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Stochastic matrices and a property of the infinite sequences of linear functionals

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ABSTRACT

Our starting point is the proof of the following property of a particular class of matrices. Let $T = \{T_{i,j}\}$ be a $n \times m$ non-negative matrix such that $\sum_j T_{i,j} = 1$ for each i. Suppose that for every pair of indices (i, j), there exists an index l such that $T_{i,l} \neq T_{j,l}$. Then, there exists a real vector $\mathbf{k} = (k_1, k_2, \dots, k_m)^T$, $k_i \neq k_j$, $i \neq j$; $0 < k_i \leqslant 1$, such that, $(T\,\mathbf{k})_i \neq (T\,\mathbf{k})_j$ if $i \neq j$.

Then, we apply that property of matrices to probability theory. Let us consider an infinite sequence of linear functionals $\{T_i\}_{i\in\mathbb{N}}$, $T_i f = \int f(t)\, d\mu_t(i)$, corresponding to an infinite sequence of probability measures $\{\mu_{(\cdot)}(i)\}_{i\in\mathbb{N}}$, on the Borel σ-algebra $\mathcal{B}([0, 1])$ such that, $\mu_{(\cdot)}(i) \neq \mu_{(\cdot)}(j)$, $i, j \in \mathbb{N}$, $i \neq j$. The property of matrices described above allows us to construct a real bounded one-to-one piecewise continuous and continuous from the left function f such that

$$T_i f = \int f(t)\, d\mu_t(i) \neq \int f(t)\, d\mu_t(j) = T_j f, \quad i, j \in \mathbb{N},\ i \neq j.$$

The relevance to quantum mechanics is showed.
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1. Introduction

In the present paper a $n \times m$ non-negative matrix such that the sum of the elements of each row is one will be called rectangular stochastic. A rectangular stochastic matrix such that $n = m$ is a stochastic matrix. In the first part of the present work we prove the following property of rectangular stochastic

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matrices. Let $T = \{T_{i,j}\}$ be a $n \times m$ rectangular stochastic matrix such that for every pair of indices (i,j), there exists an index l such that $T_{i,l} \neq T_{j,l}$. Then, there exists a real vector $\mathbf{k} = (k_1, k_2, \ldots, k_n)^T$, $k_i \neq k_j, i \neq j$; $0 < k_i \leqslant 1$, such that, $(T\,\mathbf{k})_i \neq (T\,\mathbf{k})_j$ if $i \neq j$.

In the second part of the paper, we take into account an infinite sequence of real functionals $\{T_i\}_{i\in\mathbb{N}}$,

$$T_i f = \int f(t)\, d\mu_t(i) =: G_f(i),$$

corresponding to a sequence of probability measures $\{\mu_{(\cdot)}(i)\}_{i\in\mathbb{N}}$, on the Borel σ-algebra $\mathcal{B}([0,1])$, such that $\mu_{(\cdot)}(i) \neq \mu_{(\cdot)}(j),\ i,j \in \mathbb{N},\ i \neq j$. Then, by means of the property of rectangular stochastic matrices described above we prove constructively (Theorem 3) the existence of a real bounded one-to-one function f such that, for every $i,j \in \mathbb{N}, i \neq j$,

$$G_f(i) := \int f(t)\, d\mu_t(i) \neq \int f(t)\, d\mu_t(j) =: G_f(j). \tag{1}$$

In other words, we construct a one-to-one function f such that

$$T_i f \neq T_j f,\ \ i,j \in \mathbb{N},\ \ i \neq j.$$

Moreover, we prove that f is piecewise continuous and continuous from the left.
It is worth remarking that the existence of f is proved by construction.[1] In particular we give an algorithmic procedure for the construction of f.
Eq. (1) implies that the function $G_f : \mathbb{N} \to \mathbb{R}$ is one-to-one. The fact that both f and G_f can be one-to-one plays a key role in the application of that result to quantum mechanics [7].

We note that both the property of rectangular stochastic matrices and the mathematical result on the infinite sequences of linear functionals presented here could conceivably be of interest in other areas of mathematics. For instance, they find relevant applications to the theory of positive operator valued measures and to quantum mechanics [4–7] where, it is useful to have an algorithmic procedure for the construction of the function f which can be used to get the sharp reconstruction of a given positive operator valued measure [2,3,7]. A brief description of the applications of the results of the present paper to the theory of positive operator valued measures and to quantum mechanics can be found in Section 4.

The work is organized as follows: Section 2 deals with rectangular stochastic matrices. In particular we prove Theorem 1. In Section 3, we prove constructively Theorem 3 which describes the properties of the infinite sequences of linear functionals described above. In particular, the construction of the function f is based on Theorem 1. In Section 4, we apply Theorem 3 to the theory of positive operator valued measures and to quantum mechanics. In the Appendix A we prove a lemma useful in the proof of Theorem 3.

2. On a property of rectangular stochastic matrices

In what follows a $n \times m$ non-negative matrix $\{T_{i,j}\}$ such that $\sum_{j=1}^{m} T_{i,j} = 1, i = 1\ldots, n$ will be called rectangular stochastic. Notice that a rectangular stochastic matrix such that $n = m$ is a stochastic matrix. Then, the class of stochastic matrices is a subclass of the class of rectangular stochastic matrices. The following theorem on rectangular stochastic matrices is the starting point of the present work. In Section 3 it will be applied in the framework of probability theory. In Section 4 it will be applied to the theory of positive operator valued measures and to quantum mechanics. In Ref. [5] a more general version of the theorem was applied to quantum mechanics.

Theorem 1. *A matrix of non-negative real numbers*:

$$\begin{pmatrix} \lambda_{1,1} & \lambda_{1,2} & \ldots & \lambda_{1,m} \\ \lambda_{2,1} & \lambda_{2,2} & \ldots & \lambda_{2,m} \\ \ldots & \ldots & \ldots & \ldots \\ \lambda_{N,1} & \lambda_{N,2} & \ldots & \lambda_{N,m} \end{pmatrix} \tag{2}$$

[1] It is possible to prove [7] the existence of a one-to-one function f such that, $T_i f \neq T_j f, i \neq j$, by means of the Baire category theorem but the aim of the present paper is the construction of that function.

such that:

(*i*) *for every pair of indices* $(i, j), i, j = 1, \dots, N$, *there exists an index* $l \in \{1, \dots, m\}$ *such that* $\lambda_{i,l} \neq \lambda_{j,l}$;
(*ii*) *the matrix is rectangular stochastic, i.e.,* $\sum_{j=1}^{m} \lambda_{i,j} = 1, i = 1, \dots, N$,
defines an operator $T : \mathbb{C}^m \to \mathbb{C}^N$

$$T\mathbf{k} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix} := \begin{pmatrix} \lambda_{1,1} & \lambda_{1,2} & \dots & \lambda_{1,m} \\ \lambda_{2,1} & \lambda_{2,2} & \dots & \lambda_{2,m} \\ \dots & \dots & \dots & \dots \\ \lambda_{N,1} & \lambda_{N,2} & \dots & \lambda_{N,m} \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_m \end{pmatrix} \tag{3}$$

with the property that there exists a real vector $\mathbf{k} = (k_1, k_2, \dots, k_m)^T;\ k_i \neq k_j, i \neq j; 0 < k_i \leqslant 1$, *such that,* $(T\,\mathbf{k})_i \neq (T\,\mathbf{k})_j$ *if* $i \neq j$.

Proof. We proceed by steps.

Step 1: An arbitrary vector $\mathbf{k}^{(1)} = (k_1^{(1)}, \dots, k_m^{(1)})^T,\ 0 < k_i^{(1)} \leqslant 1,\ k_i^{(1)} \neq k_j^{(1)}$, is chosen as the first vector of the sequence.
Step 2: If $(T\,\mathbf{k}^{(1)})_2 \neq (T\,\mathbf{k}^{(1)})_1$ we set $\mathbf{k}^{(2)} = \mathbf{k}^{(1)}$ and proceed to the next step. If instead, $(T\,\mathbf{k}^{(1)})_2 = (T\,\mathbf{k}^{(1)})_1$ then, by item (i), there exists an index q_2 such that $\lambda_{2,q_2} \neq \lambda_{1,q_2}$. We define $\mathbf{k}^{(2)} = \left(k_1^{(2)} = k_1^{(1)}, \dots, k_{q_2}^{(2)}, k_{q_2+1}^{(2)} = k_{q_2+1}^{(1)}, \dots, k_m^{(2)} = k_m^{(1)}\right)^T$, where $k_{q_2}^{(2)} \in \mathbb{R}$ is such that

$$\begin{cases} k_{q_2}^{(2)} \neq k_j^{(1)}, & 1 \leqslant j \leqslant m \\ 0 < k_{q_2}^{(2)} \leqslant 1 \end{cases} \tag{4}$$

We have $(T\,\mathbf{k}^{(2)})_2 \neq (T\,\mathbf{k}^{(2)})_1$. Indeed,

$$\begin{aligned} (T\,\mathbf{k}^{(2)})_2 - (T\,\mathbf{k}^{(2)})_1 &= (T\,\mathbf{k}^{(1)})_2 - (T\,\mathbf{k}^{(1)})_1 + (k_{q_2}^{(2)} - k_{q_2}^{(1)})(\lambda_{2,q_2} - \lambda_{1,q_2}) \\ &= (k_{q_2}^{(2)} - k_{q_2}^{(1)})(\lambda_{2,q_2} - \lambda_{1,q_2}) \neq 0. \end{aligned}$$

Step n $(n < N)$: If $(T\,\mathbf{k}^{(n-1)})_n \neq (T\,\mathbf{k}^{(n-1)})_l$ for every $l < n$, we set $\mathbf{k}^{(n)} = \mathbf{k}^{(n-1)}$ and proceed to the next step. If instead, there exists an index $l < n$ such that $(T\,\mathbf{k}^{(n-1)})_n = (T\,\mathbf{k}^{(n-1)})_l$ then, by item (i), there exists an index q_n such that, $\lambda_{n,q_n} \neq \lambda_{l,q_n}$. Therefore, we define $\mathbf{k}^{(n)} = (k_1^{(n)} = k_1^{(n-1)}, \dots, k_{q_n}^{(n)}, k_{q_n+1}^{(n)} = k_{q_n+1}^{(n-1)}, \dots, k_m^{(n)} = k_m^{(n-1)})^T$, where $k_{q_n}^{(n)} \in \mathbb{R}$ is such that, for any $i, j \in \{1, \dots, m\}$,

$$\begin{cases} (1)\ 0 < k_{q_n}^{(n)} \leqslant 1 \\ (2)\ k_{q_n}^{(n)} \neq k_j^{(n-1)} \\ (3)\ k_{q_n}^{(n)} \neq k_{q_n}^{(n-1)} - \frac{(T\,\mathbf{k}^{(n-1)})_j - (T\,\mathbf{k}^{(n-1)})_n}{(\lambda_{j,q_n} - \lambda_{n,q_n})}, \quad \text{if } \lambda_{j,q_n} \neq \lambda_{n,q_n}, j \neq l, j < n \\ (4)\ |k_{q_n}^{(n)} - k_{q_n}^{(n-1)}| \leqslant \frac{\min_{p=j,\dots,n-1}\{|(T\mathbf{k}^{(p)})_j - (T\mathbf{k}^{(p)})_i|\}}{8 \cdot 2^n},\ i < j < n \end{cases}$$

