) + Y^A_{(1)}(\sigma_{(o)}(s)) + \frac{\partial\, Y^A_{(o)}(\sigma_{(o)}(s))}{\partial\, \sigma^E_{(o)}}\, \sigma^E_{(1)}(s) + O(\zeta^2), \\
\partial_B\, Y^A(\sigma(s)) &= \frac{\partial\, Y^A_{(o)}(\sigma_{(o)}(s))}{\partial\, \sigma^B_{(o)}} + \frac{\partial\, Y^A_{(1)}(\sigma_{(o)}(s))}{\partial\, \sigma^B_{(o)}} + \\
&+ \frac{\partial^2\, Y^A_{(o)}(\sigma_{(o)}(s))}{\partial\, \sigma^B_{(o)}\, \partial\, \sigma^E_{(o)}}\, \sigma^E_{(1)}(s) + O(\zeta^2), \\
\partial_C\, \partial_B\, Y^A(\sigma(s)) &= \frac{\partial^2\, Y^A_{(o)}(\sigma_{(o)}(s))}{\partial\, \sigma^C_{(o)}\, \partial\, \sigma^B_{(o)}} + \frac{\partial^2\, Y^A_{(1)}(\sigma_{(o)}(s))}{\partial\, \sigma^C_{(o)}\, \partial\, \sigma^B_{(o)}} + \\
&+ \frac{\partial^3\, Y^A_{(o)}(\sigma_{(o)}(s))}{\partial\, \sigma^C_{(o)}\, \partial\, \sigma^B_{(o)}\, \partial\, \sigma^E_{(o)}}\, \sigma^E_{(1)}(s) + O(\zeta^2). \qquad (4.14)
\end{aligned}$$

By using $p^A(s) = b^A + p^A_{(1)}(s)$, as implied by Eq.(4.2), Eq.(4.9) becomes

$$\begin{aligned}
\frac{D\, Y^A(\sigma(s))}{ds} &= b^B\, \Big[\partial_B\, Y^A_{(o)}(\sigma_{(o)}(s)) + \partial_B\, Y^A_{(1)}(\sigma_{(o)}(s)) + \\
&+ \partial_E\, \partial_B\, Y^A_{(o)}(\sigma_{(o)}(s))\, \sigma^E_{(1)} + {}^4\Gamma^A_{(1)BC}(\sigma_{(o)}(s))\, Y^C_{(o)}(\sigma_{(o)}(s))\Big] + \\
&+ p^B_{(1)}(s)\, \partial_B\, Y^A_{(o)}(\sigma_{(o)}(s)) + O(\zeta^2) = \\
&= \frac{D\, Y^A_{(o)}(\sigma_{(o)}(s))}{ds} + \frac{D\, Y^A_{(1)}(\sigma_{(o)}(s))}{ds}. \qquad (4.15)
\end{aligned}$$

As a consequence, the geodesic deviation equation (4.11) becomes

$$\begin{aligned}
& b^B\, b^c\, \Big(\partial_B\, \partial_C\, Y^A_{(o)}(\sigma_{(o)}(s)) + \partial_B\, \partial_C\, Y^A_{(1)}(\sigma_{(o)}(s)) + \partial_B\, \partial_C\, \partial_E\, Y^A_{(o)}(\sigma_{(o)}(s))\, \sigma^E_{(1)}(s) + \\
& + \partial_B\, {}^4\Gamma^A_{(1)CE}(\sigma_{(o)}(s))\, Y^E_{(o)}(\sigma_{(o)}(s)) + 2\, {}^4\Gamma^A_{(1)BE}(\sigma_{(o)}(s))\, \partial_C\, Y^E_{(o)}(\sigma_{(o)}(s))\Big) + \\
& + b^B\, p^C_{(1)}(s)\, \partial_B\, \partial_C\, Y^A_{(o)}(\sigma_{(o)}(s)) = \\
& = {}^4\eta^{AD}\, {}^4R_{(1)DBCE}(\sigma_{(o)}(s))\, b^B\, b^C\, Y^E_{(o)}(\sigma_{(o)}(s)), \qquad (4.16)
\end{aligned}$$

with the Christoffel symbols and the Riemann tensor of Eqs. (2.15) and (2.16).

Therefore we have to solve the following two equations (the dependence upon $\sigma^E_{(1)}(s)$ is eliminated by the first equation)

$$
\begin{aligned}
b^B\, b^C\, \partial_B\, \partial_C\, Y^A_{(o)}(\sigma_{(o)}(s)) \;&=\; 0,\\
b^B\, b^C\, \partial_B\, \partial_C\, Y^A_{(1)}(\sigma_{(o)}(s)) \;&=\; b^B\, b^C \Big(\Big[{}^4\eta^{AD}\, {}^4R_{(1)DBCE}(\sigma_{(o)}(s)) -\\
&-\; \partial_B\, {}^4\Gamma^A_{(1)CE}(\sigma_{(o)}(s)) \Big]\, Y^E_{(o)}(\sigma_{(o)}(s)) -\\
&-\; 2\, {}^4\Gamma^A_{(1)BE}(\sigma_{(o)}(s))\, \partial_C\, Y^E_{(o)}(\sigma_{(o)}(s)) \Big) -\\
&-\; b^B\, p^C_{(1)}(s)\, \partial_B\, \partial_C\, Y^A_{(o)}(\sigma_{(o)}(s)).
\end{aligned}
\tag{4.17}
$$

From Eqs.(4.10) the initial conditions are

$$
\begin{aligned}
Y^A_{(o)}(\sigma_o) \;&=\; Y^A_{(1)}(\sigma_o) = 0,\\
\Big(\frac{D\, Y^A_{(o)}(\sigma_{(o)}(s))}{ds}\Big)|_{s=0} \;&=\; \Big(b^B\, \partial_B\, Y^A_{(o)}(\sigma_{(o)}(s))\Big)|_{s=} = \sum_{\lambda=1,2} A_{(\lambda)}\, E^A_{(o)S(\lambda)},\\
\Big(\frac{D\, Y^A_{(1)}(\sigma_{(o)}(s))}{ds}\Big)|_{s=0} \;&=\; \Big(b^B\, \partial_B\, Y^A_{(1)}(\sigma_{(o)}(s))\Big)|_{s=0} + c_{(1)}(\sigma_o)\, \partial_{\tau_o}\, Y^A_{(o)}(\sigma_o) = \sum_{\lambda=1,2} A_{(\lambda)}\, E^A_{(1)S(\lambda)}.
\end{aligned}
\tag{4.18}
$$

Since at the zero order we have ${}^4\eta_{AB}\, b^A\, b^B = 0$, ${}^4\eta_{AB}\, b^A\, E^B_{(o)S(\lambda)} = 0$ and $\epsilon\, {}^4\eta_{AB}\, u^A_{(o)S}\, b^B = 1$, due to Eqs.(4.7), the solution of the first equation, satisfying the initial conditions (4.18), is

$$
\begin{aligned}
Y^A_{(o)}(\sigma_{(o)}(s)) \;&=\; \Big(\epsilon\, {}^4\eta_{BC}\, u^B_{(o)S}\, (\sigma^C_{(o)}(s) - \sigma^C_o)\Big) \sum_{\lambda=1,2} A_{(\lambda)}\, E^A_{(o)S(\lambda)} =\\
&=\; \Big(\tau_{(o)}(s) - \tau_o\Big) \sum_{\lambda=1,2} A_{(\lambda)}\, E^A_{(o)S(\lambda)},\\
\frac{D\, Y^A_{(o)}(\sigma_{(o)}(s))}{ds} \;&=\; b^B\, \partial_B\, Y^A_{(o)}(\sigma_{(o)}(s)) = \sum_{\lambda=1,2} A_{(\lambda)}\, E^A_{(o)S(\lambda)}, \qquad independently\, from\, s.
\end{aligned}
\tag{4.19}
$$

Let us remark that $Y^A_{(o)}(\sigma_1)$ is proportional to $\tau_1 - \tau_o = \sqrt{(\vec{\sigma}_1 - \vec{\sigma}_o)^2} = d_{Euclidean}(1, 0)$ as expected at the zero order in Minkowski space-time.

Then the second of equations (4.17) and its initial conditions (4.18) become

$$
\begin{aligned}
b^B\, b^C\, \partial_B\, \partial_C\, Y^A_{(1)}(\sigma_{(o)}(s)) \,&=\, \Big(\epsilon\, {}^4\eta_{UV}\, u^U_{(o)S}\, (\sigma^V_{(o)}(s) - \sigma^V_o)\Big)\, b^B\, b^C\, \Big[{}^4\eta^{AD}\, {}^4R_{(1)DBCE}(\sigma_{(o)}(s)) \,-\\
&-\, \partial_B\, {}^4\Gamma^A_{(1)CE}(\sigma_{(o)}(s))\Big] \sum_{\lambda=1,2} A_{(\lambda)}\, E^E_{(o)S(\lambda)},\\
Y^A_{(1)}(\sigma_o) \,&=\, 0,\\
\Big(b^B\, \partial_B\, Y^A_{(1)}(\sigma_{(o)}(s))\Big)|_{s=0} \,&=\, \sum_{\lambda=1,2} A_{(\lambda)}\, \Big(E^A_{(1)S(\lambda)}(\sigma_o) - c_{(1)}(\sigma_o)\, E^A_{(o)S(\lambda)}\Big),
\end{aligned}
\tag{4.20}
$$

with $E^A_{(1)S(\lambda)}(\sigma_o)$ given in Eq.(4.7).

Since we have $\sigma_{(o)}(s) = \sigma^A_o + b^A\, s$, we get $b^B\, \frac{\partial\, Y^A_{(1)}(\sigma_{(o)}(s))}{\partial\, \sigma^B_{(o)}} = \frac{d}{ds}\, Y^A_{(1)}(\sigma_{(o)}(s))$ and $\Big(b^B\, \partial_B\, Y^A_{(1)}(\sigma_{(o)}(s))\Big)|_{s=0} = \frac{d\, Y^A_{(1)}(\sigma_{(o)}(s))}{ds}|_{s=0}$.

Therefore the solution of Eq.(4.17) with the given initial data is

$$
\begin{aligned}
Y^A_{(1)}(\sigma_{(o)}(s)) \,&=\, \Big[\sum_{\lambda=1,2} A_{(\lambda)}\, \Big(E^A_{(1)S(\lambda)}(\sigma_o) - c_{(1)}(\sigma_o)\, E^A_{(o)S(\lambda)}\Big)\Big]\, s \,+\\
&+\, \int_0^s ds_1 \int_0^{s_1} ds_2\, \Big[\Big(\epsilon\, {}^4\eta_{BC}\, u^B_{(o)S}\, (\sigma^C_{(o)}(s_2) - \sigma^C_o)\Big)\\
&b^B\, b^C\, \Big({}^4\eta^{AD}\, {}^4R_{(1)DBCE}(\sigma_{(o)}(s_2)) - \partial_B\, {}^4\Gamma^A_{(1)CE}(\sigma_{(o)}(s_2))\Big)\Big]\\
&\sum_{\lambda=1,2} A_{(\lambda)}\, E^E_{(o)S(\lambda)}.
\end{aligned}
\tag{4.21}
$$

By using Eqs. (4.14), (4.19) and (4.20) the last line of Eq.(4.12) becomes [9]

---

[9] We also use $b^A = (1; 0, 0, 1)$, $\epsilon\, {}^4g_{(1)FA(\sigma_1)}\, F^F_{E(\lambda_1)}(\sigma_1)\, E^A_{(1)S(\lambda)}(\sigma_o) = -\sum_{r=1,2}[1 + (\Gamma^{(1)}_r + 2\,\phi_{(1)})(\sigma_1)]\, g^r_{(o)(\lambda_1)}\, g^r_{(o)(\lambda)}$, $\epsilon\, {}^4g_{(1)FA}(\sigma_1)\, F^F_{(o)E(\lambda_1)}\, E^A_{(1)S(\lambda)}(\sigma_o) = \sum_{r=1,2} (\Gamma^{(1)}_r + 2\,\phi_{(1)})(\sigma_o)\, g^r_{(o)(\lambda_1)}\, g^r_{(o)(\lambda)}$ and $\omega_S(\sigma_o)/c = 1 + (c_{(1)} + n_{(1)} - \bar{n}_{(1)(3)})(\sigma_o) = 1 - (n_{(1)} + \Gamma^{(1)}_3 + 2\,\phi_{(1)} + \frac{1}{2}\,\bar{n}_{(1)(3)})(\sigma_o)$ (we used Eq.(4.1) for $c_{(1)}(\sigma_o)$).

$$
\begin{aligned}
& -\epsilon\, {}^4g_{(1)AC}(\sigma_1)\, Y^A(\sigma_1)\, F^C_{E(\lambda_1)}(\sigma_1) = \\
& = -\epsilon\, {}^4g_{(1)AC}(\sigma_1) \Big[Y^A_{(o)}(\sigma_1) + Y^A_{(1)}(\sigma_1) + \partial_E\, Y^A_{(o)}(\sigma_1) \Big]\, F^C_{E(\lambda_1)}(\sigma_1) = \\
= & \sum_\lambda \mathcal{J}_{\lambda_1\lambda}(E,S)\, \frac{A_{(\lambda)}}{\omega_S(\sigma_o)}, \\
& \mathcal{J}(E,S)_{\lambda_1\lambda} = \Big((\tau_1 - \tau_o)\, [1 + 2\, \phi_{(1)}(\sigma_1) + (c_{(1)} + n_{(1)} - \bar{n}_{(1)(3)})(\sigma_o)] + \\
& + \tau_{(1)}(\sigma_1) - (c_{(1)} + 2\, \phi_{(1)})(\sigma_o)\, s_1 \Big)\, \delta_{\lambda_1\lambda} + \\
& + \sum_{r=1,2} \Big((\tau_1 - \tau_o)\, \Gamma^{(1)}_r(\sigma_1) - \Gamma^{(1)}_r(\sigma_o)\, s_1 \Big)\, g^r_{(o)(\lambda_1)}\, g^r_{(o)(\lambda)} - \\
& - \int_0^{s_1} ds_2 \int_0^{s_2} ds_3\, (\tau_{(o)}(s_3) - \tau_o)\, \mathcal{W}_{(1)\lambda_1\lambda}(\sigma_{(o)}(s_3)) = \mathcal{J}(E,S)_{(o)\lambda_1\lambda} + \mathcal{J}(E,S)_{(1)\lambda_1\lambda},
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{J}(S,E)_{(o)\lambda_1\lambda} &= (\tau_1 - \tau_o)\, \delta_{\lambda_1\lambda} = d_{Euclidean}(S,E)\, \delta_{\lambda_1\lambda}, \\
\mathcal{J}(S,E)_{(1)\lambda_1\lambda} &= \Big(\tau_{(1)}(\sigma_1) - (2\, n_{(1)} + \Gamma^{(1)}_3 - \frac{1}{2}\, \bar{n}_{(1)(3)})(\sigma_o)\, s_1 + \\
&+ d_{Euclidean}(S,E)\, (n_{(1)} + \Gamma^{(1)}_3 + 2\, \phi_{(1)} - \frac{1}{2}\, \bar{n}_{(1)(3)}) \Big)\, \delta_{\lambda_1\lambda} + \\
&+ \sum_{r=1,2} \Big(d_{Euclidean}(S,E)\, \Gamma^{(1)}_r(\sigma_1) - \Gamma^{(1)}_r(\sigma_o)\, s_1 \Big)\, g^r_{(o)(\lambda_1)}\, g^r_{(o)(\lambda)} - \\
&- \int_0^{s_1} ds_2 \int_0^{s_2} ds_3\, (\tau_{(o)}(s_3) - \tau_o)\, \mathcal{W}_{(1)\lambda_1\lambda}(\sigma_{(o)}(s_3)),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{W}_{(1)\lambda_1\lambda}(\sigma_{(o)}(s_3)) &= \epsilon\, {}^4\eta_{FA}(\sigma_1)\, F^F_{(o)E(\lambda_1)}\, b^B\, b^C \Big[{}^4\eta^{AD}\, {}^4R_{(1)DBCK}(\sigma_{(o)}(s_3)) - \\
& - \partial_B\, {}^4\Gamma^A_{(1)CK}(\sigma_{(o)}(s_3)) \Big]\, E^K_{(o)S(\lambda)} = \\
&= \sum_{r,s=1,2} g^r_{(o)(\lambda_1)} \Big[\epsilon \Big({}^4R_{(1)r3\tau s} - {}^4R_{(1)\tau r\tau s} + {}^4R_{(1)r33s} - {}^4R_{(1)\tau r3s} \Big) + \\
&+ (\partial_\tau + \partial_3) \Big({}^4\Gamma^r_{(1)\tau s} + {}^4\Gamma^r_{(1)3s} \Big) \Big]\, g^s_{(o)(\lambda)}. \qquad (4.22)
\end{aligned}
$$

By using $\tau_1 - \tau_o = d_{Euclidean}(S,E)$ we get $\mathcal{J}_{(o)\lambda_1\lambda}(S,E) = d_{Euclidean}(S,E)\, \delta_{\lambda_1\lambda}$. As a consequence we get the following expression of the corrected luminosity distance

$$
\begin{aligned}
D_{lum}(S,E) &= \frac{d_{lum}(S,E)}{1+z(s_1)} = \sqrt{|det\, \mathcal{J}(S,E)|} = \\
&= \sqrt{\mathcal{J}(S,E)_{(o)11}\, \mathcal{J}(S,E)_{(o)22} + \mathcal{J}(S,E)_{(o)11}\, \mathcal{J}(S,E)_{(1)22} + \mathcal{J}(S,E)_{(1)11}\, \mathcal{J}(S,E)_{(o)22}} = \\
&= d_{Euclidean}(S,E)\, \sqrt{1 + \frac{\mathcal{J}(S,E)_{(1)11} + \mathcal{J}(S,E)_{(1)22}}{d_{Euclidean}(S,E)} + O(\zeta^2)} = \\
&= d_{Euclidean}(S,E)\, \sqrt{1 + \frac{\sum_{\lambda=1,2}\, \Big(\mathcal{J}(S,E)_{(1)\lambda\lambda({}^3K=0)} + \mathcal{J}(S,E)_{(1)\lambda\lambda({}^3K)}\Big)}{d_{Euclidean}(S,E)} + O(\zeta^2)},
\end{aligned}
$$

$$
\begin{aligned}
\sum_{\lambda=1,2} \mathcal{J}(S,E)_{(1)\lambda\lambda} &= 2\, \Big(\tau_{(1)}(\sigma_1) - (2\, n_{(1)} + \Gamma_3^{(1)} - \frac{1}{2}\, \bar{n}_{(1)(3)})(\sigma_o)\, s_1 + \\
&+ d_{Euclidean}(S,E)\, (n_{(1)} + \Gamma_3^{(1)} + 2\, \phi_{(1)} - \frac{1}{2}\, \bar{n}_{(1)(3)})(\sigma_1)\Big) + \\
&+ \sum_{\lambda,r=1,2} \Big(d_{Euclidean}(S,E)\, \Gamma_r^{(1)}(\sigma_1) - \Gamma_r^{(1)}(\sigma_o)\, s_1\Big)\, \Big(g^r_{(o)(\lambda)}\Big)^2 - \\
&- \int_0^{s_1} ds_2 \int_0^{s_2} ds_3\, (\tau_{(o)}(s_3) - \tau_o) \sum_{\lambda=1,2} \mathcal{W}_{(1)\lambda\lambda}(\sigma_{(o)}(s_3)),
\end{aligned}
$$

$$
\begin{aligned}
\sum_{\lambda=1,2} \mathcal{J}(S,E)_{(1)\lambda\lambda({}^3K)} &= s_1\, \Big((4\, \partial_\tau + \partial_3)\, {}^3\mathcal{K}_{(1)}\Big)(\sigma_o) - \frac{1}{2}\, d_{Euclidean}(S,E)\, \Big((2\, \partial_\tau + \partial_3)\, {}^3\mathcal{K}_{(1)}\Big)(\sigma_1) - \\
&- \int_0^{s_1} ds_2 \int_0^{s_2} ds_3\, (\tau_{(o)}(s_3) - \tau_o) \sum_{\lambda,r,s=1,2} g^r_{(o)(\lambda)}\, g^s_{(o)(\lambda)}\, \Big(\partial_r\, \partial_s\, {}^3\mathcal{K}_{(1)}\Big)(\sigma_{(o)}(s_3)),
\end{aligned}
\tag{4.23}
$$

where Eqs.(2.15) and (2.16) have been used to find the dependence upon the non-local York time ${}^3\mathcal{K}_{(1)}$.