Notice that, by items (2), (3) and (4) in step n,

$$(k_{q_n}^{(n)} - k_{q_n}^{(n-1)}) \neq -\frac{(T\,\mathbf{k}^{(n-1)})_j - (T\,\mathbf{k}^{(n-1)})_i}{(\lambda_{j,q_n} - \lambda_{i,q_n})}, \tag{5}$$

for every $i, j = 1, \dots, n$, such that $(\lambda_{j,q_n} - \lambda_{i,q_n}) \neq 0$.
Indeed, by items (2) and (3), Eq. (5) holds for every $j = 1, \dots, n-1, i = n$ and, by items (4), we have:

$$|k_{q_n}^{(n)} - k_{q_n}^{(n-1)}| \leqslant \frac{|(T\mathbf{k}^{(n-1)})_j - (T\mathbf{k}^{(n-1)})_i|)}{8 \cdot 2^n} < \frac{|(T\mathbf{k}^{(n-1)})_j - (T\mathbf{k}^{(n-1)})_i|}{|\lambda_{j,\,q_n} - \lambda_{i,\,q_n}|}$$

for all $i, j < n$, such that $(\lambda_{j,\,q_n} - \lambda_{i,\,q_n}) \neq 0$.
By Eq. (5),

$$(T\,\mathbf{k}^{(n)})_j \neq (T\,\mathbf{k}^{(n)})_i, \quad i, j = 1, \ldots, n.$$

Indeed,

$$(T\,\mathbf{k}^{(n)})_i - (T\,\mathbf{k}^{(n-1)})_i = (k_{q_n}^{(n)} - k_{q_n}^{(n-1)})\lambda_{i,\,q_n} \tag{6}$$

and, by subtracting Eq. (6) from

$$(T\,\mathbf{k}^{(n)})_j - (T\,\mathbf{k}^{(n-1)})_j = (k_{q_n}^{(n)} - k_{q_n}^{(n-1)})\lambda_{j,\,q_n},$$

we get

$$\begin{aligned}&(T\,\mathbf{k}^{(n)})_j - (T\,\mathbf{k}^{(n)})_i \\ &= (k_{q_n}^{(n)} - k_{q_n}^{(n-1)})(\lambda_{j,\,q_n} - \lambda_{i,\,q_n}) + (T\,\mathbf{k}^{(n-1)})_j - (T\,\mathbf{k}^{(n-1)})_i \neq 0\end{aligned}$$

for every $i, j = 1, \ldots, n$. Therefore, the vector $\mathbf{k}^{(n)} = (k_1^{(n)}, \ldots, k_m^{(n)})^T$ is such that $(T\,\mathbf{k}^{(n)})_j - (T\,\mathbf{k}^{(n)})_i \neq 0,\ i, j \in \{1, \ldots, n\}, i \neq j$.
At step n=N, we get a vector $\mathbf{k}^{(N)} = (k_1^{(N)}, \ldots, k_m^{(N)})^T$ such that $0 < k_i^{(N)} \leqslant 1, k_i^{(N)} \neq k_j^{(N)}, i, j = 1\ldots, m$, and $(T\,\mathbf{k}^{(N)})_j - (T\,\mathbf{k}^{(N)})_i \neq 0,\ i, j \in \{1, \ldots, N\}, i \neq j$. □

3. Stochastic matrices and infinite sequences of probability measures

In the present section, we apply Theorem 3 in the framework of probability theory.

In what follows, by a measurable function we mean a Borel measurable function [16] and by the symbol $\{\mu_{(\cdot)}(i)\}_{i\in\mathbb{N}}$ we denote a sequence of probability measures on the Borel σ-algebra $\mathcal{B}([0, 1])$. In particular, we focus on sequences of probability measures such that for every non-ordered couple of indexes $(i, j) = (j, i),\ i, j \in \mathbb{N},\ i \neq j$, there exists a Borel set $\Delta_{i,j}$ such that $\mu_{\Delta_{i,j}}(j) \neq \mu_{\Delta_{i,j}}(i)$. Moreover, we choose a one-to-one correspondence $n : (i, j) \mapsto n(i, j)$ from the set of the non-ordered couples $(i, j),\ i, j \in \mathbb{N},\ i \neq j$, to the set of natural numbers $\mathbb{N}$, and we set $\Delta_{i,j} =: \Delta_n, n = n(i, j)$.

Definition 1. A sequence of probability measures $\{\mu_{(\cdot)}(i)\}_{i\in\mathbb{N}}$ on $\mathcal{B}([0, 1])$ such that, for every non-ordered pair of indices $(i, j),\ i, j \in \mathbb{N},\ i \neq j$, there exists a Borel set $\Delta_{i,j}$ such that $\mu_{\Delta_{i,j}}(j) \neq \mu_{\Delta_{i,j}}(i)$ is called a sequence of distinct probability measures.

We briefly recall some results in the theory of family of sets.

Definition 2. A nonempty family $\mathcal{D}$ of subsets of a set X is said to be a Dynkin system or a σ-class if $\mathcal{D}$ is closed under complements and countable disjoint unions.

It is worth remarking that σ-class of sets were introduced by Suppes [17] who showed that quantum mechanical phenomena are suitably described by them. In the context of quantum mechanics they are indeed known as quantum probability spaces. They are an interesting example of a non-classical logic. Later, Gudder [10] began the study of the mathematical properties of these spaces.

Theorem 2 ([11,14,15,18])**.** *Let $\mathcal{D}(\mathbb{R}^n)$ and $\mathcal{B}(\mathbb{R}^n)$ be, respectively, the Dynkin system and the Borel σ-algebra generated by the open balls in $\mathbb{R}^n$. Then $\mathcal{D}(\mathbb{R}^n) = \mathcal{B}(\mathbb{R}^n)$. Let $\mathcal{D}(\mathbb{P}^1)$ and $\mathcal{B}(\mathbb{P}^1)$ be, respectively, the Dynkin system and the Borel σ-algebra generated by the half-open intervals $(a, b]$ in $[0, 1]$. Then, $\mathcal{D}(\mathbb{P}^1) = \mathcal{B}(\mathbb{P}^1)$.*

The importance of Theorem 2 derives from the fact that two probability measures $\mu_{(\cdot)}(1)$ and $\mu_{(\cdot)}(2)$ which agree on each open ball must agree on $\mathcal{D}(\mathbb{R}^n)$, so that if we know that $\mathcal{D}(\mathbb{R}^n)=\mathcal{B}(\mathbb{R}^n)$ then, we can conclude that the two probability measures are the same. In other words, if $\mu_{(\cdot)}(1)\neq\mu_{(\cdot)}(2)$ then, there must exist an open ball Δ such that $\mu_{(\Delta)}(1)\neq\mu_{(\Delta)}(2)$. In the case of probability measures defined on $\mathcal{B}([0,1])$, if $\mu_{(\cdot)}(1)\neq\mu_{(\cdot)}(2)$ then, there must exist a half-open interval Δ such that $\mu_{(\Delta)}(1)\neq\mu_{(\Delta)}(2)$.

Now, let $\{\mu_{(\cdot)}(i)\}_{i\in\mathbb{N}}$ be a sequence of distinct probability measures on $\mathcal{B}([0,1])$. By Theorem 2, for every non-ordered pair of indices (i,j), there exists a half-open interval $\Delta_n=(a_n,b_n]$ such that $\mu_{\Delta_n}(i)\neq\mu_{\Delta_n}(j)$ where, $n=n(i,j)$. Let us denote by $\mathcal{N}$ the family $\{\Delta_n\}_{n\in\mathbb{N}}$. Notice that such a family is not generally unique. In the following we assume $\mathcal{N}$ to be chosen once and for all. Moreover, we assume that to a partition $\sigma=\{\gamma_1,\ldots,\gamma_n\}$ of [0, 1] there corresponds the family of intervals $\{[0,\gamma_1],(\gamma_1,\gamma_2],\ldots,(\gamma_{n-1},\gamma_n]\}$.

The following theorem is a consequence of Theorem 1 on stochastic matrices.

Theorem 3. *Let $\{\mu_{(\cdot)}(i)\}_{i\in\mathbb{N}}$ be a sequence of distinct probability measures on $\mathcal{B}([0,1])$. Let us consider the infinite system of linear functionals $\{T_i\}_{i\in\mathbb{N}}$ defined as follows*

$$T_if:=\int f(t)\,d\mu_t(i)=:G_f(i),\quad i\in\mathbb{N}$$

where, $f:[0,1]\to\mathbb{R}$, is a bounded measurable function and the integration is in the sense of Lebesgue–Stieltjes.
There exists a one-to-one function $f(t)$ such that G_f is one-to-one

$$G_f(i)=\int f(t)\,d\mu_t(i)\neq\int f(t)\,d\mu_t(j)=G_f(j),\quad i,j\in\mathbb{N},\ i\neq j.$$

Moreover, f is piecewise continuous and continuous from the left.