Let us remark that Eq. (4.6) implies that the frequency $\omega(0)$ of the light emitted from the star is $\omega(0) = (1 + z(s_1))\, \omega(s_1)$, where $\omega(s_1)$ is the frequency absorbed on the Earth. Since $\omega(0) = c\, v_{S\mu}(0)\, \epsilon^\mu_A\, p^A(0) = v_{rec}(S,E)$ [10] is also the radial (i.e. along the line of sight) recessional velocity of the star, we have that the recessional velocity is proportional to the red-shift (i.e. it is a red-shift-velocity $c\, z$). On the other hand, for small deviations from the Euclidean distance, Eq.(4.23) can be written as

$$
D_{lum}(S,E) \approx d_{Euclidean}(S,E) + \frac{1}{2} \sum_{\lambda=1,2} \mathcal{J}(S,E)_{(1)\lambda\lambda} = \alpha + \beta\, (1 + z(s_1)), \tag{4.24}
$$

[10] Due to the use of proper time $c\, v^\mu_S$ has the dimension of an ordinary velocity with respect to $t = \tau/c$.

because the term $-(2\,n_{(1)} + \Gamma_3^{(1)} - \frac{1}{2}\,\bar{n}_{(1)(3)})(\sigma_o)\,s_1$ contains $\omega(0)$, i.e. a linear dependence on the red-shift.

These two results imply that the recessional velocity of the star is proportional to its luminosity distance from the Earth ($V_{rec}(S, E) = A\,z(s_1) + B$) at least for small distances. This is in accord with the Hubble old redshift-distance relation which is formalized in the Hubble law (velocity-distance relation) when the standard cosmological model is used (see for instance Ref.[26] on these topics). Again these results have a dependence on the trace of the non-local York time, which could play a role in giving a different interpretation of the data from super-novae, which are used as a support for dark energy [27].

V. THE PM EQUATIONS OF MOTION OF THE PARTICLES AND THEIR PN EXPANSION

In this Section we study the HPM equations of motion for the particles. We formulate the problem of the definition of the center of mass and relative variables first in general relativity and then in PM space-times, using the two-body problem as an example. Then we study the PN expansion of the equations of motion and we consider the 1PN limit of PM binaries for vanishing York time.

A. The PM Equations of Motion for the Particles

From Eqs.(5.2) and (5.3) of paper II, by using Eqs.(2.1), we get the following expression for the momenta and the equations of motion of the particles ($\eta^r_i(\tau), \dot{\eta}^r_i(\tau) = O(1)$, $m_i = M\, O(\zeta)$, with M the ultraviolet cutoff)

$$\frac{\kappa_{ir}(\tau)}{m_i\, c} = \frac{\dot{\eta}^r_i(\tau)}{\sqrt{1 - \dot{\vec{\eta}}_i(\tau)}} + \frac{M}{m_i}\, O(\zeta),$$

$$\begin{aligned}
\eta_i\, \frac{d}{d\tau} \quad & \Big[\frac{\dot{\eta}^r_i(\tau)}{\sqrt{1 - \dot{\vec{\eta}}^2_i(\tau)}} \Big(1 + 2\, (\Gamma^{(1)}_r + 2\, \phi_{(1)}) - \\
& - \frac{n_{(1)} - \sum_c \dot{\eta}^c_i(\tau)\, [\bar{n}_{(1)(c)} + (\Gamma^{(1)}_c + 2\, \phi_{(1)})\, \dot{\eta}^c_i(\tau)]}{1 - \dot{\vec{\eta}}^2_i(\tau)}\Big) + \\
& + \frac{\bar{n}_{(1)(r)}}{\sqrt{1 - \dot{\vec{\eta}}^2_i(\tau)}}\Big]|_{\vec{\kappa}_i = \frac{m_i c\, \dot{\vec{\eta}}_i}{\sqrt{1 - \dot{\vec{\eta}}^2_i}}} (\tau, \vec{\eta}_i(\tau)) \stackrel{\circ}{=} \\
& \stackrel{\circ}{=} \eta_i\, \frac{1}{\sqrt{1 - \dot{\vec{\eta}}^2_i(\tau)}}\, \Big[\sum_a \dot{\eta}^a_i(\tau)\, \Big(\frac{\partial\, \bar{n}_{(1)(a)}}{\partial\, \eta^r_i} + \frac{\partial\, (\Gamma^{(1)}_a + 2\, \phi_{(1)})}{\partial\, \eta^r_i}\, \dot{\eta}^a_i(\tau)\Big) - \\
& - \frac{\partial\, n_{(1)}}{\partial\, \eta^r_i}\Big]|_{\vec{\kappa}_i = \frac{m_i c\, \dot{\vec{\eta}}_i}{\sqrt{1 - \dot{\vec{\eta}}^2_i}}} (\tau, \vec{\eta}_i(\tau)), \\
\Rightarrow \quad & \ddot{\eta}^r_i(\tau) \stackrel{\circ}{=} O(\zeta). \qquad (5.1)
\end{aligned}$$

The last line is a consequence of the ultraviolet cutoff, which allows the definition of the HPM linearization.

Eqs(5.1), being implied by Hamilton equations derived from a standard relativistic particle Lagrangian (see the action (3.1) in paper I), are equal to the geodesic equations for point-like scalar particles notwithstanding these particles are dynamical and not test objects (for spinning particles this is not true due to spin-curvature couplings, see for instance Ref.[28]).

Eqs.(5.1) may be rewritten by putting all the terms involving the accelerations at the first member. Since Eqs.(2.2)-(2.4) and $\vec{\kappa}_i = \frac{m_i c\, \dot{\vec{\eta}}_i}{\sqrt{1 - \dot{\vec{\eta}}^2_i}} + M\, O(\zeta)$ imply that the functions

$f_{(1)}(\tau ,\vec{\sigma}) = \phi_{(1)}(\tau ,\vec{\sigma}), n_{(1)}(\tau ,\vec{\sigma}), \bar{n}_{(1)(r)}(\tau ,\vec{\sigma})$ depend on $\eta^r_k(\tau)$ and $\dot{\eta}^r_k(\tau)$ with $k = 1,..,N$, for each of these functions we have $\frac{d}{d\tau}\, f_{(1)}(\tau ,\vec{\sigma}) = \sum_k^{1..N}\, \sum_s\, \Big(\dot{\eta}^s_k(\tau)\, \frac{\partial\, f_{(1)}(\tau ,\vec{\sigma})}{\partial\, \eta^s_k} + \ddot{\eta}^s_k(\tau)\, \frac{\partial\, f_{(1)}(\tau ,\vec{\sigma})}{\partial\, \dot{\eta}^s_k}\Big)$. Due to the result $\ddot{\eta}^r_i(\tau) \,{\buildrel \circ \over =}\, O(\zeta)$, we get $\frac{d}{d\tau}\, f_{(1)}(\tau ,\vec{\sigma}) = \sum_k^{1..N}\, \sum_s\, \dot{\eta}^s_k(\tau)\, \frac{\partial\, f_{(1)}(\tau ,\vec{\sigma})}{\partial\, \eta^s_k} + O(\zeta^2)$. Therefore, the terms involving the accelerations have the following expression

$$
\begin{aligned}
&\eta_i\, \Big(\frac{\ddot{\eta}^r_i(\tau)}{\sqrt{1 - {\dot{\vec{\eta}}}^2_i(\tau)}}\, \Big[1 + 2\, (\Gamma_r^{(1)} + 2\, \phi_{(1)}) - \frac{n_{(1)} - \sum_c\, \dot{\eta}^c_i(\tau)\, \bar{n}_{(1)(c)} - \sum_c\, (\dot{\eta}^c_i(\tau))^2\, (\Gamma_c^{(1)} + 2\, \phi_{(1)})}{1 - {\dot{\vec{\eta}}}^2_i(\tau)} \Big] + \\
&+\ \frac{\dot{\vec{\eta}}_i(\tau) \cdot \ddot{\vec{\eta}}_i(\tau)}{(1 - {\dot{\vec{\eta}}}^2_i(\tau))^{3/2}}\, \bar{n}_{(1)(r)} + \frac{\dot{\eta}^r_i(\tau)}{(1 - {\dot{\vec{\eta}}}^2_i(\tau))^{3/2}}\, \Big[\sum_c\, \ddot{\eta}^c_i(\tau)\, \Big(\bar{n}_{(1)(c)} + 2\, \dot{\eta}^c_i(\tau)\, (\Gamma_c^{(1)} + 2\, \phi_{(1)}) \Big) + \\
&+\ \dot{\vec{\eta}}_i(\tau) \cdot \ddot{\vec{\eta}}_i(\tau)\, \Big(1 + 2\, (\Gamma_r^{(1)} + 2\, \phi_{(1)}) - 3\, \frac{n_{(1)} - \sum_c\, \dot{\eta}^c_i(\tau)\, \bar{n}_{(1)(c)} - \sum_c\, (\dot{\eta}^c_i(\tau))^2\, (\Gamma_c^{(1)} + 2\, \phi_{(1)})}{1 - {\dot{\vec{\eta}}}^2_i(\tau)} \Big] + \\
&+\ \frac{1}{\sqrt{1 - {\dot{\vec{\eta}}}^2_i(\tau)}}\, \sum_u\, \sum_{j \not= i}\, \ddot{\eta}^u_j(\tau)\, \Big[\frac{\partial\, \bar{n}_{(1)(r)}}{\partial\, \dot{\eta}^u_j} + \dot{\eta}^r_i(\tau)\, \Big(2\, \frac{\partial\, (\Gamma_r^{(1)} + 2\, \phi_{(1)})}{\partial\, \dot{\eta}^u_j} - \\
&-\ \frac{1}{1 - {\dot{\vec{\eta}}}^2_i(\tau)}\, \Big[\frac{\partial\, n_{(1)}}{\partial\, \dot{\eta}^u_j} + \sum_c\, \dot{\eta}^c_i(\tau)\, \frac{\partial\, \bar{n}_{(1)(c)}}{\partial\, \dot{\eta}^u_j} + \sum_c\, (\dot{\eta}^c_i(\tau))^2\, \frac{\partial\, (\Gamma_c^{(1)} + 2\, \phi_{(1)})}{\partial\, \dot{\eta}^u_j} \Big] \Big) \Big] \Big) (\tau, \vec{\eta}_i(\tau)) = \\
&=\ \frac{\eta_i}{\sqrt{1 - {\dot{\vec{\eta}}}^2_i(\tau)}}\, \Big(\ddot{\eta}^r_i(\tau) + \frac{\dot{\eta}^r_i(\tau)\, \dot{\vec{\eta}}_i(\tau) \cdot \ddot{\vec{\eta}}_i(\tau)}{1 - {\dot{\vec{\eta}}}^2_i(\tau)} \Big) + O(\zeta^2).
\end{aligned}
\tag{5.2}
$$

As a consequence, after having rewritten the lapse and shift functions in the form $n_{(1)} = \check{n}_{(1)} - \partial_\tau\, {}^3\mathcal{K}_{(1)}$, $\bar{n}_{(1)(r)} = \check{\bar{n}}_{(1)(r)} + \partial_r\, {}^3\mathcal{K}_{(1)}$, to display their dependence on the inertial gauge variable ${}^3\mathcal{K}_{(1)} = \frac{1}{\triangle}\, {}^3K_{(1)}$, we get the following form of the PM equations of motion of the particles

$$
\frac{\eta_i}{\sqrt{1 - {\dot{\vec{\eta}}}^2_i(\tau)}}\, \Big(\ddot{\eta}^r_i(\tau) + \frac{\dot{\eta}^r_i(\tau)\, \dot{\vec{\eta}}_i(\tau) \cdot \ddot{\vec{\eta}}_i(\tau)}{1 - {\dot{\vec{\eta}}}^2_i(\tau)} \Big) {\buildrel \circ \over =}
$$

$$
\begin{aligned}
&\stackrel{\circ}{=} \frac{\eta_i}{\sqrt{1-\dot{\vec{\eta}}_i^2(\tau)}} \Big(- \frac{\partial\, \check{n}_{(1)}(\tau, \vec{\eta}_i(\tau))}{\partial\, \eta_i^r} + \frac{\dot{\eta}_i^r(\tau)}{1-\dot{\vec{\eta}}_i^2(\tau)} \sum_u \Big[\dot{\eta}_i^u(\tau)\, \frac{\partial\, \check{n}_{(1)}}{\partial\, \eta_i^u} + \sum_{j\neq i} \dot{\eta}_j^u(\tau)\, \frac{\partial\, \check{n}_{(1)}}{\partial\, \eta_j^u} \Big](\tau, \vec{\eta}_i(\tau)) + \\
&+ \Big(\sum_u \dot{\eta}_i^u(\tau) \Big[\frac{\partial\, \check{\bar{n}}_{(1)(u)}}{\partial\, \eta_i^r} - \frac{\partial\, \check{\bar{n}}_{(1)(r)}}{\partial\, \eta_i^u} \Big] - \sum_{j\neq i} \sum_u \dot{\eta}_j^u(\tau)\, \frac{\partial\, \check{\bar{n}}_{(1)(r)}}{\partial\, \eta_j^u} - \\
&- \frac{\dot{\eta}_i^r(\tau)}{1-\dot{\vec{\eta}}_i^2(\tau)} \sum_u \dot{\eta}_i^u(\tau) \sum_s \Big[\dot{\eta}_i^s(\tau)\, \frac{\partial\, \check{\bar{n}}_{(1)(u)}}{\partial\, \eta_i^s} + \sum_{j\neq i} \dot{\eta}_j^s(\tau)\, \frac{\partial\, \check{\bar{n}}_{(1)(u)}}{\partial\, \eta_j^s} \Big] \Big)(\tau, \vec{\eta}_i(\tau)) + \\
&+ \Big(\sum_u (\dot{\eta}_i^u(\tau))^2\, \frac{\partial\, (\Gamma_u^{(1)} + 2\,\phi_{(1)})}{\partial\, \eta_i^r} - \\
&- \dot{\eta}_i^r(\tau) \sum_u \Big[\dot{\eta}_i^u(\tau) \Big(2\, \frac{\partial\, (\Gamma_r^{(1)} + 2\,\phi_{(1)})}{\partial\, \eta_i^u} + \sum_c \frac{(\dot{\eta}_i^c(\tau))^2}{1-\dot{\vec{\eta}}_i^2(\tau)}\, \frac{\partial\, (\Gamma_c^{(1)} + 2\,\phi_{(1)})}{\partial\, \eta_i^u} \Big) + \\
&+ \sum_{j\neq i} \dot{\eta}_j^u(\tau) \Big(2\, \frac{\partial\, (\Gamma_r^{(1)} + 2\,\phi_{(1)})}{\partial\, \eta_j^u} + \sum_c \frac{(\dot{\eta}_i^c(\tau))^2}{1-\dot{\vec{\eta}}_i^2(\tau)}\, \frac{\partial\, (\Gamma_c^{(1)} + 2\,\phi_{(1)})}{\partial\, \eta_j^u} \Big) \Big] \Big)(\tau, \vec{\eta}_i(\tau)) - \\
&- \frac{\dot{\eta}_i^r(\tau)}{1-\dot{\vec{\eta}}_i^2(\tau)} \Big[\partial_\tau^2|_{\vec{\eta}_i}\, {}^3\mathcal{K}_{(1)} + 2 \sum_s \dot{\eta}_i^s(\tau)\, \frac{\partial\, \partial_\tau|_{\vec{\eta}_i}\, {}^3\mathcal{K}_{(1)}}{\partial\, \eta_i^s} + \sum_{su} \dot{\eta}_i^s(\tau)\, \dot{\eta}_i^u(\tau)\, \frac{\partial^2\, {}^3\mathcal{K}_{(1)}}{\partial\, \eta_i^u\, \partial\, \eta_i^s} \Big](\tau, \vec{\eta}_i(\tau)) \Big) + O(\zeta^2) = \\
&\stackrel{def}{=} \eta_i\, \frac{\mathcal{F}_i^r(\tau | \vec{\eta}_i(\tau) | \vec{\eta}_{k\neq i}(\tau))}{m_i} + O(\zeta^2). \qquad (5.3)
\end{aligned}
$$

Since Eqs.(5.3) imply $\eta_i\, \dot{\vec{\eta}}_i(\tau) \cdot \ddot{\vec{\eta}}_i(\tau) \stackrel{\circ}{=} \eta_i\, m_i^{-1}\, (1-\dot{\vec{\eta}}_i^2(\tau))^{3/2}\, \dot{\vec{\eta}}_i(\tau) \cdot \vec{\mathcal{F}}_i(\tau | \vec{\eta}_i(\tau) | \vec{\eta}_{k\neq i}(\tau))$, the final form of the equation of motion of the particles is

$$
\begin{aligned}
m_i\, \eta_i\, \ddot{\eta}_i^r(\tau) &\stackrel{\circ}{=} \eta_i\, \sqrt{1-\dot{\vec{\eta}}_i^2(\tau)}\, \Big(\mathcal{F}_i^r - \dot{\eta}_i^r(\tau)\, \dot{\vec{\eta}}_i(\tau) \cdot \vec{\mathcal{F}}_i \Big)(\tau | \vec{\eta}_i(\tau) | \vec{\eta}_{k\neq i}(\tau)) = \\
&\stackrel{def}{=} \eta_i\, F_i^r(\tau | \vec{\eta}_i(\tau) | \vec{\eta}_{k\neq i}(\tau)). \qquad (5.4)
\end{aligned}
$$

The second member of Eqs.(5.4) defines the effective force F_i^r acting on particle i. It contains:

a) the contribution of the lapse function $\check{n}_{(1)}$, which generalizes the Newton force;

b) the contribution of the shift functions $\check{\bar{n}}_{(1)(r)}$, which gives the gravito-magnetic effects [11];

c) the retarded contribution of GW's, described by the functions $\Gamma_r^{(1)}$ of Eqs.(2.6), whose contribution at the order $O(\zeta)$ is given in Eq.(2.7);

[11] Since we have $\dot{\vec{\eta}}_i(\tau = ct) = \frac{\vec{v}_i(t)}{c}$ (see Eq.(5.5)), the term $\sum_a \frac{v_i^a}{c} \Big(\frac{\partial\, \check{\bar{n}}_{(1)(a)}}{\partial\, \eta^r} - \frac{\partial\, \check{\bar{n}}_{(1)(r)}}{\partial\, \eta^a} \Big)$ in Eqs.(5.3) is proportional to $\frac{\vec{v}}{c} \times \vec{B}_G$, where $\vec{B}_G$ is the gravito-magnetic field. It is of order $(\frac{v}{c})^2$ as shown in Eq.(5.6).

d) the contribution of the volume element $\phi_{(1)}$ ($\tilde{\phi} = 1 + 6\,\phi_{(1)} + O(\zeta^2)$), always summed to the GW's, giving forces of Newton type;

e) the contribution of the inertial gauge variable (the non-local York time) ${}^3\mathcal{K}_{(1)} = \frac{1}{\triangle}\,{}^3K_{(1)}$.

Let us remark that in the gravitational case the regularization with Grassmann-valued signs of the particle energies leads to equations of motion for the particles of the type $\eta_i\,\ddot{\eta}_i(\tau) \circeq \eta_i....$ with $\eta_i^2 = 0$ with instantaneous action-at-a-distance effects coming from the lapse $n_{(1)}$ and shift $\bar{n}_{(1)(r)}$ functions and from the volume 3-element $\tilde{\phi}_{(1)} = 1 + 6\,\phi_{(1)}$. However the retardation present in the solution (2.6) for the GW's is not eliminable and formally the equations of motion of the particles are of integro-differential type for the N functions $\eta_i^r(\tau)$, $i = 1,..,N$. However, since $\ddot{\eta}_i^r(\tau) = O(\zeta)$, the retardation effects in the GW's are pushed to higher HPM order as shown in Eqs.(2.7), so that at the lowest order we have coupled differential equations for the particles. This shows that our semi-classical approximation, obtained with our Grassmann regularization, of a unspecified "quantum gravity" theory does not take into account only a "one-graviton exchange diagram" but also more complex structures already present at the tree level (namely they are not radiative corrections) but showing up only at higher HPM orders.