Proof. In order to construct the one-to-one function f we proceed as follows.
Step 1. Let us consider the first $h>1$ probability measures, $\{\mu_{(\cdot)}(i)\}_{i=1,\ldots,h}$ and the subfamily $\mathcal{D}_1:=\{\Delta_n=\Delta_{n(i,j)}=:(\alpha_{i,j},\beta_{i,j}]\}_{i,j\leqslant h}\subset\mathcal{N}$. The family $\mathcal{D}_1$ is such that, for every non-ordered couple (i,j), $i,j\leqslant h$, there exists an interval $\Delta_n=\Delta_{n(i,j)}\in\mathcal{D}_1$ such that $\mu_{(\Delta_n)}(i)\neq\mu_{(\Delta_n)}(j)$. Moreover, $\mathcal{D}_1$ defines a partition $\sigma^{(1)}$ of [0, 1]. Indeed, if we arrange the numbers $\alpha_{i,j}$ and $\beta_{i,j}$ in increasing order we get a sequence $\gamma_1^{(1)}<\gamma_2^{(1)}<\ldots<\gamma_{s_1-1}^{(1)}$ which decomposes the interval [0, 1] into the family of sets $\mathcal{A}_1=\{\Delta_1^{(1)}:=[0,\gamma_1^{(1)}],\ \Delta_2^{(1)}:=(\gamma_1^{(1)},\gamma_2^{(1)}],\ldots,\Delta_{s_1}^{(1)}:=(\gamma_{s_1-1}^{(1)},1]\}$ where, s_1-1 denotes the number of distinct elements in the set $\{\alpha_{i,j},\beta_{i,j}\}_{i<j\leqslant h}=\{\alpha_{i,j},\beta_{i,j}\}_{i,j\leqslant h}$. Notice that, each interval $(\alpha_{i,j},\beta_{i,j}]\in\mathcal{D}_1$ is the union of a finite number of half-open intervals in $\mathcal{A}_1$, so that, we write $\mathcal{D}_1\prec\mathcal{A}_1$. Now, let us consider the rectangular stochastic matrix

$$T^{(1)}:=\begin{pmatrix}\lambda_{1,1}^{(1)} & \lambda_{1,2}^{(1)} & \ldots & \lambda_{1,s_1}^{(1)}\\ \lambda_{2,1}^{(1)} & \lambda_{2,2}^{(1)} & \ldots & \lambda_{2,s_1}^{(1)}\\ \ldots & \ldots & \ldots & \ldots\\ \lambda_{h,1}^{(1)} & \lambda_{h,2}^{(1)} & \ldots & \lambda_{h,s_1}^{(1)}\end{pmatrix}\tag{7}$$

where $\lambda_{i,j}^{(1)}:=\mu_{\Delta_j^{(1)}}(i)$.

Since $\mathcal{D}_1\prec\mathcal{A}_1$, $T^{(1)}$ satisfies item (i) in Lemma 1. Therefore, there exists a vector $\mathbf{k}^{(1)}\in\mathbb{R}^{s_1}$ such that $[T^{(1)}\mathbf{k}^{(1)}]_i\neq[T^{(1)}\mathbf{k}^{(1)}]_j$, if $i\neq j$. Moreover $\mathbf{k}^{(1)}$ can be chosen such that $0<k_i^{(1)}\leqslant1$, $k_i^{(1)}\neq k_j^{(1)}$, $i=1,\ldots,s_1$, $i\neq j$.
Step 2. Let us set $2_h:=h+2-1$, $s_2:=s_1[2(2_h)+1]$ and consider the probability measure $\mu_{(\cdot)}(h+1)$, and the h half-open intervals $\{(\alpha_j^{(2)},\beta_j^{(2)}]:=\Delta_{h+1,j}\}_{j=1,\ldots,h}$ such that $\mu_{(\Delta_{h+1,j})}(h+1)\neq\mu_{(\Delta_{h+1,j})}(j)$, $j=1,\ldots,h$. Now, let us define an arbitrary partition $\sigma^{(2)}\supset\sigma^{(1)}$ of [0, 1] which is obtained from $\sigma^{(1)}$ by dividing each interval $\Delta_i^{(1)}$ into $2(2_h)+1$ intervals in such a way that $\{\alpha_j^{(2)},\beta_j^{(2)}\}_{j=1,\ldots,h}\subset\sigma^{(2)}$. Let

$\sigma^{(2)} = \{\gamma_1^{(2)}, \gamma_2^{(2)}, \ldots, \gamma_{s_2-1}^{(2)}\}$ be such a partition. Then, the family of intervals corresponding to $\sigma^{(2)}$ is $\mathcal{A}_2 = \{\Delta_1^{(2)} := [0, \gamma_1^{(2)}], \ldots, \Delta_{j+1}^{(2)} := (\gamma_j^{(2)}, \gamma_{j+1}^{(2)}], \ldots, \Delta_{s_2} := (\gamma_{s_2-1}^{(2)}, 1]\}$. Notice that $\mathcal{A}_2$ decomposes $[0, 1]$ in such a way that each half-open interval in $\mathcal{A}_1$ is decomposed into $2(2_h) + 1$ half-open intervals in $\mathcal{A}_2$, so that, we write $\mathcal{A}_1 \prec \mathcal{A}_2$.

Now, let us consider the rectangular stochastic matrix

$$T^{(2)} := \begin{pmatrix} \lambda_{1,1}^{(2)} & \lambda_{1,2}^{(2)} & \ldots & \lambda_{1,s_2}^{(2)} \\ \lambda_{2,1}^{(2)} & \lambda_{2,2}^{(2)} & \ldots & \lambda_{2,s_2}^{(2)} \\ \ldots & \ldots & \ldots & \ldots \\ \lambda_{h+1,1}^{(2)} & \lambda_{h+1,2}^{(2)} & \ldots & \lambda_{h+1,s_2}^{(2)} \end{pmatrix} \tag{8}$$

where $\lambda_{i,j}^{(2)} := \mu_{\Delta_j^{(2)}}(i)$.

Since $\mathcal{A}_1 \prec \mathcal{A}_2$, $T^{(2)}$ satisfies item (i) in Lemma 1. Therefore, by Lemma 1, there is a vector $\mathbf{k}^{(2)}$ such that $[T^{(2)}\mathbf{k}^{(2)}]_i \neq [T^{(2)}\mathbf{k}^{(2)}]_j$, $i, j \in \{1, \ldots, h+1\}$, $i \neq j$. Now, we show a particular construction of $\mathbf{k}^{(2)}$:

Step 2.1. We start from the vector $\mathbf{k}^{(1,1)} = (k_1^{(1,1)}, k_2^{(1,1)}, \ldots, k_{s_2}^{(1,1)})^T$, where $k_i^{(1,1)} = k_l^{(1)} + a_i^{(1)}$ if $(l-1)[2(2_h)+1] < i \leqslant l[2(2_h)+1]$, $l = 1, \ldots, s_1$, and $a_i^{(1)}$ are real numbers such that (see Lemma 1 in Appendix B), for any $l, q \in \{1, \ldots, s_1\}$,

$$\begin{cases} (1)\ a_r^{(1)} = 0 & r = d_l^{(2)} \\ (2)\ a_r^{(1)}(k_{l+1}^{(1)} - k_l^{(1)}) > 0, & r \in (d_l^{(2)}, D_l^{(2)}],\ \ l < s_1 \\ (3)\ |a_r^{(1)}| \leqslant b_{(i,j)}^{(2,r)}, & 1 \leqslant i < j \leqslant 2 \\ (4)\ |a_r^{(1)}| \leqslant \delta^{(2,r)} & \\ (5)\ a_r^{(1)} \neq -k_l^{(1)}, & r \in (d_l^{(2)}, D_l^{(2)}] \\ (6)\ a_j^{(1)} - a_i^{(1)} \neq -(k_q^{(1)} - k_l^{(1)}), & i \in (d_l^{(2)}, D_l^{(2)}] \\ & j \in (d_q^{(2)}, D_q^{(2)}] \end{cases}$$

where,

$$\begin{cases} d_l^{(2)} := (l-1)[2(2_h)+1] + 1 \\ D_l^{(2)} := l[2(2_h)+1] \\ b_{(i,j)}^{(2,r)} = \frac{|\sum_{l=1}^{s_1} k_l^{(1)}(\lambda_{j,l}^{(1)} - \lambda_{i,l}^{(2)})|}{32 \cdot 2^2 \cdot 2^r} \\ \delta^{(2,r)} = \frac{\min\left\{|k_j^{(1)} - k_i^{(1)}|, |1 - k_j^{(1)}|;\ i < j \leqslant s_1\right\}}{32 \cdot 2^2 \cdot 2^r} \end{cases}$$

In what follows we will use the expression item 2.1.n to denote item (n) in step 2.1 and, more generally, we will use the expression item $n.m.$i to denote item (i) in step $n.m$.

Notice that (see item 2.1.5) $k_i^{(1,1)} \neq 0$ and (see item 2.1.6) $k_i^{(1,1)} \neq k_j^{(1,1)}$ for every $i, j = 1, \ldots, s_2$. Moreover (see items 2.1.2 and 2.1.4), $0 < k_j^{(1,1)} \leqslant 1$, $j = 1, \ldots, s_2$.

Step 2.2. if $(T^{(2)}\mathbf{k}^{(1,1)})_2 \neq (T^{(2)}\mathbf{k}^{(1,1)})_1$, we set $\mathbf{k}^{(1,2)} = \mathbf{k}^{(1,1)}$ and proceed to the next step. If instead, $(T^{(2)}\mathbf{k}^{(1,1)})_2 = (T^{(2)}\mathbf{k}^{(1,1)})_1$ then, by item (i) in Lemma 1, there exists an index $q_{2,2}$ such that, $\lambda_{1,q_{2,2}}^{(2)} \neq \lambda_{2,q_{2,2}}^{(2)}$. Therefore, we define

$$\mathbf{k}^{(1,2)} = (k_1^{(1,2)} = k_1^{(1,1)}, \ldots, k_{q_{2,2}}^{(1,2)}, k_{q_{2,2}+1}^{(1,2)} = k_{q_{2,2}+1}^{(1,1)}, \ldots, k_{s_2}^{(1,2)} = k_{s_2}^{(1,1)})^T,$$

where, $k_{q_{2,2}}^{(1,2)} \in \mathbb{R}$ is such that, for any $i, j \in \{1, \ldots, s_2\}$,

$$\begin{cases} (1)\; 0 < k^{(1,2)}_{q_{2,2}} \leqslant 1 \\ (2)\; k^{(1,2)}_{q_{2,2}} \neq k^{(1,1)}_j \\ (3)\; |(k^{(1,2)}_{q_{2,2}} - k^{(1,1)}_{q_{2,2}})| \leqslant \beta^{(2,2)}_{i,j}, & 1 \leqslant i < j \leqslant s_2 \\ (4)\; (k^{(1,2)}_{q_{2,2}} - k^{(1,1)}_{q_{2,2}})(k^{(1,1)}_{q_{2,2}+1} - k^{(1,1)}_{q_{2,2}}) > 0, & \text{if} \quad q_{2,2} < s_2 \\ (5)\; (k^{(1,2)}_{q_{2,2}} - k^{(1,1)}_{q_{2,2}})(k^{(1,1)}_{q_{2,2}} - k^{(1,1)}_{q_{2,2}-1}) < 0, & \text{if} \quad q_{2,2} = s_2 \\ (6)\; |(k^{(1,2)}_{q_{2,2}} - k^{(1,1)}_{q_{2,2}})| \leqslant \gamma^{(2,2)}_{i,j}, & 1 \leqslant i < j \leqslant 2 \\ (7)\; |(k^{(1,2)}_{q_{2,2}} - k^{(1,1)}_{q_{2,2}})| \leqslant \frac{|k^{(1)}_j - k^{(1)}_i|}{32 \cdot 2^2 \cdot 2^2}, & 1 \leqslant i < j \leqslant s_1 \end{cases}$$

where,

$$\begin{cases} \beta^{(2,2)}_{i,j} = \frac{|k^{(1,1)}_j - k^{(1,1)}_i|}{8 \cdot 2^2}, \\ \gamma^{(2,2)}_{i,j} = \frac{\left|\sum_{l=1}^{s_1} k^{(1)}_l \left(\lambda^{(1)}_{j,l} - \lambda^{(1)}_{i,l}\right)\right|}{32 \cdot 2^2 \cdot 2^2 \cdot 2^{q_{2,2}}}. \end{cases}$$

By proceeding as in step 1 of the proof of Lemma 1, one can prove that

$$[T^{(2)}\mathbf{k}^{(1,2)}]_2 \neq [T^{(2)}\mathbf{k}^{(1,2)}]_1.$$

Step 2.n ($n < 2_h$). If $(T^{(2)}\,\mathbf{k}^{(1,n-1)})_n \neq (T^{(2)}\,\mathbf{k}^{(1,n-1)})_l$ for every $l < n$, we set $\mathbf{k}^{(1,n)} = \mathbf{k}^{(1,n-1)}$ and proceed to the next step. If instead, there exists an index $l < n$ such that $(T^{(2)}\,\mathbf{k}^{(1,n-1)})_n = (T^{(2)}\,\mathbf{k}^{(1,n-1)})_l$ then, by item (i) in Lemma 1, there exists an index $q_{2,n}$ such that, $\lambda^{(2)}_{l,q_{2,n}} \neq \lambda^{(2)}_{n,q_{2,n}}$. Therefore, we define

$$\mathbf{k}^{(1,n)} = (k^{(1,n)}_1 = k^{(1,n-1)}_1, \ldots, k^{(1,n)}_{q_{2,n}}, k^{(1,n)}_{q_{2,n}+1} = k^{(1,n-1)}_{q_{2,n}+1}, \ldots, k^{(1,n)}_{s_2} = k^{(1,n-1)}_{s_2})^T,$$

where, $k^{(1,n)}_{q_{2,n}} \in \mathbb{R}$ is such that, for any $i, j \in \{1, \ldots, s_2\}$,

$$\begin{cases} (1)\; 0 < k^{(1,n)}_{q_{2,n}} \leqslant 1 \\ (2)\; k^{(1,n)}_{q_{2,n}} \neq k^{(1,n-1)}_j \\ (3)\; k^{(1,n)}_{q_{2,n}} \neq k^{(1,n-1)}_{q_{2,n}} - \frac{(T^{(2)}\,\mathbf{k}^{(1,n-1)})_j - (T^{(2)}\,\mathbf{k}^{(1,n-1)})_n}{\lambda^{(2)}_{j,\,q_{2,n}} - \lambda^{(2)}_{n,\,q_{2,n}}}, & \lambda^{(2)}_{j,\,q_{2,n}} \neq \lambda^{(2)}_{n,\,q_{2,n}}, \\ & j \neq l,\; j < n \\ (4)\; |(k^{(1,n)}_{q_{2,n}} - k^{(1,n-1)}_{q_{2,n}})| \leqslant \alpha^{(2,n)}_{i,j}, & 1 \leqslant i < j < n \\ (5)\; |(k^{(1,n)}_{q_{2,n}} - k^{(1,n-1)}_{q_{2,n}})| \leqslant \beta^{(2,n)}_{i,j}, & 1 \leqslant i < j \leqslant s_2 \\ (6)\; (k^{(1,n)}_{q_{2,n}} - k^{(1,n-1)}_{q_{2,n}})(k^{(1,n-1)}_{q_{2,n}+1} - k^{(1,n-1)}_{q_{2,n}}) > 0, & \text{if} \quad q_{2,n} < s_2 \\ (7)\; (k^{(1,n)}_{q_{2,n}} - k^{(1,n-1)}_{q_{2,n}})(k^{(1,n-1)}_{q_{2,n}} - k^{(1,n-1)}_{q_{2,n}-1}) < 0, & \text{if} \quad q_{2,n} = s_2 \\ (8)\; |(k^{(1,n)}_{q_{2,n}} - k^{(1,n-1)}_{q_{2,n}})| \leqslant \gamma^{(2,n)}_{i,j}, & 1 \leqslant i < j \leqslant 2 \\ (9)\; |(k^{(1,n)}_{q_{2,n}} - k^{(1,n-1)}_{q_{2,n}})| \leqslant \frac{|k^{(1)}_j - k^{(1)}_i|}{32 \cdot 2^n \cdot 2^2}, & 1 \leqslant i < j \leqslant s_1 \end{cases}$$

where,

$$\begin{cases} \alpha^{(2,n)}_{i,j} = \frac{\min_{p=j,\ldots,n-1}\{|(T^{(2)}\mathbf{k}^{(1,p)})_j - (T^{(2)}\mathbf{k}^{(1,p)})_i|\}}{8 \cdot 2^n}, \\ \beta^{(2,n)}_{i,j} = \frac{\min_{p=1,\ldots,n-1}\{|k^{(1,p)}_j - k^{(1,p)}_i|\}}{8 \cdot 2^n}, \\ \gamma^{(2,n)}_{i,j} = \frac{\left|\sum_{l=1}^{s_1} k^{(1)}_l \left(\lambda^{(1)}_{j,l} - \lambda^{(1)}_{i,l}\right)\right|}{32 \cdot 2^n \cdot 2^2 \cdot 2^{q_{2,n}}}. \end{cases}$$

By items 2.n.1 and 2.n.2, it follows that the vector $\mathbf{k}^{(1,n)}$ is such that $0 < k_i^{(1,n)} \leqslant 1, k_i^{(1,n)} \neq k_j^{(1,n)}, i, j = 1, \ldots, s_2, i \neq j$. Moreover, by proceeding as in step n of the proof of Lemma 1 (see items 2.n.3 and 2.n.4 above), one can prove that $[T^{(2)}\mathbf{k}^{(1,n)}]_j \neq [T^{(2)}\mathbf{k}^{(1,n)}]_i, i, j \in \{1, \ldots, n\}, i \neq j$.

Step 2.2$_h$. For $n = 2_h = h + 1$, we get a vector $\mathbf{k}^{(2)} := \mathbf{k}^{(1,h+1)}$ such that $0 < k_i^{(2)} \leqslant 1, k_i^{(2)} \neq k_j^{(2)}, i, j = 1, \ldots, s_2, i \neq j$. Moreover, $[T^{(2)}\mathbf{k}^{(2)}]_j \neq [T^{(2)}\mathbf{k}^{(2)}]_i, i, j \in \{1, \ldots, h + 1\}, i \neq j$.

Step n $(n > 1)$. Let us set $n_h := h + n - 1, s_n := s_{n-1}(2n_h + 1)$, and consider the probability measure $\mu_{(\cdot)}(n_h)$, and the $n_h - 1$ open intervals $\{(\alpha_j^{(n)}, \beta_j^{(n)}) := \Delta_{n_h,j}\}_{j=1,\ldots,n_h-1}$ such that $\mu_{(\Delta_{n_h,j})}(n_h) \neq \mu_{(\Delta_{n_h,j})}(j), j = 1, \ldots, n_h - 1$. Now, let us define an arbitrary partition $\sigma^{(n)} \supset \sigma^{(n-1)}$ of [0, 1] which is obtained from $\sigma^{(n-1)}$ by dividing each interval $\Delta_i^{(n-1)}$ into $2n_h + 1$ intervals in such a way that $\{\alpha_j^{(n)}, \beta_j^{(n)}\}_{j=1,\ldots,n_h-1} \subset \sigma^{(n)}$. Let $\sigma^{(n)} = \{\gamma_1^{(n)}, \gamma_2^{(n)}, \ldots, \gamma_{s_n-1}^{(n)}\}$ be such a partition. Then, the family of intervals corresponding to $\sigma^{(n)}$ is

$$\mathcal{A}_n = \{\Delta_1^{(n)} := [0, \gamma_1^{(n)}], \ldots, \Delta_{j+1}^{(n)} := (\gamma_j^{(n)}, \gamma_{j+1}^{(n)}], \ldots, \Delta_{s_n} := (\gamma_{s_n-1}, 1]\}.$$

Notice that $\mathcal{A}_n$ decomposes [0, 1] in such a way that each half-open interval in $\mathcal{A}_{n-1}$ is decomposed into $2n_h + 1$ half-open intervals.

Now, let us consider the rectangular stochastic matrix

$$T^{(n)} := \begin{pmatrix} \lambda_{1,1}^{(n)} & \lambda_{1,2}^{(n)} & \ldots & \lambda_{1,s_n}^{(n)} \\ \lambda_{2,1}^{(n)} & \lambda_{2,2}^{(n)} & \ldots & \lambda_{2,s_n}^{(n)} \\ \ldots & \ldots & \ldots & \ldots \\ \lambda_{n_h,1}^{(n)} & \lambda_{n_h,2}^{(n)} & \ldots & \lambda_{n_h,s_n}^{(n)} \end{pmatrix}$$

where $\lambda_{i,j}^{(n)} := \mu_{\Delta_j^{(n)}}(i)$.

Since $\mathcal{A}_{n-1} \prec \mathcal{A}_n$, $T^{(n)}$ satisfies item (i) in Lemma 1. Therefore, by Lemma 1, there is a vector $\mathbf{k}^{(n)}$ such that $[T^{(n)}\mathbf{k}^{(n)}]_i \neq [T^{(n)}\mathbf{k}^{(n)}]_j, i, j \in \{1, \ldots, n_h\}, i \neq j$.