By comparison in electro-magnetism the coupled equations of motion for the charged particles and the transverse electro-magnetic field in the radiation gauge, containing the Coulomb potential, allow to find the Lienard-Wiechert solution [29] for the transverse vector potential with the no-incoming radiation condition. The regularization with Grassmann-valued electric charges [7] implies equations of motion of the type $\ddot{\eta}_i(\tau) \circeq Q_i......$ with $Q_i^2 = 0$, so that we get the elimination of retardation effects, i.e. $Q_i\,\dot{\eta}_i(\tau)(\tau - |\vec{\sigma}|) \circeq Q_i\,\dot{\eta}_i(\tau)$. In this way the difference between retarded and advanced (or symmetric) Lienard-Wiechert solutions is killed and it is possible to identify the hidden common action-at-a-distance part of such solutions and to express the resulting semi-classical Lienard-Wiechert transverse electro-magnetic fields in terms of the canonical variables $\vec{\eta}_i(\tau)$, $\vec{\kappa}_i(\tau)$ of the particles. This implies that the electro-magnetic retardation effects are to be described as radiative corrections to the one-photon exchange diagram of QED, which is replaced by a Cauchy problem with a well defined action-at-a-distance potential. As a consequence, the final equations for the particles are second order coupled ordinary differential equations. To reduce the original phase space containing the charged particles and the electro-magnetic field in the radiation gauge, one added the second class constraints identifying the transverse electro-magnetic field with the Lienard-Wiechert solution and one evaluated the Dirac brackets. It turned out that the resulting reduced phase space containing only particles has a canonical basis spanned by new particle variables $\hat{\eta}_i^r$, $\hat{\kappa}_{ir}$ [interpretable as the old ones η_i^r, κ_{ir}, (no more canonical with respect to the Dirac brackets) dressed with a Coulomb cloud] with a mutual action-at-a-distance interaction governed by the sum of the *Coulomb and Darwin* potentials. In the rest-frame instant form of dynamics [4, 30–32],one can find the expression of the internal Poincare' generators: $p^o = Mc$, $p^r \approx 0$, j^{rs}, $j^{\tau r} \approx 0$ (the potentials appear in the energy p^o and in the Lorentz boosts $J^{\tau r}$). Then, after having gone from the canonical basis $\hat{\eta}_i^r$, $\hat{\kappa}_{ir}$, to a canonical basis containing internal center-of-mass variables $\vec{\eta} = \frac{\sum_i m_i\,\vec{\hat{\eta}}_i}{\sum_i m_i}$, $\vec{p} = \sum_i \vec{\hat{\kappa}}_i$ and relative ones $\vec{\rho}_a$, $\vec{\pi}_a$, $a = 1,..,N-1$, (see Eqs. (2.1), (2.2) of Ref.[31]), the rest-frame conditions $p^r \approx 0$, $j^{\tau r} \approx 0$, eliminated the collective variables: $\vec{p} \approx 0$, $\vec{\eta} \approx \vec{f}(\vec{\rho}_a, \vec{\pi}_a)$. As a consequence, in the reduced phase space there are second order equations of motion only for the relative variables $\vec{\rho}_a$, $\vec{\pi}_a$.

In the HPM gravitational case the analogue of the Hamiltonian action-at-a-distance Lienard-Wiechert transverse electromagnetic fields are the action-at-a-distance fields $\phi_{(1)}(\tau, \vec{\sigma})$, $n_{(1)}(\tau, \vec{\sigma})$, $\bar{n}_{(1)(r)}(\tau, \vec{\sigma})$, $\sigma_{(1)(a)(a)}|_{a \neq b}(\tau, \vec{\sigma})$, of Eqs.(2.2)-(2.5) and the tidal fields $R_{\bar{a}}(\tau, \vec{\sigma})$ of Eqs.(2.7). To find a reduced phase space containing only particles (how it was done in the electro-magnetic case), we have to add the second class constraints which identify the tidal fields $R_{\bar{a}}(\tau, \vec{\sigma})$ and $\Pi_{\bar{a}}(\tau, \vec{\sigma})$ with the HPM solution of Eqs.(2.7) and ((2.9) in our family of 3-orthogonal gauges. To get a set of second class constraints for the elimination of the gravitational field we must add to the existing first class constraints: 1) $\pi_{\tilde{\phi}}(\tau, \vec{\sigma}) - \frac{c^3}{12\pi G}\, {}^3K_{(1)}(\tau, \vec{\sigma}) \approx 0$ to the super-Hamiltonian constraint written in the form $\tilde{\phi}(\tau, \vec{\sigma}) - [1 + 6\, \phi_{(1)}(\tau, \vec{\sigma})] \approx 0$; 2) $\theta^i(\tau, \vec{\sigma}) \approx 0$ to the super-momentum constraints written in the form $\sigma_{(a)(b)}|_{a \neq b}(\tau, \vec{\sigma}) - \sigma_{(1)(a)(b)}|_{a \neq b}(\tau, \vec{\sigma}) \approx 0$; 3) $n(\tau, \vec{\sigma}) - n_{(1)}(\tau, \vec{\sigma}) \approx 0$ to $\pi_n(\tau, \vec{\sigma}) \approx 0$; 4) $\bar{n}_{(a)}(\tau, \vec{\sigma}) - \bar{n}_{(1)(a)}(\tau, \vec{\sigma}) \approx 0$ to $\pi_{\bar{n}_{(a)}}(\tau, \vec{\sigma}) \approx 0$; 5) $R_{\bar{a}}(\tau, \vec{\sigma}) - R_{(1)\bar{a}}(\tau, \vec{\sigma}) \approx 0$ and $\Pi_{\bar{a}}(\tau, \vec{\sigma}) - \Pi_{(1)\bar{a}}(\tau, \vec{\sigma}) \approx 0$. Differently from the electro-magnetic case, the evaluation of the Dirac brackets for the reduced phase space containing only particles at the lowest HPM order is trivial, because all the linearized solutions are sums of terms proportional to $G\, m_i$, $i = 1, .., N$ and the gauge variable ${}^3\mathcal{K}_{(1)}(\tau, \vec{\sigma})$ is a numerical function. Therefore in the evaluation of the Dirac brackets of the variables $\eta^r_i(\tau)$, $\kappa_{ir}(\tau)$, all the extra terms added to the ordinary Poisson bracket are quadratic in $[G m_i\, G m_j]_{j \neq i}$ and can be discarded being of order $O(\zeta^2)$ due to the ultraviolet cutoff $m_i = M\, O(\zeta)$. As a consequence the variables η^r_i, κ_{ir}, are also a canonical basis of the Dirac brackets at the lowest order: the analogue of the electro-magnetic Coulomb dressing is pushed to higher HPM order.

B. The Center-of-Mass Problem in General Relativity and in the HPM Linearization.

As we have seen Eqs.(5.4), with the gravitational waves $\Gamma^{(1)}_r(\tau, \vec{\sigma})$ given in Eqs.(2.7), are the final equations of motion for the particles in a reduced phase space containing only particles described by the canonical variables $\eta^r_i(\tau)$, $\kappa_{ir}(\tau)$. The forces appearing in Eqs.(5.4) are the gravitational analogue of the electro-magnetic mutual interaction produced by the Coulomb and Darwin potentials.

As said in papers I and II, the instantaneous (non-Euclidean) 3-spaces Σ_τ are non-inertial rest frames of the 3-universe (an isolated system including the gravitational field in the chosen family of space-times) due to the rest-frame conditions $\hat{P}^r_{ADM} \approx 0$. This remains true when the gravitational field is expressed in terms of the particles by means of the solution (2.7). The gauge fixings to these three first class constraints, eliminating the collective 3-variable of the 3-universe, are $\hat{J}^{\tau r}_{ADM} \stackrel{def}{=} \hat{K}^r_{ADM} \approx 0$. If we introduce the definition $\hat{K}^r_{ADM} \stackrel{def}{=} -(\frac{1}{c}\, \hat{E}_{ADM})\, R^r_{ADM}$, with R^r_{ADM} being the gravitational analogue of the (neither covariant nor canonical) Møller 3-center of energy, the conditions $\hat{K}^r_{ADM} \approx 0$ imply $R^r_{ADM} \approx 0$. Therefore the 3-center of energy is put in the origin of the 3-coordinates on Σ_τ (so that there is only an external decoupled center of mass of the 3-universe which can be built in terms of the ADM Poincare' generators and of the embedding of the 3-space into space-time as shown in Refs.[4, 30–32]) and is carried by the reference time-like observer using the radar

4-coordinates. In the inertial rest frames of special relativity [4, 30–32] [12] this implies that the reference observer has to be identified with the covariant non-canonical Fokker-Pryce center of inertia of the isolated system. This is true also in the non-inertial rest frames of special relativity as shown in Ref.[4]. As a consequence the isolated system can be seen as an external (unobservable) decoupled (canonical non-covariant) Newton-Wigner center of mass carrying a pole-dipole structure (the relative motion of the components of the isolated system inside the 3-space Σ_τ) identified by the rest mass M and the rest spin S^r of the system. This is also true in the gravitational case with Mc^2 replaced by $\hat{E}_{ADM}$ and with the rest spin S^r replaced by $\hat{J}^{rs}_{ADM}$. The reference observer defining the radar 4-coordinates should be replaced also in this case by an ADM Fokker-Pryce center of inertia (a non-geodesic time-like observer corresponding to the the asymptotic inertial observers existing in our class of space-times, whose spatial axes are determined by the fixed stars of star catalogues).

This is our way out from the the problem of the center of mass in general relativity and of its world-line, a still open problem in generic space-times as can be seen from Refs. [33, 34] (and Ref.[35] for the PN approach). Usually, by means of some supplementary condition, the center of mass is associated to the monopole of a multipolar expansion of the energy-momentum of a small body (see Ref.[36] for the special relativistic case).

Another open problem in general relativity is the replacement of the particle canonical variables $\eta^r_i(\tau)$, $\kappa_{ir}(\tau)$ $(i = 1, .., N)$ with relative canonical variables after the elimination of the internal center of mass. In the special relativistic electro-magnetic case (see Eqs.(2.1) of Ref.[31]) one replaces them with a naif (Newton mechanics oriented) canonical basis $\eta^r_+(\tau)$, $\kappa_{+r}(\tau)$, $\rho^r_a(\tau)$, $\pi_{ar}(\tau)$ $(a = 1, .., N-1)$ with the relative variables in the rest frame being Wigner spin-1 3-vectors. The rest-frame conditions imply $\kappa_{+r}(\tau) \approx 0$ and their gauge fixings $K^r \stackrel{def}{=} - Mc\, R^r_+ \approx 0$ allow to express $\eta^r_+(\tau)$ as a function only of the relative variables, $\eta^r_+ \approx f^r(\vec{\rho}_a, \vec{\pi}_a)$, as shown in Refs. [4, 30–32]. In these papers it is also shown how to reconstruct the world-lines of the particles by using the embedding of the 3-space Σ_τ into the space-time.

However relative variables do not exist in the non-Euclidean 3-spaces of curved space-times, where flat objects like $\vec{r}_{ij}(\tau) = \vec{\eta}_i(\tau) - \vec{\eta}_j(\tau)$ have to be replaced with a quantity proportional to the tangent vector to the space-like 3-geodesics joining the two particles in the non-Euclidean 3-space Σ_τ (see Ref. [37] for an implementation of this idea). This problem is another reason why extended objects tend to be replaced with point-like multipoles, which, however, do not span a canonical basis of phase space.

However, at the HPM order in our family of space-times we can rely on special relativity in the inertial rest frame by using the asymptotic inertial frame (with the asymptotic background Minkowski metric) like it has been done in the solution of the constraints and of the wave equation for the tidal variables [13]. This allows to define HPM collective and relative canonical variables for the particles. Now the collective variables are eliminated with Eqs. (2.11) and (2.13), which at the lowest order become $p^r_{(1)} \approx 0$ and $j^{\tau r}_{(1)} \stackrel{def}{=} - M_{(1)} c\, R^r \approx 0$ like in special relativity. If we take into account also the terms of order $O(\zeta^2)$ in Eqs.(2.11) and

[12] In these papers there is a complete clarification of the center-of-mass problem in special relativity.

[13] In the solutions $|\vec{\eta}_i(\tau) - \vec{\eta}_j(\tau)|$ is the Euclidean 3-distance between the two particles, which differs by quantities of order $O(\zeta)$ from the real non-Euclidean 3-distance on Σ_τ as shown in Eq.(3.2).

(2.13), we can find the first HPM deviation (depending also on tidal terms) from the special relativistic solution and the HPM equations of motion for the relative variables implied by Eqs.(5.4).

As an example let us consider the two-body case in presence of GW's, namely without using the retarded solution (2.7). Instead of defining the overall collective variable of the two particles and of GW's (like it was done in Ref.[31], where the transverse electro-magnetic field was replacing GW's), let us define the naive (Newton mechanics oriented) collective and relative canonical variables only of the particles (with masses m_1 and m_2; $M = m_1 + m_2$ and $\mu = \frac{m_1 m_2}{M}$ are the total and reduced masses respectively; we choose positive energy particles so that we can replace the Grassmann variables η_1 and η_2 with their mean value $< \eta_i >= +1$, after having done the regularization, as said in footnote 17 of paper I)

$$
\begin{aligned}
&\vec{\eta}_{12}(\tau) = \frac{m_1\, \vec{\eta}_1(\tau) + m_2\, \vec{\eta}_2}{M}, \qquad \vec{\rho}_{12}(\tau) = \vec{\eta}_1(\tau) - \vec{\eta}_2(\tau),\\
&\vec{\kappa}_{12}(\tau) = \vec{\kappa}_1(\tau) + \vec{\kappa}_2(\tau), \qquad \vec{\pi}_{12}(\tau) = \frac{m_2\, \vec{\kappa}_1(\tau) - m_1\, \vec{\kappa}_2(\tau)}{M},\\
&\vec{\eta}_1(\tau) = \vec{\eta}_{12}(\tau) + \frac{m_2}{M}\, \vec{\rho}_{12}(\tau), \qquad \vec{\eta}_2(\tau) = \vec{\eta}_{12}(\tau) - \frac{m_1}{M}\, \vec{\rho}_{12}(\tau),\\
&\vec{\kappa}_1(\tau) = \frac{m_1}{M}\, \vec{\kappa}_{12}(\tau) + \vec{\pi}_{12}(\tau), \qquad \vec{\kappa}_2(\tau) = \frac{m_2}{M}\, \vec{\kappa}_{12}(\tau) - \vec{\pi}_{12}(\tau).
\end{aligned} \tag{5.5}
$$

All these quantities are of the type $a = a_{(o)} + a_{(1)} + a_{(2)} + O(\zeta^3)$: the coordinates start with $O(1)$ terms, while the momenta start with $O(\zeta)$ terms due to the ultravilet cutoff.

The rest-frame condition (2.11) implies $\vec{\kappa}_{(1)12}(\tau) \approx 0$ and a certain expression for $\vec{\kappa}_{(2)12}(\tau)$ in terms of GW's, $\vec{\eta}_{(o)12}(\tau)$, $\vec{\rho}_{(o)12}(\tau)$, $\vec{\pi}_{(1)12}(\tau)$ and of the non-local York time.

Instead the condition (2.13) determines $\vec{\eta}_{(o)12}(\tau)$ in terms of $\vec{\rho}_{(o)12}(\tau)$ and $\vec{\pi}_{(1)12}(\tau)$

$$
\begin{aligned}
&\vec{\eta}_{(o)12}(\tau) \;\approx\; -A_{(o)}(\tau)\; \vec{\rho}_{(o)12}(\tau),\\
&A_{(o)}(\tau) = \frac{\frac{m_2}{M}\, \sqrt{m_1^2\, c^2 + \vec{\pi}^2_{(1)12}(\tau)} - \frac{m_1}{M}\, \sqrt{m_2^2\, c^2 + \vec{\pi}^2_{(1)12}(\tau)}}{\sqrt{m_1^2\, c^2 + \vec{\pi}^2_{(1)12}(\tau)} + \sqrt{m_2^2\, c^2 + \vec{\pi}^2_{(1)12}(\tau)}}.
\end{aligned} \tag{5.6}
$$

It vanishes in the equal mass case. As a consequence $\vec{\kappa}_{(2)12}(\tau)$ depends only on relative variables.

Then $\vec{\eta}_{(1)12}(\tau)$ is determined in terms of GW's, of particle relative variables and of both the local and non-local York times: $\vec{\eta}_{(1)12}(\tau) \approx \vec{f}_{(1)}(\tau)[rel.var.]$. As a consequence the particle 3-coordinates depend only on relative variables

$$
\begin{aligned}
\vec{\eta}_1(\tau) &= \vec{\eta}_{(o)1}(\tau) + \vec{\eta}_{(1)1}(\tau) \approx \\
&\approx \Big(\frac{m_2}{M} - A_{(o)}(\tau)\Big)\, \vec{\rho}_{(o)12}(\tau) + \frac{m_2}{M}\, \vec{\rho}_{(1)12}(\tau) + \vec{f}_{(1)}(\tau)[rel.var.],\\
\vec{\eta}_2(\tau) &= \vec{\eta}_{(o)2}(\tau) + \vec{\eta}_{(1)2}(\tau) \approx \\
&\approx -\Big(\frac{m_1}{M} + A_{(o)}(\tau)\Big)\, \vec{\rho}_{(o)12}(\tau) - \frac{m_1}{M}\, \vec{\rho}_{(1)12}(\tau) + \vec{f}_{(1)}(\tau)[rel.var.],
\end{aligned}
\tag{5.7}
$$

and the world-lines can be reconstructed by using the embedding of 3-spaces into space-time: $x^\mu_i(\tau) = z^\mu(\tau, \vec{\eta}_i(\tau))$ [14].

At the lowest order the velocities $\dot{\vec{\eta}}_{(o)i}(\tau)$ depend on $\dot{\vec{\rho}}_{(o)12}(\tau)$, $\vec{\rho}_{(o)12}(\tau)$, and on the function $A_{(o)}(\tau)$ and its τ-derivative.

The ADM energy and angular momentum of Eqs.(2.10) and (2.12) become

$$
\begin{aligned}
\hat{E}_{ADM} &\approx c\,\Big(\sqrt{m_1^2\, c^2 + \vec{\pi}^2_{(1)12}(\tau)} + \sqrt{m_2^2\, c^2 + \vec{\pi}^2_{(1)12}(\tau)}\Big) - \\
&- \frac{G}{c^3}\, \frac{\vec{\pi}^2_{(1)12}(\tau)}{|\vec{\rho}_{(o)12}(\tau)|}\, \Big(\frac{\sqrt{m_2^2\, c^2 + \vec{\pi}^2_{(1)12}(\tau)}}{\sqrt{m_1^2\, c^2 + \vec{\pi}^2_{(1)12}(\tau)}} + \frac{\sqrt{m_1^2\, c^2 + \vec{\pi}^2_{(1)12}(\tau)}}{\sqrt{m_2^2\, c^2 + \vec{\pi}^2_{(1)12}(\tau)}}\Big) - \\
&- \frac{1}{2}\, \vec{\pi}^2_{(1)12}(\tau)\, \sum_c \frac{\partial_c^2}{\triangle}\, \Big(\frac{\Gamma_c^{(1)}(\tau, \vec{\eta}_1(\tau))}{\sqrt{m_1^2\, c^2 + \vec{\pi}^2_{(1)12}(\tau)}} + \frac{\Gamma_c^{(1)}(\tau, \vec{\eta}_2(\tau))}{\sqrt{m_2^2\, c^2 + \vec{\pi}^2_{(1)12}(\tau)}}\Big) - \\
&- \sum_c (\pi^c_{(1)12}(\tau))^2\, \Big(\frac{\Gamma_c^{(1)}(\tau, \vec{\eta}_1(\tau))}{\sqrt{m_1^2\, c^2 + \vec{\pi}^2_{(1)12}(\tau)}} + \frac{\Gamma_c^{(1)}(\tau, \vec{\eta}_2(\tau))}{\sqrt{m_2^2\, c^2 + \vec{\pi}^2_{(1)12}(\tau)}}\Big) - \\
&- \frac{G}{c^2}\, \frac{\sqrt{m_1^2\, c^2 + \vec{\pi}^2_{(1)12}(\tau)}\, \sqrt{m_2^2\, c^2 + \vec{\pi}^2_{(1)12}(\tau)}}{|\vec{\rho}_{(o)12}|} - \frac{G}{c^2}\, \sum_{rs} \pi^r_{(1)12}(\tau)\, \pi^s_{(1)12}(\tau)\, \Big(\frac{7}{2}\, \frac{\delta^{rs}}{|\vec{\rho}_{(o)12}(\tau)|} + \\
&+ \frac{1}{2}\, \frac{\rho^r_{(o)12}(\tau)\, \rho^s_{(o)12}(\tau)}{|\vec{\rho}_{(o)12}(\tau)|^3}\Big) + \\
&+ \frac{c^4}{16\pi\, G}\, \sum_{\bar{a}\bar{b}} \int d^3\sigma\, \Big[\partial_\tau\, R_{\bar{a}}\, M_{\bar{a}\bar{b}}\, \partial_\tau\, R_{\bar{b}} + \sum_a \partial_a\, R_{\bar{a}}\, M_{\bar{a}\bar{b}}\, \partial_a\, R_{\bar{b}}\Big]\Big)(\tau, \vec{\sigma}) + M c\, O(\zeta^3),
\end{aligned}
$$

[14] We have the following reconstruction of the particle world-lines in the preferred adapted world 4-coordinate system defined in the Introduction ($\tilde{\eta}^r_i(t) = \eta^r_i(\tau = ct)$)

$$
x^\mu_i(\tau) = z^\mu(\tau, \vec{\eta}(\tau)) = \tilde{x}^\mu_i(t) = x^\mu_o + \epsilon^\mu_\tau\, \tau + \epsilon^\mu_r\, \eta^r_i(\tau) = x^\mu_o + \epsilon^\mu_\tau\, c\, t + \epsilon^\mu_r\, \tilde{\eta}^r_i(t).
$$

$$
\begin{aligned}
\hat{J}^{rs}_{ADM} &\approx \Big(\rho^r_{(o)12}(\tau) + \rho^r_{(1)12}(\tau)\Big)\, \pi^s_{(1)12}(\tau) - \Big(\rho^s_{(o)12}(\tau) + \rho^s_{(1)12}(\tau)\Big)\, \pi^r_{(1)12}(\tau) + \\
&+\ \rho^r_{(o)12}(\tau)\,\Big(\pi^s_{(2)12}(\tau) - A_{(o)}(\tau)\,\kappa^s_{(2)12}(\tau)\Big) - \rho^s_{(o)12}(\tau)\,\Big(\pi^r_{(2)12}(\tau) - A_{(o)}(\tau)\,\kappa^r_{(2)12}(\tau)\Big) + \\
&+\ \sqrt{m_1^2\, c^2 + \vec{\pi}^2_{(1)12}(\tau)}\, (\frac{m_2}{M} - A_{(o)}(\tau))\,\Big(\rho^r_{(o)12}(\tau)\,\frac{\partial}{\partial\, \eta^s_{(o)1}} - \rho^s_{(o)12}(\tau)\,\frac{\partial}{\partial\, \eta^r_{(o)1}}\Big)\, {}^3\mathcal{K}_{(1)}(\tau, \vec{\eta}_{(o)1}(\tau)) - \\
&-\ \sqrt{m_2^2\, c^2 + \vec{\pi}^2_{(1)12}(\tau)}\, (\frac{m_1}{M} + A_{(o)}(\tau))\,\Big(\rho^r_{(o)12}(\tau)\,\frac{\partial}{\partial\, \eta^s_{(o)2}} - \rho^s_{(o)12}(\tau)\,\frac{\partial}{\partial\, \eta^r_{(o)2}}\Big)\, {}^3\mathcal{K}_{(1)}(\tau, \vec{\eta}_{(o)2}(\tau)) -
\end{aligned}
$$

$$
\begin{aligned}
&-\ 2\sum_u \pi^u_{(1)12}(\tau) \\
&\Big[(\frac{m_2}{M} - A_{(o)}(\tau))\,\Big(\rho^r_{(o)12}(\tau)\,\frac{\partial}{\partial\, \eta^s_{(o)1}} - \rho^s_{(o)12}(\tau)\,\frac{\partial}{\partial\, \eta^r_{(o)1}}\Big)\,\frac{\partial_u}{\triangle}\,\Big(\Gamma^{(1)}_u - \frac{1}{4}\sum_c \frac{\partial_c^2}{\triangle}\,\Gamma^{(1)}_c\Big)(\tau, \vec{\eta}_{(o)1}(\tau)) - \\
&-\ (\frac{m_1}{M} + A_{(o)}(\tau))\,\Big(\rho^r_{(o)12}(\tau)\,\frac{\partial}{\partial\, \eta^s_{(o)2}} - \rho^s_{(o)12}(\tau)\,\frac{\partial}{\partial\, \eta^r_{(o)2}}\Big)\,\frac{\partial_u}{\triangle}\,\Big(\Gamma^{(1)}_u - \frac{1}{4}\sum_c \frac{\partial_c^2}{\triangle}\,\Gamma^{(1)}_c\Big)(\tau, \vec{\eta}_{(o)2}(\tau))\Big] + \\
&+\ 2\,\Big[\pi^r_{(1)12}(\tau)\,\sum_i \frac{\partial_s}{\triangle}\,\Big(\Gamma^{(1)}_s - \frac{1}{4}\sum_c \frac{\partial_c^2}{\triangle}\,\Gamma^{(1)}_c\Big)(\tau, \vec{\eta}_{(o)i}(\tau)) - \\
&-\ \pi^s_{(1)12}(\tau)\,\sum_i \frac{\partial_r}{\triangle}\,\Big(\Gamma^{(1)}_r - \frac{1}{4}\sum_c \frac{\partial_c^2}{\triangle}\,\Gamma^{(1)}_c\Big)(\tau, \vec{\eta}_{(o)i}(\tau))\Big] -
\end{aligned}
$$

$$
\begin{aligned}
&-\ \frac{c^3}{8\pi\, G}\int d^3\sigma\,\Big[\sum_{\bar{a}\bar{b}} (\sigma^r\,\partial_s - \sigma^s\,\partial_r)\, R_{\bar{a}}\, M_{\bar{a}\bar{b}}\,\partial_\tau\, R_{\bar{b}} + 2\,{}^3K_{(1)}\,\partial_r\,\partial_s\,(\Gamma^{(1)}_s - \Gamma^{(1)}_r) + \\
&+\ 2\,(\partial_\tau\,\Gamma^{(1)}_r + \partial_\tau\,\Gamma^{(1)}_s - \frac{1}{2}\sum_c \frac{\partial_c^2}{\triangle}\,\partial_\tau\,\Gamma^{(1)}_c)\,\frac{\partial_r\,\partial_s}{\triangle}\,(\Gamma^{(1)}_s - \Gamma^{(1)}_r)\Big](\tau, \vec{\sigma}) + M c\, L\, O(\zeta^3).
\end{aligned}
\tag{5.8}
$$

The dependence of the rest-frame energy on the York time is pushed to order $O(\zeta^3)$. Instead the rest-frame angular momentum (or spin) depends on both local and non-local York time.

Let us now consider the equations of motion (5.4). By using Eqs.(5.5) and the previous results, they can be written in the following form

$$
\begin{aligned}
M\,\ddot{\eta}^r_{12}(\tau) &\approx M\,\Big[\frac{d^2}{d\tau^2}\,\Big(A_{(o)}(\tau)\,\rho^r_{(o)12}(\tau) + f_{(1)}(\tau)[rel.var.]\Big)\Big] \overset{\circ}{=} \\
&\overset{\circ}{=} F_1^r(\tau|\vec{\eta}_{(o)1}(\tau)|\vec{\eta}_{(o)2}(\tau)) + F_2^r(\tau|\vec{\eta}_{(o)2}(\tau)|\vec{\eta}_{(o)1}(\tau)), \\
\mu\,\ddot{\rho}^r_{12}(\tau) &\approx \mu\,\Big(\ddot{\rho}^r_{(o)12}(\tau) + \ddot{\rho}^r_{(1)12}(\tau)\Big) \overset{\circ}{=} \frac{m_2}{M}\,F_1^r(\tau|\vec{\eta}_{(o)1}(\tau)|\vec{\eta}_{(o)2}(\tau)) - \frac{m_1}{M}\,F_2^r(\tau|\vec{\eta}_{(o)2}(\tau)|\vec{\eta}_{(o)1}(\tau)).
\end{aligned}
\tag{5.9}
$$

Since the forces (depending on $\vec{\rho}_{(o)12}(\tau)$ and $\dot{\vec{\eta}}_{(o)i}(\tau)$) are of order $O(\zeta)$, since $\frac{d}{d\tau}\, A_{(o)}(\tau)$ is of order $O(\zeta)$ involving $\dot{\pi}^r_{(1)}(\tau)$ and since $\ddot{\rho}^r_{(1)12}(\tau)$ is of higher order with respect to $\ddot{\rho}^r_{(o)12}(\tau)$, Eqs.(5.9) imply the following equations of motion

$$\mu\, \ddot{\rho}^r_{(o)12}(\tau) \,\stackrel{\circ}{=}\, \frac{m_2}{M}\, F^r_1(\tau|\vec{\eta}_{(o)1}(\tau)|\vec{\eta}_{(o)2}(\tau)) - \frac{m_1}{M}\, F^r_2(\tau|\vec{\eta}_{(o)2}(\tau)|\vec{\eta}_{(o)1}(\tau)),$$

$$\begin{aligned} M \quad & \Big[\ddot{\eta}^r_{(1)12}(\tau) - \frac{d^2}{d\tau^2}\,\Big(A_{(o)}(\tau)\,\rho^r_{(o)12}(\tau)\Big)\Big] \stackrel{\circ}{=} \\ & \stackrel{\circ}{=}\, F^r_1(\tau|\vec{\eta}_{(o)1}(\tau)|\vec{\eta}_{(o)2}(\tau)) + F^r_2(\tau|\vec{\eta}_{(o)2}(\tau)|\vec{\eta}_{(o)1}(\tau)). \end{aligned} \tag{5.10}$$

While the first equation determines the relative motion, the second equation is a consistency relation connecting the motion of the particle 3-center of mass to GW's and expressing the fact that the overall 3-center mass of the isolated system "particles plus GW's" is in free motion.

If the previous discussion is formulated in the reduced phase space containing only particles after the elimination of GW's, by forcing them to coincide with the solution (2.7), nothing changes except that the second of Eqs.(5.10) should become the time derivative of the rest-frame condition $\vec{\eta}_{(1)12}(\tau) \approx \vec{f}_{(1)}(\tau)[rel.var.]$.

C. The PN Expansion at all Orders in the Slow Motion Limit.

Due to our ultraviolet cutoff M we have been able to obtain a HPM linearization without never making PN expansions. However, if all the particles are contained in a compact set of radius l_c, we can add the slow motion condition in the form $\sqrt{\epsilon} = \frac{v}{c} \approx \sqrt{\frac{R_{m_i}}{l_c}}$, $i = 1, ..N$ ($R_{m_i} = \frac{2\,G\,m_i}{c^2}$ is the gravitational radius of particle i) with $l_c \geq R_M$ and $\lambda >> l_c$ (see the Introduction). In this case we can do the PN expansion of Eqs.(5.4).

Since we have $\tau = c\,t$, we make the following change of notation

$$\begin{aligned} \vec{\eta}_i(\tau) = \tilde{\vec{\eta}}_i(t), \qquad \vec{v}_i(t) = \frac{d\tilde{\vec{\eta}}_i(t)}{dt}, \qquad \vec{a}_i(t) = \frac{d^2\tilde{\vec{\eta}}_i(t)}{dt^2}, \\ \dot{\vec{\eta}}_i(\tau) = \frac{\vec{v}_i(t)}{c}, \qquad \ddot{\vec{\eta}}_i(\tau) = \frac{\vec{a}_i(t)}{c^2}. \end{aligned} \tag{5.11}$$

For the non-local York time we use the notation ${}^3\tilde{\mathcal{K}}_{(1)}(t, \vec{\sigma}) = {}^3\mathcal{K}_{(1)}(\tau, \vec{\sigma}) = \frac{1}{\triangle}\, {}^3K_{(1)}(\tau, \vec{\sigma}) = \frac{1}{\triangle}\, {}^3\tilde{K}(t, \vec{\sigma})$.

By using $(1-x)^{-1/2} = \sum_{k=0}^{\infty} (-)^k\, \frac{(2k-1)!!}{(2k)!!}\, x^k$ (valid for $x^2 < 1$), Eqs.(2.7) and (2.2)-(2.4) can be written in the form (here $A_{[(k)]} = O(\epsilon^{k/2} = (\frac{v}{c})^k)$ is of order $\frac{k}{2}\, PN$)

$$\eta_i\,\Gamma^{(1)}_r(\tau,\vec{\eta}_i(\tau)) \;=\; -\frac{2G}{c^2}\,\sum_b\,\tilde{M}^{-1}_{rs}(\vec{\tilde{\eta}}_i(t))\,\sum_{j\neq i}\eta_j\,m_j\,\sum_{uv}\mathcal{P}_{ssuv}(\vec{\tilde{\eta}}_i(t))\,\frac{\frac{v^u_j(t)}{c}\,\frac{v^v_j(t)}{c}}{\sqrt{1-(\frac{\vec{v}_j(t)}{c})^2}}$$

$$\Big[|\vec{\tilde{\eta}}_i(t)-\vec{\tilde{\eta}}_j(t)|^{-1}+\sum_{m=1}^{\infty}\,\frac{1}{(2m)!}\,\Big(\frac{\vec{v}_j(t)}{c}\cdot\frac{\partial}{\partial\,\vec{\tilde{\eta}}_i}\Big)^{2m}\,|\vec{\tilde{\eta}}_i(t)-\vec{\tilde{\eta}}_j(t)|^{2m-1}\Big] =$$

$$=\;\eta_i\,\tilde{\Gamma}^{(1)}_r(t,\vec{\tilde{\eta}}_i(t))=\eta_i\,\sum_{j\neq i}\,\eta_j\,\frac{G\,m_j}{c^2}\,\sum_{k=1}^{\infty}\,\hat{\Gamma}^{(1)}_{rj[(2k)]}(t,\vec{\tilde{\eta}}_i(t)|\vec{\tilde{\eta}}_j(t)),$$

$$\hat{\Gamma}^{(1)}_{jr[(2k)]}(t,\vec{\tilde{\eta}}_i(t)|\vec{\tilde{\eta}}_j(t))=\sum_s\,\tilde{M}^{-1}_{rs}(\vec{\tilde{\eta}}_i(t))\,\sum_{uv}\,\mathcal{P}_{ssuv}(\vec{\tilde{\eta}}_i(t))\,\frac{v^u_j(t)}{c}\,\frac{v^v_j(t)}{c}$$

$$\sum_{h=1}^{k}\,\frac{(-)^h(2h-3)!!}{(2h-2)!!\,[2(k-h)]!}\,\Big(\frac{\vec{v}_j(t)}{c}\Big)^{2\,(h-1)}\,\Big(\frac{\vec{v}_j(t)}{c}\cdot\frac{\partial}{\partial\,\vec{\tilde{\eta}}_i}\Big)^{2(k-h)}\,|\vec{\tilde{\eta}}_i(t)-\vec{\tilde{\eta}}_j(t)|^{2(k-h)-1},$$

$$\eta_i\,\phi_{(1)}(\tau,\vec{\eta}_i(\tau)) \;=\; \eta_i\,\sum_{j\neq i}\,\eta_j\,\frac{G\,m_j}{c^2}\,\Big(\frac{1}{2\,|\vec{\tilde{\eta}}_i(t)-\vec{\tilde{\eta}}_j(t)|}+\sum_{k=1}^{\infty}\,\hat{\phi}_{(1)j[(2k)]}(t,\vec{\tilde{\eta}}_i(t)|\vec{\tilde{\eta}}_j(t))\Big),$$

$$\hat{\phi}_{(1)j[(2k)]}(t,\vec{\tilde{\eta}}_i(t)|\vec{\tilde{\eta}}_j(t))=(-)^k\,\frac{(2k-1)!!}{(2k)!!}\,\frac{(\frac{\vec{v}_j(t)}{c})^{2k}}{2\,|\vec{\tilde{\eta}}_i(t)-\vec{\tilde{\eta}}_j(t)|}\,-$$

$$-\;\int d^3\sigma\,\frac{\sum_r\,\partial^2_r\,\hat{\Gamma}^{(1)}_{jr[(2k)]}(t,\vec{\sigma}|\vec{\tilde{\eta}}_j(t))}{16\,|\vec{\tilde{\eta}}_i(t)-\vec{\sigma}|},$$

$$\eta_i\,n_{(1)}(\tau,\vec{\eta}_i(\tau)) \;=\; \eta_i\,\Big[-\partial_\tau|_{\vec{\eta}_i}\,{}^3\mathcal{K}_{(1)}(\tau,\vec{\eta}_i(\tau))+\check{n}_{(1)}(\tau,\vec{\eta}_i(\tau))\Big] =$$

$$=\;\eta_i\,\Big[-\frac{1}{c}\,\partial_t|_{\vec{\tilde{\eta}}_i}\,{}^3\tilde{\mathcal{K}}_{(1)}(t,\vec{\tilde{\eta}}_i(t))-\sum_{j\neq i}\,\eta_j\,\frac{G\,m_j}{c^2}\,\frac{1}{|\vec{\tilde{\eta}}_i(t)-\vec{\tilde{\eta}}_j(t)|}\,+$$

$$+\;\sum_{j\neq i}\,\eta_j\,\frac{G\,m_j}{c^2}\,\sum_{k=1}^{\infty}\,\hat{n}_{(1)j[(2k)]}(t,\vec{\tilde{\eta}}_i(t)|\vec{\tilde{\eta}}_j(t))\Big],$$

$$\hat{n}_{(1)j[(2)]}(t,\vec{\tilde{\eta}}_i(t)|\vec{\tilde{\eta}}_j(t))=-\frac{\frac{3}{2}\,(\frac{\vec{v}_j(t)}{c})^2}{|\vec{\tilde{\eta}}_i(t)-\vec{\tilde{\eta}}_j(t)|},$$

$$\hat{n}_{(1)j[(2k)]}(t,\vec{\tilde{\eta}}_i(t)|\vec{\tilde{\eta}}_j(t))=-\Big((-)^k\,\frac{(2k-1)!!}{(2k)!!}+(-)^{k-1}\,\frac{(2k-3)!!}{(2k-2)!!}\Big)\,\times$$

$$\times\frac{(\frac{\vec{v}_i(t)}{c})^{2k}}{|\vec{\tilde{\eta}}_i(t)-\vec{\tilde{\eta}}_j(t)|},\qquad k\geq 2,$$

$$
\begin{aligned}
\eta_i\, \bar{n}_{(1)(r)}(\tau, \vec{\eta}_i(\tau)) \;&=\; \eta_i \Big[\partial_r\, {}^3\mathcal{K}_{(1)}(\tau, \vec{\eta}_i(\tau)) + \check{\bar{n}}_{(1)(r)}(\tau, \vec{\eta}_i(\tau)\Big] =\\
&=\; \eta_i \Big[\partial_r\, {}^3\tilde{\mathcal{K}}_{(1)}(t, \vec{\tilde{\eta}}_i(t)) + \sum_{j\neq i} \eta_j\, \frac{G\, m_j}{c^2}\, \hat{\bar{n}}_{(1)(r)j[(1)]}(t, \vec{\tilde{\eta}}_i(t)|\vec{\tilde{\eta}}_j(t) +\\
&+\; \sum_{j\neq i} \eta_j\, \frac{G\, m_j}{c^2} \sum_{k=1}^{\infty} \Big(\hat{\bar{n}}_{(1)(r)j[(2k)]} + \hat{\bar{n}}_{(1)(r)j[(2k+1)]}(t, \vec{\tilde{\eta}}_i(t)|\vec{\tilde{\eta}}_j(t))\Big)\Big],
\end{aligned}
$$