Now, we show a particular construction of $\mathbf{k}^{(n)}$:

Step n.1. We start from the vector $\mathbf{k}^{(n-1,1)} = (k_1^{(n-1,1)}, k_2^{(n-1,1)}, \ldots, k_{s_n}^{(n-1,1)})^T$ where,

$$k_i^{(n-1,1)} = k_l^{(n-1)} + a_i^{(n-1)} \quad \text{if} \quad (l-1)(2n_h + 1) < i \leqslant l(2n_h + 1), \; l = 1, \ldots, s_{n-1},$$

and $a_i^{(n-1)}$ are real numbers such that (see Lemma 1 in Appendix B) for any $q, l \in \{1, \ldots, s_{n-1}\}$,

$$\begin{cases} (1)\; a_r^{(n-1)} = 0 & r = d_l^{(n)} \\ (2)\; a_r^{(n-1)}(k_{l+1}^{(n-1)} - k_l^{(n-1)}) > 0, & r \in (d_l^{(n)}, D_l^{(n)}], \; l < s_{n-1} \\ (3)\; |a_r^{(n-1)}| \leqslant b_{(i,j)}^{(n,r)}, & 1 \leqslant i < j \leqslant n \\ (4)\; |a_r^{(n-1)}| \leqslant \delta^{(n,r)} & \\ (5)\; a_r^{(n-1)} \neq -k_l^{(n-1)}, & r \in (d_l^{(n)}, D_l^{(n)}] \\ (6)\; a_j^{(n-1)} - a_i^{(n-1)} \neq -(k_q^{(n-1)} - k_l^{(n-1)}), & i \in (d_l^{(n)}, D_l^{(n)}] \\ & j \in (d_q^{(n)}, D_q^{(n)}] \end{cases}$$

where,

$$\begin{cases} d_l^{(n)} := (l-1)(2n_h + 1) + 1 \\ D_l^{(n)} := l(2n_h + 1) \\ b_{i,j}^{(n,r)} = \frac{\min_{p=j-1,\ldots,n-1}\left\{\left|\sum_{l=1}^{s_p} k_l^{(p)}(\lambda_{j,l}^{(p)} - \lambda_{i,l}^{(p)}\right|\right\}}{32 \cdot 2^n \cdot 2^r} \\ \delta^{(n,r)} = \frac{\min_{p=1,\ldots,n-1}\left\{|k_j^{(p)} - k_i^{(p)}|, |1 - k_j^{(p)}|; \; i < j \leqslant s_p\right\}}{32 \cdot 2^n \cdot 2^r} \end{cases}$$

Notice that (item n.1.5) $k_i^{(n-1,1)} \neq 0$, $i = 1, \ldots, s_n$, and (item n.1.6) $k_i^{(n-1,1)} \neq k_j^{(n-1,1)}$, $i \neq j$. Moreover, (items n.1.2 and n.1.4) $0 < k_j^{(n-1,1)} \leqslant 1$, $j = 1, \ldots, s_n$.

Step n.2. if $(T^{(n)}\,\mathbf{k}^{(n,1)})_2 \neq (T^{(n)}\,\mathbf{k}^{(n,1)})_1$, we set $\mathbf{k}^{(n,2)} = \mathbf{k}^{(n,1)}$ and proceed to the next step. If instead, $(T^{(n)}\,\mathbf{k}^{(n,1)})_2 = (T^{(n)}\,\mathbf{k}^{(n,1)})_1$ then, by item (i) in Lemma 1, there exists an index $q_{n,2}$ such that, $\lambda^{(n)}_{1,\,q_{n,2}} \neq \lambda^{(n)}_{2,\,q_{n,2}}$. Therefore, we define,

$$\mathbf{k}^{(n,2)} = (k_1^{(n,2)} = k_1^{(n,1)}, \ldots, k_{q_{n,2}}^{(n,2)}, k_{q_{n,2}+1}^{(n,2)} = k_{q_{n,2}+1}^{(n,1)}, \ldots, k_{s_n}^{(n,2)} = k_{s_n}^{(n,1)})^T,$$

where, $k_{q_{n,2}}^{(n,2)} \in \mathbb{R}$ is such that, for any $i, j \in \{1, \ldots, s_n\}$,

$$\begin{cases} (1)\ 0 < k_{q_{n,2}}^{(n-1,2)} \leqslant 1 & \\ (2)\ k_{q_{n,2}}^{(n-1,2)} \neq k_j^{(n-1,1)} & \\ (3)\ |(k_{q_{n,2}}^{(n-1,2)} - k_{q_{n,2}}^{(n-1,1)})| \leqslant \beta_{i,j}^{(n,2)}, & 1 \leqslant i < j \leqslant s_n \\ (4)\ (k_{q_{n,2}}^{(n-1,2)} - k_{q_{n,2}}^{(n-1,1)})(k_{q_{n,2}+1}^{(n-1,1)} - k_{q_{n,2}}^{(n-1,1)}) > 0, & \text{if} \quad q_{n,2} < s_n \\ (5)\ (k_{q_{n,2}}^{(n-1,2)} - k_{q_{n,2}}^{(n-1,1)})(k_{q_{n,2}}^{(n-1,1)} - k_{q_{n,2}-1}^{(n-1,1)}) < 0, & \text{if} \quad q_{n,2} = s_n \\ (6)\ |(k_{q_{n,2}}^{(n-1,2)} - k_{q_{n,2}}^{(n-1,1)})| \leqslant \gamma_{i,j}^{(n,2)}, & 1 \leqslant i < j \leqslant n \\ (7)\ |(k_{q_{n,2}}^{(n-1,2)} - k_{q_{n,2}}^{(n-1,1)})| \leqslant \bar{\delta}^{(n,2)} & \end{cases}$$

where,

$$\begin{cases} \beta_{i,j}^{(n,2)} = \frac{|k_j^{(n-1,1)} - k_i^{(n-1,1)}|}{8 \cdot 2^2}, \\ \gamma_{i,j}^{(n,2)} = \frac{\min_{p=j-1,\ldots,n-1}\left\{\left|\sum_{l=1}^{s_p} k_l^{(p)}(\lambda_{j,l}^{(p)} - \lambda_{i,l}^{(p)})\right|\right\}}{32 \cdot 2^n \cdot 2^2 \cdot 2^{q_{n,2}}}, \\ \bar{\delta}^{(n,2)} = \frac{\min_{p=1,\ldots,n-1}\{|k_j^{(p)} - k_i^{(p)}|,\ i<j \leqslant s_p\}}{32 \cdot 2^n \cdot 2^2}. \end{cases}$$

By proceeding as in step 1 of the proof of Lemma 1, one can prove that

$$[T^{(n)}\mathbf{k}^{(n-1,2)}]_2 \neq [T^{(n)}\mathbf{k}^{(n-1,2)}]_1$$

Step n.m ($m < n_h$). If $(T^{(n)}\,\mathbf{k}^{(n-1,m-1)})_m \neq (T^{(n)}\,\mathbf{k}^{(n-1,m-1)})_l$, for every $l < m$, we set $\mathbf{k}^{(n-1,m)} = \mathbf{k}^{(n-1,m-1)}$ and proceed to the next step. If instead, there exists an index $l < m$ such that $(T^{(n)}\,\mathbf{k}^{(n-1,m-1)})_m = (T^{(n)}\,\mathbf{k}^{(n-1,m-1)})_l$ then, by item (i), there exists an index $q_{n,m}$ such that, $\lambda^{(n)}_{l,\,q_{n,m}} \neq \lambda^{(n)}_{m,\,q_{n,m}}$.

Hence, we define $\mathbf{k}^{(n-1,m)} = (k_1^{(n-1,m)} = k_1^{(n-1,m-1)}, \ldots, k_{q_{n,m}}^{(n-1,m)}, k_{q_{n,m}+1}^{(n-1,m)} = k_{q_{n,m}+1}^{(n-1,m-1)}, \ldots, k_{s_n}^{(n-1,m)} = k_{s_n}^{(n-1,m-1)})^T$, with $k_{q_{n,m}}^{(n-1,m)} \in \mathbb{R}$ such that, for any $i, j \in \{1, \ldots, s_n\}$,

$$\begin{cases} (1)\ 0 < k_{q_{n,m}}^{(n-1,m)} \leqslant 1 & \\ (2)\ k_{q_{n,m}}^{(n-1,m)} \neq k_j^{(n-1,m-1)} & \\ (3)\ k_{q_{n,m}}^{(n-1,m)} \neq a_{q_{n,m}}^{(n,m)}, & \mu_{\Delta^{(n)}_{q_{n,m}}}(j) \neq \mu_{\Delta^{(n)}_{q_{n,m}}}(m) \\ & j \neq l,\ j < m \\ (4)\ |(k_{q_{n,m}}^{(n-1,m)} - k_{q_{n,m}}^{(n-1,m-1)})| \leqslant \alpha_{i,j}^{(n,m)}, & 1 \leqslant i < j < m \\ (5)\ |(k_{q_{n,m}}^{(n-1,m)} - k_{q_{n,m}}^{(n-1,m-1)})| \leqslant \beta_{i,j}^{(n,m)}, & 1 \leqslant i < j \leqslant s_n \\ (6)\ (k_{q_{n,m}}^{(n-1,m)} - k_{q_{n,m}}^{(n-1,m-1)}) \cdot b_{q_{n,m}}^{(n,m)} > 0, & \text{if} \quad q_{n,m} < s_n \\ (7)\ (k_{q_{n,m}}^{(n-1,m)} - k_{q_{n,m}}^{(n-1,m-1)}) \cdot b_{q_{n,m}-1}^{(n,m)} < 0, & \text{if} \quad q_{n,m} = s_n \\ (8)\ |(k_{q_{n,m}}^{(n-1,m)} - k_{q_{n,m}}^{(n-1,m-1)})| \leqslant \gamma_{i,j}^{(n,m)}, & 1 \leqslant i < j \leqslant n \\ (9)\ |(k_{q_{n,m}}^{(n-1,m)} - k_{q_{n,m}}^{(n-1,m-1)})| \leqslant \bar{\delta}^{(n,m)} & \end{cases}$$

where,

$$\begin{cases} a^{(n,m)}_{q_{n,m}} = k^{(n-1,m-1)}_{q_{n,m}} - \frac{(T^{(n)}\,\mathbf{k}^{(n-1,m-1)})_j - (T^{(n)}\,\mathbf{k}^{(n-1,m-1)})_m}{\lambda^{(n)}_{j,\,q_{n,m}} - \lambda^{(n)}_{m,\,q_{n,m}}} \\ \alpha^{(n,m)}_{i,j} = \frac{\min_{p=j,\ldots,m-1}\{|(T^{(n)}\mathbf{k}^{(n-1,p)})_j - (T^{(n)}\mathbf{k}^{(n-1,p)})_i|\}}{8\cdot 2^m} \\ \beta^{(n,m)}_{i,j} = \frac{\min_{p=1,\ldots,m-1}\{|k^{(n-1,p)}_j - k^{(n-1,p)}_i|\}}{8\cdot 2^m} \\ b^{(n,m)}_{q_{n,m}} = (k^{(n-1,m-1)}_{q_{n,m}+1} - k^{(n-1,m-1)}_{q_{n,m}}) \\ \gamma^{(n,m)}_{i,j} = \frac{\min_{p=j-1,\ldots,n-1}\left\{\left|\sum_{l=1}^{s_p} k^{(p)}_l(\lambda^{(p)}_{j,l} - \lambda^{(p)}_{i,l})\right|\right\}}{32\cdot 2^n\cdot 2^m\cdot 2^{q_{n,m}}} \\ \bar{\delta}^{(n,m)} = \frac{\min_{p=1,\ldots,n-1}\{|k^{(p)}_j - k^{(p)}_i|;\ i<j\leqslant s_p\}}{32\cdot 2^n\cdot 2^m} \end{cases}$$

By items $n.m.1$ and $n.m.2$, it follows that the vector $\mathbf{k}^{(n-1,m)}$ is such that $0 < k_i^{(n-1,m)} \leqslant 1$, $k_i^{(n-1,m)} \neq k_j^{(n-1,m)}$, $i, j = 1, \ldots, s_n$, $i \neq j$. Moreover, by proceeding as in step n of the proof of Lemma 1 (see items n.m.3 and n.m.4 above), one can prove that $[T^{(n)}\mathbf{k}^{(n-1,m)}]_j \neq [T^{(n)}\mathbf{k}^{(n-1,m)}]_i$, $i, j \in \{1, \ldots, m\}$, $i \neq j$.

Step $n.n_h$. For $m = n_h$, we get a vector $\mathbf{k}^{(n)} := \mathbf{k}^{(n-1,n_h)}$ such that $0 \neq k_i^{(n)} \leqslant 1$, $k_i^{(n)} \neq k_j^{(n)}$, $i, j = 1, \ldots, s_n$, $i \neq j$. Moreover, $[T^{(n)}\mathbf{k}^{(n)}]_j \neq [T^{(n)}\mathbf{k}^{(n)}]_i$, $i, j \in \{1, \ldots, n_h\}$, $i \neq j$.

The procedure outlined above defines inductively a sequence of real vectors $\{\mathbf{k}^{(n)}\}_{n\in\mathbb{N}}$. Now, let us consider the sequence of uniformly bounded functions $\{f_n(t)\}_{n\in\mathbb{N}}$ defined as follows

$$f_n(t) := \sum_{i=1}^{s_n} k_i^{(n)} \chi_{\Delta_i^{(n)}}(t) \tag{9}$$

where, $\chi_\Delta(t)$ denotes the characteristic function of the Borel set Δ.

Clearly, $\|f_n\|_\infty \leqslant 1$, $\forall n \in \mathbb{N}$. Now, we prove that

(a) $\{f_n(t)\}_{n\in\mathbb{N}}$ is point-wise convergent

In order to prove item (a), we prove that, for any $t \in [0, 1]$, the sequence $f_n(t)$ is Cauchy. We proceed as follows. For every $t \in [0, 1]$ and $i \in \mathbb{N}$, let us denote by $\Delta^{(i)}_{i(t)}$ the set in $\mathcal{A}_i$ such that $t \in \Delta^{(i)}_{i(t)}$. We have (see items $n.1.4$ and $n.m.9$),

$$\begin{aligned} |f_l(t) - f_{l-1}(t)| &= \left|\sum_{i=1}^{s_l} k_i^{(l)}\chi_{\Delta_i^{(l)}}(t) - \sum_{i=1}^{s_{l-1}} k_i^{(l-1)}\chi_{\Delta_i^{(l-1)}}(t)\right| \\ &= \left|\sum_{i=1}^{s_l}(k_i^{(l)} - \widetilde{k}_i^{(l-1)})\chi_{\Delta_i^{(l)}}(t)\right| = \left|k^{(l)}_{l(t)} - \widetilde{k}^{(l-1)}_{l(t)}\right| \\ &= \left|k^{(l-1,l_h)}_{l(t)} - k^{(l-1,1)}_{l(t)} + k^{(l-1,1)}_{l(t)} - \widetilde{k}^{(l-1)}_{l(t)}\right| \\ &\leqslant \left|k^{(l-1,l_h)}_{l(t)} - k^{(l-1,1)}_{l(t)}\right| + \left|k^{(l-1,1)}_{l(t)} - \widetilde{k}^{(l-1)}_{l(t)}\right| \\ &= \left|\sum_{r=2}^{l_h}(k^{(l-1,r)}_{l(t)} - k^{(l-1,r-1)}_{l(t)})\right| + \left|k^{(l-1,1)}_{l(t)} - \widetilde{k}^{(l-1)}_{l(t)}\right| \\ &\leqslant \sum\nolimits_{r=2}^{l_h}\left|(k^{(l-1,r)}_{l(t)} - k^{(l-1,r-1)}_{l(t)})\right| + \left|a^{(l-1)}_{l(t)}\right| < \frac{1}{8\cdot 2^l} \end{aligned} \tag{10}$$

where, for every, $i \in [d_j^{(l)}, D_j^{(l)}]$, $j = 1, \ldots, l_h$, we have defined

$$\widetilde{k}_i^{(l-1)} = k_j^{(l-1)}$$

so that,

$$f_{l-1}(t) = \sum_{i}^{s_{l-1}} k_i^{(l-1)} \chi_{\Delta_i^{(l-1)}}(t) = \sum_{i=1}^{s_l} \tilde{k}_i^{(l-1)} \chi_{\Delta_i^{(l)}}(t).$$

By Eq. (10) the sequence $f_n(t)$ is Cauchy and then convergent for any $t \in [0, 1]$. Indeed, for any $\epsilon > 0$ there exists an index $\bar{n}$ such that $\sum_{i=\bar{n}}^{\infty} \frac{1}{2^i} \leqslant \epsilon$ so that, for any pair of indices n, m with, $n > m > \bar{n}$, one has

$$|f_n(t) - f_m(t)| = \left| \sum_{i=m+1}^{n} f_i(t) - f_{i-1}(t) \right| \leqslant \sum_{i=m+1}^{n} |f_i(t) - f_{i-1}(t)| \leqslant \sum_{i=\bar{n}}^{\infty} \frac{1}{2^i} \leqslant \epsilon. \tag{11}$$

Therefore, there exists a function $f(t)$ such that $\lim_{n\to\infty} f_n(t) = f(t)$. Notice that f is Borel measurable because it is the limit of a sequence of Borel measurable functions [16]. We can say more. Indeed, since the inequality in (11) does not depend on $t, f_n(t)$ converges uniformly to $f(t)$. This implies that f is piecewise continuous and continuous from the left since the space of left continuous step functions with the uniform norm is dense in the space of piecewise continuous functions which are continuous from the left [16]. It remains to prove that

(b) f is one-to-one
(c) G_f is one-to-one

In order to prove item (b) we proceed as follows. For every $t, \bar{t} \in [0, 1]$, there exists an index s such that $t \in \Delta_{s(t)}^{(s)}, \bar{t} \in \Delta_{s(\bar{t})}^{(s)}, \Delta_{s(t)}^{(s)} \cap \Delta_{s(\bar{t})}^{(s)} = \emptyset$. Let j be the smallest index such that $t \in \Delta_{j(t)}^{(j)}, \bar{t} \in \Delta_{j(\bar{t})}^{(j)}, \Delta_{j(t)}^{(j)} \cap \Delta_{j(\bar{t})}^{(j)} = \emptyset$. Moreover, let us suppose, without loss of generality, $j(t) > j(\bar{t})$ (notice that, for every $s > j, s(t) > s(\bar{t}), s(t) > j(t), s(\bar{t}) > j(\bar{t})$).
For every $n > j, j(t), j(\bar{t})$,

$$\begin{aligned} |f_n(t) - f_n(\bar{t})| &= \left| f_j(t) - f_j(\bar{t}) + \sum_{l=j+1}^{n} [f_l(t) - f_l(\bar{t})] - [f_{l-1}(t) - f_{l-1}(\bar{t})] \right| \\ &= \left| f_j(t) - f_j(\bar{t}) + \sum_{l=j+1}^{n} [f_l(t) - f_{l-1}(t)] - [f_l(\bar{t}) - f_{l-1}(\bar{t})] \right| \\ &= \left| (k_{j(t)}^{(j)} - k_{j(\bar{t})}^{(j)}) + \sum_{l=j+1}^{n} [k_{l(t)}^{(l)} - \widetilde{k}_{l(t)}^{(l-1)}] - [k_{l(\bar{t})}^{(l)} - \widetilde{k}_{l(\bar{t})}^{(l-1)}] \right| \end{aligned}$$

Moreover (see items n.1.4 and $n.m$.9),

$$\begin{aligned} \left| k_{l(t)}^{(l)} - \widetilde{k}_{l(t)}^{(l-1)} \right| &= \left| k_{l(t)}^{(l-1,l_h)} - k_{l(t)}^{(l-1,1)} + k_{l(t)}^{(l-1,1)} - \widetilde{k}_{l(t)}^{(l-1)} \right| \\ &\leqslant \left| k_{l(t)}^{(l-1,l_h)} - k_{l(t)}^{(l-1,1)} \right| + \left| k_{l(t)}^{(l-1,1)} - \widetilde{k}_{l(t)}^{(l-1)} \right| \\ &= \left| \sum_{r=2}^{l_h} (k_{l(t)}^{(l-1,r)} - k_{l(t)}^{(l-1,r-1)}) \right| + \left| k_{l(t)}^{(l-1,1)} - \widetilde{k}_{l(t)}^{(l-1)} \right| \\ &\leqslant \textstyle\sum_{r=2}^{l_h} \left| (k_{l(t)}^{(l-1,r)} - k_{l(t)}^{(l-1,r-1)}) \right| + \left| a_{l(t)}^{(l-1)} \right| < \dfrac{|(k_{j(t)}^{(j)} - k_{j(\bar{t})}^{(j)})|}{8 \cdot 2^l} \end{aligned}$$