$$
\begin{aligned}
&\hat{\bar{n}}_{(1)(r)j[(1)]}(t, \vec{\tilde{\eta}}_i(t)|\vec{\tilde{\eta}}_j(t)) = -\frac{1}{|\vec{\tilde{\eta}}_i(t) - \vec{\tilde{\eta}}_j(t)|}\Big(\frac{7}{2}\, \frac{v^r_j(t)}{c} +\\
-&\;\frac{1}{2}\frac{(\tilde{\eta}^r_i(t) - \tilde{\eta}^r_j(t))\, \frac{\vec{v}_j(t)}{c}\, \cdot (\vec{\tilde{\eta}}_i(t) - \vec{\tilde{\eta}}_j(t))}{|\vec{\tilde{\eta}}_i(t) - \vec{\tilde{\eta}}_j(t)|^2}\Big),\\
&\hat{\bar{n}}_{(1)(r)j[(2k+1)]}(t, \vec{\tilde{\eta}}_i(t)|\vec{\tilde{\eta}}_j(t)) = \frac{(2k-1)!!}{(2k)!!}\, \frac{(-)^{k+1}}{|\vec{\tilde{\eta}}_i(t) - \vec{\tilde{\eta}}_j(t)|}\Big(\frac{7}{2}\frac{v^r_j(t)}{c} +\\
-&\;\frac{1}{2}\frac{(\tilde{\eta}^r_i(t) - \tilde{\eta}^r_j(t))\, \frac{\vec{v}_j(t)}{c}\, \cdot (\vec{\tilde{\eta}}_i(t) - \vec{\tilde{\eta}}_j(t))}{|\vec{\tilde{\eta}}_i(t) - \vec{\tilde{\eta}}_j(t)|^2}\Big)\Big(\frac{\vec{v}_i(t)}{c}\Big)^{2k},\\
&\hat{\bar{n}}_{(1)(r)j[(2k)]}(t, \vec{\tilde{\eta}}_i(t)|\vec{\tilde{\eta}}_j(t)) = -\int \frac{d^3\sigma_1}{4\pi\, |\vec{\tilde{\eta}}_i(t) - \vec{\sigma}_1|}\, \frac{\partial_{1r}\, \partial_t|_{\vec{\tilde{\eta}}_j}}{c}\, \Big[2\, \hat{\Gamma}^{(1)}_{jr[(2k)]}(t, \vec{\sigma}_1|\vec{\tilde{\eta}}_j(t)) -\\
-&\;\int d^3\sigma_2\, \frac{\sum_c \partial^2_{2c}\, \hat{\Gamma}^{(1)}_{jc[(2k)]}(t, \vec{\sigma}_2|\vec{\tilde{\eta}}_j(t))}{8\pi\, |\vec{\sigma}_1 - \vec{\sigma}_2|}\Big].
\end{aligned}
\tag{5.12}
$$

All the quantities are even in $\frac{v}{c}$ except the shift functions which have both odd and even terms. As a consequence, ${}^4g_{(1)\tau\tau}$ and ${}^4g_{(1)rs}$ are even in $\frac{v}{c}$, but this is not true for ${}^4g_{(1)\tau r}$.

By using Eqs.(5.12), Eqs.(5.4) can be written in the following form after having being multiplied by c^2 and m_i (we use $(1-x)^{-1} = \sum_{h=0}^{\infty} x^h$, valid for $x < 1$)

$$
\begin{aligned}
m_i\,\eta_i\,a^r_i(t) &\, \stackrel{\circ}{=}\, \eta_i\,\tilde F^r_i(t|\vec{\tilde\eta}_i(t)|\vec{\tilde\eta}_j(t)) =\\
&= \eta_i\,m_i\,c^2\Big[-\frac{\partial\check n_{(1)}}{\partial\tilde\eta^r_i}+2\frac{v^r_i}{c}\,\frac{\vec v_i}{c}\cdot\frac{\partial\check n_{(1)}}{\partial\vec{\tilde\eta}_i}+\sum_{j\neq i}\frac{v^r_i}{c}\,\frac{\vec v_j}{c}\cdot\frac{\partial\check n_{(1)}}{\partial\vec{\tilde\eta}_j}+\\
&\quad+\sum_u\Big(\frac{v^u_i}{c}\,\frac{\partial\check{\check n}_{(1)(u)}}{\partial\tilde\eta^r_i}-\frac{v^u_i}{c}\,\frac{\partial\check{\check n}_{(1)(r)}}{\partial\tilde\eta^u_i}-\sum_{j\neq i}\frac{v^u_j}{c}\,\frac{\partial\check{\check n}_{(1)(r)}}{\partial\tilde\eta^u_j}\Big)+\\
&\quad+\frac{v^r_i}{c}\sum_{u,v}\frac{v^u_i}{c}\frac{v^v_i}{c}\,\frac{\partial\check{\check n}_{(1)(u)}}{\partial\tilde\eta^v_i}+\\
&\quad+\sum_s\Big(\frac{v^s_i}{c}\Big)^2\Big(\frac{\partial}{\partial\tilde\eta^r_i}-2\frac{v^r_i}{c}\,\frac{\vec v_i}{c}\cdot\frac{\partial}{\partial\vec{\tilde\eta}_i}\Big)(\tilde\Gamma^{(1)}_s+2\phi_{(1)})-\\
&\quad-2\,\frac{v^r_i}{c}\Big(1-\frac{v^2_i}{c^2}\Big)\Big(\frac{\vec v_i}{c}\cdot\frac{\partial}{\partial\vec{\tilde\eta}_i}+\sum_{j\neq i}\frac{\vec v_j}{c}\cdot\frac{\partial}{\partial\vec{\tilde\eta}_j}\Big)(\tilde\Gamma^{(1)}_s+2\phi_{(1)})-\\
&\quad-\frac{v^r_i}{c}\sum_s\Big(\frac{v^s_i}{c}\Big)^2\sum_{j\neq i}\frac{\vec v_j}{c}\cdot\frac{\partial}{\partial\vec{\tilde\eta}_j}(\tilde\Gamma^{(1)}_s+2\phi_{(1)})-\\
&\quad-\frac{v^r_i}{c}\,\frac{1}{c^2}\Big(\partial^2_t|_{\vec{\tilde\eta}_i}\,{}^3\tilde{\mathcal{K}}_{(1)}+2\sum_u v^u_i(t)\,\frac{\partial\,\partial_t|_{\vec{\tilde\eta}_i}\,{}^3\tilde{\mathcal{K}}_{(1)}}{\partial\,\tilde\eta^u_i}+\\
&\qquad+\sum_{uv}v^u_i(t)\,v^v_i(t)\,\frac{\partial^2\,{}^3\tilde{\mathcal{K}}_{(1)}}{\partial\,\tilde\eta^u_i\,\partial\,\tilde\eta^v_i}\Big)\Big](t,\vec{\tilde\eta}_i(t))=\\
&= -\eta_i\sum_{j\neq i}\eta_j\,G\,m_i\,m_j\,\frac{\tilde\eta^r_i(t)-\tilde\eta^r_j(t)}{|\vec{\tilde\eta}_i(t)-\vec{\tilde\eta}_j(t)|^3}\\
&\quad-\eta_i\,\frac{v^r_i(t)}{c}\Big[\partial^2_t|_{\vec{\tilde\eta}_i}\,{}^3\tilde{\mathcal{K}}_{(1)}+2\sum_u v^u_i(t)\,\frac{\partial\,\partial_t|_{\vec{\tilde\eta}_i}\,{}^3\tilde{\mathcal{K}}_{(1)}}{\partial\,\tilde\eta^u_i}+\\
&\qquad+\sum_{uv}v^u_i(t)\,v^v_i(t)\,\frac{\partial^2\,{}^3\tilde{\mathcal{K}}_{(1)}}{\partial\,\tilde\eta^u_i\,\partial\,\tilde\eta^v_i}\Big](t,\vec{\tilde\eta}_i(t))+\\
&\quad+m_i\sum_{j\neq i}\eta_i\eta_j\sum_{k=1}^{\infty}\Big(\tilde F^r_{ij[(2k)]}+\tilde F^r_{ij[(2k+1)]}\Big)
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{F}^r_{ij[(2)]} \;=\; & -\frac{\partial \hat{n}_{(1)j[(2)]}}{\partial \tilde{\eta}^r_i} + 2\frac{v^r_i}{c}\,\frac{\vec{v}_i}{c}\cdot\frac{\partial \hat{n}_{(1)j[(0)]}}{\partial \vec{\tilde{\eta}}_i} + \sum_{j\neq i}\frac{v^r_i}{c}\,\frac{\vec{v}_j}{c}\cdot\frac{\partial \hat{n}_{(1)j[(0)]}}{\partial \vec{\tilde{\eta}}_j} + \\
& + \sum_u \Big(\frac{v^u_i}{c}\,\frac{\partial \hat{\bar{n}}_{(1)(u)j[(1)]}}{\partial \tilde{\eta}^r_i} - \frac{v^u_i}{c}\,\frac{\partial \hat{\bar{n}}_{(1)(r)j[(1)]}}{\partial \tilde{\eta}^u_i} - \frac{v^u_j}{c}\,\frac{\partial \hat{\bar{n}}_{(1)(r)j[(1)]}}{\partial \tilde{\eta}^u_j}\Big) + \\
& +2\sum_s \Big(\frac{v^s_i}{c}\Big)^2 \Big(\frac{\partial}{\partial \tilde{\eta}^r_i}\phi_{(1)j[(0)]}\Big) - \\
& -4\,\frac{v^r_i}{c}\Big(\frac{\vec{v}_i}{c}\cdot\frac{\partial}{\partial \vec{\tilde{\eta}}_i} + \frac{\vec{v}_j}{c}\cdot\frac{\partial}{\partial \vec{\tilde{\eta}}_j}\Big)\phi_{(1)j[(0)]}\big)
\end{aligned}
$$

and for $k > 1$

$$
\begin{aligned}
\tilde{F}^r_{ij[(2k)]} \;=\; & -\frac{\partial \hat{n}_{(1)j[(2k)]}}{\partial \tilde{\eta}^r_i} + 2\frac{v^r_i}{c}\,\frac{\vec{v}_i}{c}\cdot\frac{\partial \hat{n}_{(1)j[(2k-2)]}}{\partial \vec{\tilde{\eta}}_i} + \frac{v^r_i}{c}\,\frac{\vec{v}_j}{c}\cdot\frac{\partial \hat{n}_{(1)j[(2k-2)]}}{\partial \vec{\tilde{\eta}}_j} + \\
& + \sum_u \Big(\frac{v^u_i}{c}\,\frac{\partial \hat{\bar{n}}_{(1)(u)j[2k-1]}}{\partial \tilde{\eta}^r_i} - \frac{v^u_i}{c}\,\frac{\partial \hat{\bar{n}}_{(1)(r)j[(2k-1)]}}{\partial \tilde{\eta}^u_i} - \sum_{j\neq i}\frac{v^u_j}{c}\,\frac{\partial \hat{\bar{n}}_{(1)(r)j[(2k-1)]}}{\partial \tilde{\eta}^u_j}\Big) + \\
& + \frac{v^r_i}{c}\sum_{u,v}\frac{v^u_i}{c}\,\frac{v^v_i}{c}\,\frac{\partial \hat{\bar{n}}_{(1)(u)j[(2k-3)]}}{\partial \tilde{\eta}^v_i} + \\
& + \sum_s \Big(\frac{v^s_i}{c}\Big)^2 \Big(\frac{\partial}{\partial \tilde{\eta}^r_i}(\hat{\Gamma}^{(1)}_{sj[(2k-2)]} + 2\phi_{(1)j[(2k-2)]}) - \\
& \quad -2\frac{v^r_i}{c}\,\frac{\vec{v}_i}{c}\cdot\frac{\partial}{\partial \vec{\tilde{\eta}}_i}(\hat{\Gamma}^{(1)}_{sj[(2k-4)]} + 2\phi_{(1)j[(2k-4)]})\Big) - \\
& -2\,\frac{v^r_i}{c}\Big(\frac{\vec{v}_i}{c}\cdot\frac{\partial}{\partial \vec{\tilde{\eta}}_i} + \frac{\vec{v}_j}{c}\cdot\frac{\partial}{\partial \vec{\tilde{\eta}}_j}\Big)(\hat{\Gamma}^{(1)}_{sj[(2k-2)]} + 2\phi_{(1)j[(2k-2)]}) + \\
& +2\,\frac{v^r_i}{c}\,\frac{v^2_i}{c^2}\Big(\frac{\vec{v}_i}{c}\cdot\frac{\partial}{\partial \vec{\tilde{\eta}}_i} + \frac{\vec{v}_j}{c}\cdot\frac{\partial}{\partial \vec{\tilde{\eta}}_j}\Big)(\hat{\Gamma}^{(1)}_{sj[(2k-4)]} + 2\phi_{(1)j[(2k-4)]}) \\
& - \frac{v^r_i}{c}\sum_s \Big(\frac{v^s_i}{c}\Big)^2 \sum_{j\neq i}\frac{\vec{v}_j}{c}\cdot\frac{\partial}{\partial \vec{\tilde{\eta}}_j}(\hat{\Gamma}^{(1)}_{sj[(2k-4)]} + 2\phi_{(1)j[(2k-4)]})\Big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
\tilde{F}^r_{ij[(2k+1)]} \;=\; & \sum_u \Big(\frac{v^u_i}{c}\,\frac{\partial \hat{\bar{n}}_{(1)(u)j[(2k)]}}{\partial \tilde{\eta}^r_i} - \frac{v^u_i}{c}\,\frac{\partial \hat{\bar{n}}_{(1)(r)j[(2k)]}}{\partial \tilde{\eta}^u_i} - \sum_{j\neq i}\frac{v^u_j}{c}\,\frac{\partial \hat{\bar{n}}_{(1)(r)j[(2k)]}}{\partial \tilde{\eta}^u_j}\Big) + \\
& + \frac{v^r_i}{c}\sum_{u,v}\frac{v^u_i}{c}\,\frac{v^v_i}{c}\,\frac{\partial \hat{\bar{n}}_{(1)(u)j[(2k-2)]}}{\partial \tilde{\eta}^v_i}
\end{aligned}
\qquad (5.13)
$$

Let us remark that the force $\tilde{F}^r_i$ contains at the 0PN order the Newton force of Newtonian gravity. In Eqs.(5.13) it is possible to see that the terms depending on the inertial gauge variable (the non-local York time) ${}^3\tilde{\mathcal{K}}_{(1)}$, absent in the Euclidean 3-spaces of Newton gravity, are present at the order 0.5PN. Moreover the force $\tilde{F}^r_i$ contains both even and odd terms at all the orders. See Appendix A for the 1PN expression of the ADM Poincare' generators with terms till order $O(\zeta^2)$ included.

In the standard approach in harmonic gauges the first odd terms start at 2.5PN order: they are connected to the breaking of time-reversal invariance due to the choice of the no-incoming radiation condition and to the effect of back-reaction in presence of gravitational self-force with the associated (either Hadamard or dimensional) regularization (see the review in Ref.[6]). In our approach the Grassmann regularization eliminates the self-force but back-reaction is present due to the constancy of the ADM energy and produces the correct energy balance for the emission of GW's as shown in paper II.

Since we are in a non-harmonic gauge, we use a Grassmann regularization and, moreover, we are not introducing ad hoc Lagrangians for the particles, it is not possible to make comparisons with the standard results known till 3.5PN order [6] (where also the hereditary terms are present: we will need higher orders in the HPM expansion to see these terms, if them are present in our non-harmonic gauges).

D. The HPM Binaries at the 1PN Order

Let us now consider the 1PN two-body problem, which is relevant for the treatment of binary systems [15] as shown in Chapter VI of Refs.[6] based on Ref.[40].

Since there is no convincing evidence of dark matter in the Solar System and near the galactic plane of the Milky Way [41], we shall ignore the 0.5PN terms containing the non-local York time in the study of binary systems of stars in some galaxy. Instead in the next Section we will see that the non-local York time can be relevant in the 0.5PN simulation of dark matter at the level of mass and rotation curves of a whole galaxy.

If we ignore the York time, the 1PN equations of motion for the binary implied by Eqs.(5.13) and (5.12) are (we consider only positive energy, i.e. $\eta_1, \eta_2 \mapsto < \eta_1 >, < \eta_2 >= +1$)

---

[15] For binaries one assumes $\frac{v}{c} \approx \sqrt{\frac{R_m}{l_c}} << 1$, where $l_c \approx |\vec{r}|$ with $\vec{r}(t)$ being the relative separation after the decoupling of the center of mass. Often one considers the case $m_1 \approx m_2$. See chapter 4 of Ref. [6] for a review of the emission of GW's from circular and elliptic Keplerian orbits and of the induced inspiral phase implying a secular change in the semi-major axis, in the ellipticity and in the period, during which the waveform of GW's increases in amplitude and frequency producing a characteristic *chirp*. If we would add terms of higher PN order from Eqs.(5.7), we would get the analogue in the HPM linearization of the standard 3.5PN calculations for the inspiral phase before merging and ring-down (see section 5.6 of Ref.[6] and Ref.[38] for a review; see also Ref.[39]). Again the Grassmann regularization gives different results for the back-reaction. Instead the PM equations of motion Eqs.(5.3) and (5.4) should be used to take under control the relativistic recoils during the inspiral phase.

$$
\begin{aligned}
& m_1\, a^r_1(t) \stackrel{\circ}{=}\\
\stackrel{\circ}{=}\ & -G\, m_1 m_2 \Big[\frac{\tilde{\eta}^r_1(t) - \tilde{\eta}^r_2(t)}{|\vec{\tilde{\eta}}_1(t) - \vec{\tilde{\eta}}_2(t)|^3} \Big(1 + \frac{\vec{v}^2_1(t)}{c^2} + 2\frac{\vec{v}^2_2(t)}{c^2} - 4\frac{\vec{v}_1(t)\cdot\vec{v}_2(t)}{c^2} - \frac{3}{2}\frac{\Big(\frac{\vec{v}_2(t)}{c}\cdot(\vec{\tilde{\eta}}_1(t) - \vec{\tilde{\eta}}_2(t))\Big)^2}{|\vec{\tilde{\eta}}_1(t) - \vec{\tilde{\eta}}_2(t)|^2}\Big) +\\
- & \frac{v^r_1(t) - v^r_2(t)}{2}\Big(4\,\frac{\frac{\vec{v}_1(t)}{c}\cdot(\vec{\tilde{\eta}}_1(t) - \vec{\tilde{\eta}}_2(t))}{|\vec{\tilde{\eta}}_1(t) - \vec{\tilde{\eta}}_2(t)|^2} - 3\,\frac{\frac{\vec{v}_2(t)}{c}\cdot(\vec{\tilde{\eta}}_1(t) - \vec{\tilde{\eta}}_2(t))}{|\vec{\tilde{\eta}}_1(t) - \vec{\tilde{\eta}}_2(t)|^2}\Big)\Big] = \tilde{F}^r_{1(1PN)}(t),
\end{aligned}
$$

$$
\begin{aligned}
& m_2\, a^r_2(t) \stackrel{\circ}{=}\\
\stackrel{\circ}{=}\ & +G\, m_1 m_2 \Big[\frac{\tilde{\eta}^r_1(t) - \tilde{\eta}^r_2(t)}{|\vec{\tilde{\eta}}_1(t) - \vec{\tilde{\eta}}_2(t)|^3} \Big(1 + \frac{\vec{v}^2_2(t)}{c^2} + 2\frac{\vec{v}^2_1(t)}{c^2} - 4\frac{\vec{v}_1(t)\cdot\vec{v}_2(t)}{c^2} - \frac{3}{2}\frac{\Big(\frac{\vec{v}_1(t)}{c}\cdot(\vec{\tilde{\eta}}_1(t) - \vec{\tilde{\eta}}_2(t))\Big)^2}{|\vec{\tilde{\eta}}_1(t) - \vec{\tilde{\eta}}_2(t)|^2}\Big) +\\
- & \frac{v^r_2(t) - v^r_1(t)}{2}\Big(4\,\frac{\frac{\vec{v}_2(t)}{c}\cdot(\vec{\tilde{\eta}}_1(t) - \vec{\tilde{\eta}}_2(t))}{|\vec{\tilde{\eta}}_1(t) - \vec{\tilde{\eta}}_2(t)|^2} - 3\,\frac{\frac{\vec{v}_1(t)}{c}\cdot(\vec{\tilde{\eta}}_1(t) - \vec{\tilde{\eta}}_2(t))}{|\vec{\tilde{\eta}}_1(t) - \vec{\tilde{\eta}}_2(t)|^2}\Big)\Big] = \tilde{F}^r_{2(1PN)}(t).
\end{aligned}
\qquad (5.14)
$$

The last two lines in each equation correspond to the gravito-magnetic force generated by the shift functions. These equations identify the 1PN forces $\tilde{F}^r_{i(1PN)}(t)$.