By the same reasoning applied to the case $\bar{t}$ we get

$$\left| k_{l(\bar{t})}^{(l)} - \widetilde{k}_{l(\bar{t})}^{(l-1)} \right| < \frac{\left| (k_{j(t)}^{(j)} - k_{j(\bar{t})}^{(j)}) \right|}{8 \cdot 2^l}$$

Therefore,

$$\begin{aligned}
&\left|\lim_{n\to\infty}\sum_{l=j+1}^{n}[k_{l(t)}^{(l)}-\widetilde{k}_{l(t)}^{(l-1)}]-[k_{l(\bar{t})}^{(l)}-\widetilde{k}_{l(\bar{t})}^{(l-1)}]\right|\\
&=\lim_{n\to\infty}\left|\sum_{l=j+1}^{n}[k_{l(t)}^{(l)}-\widetilde{k}_{l(t)}^{(l-1)}]-[k_{l(\bar{t})}^{(l)}-\widetilde{k}_{l(\bar{t})}^{(l-1)}]\right|\\
&<\lim_{n\to\infty}\sum_{l=1}^{n}\frac{\left|(k_{j(t)}^{(j)}-k_{j(\bar{t})}^{(j)})\right|}{4\cdot 2^{l}}<\frac{\left|(k_{j(t)}^{(j)}-k_{j(\bar{t})}^{(j)})\right|}{2}
\end{aligned} \tag{12}$$

Then,

$$\lim_{n\to\infty}|f_n(t)-f_n(\bar{t})|\neq 0$$

which proves that f is one-to-one.
Now, we proceed to prove item (c).
First we show that $\lim_{n\to\infty}\left(T^{(n)}\mathbf{k}^{(n)}\right)_j\neq\lim_{n\to\infty}\left(T^{(n)}\mathbf{k}^{(n)}\right)_i$.
For every $n>j>i$,

$$\begin{aligned}
&\left|\left(T^{(n)}\mathbf{k}^{(n)}\right)_j-\left(T^{(n)}\mathbf{k}^{(n)}\right)_i\right|=\Bigg|\left(T^{(j)}\mathbf{k}^{(j)}\right)_j-\left(T^{(j)}\mathbf{k}^{(j)}\right)_i\\
&+\sum_{l=j+1}^{n}\left\{\left[\left(T^{(l)}\mathbf{k}^{(l)}\right)_j-\left(T^{(l)}\mathbf{k}^{(l)}\right)_i\right]-\left[\left(T^{(l-1)}\mathbf{k}^{(l-1)}\right)_j-\left(T^{(l-1)}\mathbf{k}^{(l-1)}\right)_i\right]\right\}\Bigg|\\
&=\left|\sum_{r=1}^{s_j}k_r^{(j)}[\lambda_{j,r}^{(j)}-\lambda_{i,r}^{(j)}]+\sum_{l=j+1}^{n}\left(\sum_{q=1}^{s_l}k_q^{(l)}(\lambda_{j,q}^{(l)}-\lambda_{i,q}^{(l)})-\sum_{q=1}^{s_{l-1}}k_q^{(l-1)}[\lambda_{j,q}^{(l-1)}-\lambda_{i,q}^{(l-1)}]\right)\right|.
\end{aligned}$$

Notice that,

$$\lambda_{j,q}^{(l-1)}=\mu_{\Delta_q^{(l-1)}}(j)=\sum_{p\in[d_q^l,D_q^l]}\mu_{\Delta_p^{(l)}}(j)=\sum_{p\in[d_q^l,D_q^l]}\lambda_{j,p}^{(l)},\quad j=1,\ldots,l_h-1$$

hence,

$$\begin{aligned}
&\left|\left(T^{(n)}\mathbf{k}^{(n)}\right)_j-\left(T^{(n)}\mathbf{k}^{(n)}\right)_i\right|\\
&=\left|\sum_{r=1}^{s_j}k_r^{(j)}[\lambda_{j,r}^{(j)}-\lambda_{i,r}^{(j)}]+\sum_{l=j+1}^{n}\left[\sum_{q=1}^{s_l}k_q^{(l)}[\lambda_{j,q}^{(l)}-\lambda_{i,q}^{(l)}]-\sum_{q=1}^{s_{l-1}}k_q^{(l-1)}\sum_{p\in[d_q^l,D_q^l]}(\lambda_{j,p}^{(l)}-\lambda_{i,p}^{(l)})\right]\right|\\
&=\left|\sum_{r=1}^{s_j}k_r^{(j)}[\lambda_{j,r}^{(j)}-\lambda_{i,r}^{(j)}]+\sum_{l=j+1}^{n}\left(\sum_{q=1}^{s_l}k_q^{(l)}[\lambda_{j,q}^{(l)}-\lambda_{i,q}^{(l)}]-\sum_{q=1}^{s_l}\widetilde{k}_q^{(l-1)}[\lambda_{j,q}^{(l)}-\lambda_{i,q}^{(l)}]\right)\right|\\
&=\left|\sum_{r=1}^{s_j}k_r^{(j)}[\lambda_{j,r}^{(j)}-\lambda_{i,r}^{(j)}]+\sum_{l=j+1}^{n}\left(\sum_{q=1}^{s_l}(k_q^{(l)}-\widetilde{k}_q^{(l-1)})[\lambda_{j,q}^{(l)}-\lambda_{i,q}^{(l)}]\right)\right|.
\end{aligned}$$

Moreover (see items $n.m.8$ and $n.1.3$),

$$\begin{aligned}
\left|k_q^{(l)}-\widetilde{k}_q^{(l-1)}\right|&=\left|k_q^{(l-1,l_h)}-k_q^{(l-1,1)}+k_q^{(l-1,1)}-\widetilde{k}_q^{(l-1)}\right|\\
&\leqslant\left|k_q^{(l-1,l_h)}-k_q^{(l-1,1)}\right|+\left|k_q^{(l-1,1)}-\widetilde{k}_q^{(l-1)}\right|
\end{aligned}$$

$$
\begin{aligned}
&= \left| \sum_{r=2}^{l_h} (k_q^{(l-1,r)} - k_q^{(l-1,r-1)}) \right| + \left| k_q^{(l-1,1)} - \widetilde{k}_q^{(l-1)} \right| \\
&\leqslant \sum_{r=2}^{l_h} \left| k_q^{(l-1,r)} - k_q^{(l-1,r-1)} \right| + \left| a_q^{(l-1)} \right| \\
&\leqslant \frac{|\sum_{s=1}^{s_j} k_s^{(j)} (\lambda_{j,s}^{(j)} - \lambda_{i,s}^{(j)})|}{8 \cdot 2^l \cdot 2^q}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\left| \lim_{n\to\infty} \sum_{l=j+1}^{n} \left(\sum_{q=1}^{s_l} (k_q^{(l)} - \widetilde{k}_q^{(l-1)}) [\lambda_{j,q}^{(l)} - \lambda_{i,q}^{(l)}] \right) \right| \\
&\leqslant \lim_{n\to\infty} \sum_{l=j+1}^{n} \sum_{q=1}^{s_l} \frac{\left| \sum_{s=1}^{s_j} k_s^{(j)} (\lambda_{j,s}^{(j)} - \lambda_{i,s}^{(j)}) \right|}{8 \cdot 2^l \cdot 2^q} \\
&= \lim_{n\to\infty} \sum_{l=j+1}^{n} \frac{\left| \sum_{s=1}^{s_j} k_s^{(j)} (\lambda_{j,s}^{(j)} - \lambda_{i,s}^{(j)}) \right|}{4 \cdot 2^l} \\
&< \frac{\left| \sum_{s=1}^{s_j} k_s^{(j)} (\lambda_{j,s}^{(j)} - \lambda_{i,s}^{(j)}) \right|}{2}
\end{aligned}
$$

which implies,

$$
\sum_{r=1}^{s_j} k_r^{(j)} [\lambda_{j,r}^{(j)} - \lambda_{i,r}^{(j)}] \neq \lim_{n\to\infty} \sum_{l=j+1}^{n} \left(\sum_{q=1}^{s_l} (k_q^{(l)} - \widetilde{k}_q^{(l-1)}) [\lambda_{j,q}^{(l)} - \lambda_{i,q}^{(l)}] \right)
$$

and then,

$$
\lim_{n\to\infty} \left(T^{(n)} \mathbf{k}^{(n)} \right)_j \neq \lim_{n\to\infty} \left(T^{(n)} \mathbf{k}^{(n)} \right)_i .
$$

By the dominated convergence Theorem [12], we get

$$
\begin{aligned}
G_f(i) &= \int f(t)\, d\mu_t(i) = \lim_{n\to\infty} \int f_n(t)\, d\mu_t(i) \\
&= \lim_{n\to\infty} \left(T^{(n)} \mathbf{k}^{(n)} \right)_i \neq \lim_{n\to\infty} \left(T^{(n)} \mathbf{k}^{(n)} \right)_j \\
&= \lim_{n\to\infty} \int f_n(t)\, d\mu_t(j) = \int f(t)\, d\mu_t(j) = G_f(j)
\end{aligned}
$$

which proves item (c) and ends the proof of the theorem. □

4. Applications to the theory of positive operator valued measures and to quantum mechanics

In the present section we show how Theorem 3 can be fruitfully applied to the theory of positive operator valued measures which are used in quantum mechanics in order to generalize the concept of observable and which are a powerful tool in quantum computation. But before we need to introduce some preliminaries.

In the following, we denote by $\mathcal{B}(\mathbb{R})$ the Borel σ-algebra of $\mathbb{R}$, by **0** and **1** the null and the identity operators respectively, by $\mathcal{L}_s(\mathcal{H})$ the space of all bounded self-adjoint linear operators acting in a Hilbert space $\mathcal{H}$ with scalar product $\langle \cdot, \cdot \rangle$, and by $\mathcal{F}(\mathcal{H}) \subset \mathcal{L}_s(\mathcal{H})$ the subspace of all positive, bounded self-adjoint operators on $\mathcal{H}$.

Definition 3. A positive operator valued measure on $\mathbb{R}$ (in short, a POV measure (on $\mathbb{R}$)) is a map $F : \mathcal{B}(\mathbb{R}) \to \mathcal{F}(\mathcal{H})$ such that:

$$F\left(\bigcup_{n=1}^{\infty} \Delta_n\right) = \sum_{n=1}^{\infty} F(\Delta_n)$$

where, $\{\Delta_n\}$ is a countable family of disjoint sets in $\mathcal{B}(\mathbb{R})$ and the series converges in the weak operator topology. It is said to be normalized if $F(\mathbb{R}) = \mathbf{1}$. It is said to be commutative if $[F(\Delta_1), F(\Delta_2)] = \mathbf{0}$ for all $\Delta_1, \Delta_2 \in \mathcal{B}(\mathbb{R})$.