Let us now reformulate this two-body problem in the canonical basis of center-of-mass and relative variables used in Eqs.(5.5)-(5.10) by restricting us to the lowest order $\vec{\eta}_i(\tau) = \vec{\tilde{\eta}}_i(t) = \vec{\tilde{\eta}}_{(o)i}(t)$. Since we have $\vec{\kappa}_{(1)i}(\tau) = \frac{m_i\,\vec{v}_i(t)}{\sqrt{1-(\frac{\vec{v}_i(t)}{c})^2}}$ with $\vec{\kappa}_{(1)1}(\tau) + \vec{\kappa}_{(1)2}(\tau) = 0$, from Eq.(5.5) we get $\vec{\pi}_{(1)12}(\tau) = \mu\Big(\frac{\vec{v}_1(t)}{\sqrt{1-(\frac{\vec{v}_1(t)}{c})^2}} - \frac{\vec{v}_2(t)}{\sqrt{1-(\frac{\vec{v}_2(t)}{c})^2}}\Big)$ with $\dot{\vec{\tilde{\eta}}}_{(o)1}(\tau) = \frac{\vec{v}_1(t)}{c} = (\frac{m_2}{M} - \tilde{A}_{(o)}(t))\,\frac{\vec{v}_{(rel)(o)12}(t)}{c}$ and $\dot{\vec{\tilde{\eta}}}_{(o)2}(\tau) = \frac{\vec{v}_2(t)}{c} = -(\frac{m_1}{M} + \tilde{A}_{(o)}(t))\,\frac{\vec{v}_{(rel)(o)12}(t)}{c}$, where we introduced the definition $\vec{v}_{(rel)(o)12}(t) = \frac{d\,\vec{\rho}_{(o)12}(t)}{dt}$. In writing $\dot{\vec{\tilde{\eta}}}_{(o)i}(\tau)$ we ignored the term $-\frac{1}{c}\,\frac{d\,\tilde{A}_{(o)}(t)}{dt}\,\vec{\rho}_{(o)12}(t)$: it is of higher order because it depends on $\dot{\vec{\pi}}_{(1)12}(\tau)$.

Then at the 1PN order we get

$$
\begin{aligned}
\vec{\pi}_{(1)12}(\tau) &= \mu\Big(\vec{v}_1(t)\Big[1 + \frac{1}{2}(\frac{\vec{v}_1(t)}{c})^2\Big] - \vec{v}_2(t)\Big[1 + \frac{1}{2}(\frac{\vec{v}_2(t)}{c})^2\Big]\Big) =\\
&= \mu\,\vec{v}_{(rel)(o)12}(t)\Big(1 + \frac{1}{2}\Big[(\frac{\mu}{m_1} - \tilde{A}_{(o)}(t))^3 + (\frac{\mu}{m_2} + \tilde{A}_{(o)}(t))^3\Big]\Big(\frac{\vec{v}_{(rel)(o)12}(t)}{c}\Big)^2\Big).
\end{aligned}
\qquad (5.15)
$$

Since the 1PN limit of the function $\tilde{A}_{(o)}(t)$ of Eq.(5.6) is $\frac{\mu\,(m_1 - m_2)}{2M^2}\,(\frac{\vec{v}_{(rel)(o)12}(t)}{c})^2$ (so that from Eq.(5.6) we have $\vec{\tilde{\eta}}_{(o)12}(t) \approx 0 + O(\frac{v^2}{c^2})$), we get $\vec{\pi}_{(1)12}(\tau) = \mu\,\vec{v}_{(rel)(o)12}(t)\Big[1 + \frac{m_1^3 + m_2^3}{2\,M^3}\,(\frac{\vec{v}_{(rel)(o)12}(t)}{c})^2\Big]$ as the 1PN limit of the relative momentum.

Then from Eqs.(5.8) the 1PN limit of the ADM energy and angular momentum in the rest-frame is (at this order there is no dependence on the York time)

$$
\begin{aligned}
\hat{E}_{ADM(1PN)} &= \sum_i m_i\, c^2 + \mu\, \Big(\frac{1}{2}\, \vec{v}^2_{(rel)(o)12}(t)\, \Big[1 + \frac{m_1^3 + m_2^3}{M^3}\, (\frac{\vec{v}_{(rel)(o)12}(t)}{c})^2 \Big] - \\
&- \frac{G\, M}{|\vec{\rho}_{(o)12}(t)|}\, \Big[1 + \frac{1}{2}\, \Big((3 + \frac{\mu}{M})\, \frac{\vec{v}^2_{(rel)(o)12}(t)}{c^2} + \frac{\mu}{M}\, (\frac{\vec{v}_{(rel)(o)12}(t)}{c} \cdot \frac{\vec{\rho}_{(o)12}(t)}{|\vec{\rho}_{(o)12}(t)|})^2 \Big) \Big] \Big), \\
\hat{J}^{rs}_{ADM(1PN)} &= \Big(\rho^r_{(o)12}(t)\, v^s_{(rel)(o)12}(t) - \rho^s_{(o)12}(t)\, v^r_{(rel)(o)12}(t) \Big) \\
&\Big[1 + \frac{m_1^3 + m_2^3}{2\, M^3}\, (\frac{\vec{v}_{(rel)(o)12}(t)}{c})^2 \Big], \qquad (5.16)
\end{aligned}
$$

while from Eqs.(5.10) the equation of motion for the relative variable is

$$
\begin{aligned}
\frac{d\, \vec{v}_{(rel)(o)12}(t)}{dt} &\,\stackrel{\circ}{=}\, \frac{1}{\mu}\, \Big(\frac{m_2}{M}\, \tilde{F}^r_{1(1PN)}(t) - \frac{m_1}{M}\, \tilde{F}^r_{2(1PN)}(t) \Big)(t, \vec{\tilde{\eta}}_{i(o)}(t), \vec{v}_i(t)) = \\
&= -G\, M\, \frac{\vec{\rho}_{(o)12}(t)}{|\vec{\rho}_{(o)12}(t)|^3} \Big[1 + \Big(1 + 3 \frac{\mu}{M} \Big) \frac{v^2_{(rel)(o)12}(t)}{c^2} - \frac{3\mu}{2M}\, \Big(\frac{\vec{v}_{(rel)(o)12}(t) \cdot \vec{\rho}_{(o)12}(t))}{|\vec{\rho}_{(o)12}(t)|} \Big)^2 \Big] + \\
&- \frac{G\, M}{|\vec{\rho}_{(o)12}(t)|^3} \Big(4 - \frac{2\mu}{M} \Big) \vec{v}_{(rel)(o)12}(t)\, \frac{\vec{v}_{(rel)(o)12}(t) \cdot \vec{\rho}_{(o)12}(t))}{|\vec{\rho}_{(o)12}(t)|}. \qquad (5.17)
\end{aligned}
$$

with the forces $\tilde{F}^r_{i(1PN)}(t)$ defined in Eqs.(5.14).

This is the result (ignoring the 0.5PN contribution of the non-local York time; for it see next Section) for the 1PN relative motion of binaries in our HPM linearization in the 3-orthogonal gauges, where the energy and angular momentum constants of motion are given by the corresponding ADM generators (implying planar motion in the plane orthogonal to the rest-frame ADM angular momentum).

Our 1PN equations (5.16) and (5.17) in the 3-orthogonal gauges coincide with Eqs. (2.5), (2.13) and (2.14) of the first paper Damour and Deruelle in Ref.[40] (without G^2 terms since they are $O(\zeta^2)$), which are obtained in the family of harmonic gauges starting from an ad hoc 1PN Lagrangian for the relative motion of two test particles (first derived by Infeld and Plebanski [42]) [16]. These equations are the starting point for studying the post-Keplerian parameters of the binaries, which, together with the Roemer, Einstein and Shapiro time delays (both near Earth and near the binary) in light propagation, allow to fit the experimental data from the binaries (see the second paper in Ref.[40] and Chapter VI of Ref.[6]). Therefore these results are reproduced also in our 3-orthogonal gauge with ${}^3K_{(1)}(\tau, \vec{\sigma}) = 0$.

[16] This is also the starting point of the effective one body description of the two-body problem of Refs. [43].

VI. DARK MATTER AS A RELATIVISTIC INERTIAL EFFECT DUE TO YORK TIME

In this Section we will see that the non-local York time can be relevant in the simulation of dark matter at the level of the mass of clusters of galaxies and of the rotation curves of galaxies.

In the first Subsection we consider the 0.5PN equations of motion. In the next two Subsections we will consider two of the main signatures of the existence of dark matter in the observed masses of galaxies and clusters of galaxies, namely the virial theorem [44, 45] and weak gravitational lensing [25, 45, 46]. Then in the fourth Subsection we will reproduce the pattern of rotation curves of spiral galaxies [47]. In a final Subsection we discuss which information we obtain on the York time from the dark matter data.

A. The 0.5 Post-Newtonian Limit of the Equations of Motion for the Particles

At the order $0.5PN$, with the non-local York time ${}^4\tilde{\mathcal{K}}_{(1)}$ (with dimension $[{}^3\mathcal{K}_{(1)}] = [l]$ since $[{}^3K_{(1)}] = [l^{-1}]$) taken into account, Eqs.(5.13) for the particles become

$$\eta_i\, m_i\, \frac{d^2\, \tilde{\eta}^r_i(t)}{dt^2} \,\stackrel{\circ}{=}\, \eta_i\, m_i\, \Big[- G\, \frac{\partial}{\partial\, \tilde{\eta}^r_i} \sum_{j\not= i} \eta_j\, \frac{m_j}{|\vec{\tilde{\eta}}_i(t) - \vec{\tilde{\eta}}_j(t)|} - \frac{1}{c}\, \frac{d\tilde{\eta}^r_i(t)}{dt} \Big(\partial^2_t|_{\vec{\tilde{\eta}}_i}\, {}^3\tilde{\mathcal{K}}_{(1)} + 2 \sum_u v^u_i(t)\, \frac{\partial\, \partial_t|_{\vec{\tilde{\eta}}_i}\, {}^3\tilde{\mathcal{K}}_{(1)}}{\partial\, \tilde{\eta}^u_i} + \sum_{uv} v^u_i(t)\, v^v_i(t)\, \frac{\partial^2\, {}^3\tilde{\mathcal{K}}_{(1)}}{\partial\, \tilde{\eta}^u_i\, \partial\, \tilde{\eta}^v_i}\Big)(t, \vec{\tilde{\eta}}_i(t))\Big]. \tag{6.1}$$

In these equations we can replace the Grassmann variables with their mean value $< \eta_i >= 1$, i = 1,..,N, for positive energy particles.

Therefore at the order 0.5PN the double rate of change in time of the trace of the extrinsic curvature, the arbitrary inertial gauge function parametrizing the family of 3-orthogonal gauges, creates PN damping terms with damping coefficients

$$\gamma_i(t, \vec{\tilde{\eta}}_i(t)) = \Big(\partial^2_t|_{\vec{\tilde{\eta}}_i} + 2 \sum_u v^u_i(t)\, \partial_u\, \partial_t|_{\vec{\tilde{\eta}}_i} + \sum_{uv} v^u_i(t)\, v^v_i(t)\, \partial_u\, \partial_v\Big)\, {}^3\tilde{\mathcal{K}}_{(1)}(t, \vec{\tilde{\eta}}_i(t)). \tag{6.2}$$

For instance the first term corresponds to a *damping* when $\partial^2_t|_{\vec{\tilde{\eta}}_i}\, {}^3\mathcal{K}_{(1)}(\tau, \vec{\eta}_i(\tau)) > 0$, but it is an *anti-damping* when $\partial^2_t|_{\vec{\tilde{\eta}}_i}\, {}^3\mathcal{K}_{(1)}(\tau, \vec{\eta}_i(\tau)) < 0$. Since we have $[c^2\, \partial^2_\tau\, {}^3K_{(1)}(\tau, \vec{\sigma})]|_{\vec{\sigma} = \vec{\eta}_i(\tau)} = [\triangle\, \partial^2_t\, {}^3\tilde{\mathcal{K}}_{(1)}(t, \vec{\sigma})]|_{\vec{\sigma} = \vec{\tilde{\eta}}_i(t)}$, the anti-damping (damping) effect is governed by the acceleration of the change in time of the convexity (concavity) of the instantaneous 3-space Σ_τ near the particle as an embedded 3-manifold of space-time. This is a inertial effect, relevant at small accelerations of the particle, not existing in Newton theory where the Euclidean 3-space is absolute and absent in all the gauges with ${}^3K(\tau, \vec{\sigma}) = 0$ (see for instance Ref.[38] for the lowest order of PN harmonic gauges). The other damping terms have similar interpretation but with an extra dependence on the velocities.

We can rewrite Eq.(6.2) in the following form

$$\gamma_i(t, \vec{\tilde{\eta}}_i(t)) = \frac{d^2}{dt^2}\, {}^3\tilde{\mathcal{K}}_{(1)}(t, \vec{\tilde{\eta}}_i(t)) + \mathcal{O}(\zeta^2), \tag{6.3}$$

by using $\frac{d}{dt} = \partial_t + v_i^u(t)\, \partial_u$ and by taking into account the fact that the accelerations $dv_i^u(t)/dt$ are of order $\mathcal{O}(\zeta^2)$. As a consequence Eq.(6.1) can be written in the form

$$\frac{d}{dt}\Big[m_i \Big(1 + \frac{1}{c}\frac{d}{dt}\, {}^3\tilde{\mathcal{K}}_{(1)}(t, \vec{\tilde{\eta}}_i(t))\Big) \frac{d\, \tilde{\eta}_i^r(t)}{dt}\Big] \;\stackrel{\circ}{=}\; -G\, \frac{\partial}{\partial\, \tilde{\eta}_i^r} \sum_{j\neq i} \eta_j\, \frac{m_i\, m_j}{|\vec{\tilde{\eta}}_i(t) - \vec{\tilde{\eta}}_j(t)|} + \\ + \;\mathcal{O}(\zeta^2) \tag{6.4}$$

We see that the term in the non-local York time can be *interpreted* as the introduction of an *effective (time-, velocity- and position-dependent) inertial mass term* for the kinetic energy of each particle: $m_i \;\mapsto\; m_i\, \Big(1 + \frac{1}{c}\frac{d}{dt}\, {}^3\tilde{\mathcal{K}}_{(1)}(t, \vec{\tilde{\eta}}_i(t))\Big)$ in each instantaneous 3-space. Instead in the Newton potential there are the gravitational masses of the particles, equal to the inertial ones in the 4-dimensional due to the equivalence principle. Therefore the effect is due to a modification of the effective inertial mass in each non-Euclidean 3-space: it is the equality of the inertial and gravitational masses of Newtonian gravity to be violated!

B. Dark Matter in Galaxy Masses from the Virial Theorem

One of the experimental signatures of the existence of dark matter comes from the use of the virial theorem for the determination of the mass of clusters of galaxies [44, 45]. For a bound system of N particles of mass m (N equal mass galaxies) at equilibrium, the virial theorem relates the average kinetic energy $< E_{kin} >$ in the system to the average potential energy $< U_{pot} >$ in the system: $< E_{kin} >= -\frac{1}{2} < U_{pot} >$ assuming Newton gravity. The equilibrium condition is supposed to be more valid for clusters of galaxies rather than for galaxies (clusters of stars). For the average kinetic energy of a galaxy in the cluster one takes $< E_{kin} >\approx \frac{1}{2} m < v^2 >$, where $< v^2 >$ is the average of the square of the radial velocity of single galaxies with respect to the center of the cluster (measured with Doppler shift methods; the velocity distribution is assumed isotropic). The average potential energy of the galaxy is assumed of the form $< U_{pot} >\approx -G\, \frac{m\, M}{\mathcal{R}}$, where $M = Nm$ is the total mass of the cluster and $\mathcal{R} = \alpha\, R$ is a "effective radius" depending on the cluster size R (the angular diameter of the cluster and its distance from Earth are needed to find R) and on the mass distribution on the cluster (usually $\alpha \approx 1/2$). Then the virial theorem implies $M \approx \frac{\mathcal{R}}{G} < v^2 >$. It turns out that this mass M of the cluster is usually at least an order of magnitude bigger that the baryonic matter of the cluster (spectroscopically determined).

If we consider the 0.5PN limit of the equations of motion for the particles given in eq.(6.1), we have to introduce a correction to final form of the virial theorem due to the extra term depending on the non local York time.

Usually the derivation of virial theorem starts assuming that in a self gravitating system at equilibrium we have

$$\frac{d^2}{dt^2} \sum_i m_i \mid \vec{\tilde{\eta}}_i(t) \mid^2 = 0. \tag{6.5}$$

This implies

$$\sum_i m_i v_i^2(t) + \sum_i m_i \vec{\tilde{\eta}}_i(t) \cdot \frac{d\vec{v}_i(t)}{dt} = 0. \tag{6.6}$$

By using Eqs.(6.1) and (6.2) we get

$$\sum_i m_i v_i^2(t) - G \sum_{i>j} \eta_j \frac{m_i m_j}{|\vec{\tilde{\eta}}_i(t) - \vec{\tilde{\eta}}_j(t)|} - \frac{1}{c} \sum_i m_i \Big(\vec{\tilde{\eta}}_i(t) \cdot \vec{v}_i(t) \Big) \gamma_i(t, \vec{\tilde{\eta}}_i(t)) = 0. \tag{6.7}$$

In the case $m_i = m$, the mean square velocity is $\langle v^2 \rangle = \frac{1}{N} \sum_i v_i^2$, and the mean gravitational potential energy for particle (with $\mathcal{R} = R/2$) has the form $\langle U_{pot} \rangle = -\frac{1}{N} \sum_{i>j} \eta_j G \frac{m_i m_j}{|\vec{\tilde{\eta}}_i(t) - \vec{\tilde{\eta}}_j(t)|} \approx G \frac{mM}{\mathcal{R}}$, with $M_{bar} = Nm$ being the baryonic mass.

Then, if we define $\langle \Big(\vec{\tilde{\eta}} \cdot \vec{v} \Big) \gamma(t, \vec{\tilde{\eta}}) \rangle = \frac{1}{N} \sum_i \Big(\vec{\tilde{\eta}}_i(t) \cdot \vec{v}_i(t) \Big) \gamma_i(t, \vec{\tilde{\eta}}_i(t))$, we get the following result from Eq.(6.7)

$$\frac{1}{2} m \langle v^2 \rangle = -\frac{1}{2} \langle U_{pot} \rangle + \frac{m}{2c} \langle \Big(\vec{\tilde{\eta}} \cdot \vec{v} \Big) \gamma(t, \vec{\tilde{\eta}}) \rangle. \tag{6.8}$$

Therefore for the measured mass M (the effective inertial mass in 3-space) we have

$$\begin{aligned} M &= \frac{\mathcal{R}}{G} < v^2 >= M_{bar} + \frac{\mathcal{R}}{G\,c} \langle \Big(\vec{\tilde{\eta}} \cdot \vec{v} \Big) \gamma(t, \vec{\tilde{\eta}}) \rangle = \\ &\stackrel{def}{=} M_{bar} + M_{DM}, \end{aligned} \tag{6.9}$$

and we see that the non-local York time can give rise to a dark matter contribution $M_{DM} = M - M_{bar}$.

C. Dark Matter in Galaxy Masses from Weak Gravitational Lensing

Another experimental signature of dark matter is the determination of the mass of a galaxy (or a cluster of galaxies) by means of weak gravitational lensing [25, 45, 46]. Let us consider a galaxy (or a cluster of galaxies) of big mass M (typically $M \approx M_{sun}^{12}$) behind which a distant, bright object (often a galaxy) is located. The light from the distant object is bent by the massive one (the lens) and arrives on the Earth deflected from the original propagation direction.

As shown in Ref.[25] we have to evaluate Einstein deflection of light, emitted by a source S at distance d_S from the observer O on the Earth, generated by the big mass at a distance d_D from the observer O. The mass M, at distance d_{DS} from the source S, is considered as a point-like mass generating a 4-metric either of the Schwarzschild type (Schwarzschild lens) or of the type of Eq.(3.7) (nearly point-like case). The ray of light is assumed to propagate in Minkowski space-time till near M, to be deflected by an angle α by the local gravitational field of M and then to propagate in Minkowski space-time till the observer O. The distances d_S, d_D, d_{DS}, are evaluated by the observer O at some reference time in some nearly-inertial Minkowski frame with nearly Euclidean 3-spaces (in the Euclidean case $d_{DS} = d_S - d_D$). If $\xi = \theta\, d_D$ is the impact parameter of the ray of light at M and if $\xi >> R_s = \frac{2\,G\,M}{c^2}$ (the gravitational radius), Einstein's deflection angle is $\alpha = \frac{2\,R_s}{\xi}$ [25] and the so-called Einstein radius (or characteristic angle) is

$$\alpha_o = \sqrt{2\,R_s\,\frac{d_{DS}}{d_D\,d_S}} = \sqrt{\frac{4\,G\,M}{c^2}\,\frac{d_{DS}}{d_D\,d_S}}. \tag{6.10}$$

A measurement of the deflection angle and of the three distances allows to get a value for the mass M of the lens, which usually turns out to be much larger of its mass inferred from the luminosity of the lens.

For the calculation of the deflection angle $\alpha = \frac{4\,G\,M}{c^2}$ one considers the propagation of ray of light in a stationary 4-metric of the type of Eq.(3.8) (with the assumption that $\partial_\tau\, {}^3\mathcal{K}_{(1)}$ and $\partial_r\, {}^3\mathcal{K}_{(1)}$ are slowly varying functions of time) and uses a version of Fermat's principle (see Sections 3.3 and 4.5 of Ref.[25]). In this description the spatial path $\vec{\sigma}(l)$ ($dl = |\vec{\sigma}|$ is the Euclidean arc length in an Euclidean 3-space) is the minimum of the variational principle(with fixed end points) $\delta\, \int_1^2 n\, dl = 0$, where the effective (position- and direction-dependent) effective index of refraction is $n = {}^4g_{\tau\tau} + \sum_r\, {}^4g_{\tau r}\, \frac{d\sigma^r}{dl}$. If one studies the Euler-Lagrange equations for the variational principle, if one ignores gravito-magnetism and if one chooses ${}^4g_{\tau\tau} = \epsilon\, [1 - 2\, \frac{w}{c^2} - 2\, \partial_\tau\, {}^3\mathcal{K}_{(1)}]$ as in Eqs.(3.8) with the choice $\frac{2\,w}{c^2} = - \frac{G\, M_{bar}}{c^2\, |\vec{\sigma}|}$ and with the definition $2\, \partial_\tau\, {}^3\mathcal{K}_{(1)} \stackrel{def}{=} - \frac{G\, M_{DM}}{c^2\, |\vec{\sigma}|}$, then the resulting Einstein deflection angle α turns out [24] to be the modulus of the vector

$$\vec{\alpha} = \frac{4\,G\,M}{c^2}\, \frac{\vec{\xi}}{|\vec{\xi}|^2} \qquad with \quad M = M_{bar} + M_{DM}. \tag{6.11}$$

Here $\vec{\xi}$ is the impact parameter if the unperturbed spatial trajectory of the ray is parametrized as $\vec{\sigma}(l) = \vec{\xi} + l\, \vec{e}$ ($\vec{e} = \frac{d\vec{\sigma}(l)}{dl}$ is the unit tangent vector of the ray satisfying $\vec{e} \cdot \vec{\xi} = 0$).

Therefore also in this case the measured mass M is the sum of a baryonic mass M_{bar} and of a dark matter mass M_{DM} induced by the non-local York time at the location of the lens.

D. The 0.5PN Two-Body Problem and the Rotation Curves of Galaxies

To study Eqs.(6.1) in the two-body case (i=1,2), we have to define center-of-mass and relative variables in the gravitational case with non-Euclidean 3-spaces, deviating from the Euclidean ones by order $O(\zeta)$, at least at the 0.5PN order.

With the notation of Eqs.(5.5) for the center-of-mass position we put $\tilde{\eta}^r_{12}(t) = \tilde{\eta}^r_{(o)12}(t) + \tilde{\eta}^r_{(1)12}(t)$, where $\vec{\tilde{\eta}}_{(o)12}(t) = \frac{m_1\, \vec{\tilde{\eta}}_1(t) + m_2\, \vec{\tilde{\eta}}_2(t)}{M}$ ($M = m_1 + m_2$; $\mu = \frac{m_1\, m_2}{M}$ is the reduced mass; $m_i = \frac{M}{2}\, (1 + (-)^{i+1}\, \sqrt{1 - 2\, \frac{\mu}{M}})$) is the non-relativistic center of mass and $\vec{\tilde{\eta}}_{(1)12}(t) = O(\zeta)$ is a small non-Euclidean correction.

The relative position variable is chosen as $\tilde{\rho}^r_{12}(t) = \vec{\tilde{\eta}}_1(t) - \vec{\tilde{\eta}}_2(t)$ [17] As a consequence we have $\tilde{\eta}^r_1(t) = \tilde{\eta}^r_{(o)12}(t) + \tilde{\eta}^r_{(1)12}(t) + \frac{m_2}{M}\,\tilde{\rho}^r_{12}(t)$ and $\tilde{\eta}^r_2(t) = \tilde{\eta}^r_{(o)12}(t) + \tilde{\eta}^r_{(1)12}(t) - \frac{m_1}{M}\,\tilde{\rho}^r_{(12}(t))$.

In Subsection VB also a decomposition $\tilde{\rho}^r_{12}(t) = \tilde{\rho}^r_{(0)12}(t) + \tilde{\rho}^r_{(1)12}(t)$ was adopted. This decomposition was motivated by the analogous decomposition of the center-of-mass position, solution of the equation $\hat{J}^{\tau r}_{ADM} \approx 0$. However the equation for the relative motion (see below) do not permit to say if the accelerations $d^2\tilde{\rho}^r_{(0)12}(t)/dt^2$ and $d^2\tilde{\rho}^r_{(1)12}(t)/dt^2$ are of different order or if they are of the same order. Therefore in this Subsection we do not use this decomposition.

As said after Eqs.(5.15), the vanishing of the ADM Lorentz boosts implies

$$\vec{\tilde{\eta}}_{(o)12}(t) \approx 0 + O(\frac{v^2}{c^2}). \tag{6.12}$$

It eliminates the non-relativistic 3-center of mass by putting it in the origin of the 3-coordinates, $\vec{\tilde{\eta}}_{NR}(t) \approx 0$. Therefore we have $\tilde{\eta}^r_{12}(t) \approx \tilde{\eta}^r_{(1)12}(t) = O(\zeta)$.

Then the sum and the difference of the two Eqs.(6.1) gives the following equations of motion for the center of mass position $\tilde{\eta}^r_{(1)12}(t)$ and for the relative variable $\tilde{\rho}^r_{12}(t)$ (we use the notation $v^r(t) = d\tilde{\rho}^r_{12}(t)/dt$)

$$\begin{aligned}
\frac{d^2\,\tilde{\eta}^r_{(1)12}(t)}{dt^2} &\,\stackrel{\circ}{=}\, -\frac{\mu}{M}\,\frac{1}{c}\,\frac{d\tilde{\rho}^r_{12}(t)}{dt}\,\gamma_-(t, \vec{\tilde{\rho}}_{12}(t), \vec{v}(t))\\
\frac{d^2\,\tilde{\rho}^r_{(1)12}(t)}{dt^2} &\,\stackrel{\circ}{=}\, -G\,M\,\frac{\tilde{\rho}^r_{12}(t)}{|\vec{\tilde{\rho}}_{12}(t)|^3} - \frac{1}{c}\,\frac{d\tilde{\rho}^r_{12}(t)}{dt}\,\gamma_+(t, \vec{\tilde{\rho}}_{12}(t), \vec{v}(t)),
\end{aligned}$$

$$\begin{aligned}
\gamma_+(t, \vec{\tilde{\rho}}_{12}(t), \vec{v}(t)) &= \frac{m_1}{M}\,\gamma_1(t, \frac{m_2}{M}\,\vec{\tilde{\rho}}_{12}(t), \vec{v}(t)) + \frac{m_2}{M}\,\gamma_2(t, -\frac{m_1}{M}\,\vec{\tilde{\rho}}_{12}(t), \vec{v}(t)),\\
\gamma_-(t, \vec{\tilde{\rho}}_{12}(t), \vec{v}(t)) &= \gamma_1(t, \frac{m_2}{M}\,\vec{\tilde{\rho}}_{12}(t), \vec{v}(t)) - \gamma_2(t, -\frac{m_1}{M}\,\vec{\tilde{\rho}}_{12}(t), \vec{v}(t)),\\
\gamma_1(t, \frac{m_2}{M}\,\vec{\tilde{\rho}}_{12}(t), \vec{v}(t)) &= \Big(\partial_t^2|_{\vec{\tilde{\eta}}_1} + 2\,\frac{m_2}{M}\,\sum_u v^u(t)\,\partial_u\,\partial_t|_{\vec{\tilde{\eta}}_1} +\\
&+ (\frac{m_2}{M})^2\,\sum_{uv} v^u(t)\,v^v(t)\,\partial_u\,\partial_v\Big)\,{}^3\tilde{\mathcal{K}}_{(1)}(t, \frac{m_2}{M}\,\vec{\tilde{\rho}}_{12}(t)),\\
\gamma_2(t, -\frac{m_1}{M}\,\vec{\tilde{\rho}}_{12}(t), \vec{v}(t)) &= \Big(\partial_t^2|_{\vec{\tilde{\eta}}_1} - 2\,\frac{m_1}{M}\,\sum_u v^u(t)\,\partial_u\,\partial_t|_{\vec{\tilde{\eta}}_1} +\\
&+ (\frac{m_1}{M})^2\,\sum_{uv} v^u(t)\,v^v(t)\,\partial_u\,\partial_v\Big)\,{}^3\tilde{\mathcal{K}}_{(1)}(t, -\frac{m_1}{M}\,\vec{\tilde{\rho}}_{12}(t)),.
\end{aligned} \tag{6.13}$$

[17] It should be defined as the tangent to the 3-geodesic of Σ_τ joining the two points (see the next Eq.(6.16)), which is parallel transported along it. See for instance Ref.[37]. At the orders $O(\zeta)$ and 0.5PN the above definition is acceptable.

We want estimate the contribution of the York time to the rotation curves. Motivated by the $1/c$ factor in front of this term, we can treat the York time term $v/c\,\gamma_+$ in the relative motion equation as a *perturbative term* added to usual Kepler problem $\frac{d^2\,\tilde{\rho}^r_{12}(t)}{dt^2} \stackrel{\circ}{=} - G\,M\,\frac{\tilde{\rho}^r_{12}(t)}{|\vec{\tilde{\rho}}_{12}(t)|^3}$. To realize this explicitly we take a solution with circular trajectory such that $\mid \vec{\tilde{\rho}}_{12}(t) \mid = R = constant$ and we make a decomposition of the velocity $\vec{v}(t) = \vec{v}_o(t) + \vec{v}_1(t)$ such that $\vec{v}_o(t) = v_o\,\hat{n}(t)$ is the keplerian velocity such that $v_o = \sqrt{\frac{G\,M}{R}} \rightarrow_{R\rightarrow\infty} 0$ and where $v_1(t)$ is the first order perturbative correction such that

$$\frac{dv^r_1(t)}{dt} = -\frac{v^r_o}{c}\,\hat{n}(t)\,\gamma_+(t, \vec{\tilde{\rho}}_{12}(t), \vec{v}^r_o(t)). \tag{6.14}$$

Then we get

$$v^r_1(t) = -\frac{v^r_o}{c}\int^t dt_1\,\hat{n}(t_1)\,\gamma_+(t_1, \vec{\tilde{\rho}}_{12}(t_1), \vec{v}_o(t_1)). \tag{6.15}$$

At the first order in the perturbation we get

$$v^2(t) = v^2_o\,\Big(1 - \frac{2}{c}\,\hat{n}(t)\cdot\int^t_o dt_1\,\hat{n}(t_1)\,\gamma_+(t_1, \vec{\tilde{\rho}}_{12}(t_1), \vec{v}_o(t_1))\Big). \tag{6.16}$$

Therefore, after having taken a mean value over a period T (the time dependence of the mass of a galaxy is not known) the effective mass of the two-body system is

$$\begin{aligned} M_{eff} &= \frac{\langle v^2\rangle\,R}{G} = M\,\Big(1 - \Big\langle \frac{2}{c}\,\hat{n}(t)\cdot\int^t dt_1\,\hat{n}(t_1)\,\gamma_+(t_1, \vec{\tilde{\rho}}_{12}(t_1), \vec{v}_o(t_1))\Big\rangle\Big)) = \\ &= M_{bar} + M_{DM}. \end{aligned} \tag{6.17}$$

Again the effective inertial mass in 3-space is the sum of the baryonic matter $M_{bar} = M$ plus a dark matter term.

Let us consider the case $m_1 >> m_2$. Let m_1 be the visible mass of a galaxy and let m_2 be the mass of either a star or a gas cloud circulating around the galaxy outside outside its visible radius. If the 3-space is Euclidean and the Keplerian orbit circular we have that the velocity goes to zero when the distance from the galaxy increases, since $\frac{d\,\vec{\tilde{r}}_{kepl,circ}(t)}{dt} = v_o\,\hat{n}(t) = \sqrt{\frac{G\,M}{R}}\hat{n}(t) \rightarrow_{R\rightarrow\infty} 0$. Instead from observations one finds that the velocity tend to a constant (till where it can be measured) and this so-called problem of the rotation curves of galaxies supports the existence of *dark matter haloes* around the galaxy (see for instance Ref.[47] for a review). Again dark matter is an inertial relativistic effect and the experimental dark matter distributions can be used to get informations on the non-local York time.

This explanation of dark matter differs:
1) from the non-relativistic MOND approach [48] (where one modifies Newton equations);

2) from modified gravity theories like the $f(R)$ ones (see for instance Refs.[49]; here one gets a modification of the Newton potential);
3) from postulating the existence of WIMP particles [50].

Let us also remark that the 0.5PN effect has origin in the lapse function and not in the shift one, as in the gravito-magnetic elimination of dark matter proposed in Ref.[51].

E. The Non-Local York Time from Dark Matter Data

As a consequence of the non-Euclidean nature of 3-space in Einstein space-times there is the possibility of describing part (or maybe all) dark matter as a *relativistic inertial effect*. As we have seen the three main experimental signatures of dark matter can be explained in terms of the non-local York time ${}^3\mathcal{K}_{(1)}(\tau,\vec{\sigma})$, the inertial gauge variable describing the general relativistic remnant of the gauge freedom in clock synchronization.

The open problem is the determination of the non-local York time from the data. From what is known from the Solar System and from inside the Milky Way near the galactic plane, it seems that it is negligible near the stars inside a galaxy. On the other hand, it is non zero near galaxies and clusters of galaxies of big mass. However only a mean value in time of time- and space-derivatives of the non-local York time can be extracted from the data. At this stage it seems that the non-local York time is relevant around the galaxies and the clusters of galaxies where there are big concentrations of mass and the dark matter haloes and that it becomes negligible inside the galaxies where there is a lower concentration of mass. Instead there is no indication on its value in the voids existing among the clusters of galaxies.

However to get an experimental determination of the York time ${}^3K_{(1)}(\tau,\vec{\sigma}) = \triangle\, {}^3\mathcal{K}_{(1)}(\tau,\vec{\sigma})$ we would need to know the non-local York time on all the 3-universe: phenomenological parametrizations of ${}^3\mathcal{K}_{(1)}(\tau,\vec{\sigma})$ will have to be devised to see the implications for ${}^3K_{(1)}(\tau,\vec{\sigma})$. As said in the Conclusions of paper II, a phenomenological determination of the York time would help in trying to get a PM extension of the Celestial reference frame (ICRS). This would be the way out from the gauge problem in general relativity: the observational conventions for matter would select a reference system of 4-coordinates for PM space-times in the associated 3-orthogonal gauge.

VII. CONCLUSIONS

In this paper we ended the study of the PM linearization of ADM tetrad gravity in the York canonical basis for asymptotically Minkowskian space-times in the family of non-harmonic 3-orthogonal gauges parametrized by the York time ${}^3K(\tau, \vec{\sigma})$, the trace of the extrinsic curvature of the 3-spaces. This inertial gauge variable, not existing in Newton gravity, describes the general relativistic remnant of the freedom in clock synchronization: its fixation gives the final identification of the instantaneous 3-spaces, after that their main structure has been dynamically determined by the solution of the Hamilton equations replacing Einstein equations. It turns out that at the PM level all the quantities depend on the spatially non-local quantity ${}^3\mathcal{K} = \frac{1}{\triangle}\, {}^3K$ (the non-local York time) with the only exception of the ADM Lorentz generators.

As matter we consider only N scalar point particles (without the transverse electro-magnetic field present in papers I and II) with a Grassmann regularization of the self-energies and with a ultraviolet cutoff making possible the HPM linearization and the evaluation of the PM solution for the gravitational field.

We studied in detail all the properties of these PM space-times emphasizing their dependence on the gauge variable ${}^3\mathcal{K} = \frac{1}{\triangle}\, {}^3K$ (the non-local York time): Riemann and Weyl tensors, 3-spaces, time-like and null geodesics, red-shift and luminosity distance. Also the Ashtekar variables of PM space-times were evaluated in the York canonical basis. We also studied the PM equations of motion of the particles, their PN expansion and the PM problem of the determination of center-of-mass and relative variables for the particles.

All the main measurable quantities turn out to have a dependence on the non-local York time. However it seems plausible that inside the Solar system and in every nearly isolated binary system this gauge quantity is negligible. This is not true at the astrophysical level especially for galaxies and clusters of galaxies with a big mass.

In Section VI we have shown that the main features of the experimental signatures for dark matter (masses of clusters of galaxies, rotation curves of spiral galaxies) can be explained in terms of the non-local York time at the 0.5PN level in the PN expansion.

This opens the possibility *to explain dark matter inside Einstein theory without modifications as a relativistic inertial effect*: the determination of ${}^3\mathcal{K}_{(1)}$ from the mass and the rotation curves of galaxies [45, 47] would give information on how to find a PM extension of the existing PN Celestial reference frame (ICRS) used as observational convention in the 4-dimensional description of stars and galaxies.

Therefore what is called dark matter would be an indicator of the non-Euclidean nature of 3-spaces as 3-sub-manifolds of space-time (extrinsic curvature effect), whose internal 3-curvature can be very small if it is induced by GW's. It is the Newtonian equality of inertial and gravitational masses in Euclidean 3-space to be violated, not their equality in the 4-dimensional space-time implied by the equivalence principle.