Definition 4. A projection valued measure E (in short, PV measure) is a normalized POV measure such that $E(\Delta)$ is a projection operator for each Δ.

In quantum mechanics, non-orthogonal normalized POV measures are also called generalised or unsharp observables and PV measures standard or sharp observables.

An important question in quantum mechanics is to look for the relationships between standard observables and generalized observables. Theorem 3 of the present work will be used to give some answers to that problem.

In what follows, we focus on the following two characterizations of POV measures. The first one, due to Naimark, applies both to commutative and non-commutative POV measures while the second one applies to commutative POV measures.

Theorem 4 (Naimark [13]). *Let F be a POV measure of the Hilbert space $\mathcal{H}$. Then, there exist a Hilbert space $\mathcal{H}^+ \supset \mathcal{H}$ and a PV measure E^+ of the space $\mathcal{H}^+$ such that*

$$F(\Delta) = P^+ E^+(\Delta)_{|\mathcal{H}}$$

where P^+ is the operator of projection onto $\mathcal{H}$.

We recall that, for each vector $x \in \mathcal{H}$, $\langle F(\cdot)x, x\rangle$ is a Lebesgue–Stieltjes measure [12] and we will use the symbol $d\langle F_t x, x\rangle$ to denote integration with respect to the measure $\langle F(\cdot)x, x\rangle$. For each bounded and measurable function f, there exists [8] a unique self-adjoint operator B such that

$$\langle Bx, x\rangle = \int f(\lambda) d\langle F_t x, x\rangle, \quad \forall x \in \mathcal{H}. \tag{13}$$

If Eq. (13) is satisfied we write $B = \int f(t) dF_t$.

Definition 5 (*see Ref. [4]*). Each operator $\int f(\lambda)\, dE^+_\lambda$, where f is a real, one-to-one, measurable function, is said to be a Naimark operator corresponding to F. The Naimark operator $\int \lambda\, dE^+_\lambda$ is denoted by A^+.

Commutative operator valued measures are characterized as follows.

Theorem 5 (see Ref. [2]). *A POV measure $F : \mathcal{B}(\mathbb{R}) \to \mathcal{F}(\mathcal{H})$ is commutative if and only if: i) there exist a self-adjoint operator A and, for every λ in the spectrum of A, a probability measure $\mu^A_{(\cdot)}(\lambda) : \mathcal{B}(\mathbb{R}) \to [0, 1]$ such that $\mu^A_\Delta(A) = F(\Delta)$, ii) if B and $\mu^B_{(\cdot)}(\lambda)$ are such that $\mu^B_\Delta(B) = F(\Delta)$ then, there exists a measurable function g such that $A = g(B)$. A is called the sharp reconstruction of F and is unique up to bijections.*

It is worth remarking that both Theorems 4 and 5 establish a relationship between a POV measure and a PV measure. In Theorem 4, the PV measure corresponding to the POV measure F acts on an extended Hilbert space while, in Theorem 5, the PV measure corresponding to F acts on the same Hilbert space on which F acts. Moreover, Theorem 5 allows us to interpret a commutative unsharp observable as a randomization of a sharp observable [1,2]. All that raises the question of what are the relationships between the PV measure introduced by the Naimark theorem and the one introduced by Theorem 5.

An answer to that question can be given by using the main result of the present paper (Theorem 3). In particular we will establish a relationship between the sharp reconstruction A and the Naimark operator A^+ corresponding to a commutative POV measure F such that the operators in the range of F are discrete (an operator is discrete if there exists a basis of eigenvectors of the operator). The following theorem is a consequence of Theorem 3.

Definition 6. Two bounded self-adjoint operators A and B are said to be equivalent if there exists a bounded, one-to-one, measurable function f such that $A = f(B)$.

Theorem 6. *Let $F : \mathcal{B}(\mathbb{R}) \to \mathcal{F}(H)$ be a commutative POV measure such that the operators in the range of F are discrete. Let A be the sharp reconstruction of F, E^+ an extension of F whose existence is asserted by Naimark's theorem and A^+ the Naimark operator $\int \lambda\, dE^+_\lambda$. Then, A is equivalent to the projection of a Naimark operator $f(A^+)$. Moreover, $A = \int f(t)dF_t$ up to bijections.*

Proof. By Theorem 10 in [4], we can restrict ourselves, without loss of generality, to the case of POV measures with spectrum in [0, 1]. Therefore, let F be a POV measure with spectrum in [0, 1] and such that $F(\Delta)$ is discrete for every $\Delta \in \mathcal{B}([0, 1])$. By Theorem 3.5 in Ref. [7], A is discrete so that we can write $A = \sum_{i=1}^{\infty} \lambda_i E_i^A$. Let $\{\mu_{(\cdot)}(\lambda_i)\}_{i\in\mathbb{N}}$ be the sequence of probability measures such that $F(\Delta) = \mu_\Delta(A)$. By Lemma 3.6 in Ref. [7], $\{\mu_{(\cdot)}(\lambda_i)\}_{i\in\mathbb{N}}$ is a sequence of distinct probability measures. Theorem 3 ensures the existence of a measurable, one-to-one function $f(t)$ such that the function

$$G_f(\lambda_i) = \int f(t)\, d\mu_t(\lambda_i)$$

is one-to-one. Theorem 4 in Ref. [5] and the fact that the sharp reconstruction is defined up to bijections end the proof. □

Theorem 6, which is a consequence of Theorem 3, establishes the equivalence between sharp reconstructions and projections of Naimark operators and generalizes some previous results [4,5].

Moreover, Theorem 3 can be used to reverse the Naimark extension process described in Theorem 4 and therefore to go back from the Naimark operator A^+ acting in the extended Hilbert space $\mathcal{H}^+$ to the sharp reconstruction A acting on $\mathcal{H}$. And this can be done concretely since we have a procedure for the construction of the function f.

Finally, we want to further remark the importance of the fact that in the present paper we give a constructive proof of Theorem 3. Indeed, the construction in Theorem 3 can be used to get a representation of the sharp reconstruction A of F as an integral with respect to F (see the ends of Theorem 6). It is also worth remarking that there exists a procedure [9,2] for the construction of the functions $\mu_\Delta(\lambda)$.

Appendix A. A useful lemma

Lemma 1. *Let us consider step n.1 in the proof of Theorem 3. There exists a sequence of real numbers $a_i^{(n-1)}$ which satisfies the items from n.1.1 to n.1.6.*

Proof. We set

$$\begin{cases} b^{(n,r)} = \dfrac{\min_{p=j-1,\dots,n-1}\left\{\left|\sum_{l=1}^{s_p} k_l^{(p)}(\lambda_{j,l}^{(p)} - \lambda_{i,l}^{(p)})\right|;\ i<j\leqslant n\right\}}{32\cdot 2^n\cdot 2^r} \\ \delta^{(n,r)} = \dfrac{\min_{p=1,\dots,n-1}\{|k_j^{(p)} - k_i^{(p)}|,\ |1-k_j^{(p)}|;\ i<j\leqslant s_p\}}{32\cdot 2^n\cdot 2^r} \\ B^{(n,r)} := \min\{b^{(n,r)}, \delta^{(n,r)}\} \\ C^{(n,l)} := \dfrac{(k_{l+1}^{(n-1)} - k_l^{(n-1)})}{|k_{l+1}^{(n-1)} - k_l^{(n-1)}|} \end{cases}$$

In order to prove the Lemma, we set, for every $l \in \{1, \ldots, s_{n-1}\}$,

$$a_r^{(n-1)} = \begin{cases} 0, & r = d_l^{(n)} \\ C^{(n,l)} B^{(n,r)} k_l^{(n-1)}, & r \in (d_l^{(n)}, D_l^{(n)}],\ l < s_{n-1} \\ B^{(n,r)} k_{s_{n-1}}^{(n-1)}, & r \in (d_{s_{n-1}}^{(n)}, D_{s_{n-1}}^{(n)}]. \end{cases}$$

Then, items n.1.1, n.1.2, n.1.3, n.1.4, n.1.5 are obviously satisfied. It remains to prove item n.1.6. We have, for every $q, l \in \{1, \ldots, s_{n-1}\}$, $l, q \neq s_{n-1}$, $q \neq l$,

$$\left| a_r^{(n-1)} - a_j^{(n-1)} \right| = \begin{cases} |C^{(n,l)} k_l^{(n-1)} (B^{(n,r)} - B^{(n,j)})| \neq 0, & j, r \in (d_l^{(n)}, D_l^{(n)}] \\ |C^{(n,l)} B^{(n,r)} k_l^{(n-1)} - C^{(n,q)} B^{(n,j)} k_q^{(n-1)}| & \\ < |k_l^{(n-1)} - k_q^{(n-1)}|, & r \in (d_l^{(n)}, D_l^{(n)}] \\ & j \in (d_q^{(n)}, D_q^{(n)}] \\ |B^{(n,r)} k_{s_{n-1}}^{(n-1)} - C^{(n,l)} B^{(n,j)} k_l^{(n-1)}| & \\ < |k_{s_{n-1}}^{(n-1)} - k_l^{(n-1)}|, & r \in (d_{s_{n-1}}^{(n)}, D_{s_{n-1}}^{(n)}] \\ & j \in (d_l^{(n)}, D_l^{(n)}] \\ |B^{(n,r)} k_{s_{n-1}}^{(n-1)} - B^{(n,j)} k_{s_{n-1}}^{(n-1)}| & \\ = k_{s_{n-1}}^{(n-1)} |B^{(n,r)} - B^{(n,j)}| \neq 0, & r, j \in (d_{s_{n-1}}^{(n)}, D_{s_{n-1}}^{(n)}] \end{cases} \tag{A1}$$

In order to explain the second and the third inequalities in (A1), let us assume $r > j$. Then (see the definition of $B^{(n,r)}$),

$$\left| C^{(n,l)} B^{(n,r)} k_l^{(n-1)} - C^{(n,q)} B^{(n,j)} k_q^{(n-1)} \right| \leqslant \frac{\left| k_l^{(n-1)} - k_q^{(n-1)} \right|}{32 \cdot 2^j \cdot 2^n} \left(\frac{k_l^{(n-1)}}{2^{r-j}} + k_q^{(n-1)} \right)$$
$$< \left| k_l^{(n-1)} - k_q^{(n-1)} \right|.$$

An analogous reasoning can be used to prove the third inequality in A1. □

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