This conclusion derives from the analysis of the *gauge problem in general relativity* done in the Conclusions of paper II. The gauge freedom of space-time 4-diffeomorphisms implies that a gauge choice is equivalent to the choice of a set of 4-coordinates in the atlas of the space-time 4-manifold and that the observables are 4-scalars. At the Hamiltonian level the gauge group is deformed and the Hamiltonian observables are the Dirac observables (DO), which generically are only 3-scalars of the 3-space. However, for the tidal variables and the electro-magnetic field there is the possibility (under investigation by using the Newman-Penrose formalism [52]) that 4-scalar DO's describing them could exist.

On the other side at the experimental level *the description of baryon matter is intrinsically coordinate-dependent*, namely is connected with the conventions used by physicists, engineers and astronomers for the modeling of space-time. As a consequence of the dependence on coordinates of the description of matter, our proposal for solving the gauge problem in our Hamiltonian framework with non-Euclidean 3-spaces is to choose a gauge (i.e. a 4-coordinate system) in non-modified Einstein gravity which is in agreement with the observational conventions in astronomy. Since ICRS [5] has diagonal 3-metric, our 3-orthogonal gauges are a good choice. We are left with the inertial gauge variable ${}^3\mathcal{K}_{(1)} = \frac{1}{\triangle}\,{}^3K_{(1)}$ not existing in Newtonian gravity. As already said the suggestion is to try to fix ${}^3\mathcal{K}_{(1)}$ in such a way to eliminate as much dark matter as possible, by reinterpreting it as a relativistic inertial effect induced by the shift from Euclidean 3-spaces to non-Euclidean ones (independently from cosmological assumptions). As a consequence, ICRS should be reformulated not as a *quasi-inertial* reference frame in Galilei space-time, but as a reference frame in a PM space-time with ${}^3K_{(1)}$ (i.e. the clock synchronization convention) deduced from the data connected to dark matter. Then automatically BCRS would be its quasi-Minkowskian approximation (quasi-inertial reference frame in Minkowski space-time) for the Solar System. This point of view could also be useful for the ESA GAIA mission (cartography of the Milky Way) [53] and for the possible anomalies inside the Solar System [14].

Moreover our approach will require further developments in the following directions:

a) Find the second order of HPM to see whether in PM space-times there is the emergence of hereditary terms (see Refs.[6, 54]) like the ones present in harmonic gauges. Like in standard approaches (see the review in Appendix A of paper II) regularization problems may arise at the higher orders.

b) Study the PM equations of motion of the transverse electro-magnetic field trying to find Lienard-Wiechert-type solutions (see Subsection VB of paper II).

c) Dark energy in cosmology [27]. Take a perfect fluid as matter in the first order of HPM expansion [55] adapting to tetrad gravity the special relativistic results of Refs.[56]. Since in our formalism all the canonical variables in the York canonical basis, except the angles θ^i, are 3-scalars, we can complete Buchert's formulation of back-reaction [57] (see also Ref.[58]) by taking the spatial average of all the PM Hamilton equations in our non-harmonic 3-orthogonal gauges. This will allow to make the transition from the PM space-time 4-metric to an inhomogeneous cosmological one (only conformally related to Minkowski space-time at spatial infinity) and to reinterpret the dark energy as a non-linear effect of inhomogeneities. The role of the York time, now considered as an inertial gauge variable, in the theory of back-reaction and in the identification of what is called dark energy [18] is completely unexplored.

Let us remark that in the Friedmann-Robertson-Walker (FLW) cosmological solution the Killing symmetries connected with homogeneity and isotropy imply (τ is the cosmic time, $a(\tau)$ the scale factor) ${}^3K(\tau) = -\frac{\dot{a}(\tau)}{a(\tau)} = -H$, namely the York time is the Hubble constant. However at the first order in cosmological perturbations we have ${}^3K = -H + {}^3K_{(1)}$ with

[18] As we have seen the red-shift and the luminosity distance depend upon the York time, and this could play a role in the interpretation of the data from super-novae.

$^3K_{(1)}$ being an inertial gauge variable. Instead in the spatial-averaging method of Ref.[57] one gets that the spatial average of the York time (a 3-scalar gauge variable) gives the effective Hubble constant of that approach. Therefore we will try to see if also dark energy can be considered as an inertial effect of the York time [19] in the transition from astrophysics to cosmology.

[19] In paper I we showed that in the York canonical basis the York time contributes with a negative term to the kinetic energy in the ADM energy. It would also play a role in study to be done on the reformulation of the Landau-Lifschitz energy-momentum pseudo-tensor as the energy-momentum tensor of a viscous pseudo-fluid. It could be possible that for certain choices of the York time the resulting effective equation of state has negative pressure, realizing in this way a simulation of dark energy.

Appendix A: The PN expansion of the Weak ADM Poincare' Generators.

The 1PN expansion of the ADM Poincare' generators of Eqs. (2.10)-(2.13) with $O(\zeta^2)$ order included is

$$
\begin{aligned}
\hat{E}_{ADM} &= \sum_i \eta_i\, m_i\, c^2 + \sum_i \eta_i\, \frac{1}{2}\, m_i \Big[1 + \frac{1}{4}\, \frac{\vec{v}_i^2(t)}{c^2}\Big]\, \vec{v}_i^2(t) + Mc^2\, O(\zeta)\, O(\frac{v^6}{c^6}) - \\
&- \sum_i \eta_i\, m_i c^2 \Big(1 - \frac{1}{2}\, \frac{\vec{v}_i^2(t)}{c^2}\Big)\, \frac{\vec{v}_i(t)}{c} \cdot \vec{\partial}\, {}^3\tilde{\mathcal{K}}_{(1)}(t, \vec{\tilde{\eta}}_i(t)) - \\
&- G \sum_{i>j} \eta_i\, \eta_j\, \frac{m_i\, m_j}{|\vec{\tilde{\eta}}_i(t) - \vec{\tilde{\eta}}_j(t)|} \Big[1 + \frac{3}{2}\, \frac{\vec{v}_j^2(t)}{c^2} - \frac{7}{2}\, \frac{\vec{v}_i(t)\cdot \vec{v}_j(t)}{c^2} - \\
&- \frac{1}{2}\, \frac{\frac{\vec{v}_i(t)}{c} \cdot (\vec{\tilde{\eta}}_i(t) - \vec{\tilde{\eta}}_j(t))\, \frac{\vec{v}_j(t)}{c} \cdot (\vec{\tilde{\eta}}_i(t) - \vec{\tilde{\eta}}_j(t))}{|\vec{\tilde{\eta}}_i(t) - \vec{\tilde{\eta}}_j(t)|^2}\Big] + \\
&+ Mc^2\, O(\zeta^2)\, O(\frac{v^4}{c^4}) + Mc^2\, O(\zeta^3),
\end{aligned}
$$

$$
\begin{aligned}
\hat{J}^{rs}_{ADM} &= \sum_i \eta_i\, m_i \Big(1 + \frac{1}{2}\, \frac{\vec{v}_i^2(t)}{c^2}\Big) \Big(\tilde{\eta}_i^r(t)\, v_i^s(t) - \tilde{\eta}_i^s(t)\, v_i^r(t)\Big) + Mc\, L\, O(\zeta)\, O(\frac{v^5}{c^5}) + \\
&+ \sum_i \eta_i \Big(\tilde{\eta}_i^r(t)\, \frac{\partial}{\partial\, \tilde{\eta}_i^s} - \tilde{\eta}_i^s(t)\, \frac{\partial}{\partial\, \tilde{\eta}_i^r}\Big)\, m_i \Big(c\, (1 + \frac{\vec{v}_i^2(t)}{2\, c^2})\, {}^3\tilde{\mathcal{K}}_{(1)} - \\
&- 2 \sum_u v_i^u(t)\, \frac{\partial_u}{\triangle} \Big[\tilde{\Gamma}_u^{(1)} - \frac{1}{4} \sum_c \frac{\partial_c^2}{\triangle}\, \tilde{\Gamma}_c^{(1)}\Big]\Big)(t, \vec{\tilde{\eta}}_i(t)) + \\
&+ 2 \sum_i \eta_i\, m_i \Big[v_i^r(t)\, \frac{\partial_s}{\triangle} \Big(\tilde{\Gamma}_s^{(1)} - \frac{1}{4} \sum_c \frac{\partial_c^2}{\triangle}\, \tilde{\Gamma}_c^{(1)}\Big) - \\
&- v_i^s(t)\, \frac{\partial_r}{\triangle} \Big(\tilde{\Gamma}_r^{(1)} - \frac{1}{4} \sum_c \frac{\partial_c^2}{\triangle}\, \tilde{\Gamma}_c^{(1)}\Big)\Big](t, \vec{\tilde{\eta}}_i(t)) + \\
&+ \frac{C^3}{4\pi\, G} \int d^3\sigma\, \Big({}^3\tilde{K}_{(1)}\, \partial_r\, \partial_s\, (\Gamma_r^{(1)} - \Gamma_s^{(1)})\Big)(t, \vec{\sigma}) + Mc\, L\, O(\zeta^2)\, O(\frac{v^5}{c^5}) + Mc\, L\, O(\zeta^3),
\end{aligned}
$$

$$
\begin{aligned}
\hat{P}^r_{ADM} &= \sum_i \eta_i\, m_i \Big(\Big[\Big(1 + \frac{\vec{v}_i^2(t)}{2\, c^2}\Big)\, v_i^r(t) + Mc\, O(\zeta)\, O(\frac{v^5}{c^5}) - \\
&- \sum_i \eta_i\, m_i \sum_a v_i^a(t)\, \frac{\partial_r\, \partial_a}{\triangle} \Big(2\, \tilde{\Gamma}_a^{(1)} - \frac{1}{2} \sum_c \frac{\partial_c^2}{\triangle}\, \tilde{\Gamma}_c^{(1)}\Big)(\tau, \vec{\tilde{\eta}}_i(t)) - \\
&- \sum_i \eta_i\, m_i\, c\, (1 + \frac{\vec{v}_i^2(t)}{2\, c^2})\, \partial_r\, {}^3\tilde{\mathcal{K}}_{(1)}(\tau, \vec{\tilde{\eta}}_i(t)) + +Mc\, O(\zeta^2)\, O(\frac{v^5}{c^5}) + Mc\, O(\zeta^3) \approx 0,
\end{aligned}
$$

$$
\begin{aligned}
\hat{J}^{\tau r}_{ADM} \;=\; & -\sum_i \eta_i\, \tilde{\eta}^r_i(t)\, m_i c \Big(1 + \frac{\vec{v}^2_i(t)}{2\, c^2} + \frac{\vec{v}^2_i(t)}{c^2}\, \frac{G}{c^2} \sum_{j\not= i} \eta_j\, \frac{m_j}{|\vec{\tilde{\eta}}_i(t) - \vec{\tilde{\eta}}_j(t)|}\Big) - \\
& - \int d^3\sigma\, \sigma^r \Big[\frac{1}{2} \sum_a \sum_i \eta_i\, m_i c\, \frac{\sigma^a - \tilde{\eta}^a_i(t)}{4\pi \mid \vec{\sigma} - \vec{\tilde{\eta}}_i(t) \mid^3}\, \partial_a \Big(\tilde{\Gamma}^{(1)}_a - \frac{1}{2} \sum_c \frac{\partial^2_c}{\Delta}\, \tilde{\Gamma}^{(1)}_c\Big) + \\
& + \frac{32\pi G}{c} \sum_{i\not= j} \eta_i \eta_j\, m_i m_j\, \Big(1 + \frac{v^2_i(t) + v^2_j(t)}{2c^2}\Big) \frac{(\vec{\sigma} - \vec{\tilde{\eta}}_i(t)) \cdot (\vec{\sigma} - \vec{\tilde{\eta}}_j(t))}{16\pi^2 \mid \vec{\sigma} - \vec{\tilde{\eta}}_i(t) \mid^3 \mid \vec{\sigma} - \vec{\tilde{\eta}}_j(t) \mid^3} - \\
& - \frac{2}{c} \sum_{a,b} \Big(\widetilde{M}_{ab}\, \partial_t\, \tilde{\Gamma}^{(1)}_b\Big) \sum_i \eta_i\, \frac{m_i\, v^a_i(t)\, (\sigma^a - \tilde{\eta}^a_i(t))}{4\pi \mid \vec{\sigma} - \vec{\tilde{\eta}}_i(t) \mid^3} + \\
& + \frac{2}{c} \sum_{a\not= b} \frac{\partial_a \partial_b \partial_t}{\Delta} \Big(\tilde{\Gamma}^{(1)}_a + \tilde{\Gamma}^{(1)}_b - \frac{1}{2} \sum_c \frac{\partial^2_c}{\Delta}\, \tilde{\Gamma}^{(1)}_c\Big) \sum_i \eta_i\, \frac{m_i\, v^a_i(t)\, (\sigma^a - \tilde{\eta}^a_i(t))}{4\pi \mid \vec{\sigma} - \vec{\tilde{\eta}}_i(t) \mid^3} - \\
& - \frac{c^2}{8\pi G} \sum_{a,b} \Big(\widetilde{M}_{ab}\, \partial_t\, \tilde{\Gamma}^{(1)}_b\Big) \frac{\partial^2_a}{\Delta} \Big({}^3\tilde{K}_{(1)} - \frac{4\pi G}{c^3} \sum_i \eta_i\, \frac{m_i\, \vec{v}_i(t) \cdot (\vec{\sigma} - \vec{\tilde{\eta}}_i(t))}{4\pi \mid \vec{\sigma} - \vec{\tilde{\eta}}_i(t) \mid^3}\Big) + \\
& + \frac{c^2}{8\pi G} \sum_{a\not= b} \frac{\partial_a \partial_b \partial_t}{\Delta} \Big(\tilde{\Gamma}^{(1)}_a + \tilde{\Gamma}^{(1)}_b - \frac{1}{2} \sum_c \frac{\partial^2_c}{\Delta}\, \tilde{\Gamma}^{(1)}_c\Big) \\
& \quad \frac{\partial_a \partial_b}{\Delta} \Big({}^3\tilde{K}_{(1)} - \frac{4\pi G}{c^3} \sum_i \eta_i\, \frac{m_i\, \vec{v}_i(t) \cdot (\vec{\sigma} - \vec{\tilde{\eta}}_i(t))}{4\pi \mid \vec{\sigma} - \vec{\tilde{\eta}}_i(t) \mid^3}\Big) + \\
& + \frac{c^3}{16\pi G} \sum_{a,b} \Big(\frac{\partial_a \partial_b}{\Delta}\, {}^3\tilde{K}_{(1)}\Big)^2 + \\
& + \frac{G}{16\pi c^3} \sum_{a,b} \sum_{i\not= j} \eta_i \eta_j\, m_i m_j \frac{\partial_a \partial_b}{\Delta} \Big(\frac{\vec{v}_i(t) \cdot (\vec{\sigma} - \vec{\tilde{\eta}}_i(t))}{\mid \vec{\sigma} - \vec{\tilde{\eta}}_i(t) \mid^3}\Big) \frac{\partial_a \partial_b}{\Delta} \Big(\frac{\vec{v}_j(t) \cdot (\vec{\sigma} - \vec{\tilde{\eta}}_j(t))}{\mid \vec{\sigma} - \vec{\tilde{\eta}}_j(t) \mid^3}\Big) - \\
& - \frac{1}{8\pi} \sum_{a,b} \Big(\frac{\partial_a \partial_b}{\Delta}\, {}^3\tilde{K}_{(1)}\Big) \sum_i \eta_i\, m_i \Big(1 + \frac{\vec{v}^2_i(t)}{2c^2}\Big) \frac{\partial_a \partial_b}{\Delta} \Big(\frac{\vec{v}_i(t) \cdot (\vec{\sigma} - \vec{\tilde{\eta}}_i(t))}{\mid \vec{\sigma} - \vec{\tilde{\eta}}_i(t) \mid^3}\Big) + \\
& + \frac{1}{2\pi} \sum_{a,b} \Big(\frac{\partial_a \partial_b}{\Delta}\, {}^3\tilde{K}_{(1)}\Big) \sum_i \eta_i\, m_i \Big(1 + \frac{\vec{v}^2_i(t)}{2c^2}\Big) \frac{v^b_i(t)\, (\sigma^a - \tilde{\eta}^a_i(t))}{\mid \vec{\sigma} - \vec{\tilde{\eta}}_i(t) \mid^3} - \\
& - \frac{G}{2\pi c^3} \sum_{i\not= j} \eta_i \eta_j\, m_i m_j \frac{v^b_i(t)\, (\sigma^a - \tilde{\eta}^a_i(t))}{\mid \vec{\sigma} - \vec{\tilde{\eta}}_i(t) \mid^3}\, \frac{\partial_a \partial_b}{\Delta} \Big(\frac{\vec{v}_j(t) \cdot (\vec{\sigma} - \vec{\tilde{\eta}}_j(t))}{\mid \vec{\sigma} - \vec{\tilde{\eta}}_j(t) \mid^3}\Big) - \\
& - \frac{c^3}{72\pi G} \Big({}^3\tilde{K}_{(1)}\Big)^2 - \frac{1}{8\pi}\, {}^3\tilde{K}_{(1)} \sum_i \eta_i\, m_i \Big(1 + \frac{\vec{v}^2_i(t)}{2c^2}\Big) \frac{\vec{v}_i(t) \cdot (\vec{\sigma} - \vec{\tilde{\eta}}_i(t))}{\mid \vec{\sigma} - \vec{\tilde{\eta}}_i(t) \mid^3} +
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{3}{16}\frac{G}{\pi c^3}\sum_{i\neq j}\eta_i\eta_j\, m_i m_j\, \frac{\vec{v}_i(t)\cdot(\vec{\sigma}-\vec{\tilde{\eta}}_i(t))}{\mid\vec{\sigma}-\vec{\tilde{\eta}}_i(t)\mid^3}\,\frac{\vec{v}_j(t)\cdot(\vec{\sigma}-\vec{\tilde{\eta}}_i(t))}{\mid\vec{\sigma}-\vec{\tilde{\eta}}_j(t)\mid^3} + \\
&+ \frac{8\pi G}{c^3}\sum_{i\neq j}\eta_i\eta_j\, m_i m_j \\
&\quad \frac{\vec{v}_i(t)\cdot\vec{v}_j(t)\,(\vec{\sigma}-\vec{\tilde{\eta}}_j(t))\cdot(\vec{\sigma}-\vec{\tilde{\eta}}_i(t)) + \vec{v}_i(t)\cdot(\vec{\sigma}-\vec{\tilde{\eta}}_j(t))\,\vec{v}_j(t)\cdot(\vec{\sigma}-\vec{\tilde{\eta}}_i(t))}{16\pi^2\ \mid\vec{\sigma}-\vec{\tilde{\eta}}_i(t)\mid^3\mid\vec{\sigma}-\vec{\tilde{\eta}}_j(t)\mid^3}\Big](t,\vec{\sigma}) + \\
&+ \frac{3}{2}\int d^3\sigma\,\sum_i \eta_i \frac{m_i c}{4\pi\mid\vec{\sigma}-\vec{\tilde{\eta}}_i(t)\mid}\,\partial_r\,\tilde{\Gamma}_r^{(1)} + \\
&+ \int d^3\sigma\,\partial_r\Big[\frac{2\pi G}{c^3}\sum_{i\neq j}\eta_i\eta_j\,\frac{m_i m_j c^2}{16\pi^2\mid\vec{\sigma}-\vec{\tilde{\eta}}_i(t)\mid\mid\vec{\sigma}-\vec{\tilde{\eta}}_j(t)\mid}\Big(1+\frac{v_i^2(t)+v_j^2(t)}{2c^2}\Big)\Big](t,\vec{\sigma}) + \\
&+ Mc\,L\,O(\zeta)\,O(\frac{v^4}{c^4}) + Mc\,L\,O(\zeta^2)\,O(\frac{v^4}{c^4}) + Mc\,L\,O(\zeta^3) \approx 0. \qquad \text{(A1)}
\end{aligned}
$$

Since we have $R_{\bar{a}}, \Gamma_a^{(1)} = \frac{G}{c^2}\,O(\frac{v^2}{c^2})$ the GW kinetic term in the ADM energy is of order $Mc^2\,O(\frac{v^4}{c^4})$. The expression of the energy has been obtained by making an integration over 3-space and by using the integral given after Eq.(4.17) of paper II.

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