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Analysis of the low-energy theorem for $\gamma p \to p\pi^0$

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Abstract. The derivation of the 'classical' low-energy theorem (LET) for $\gamma p \to p\pi^0$ is re-examined and compared to chiral perturbation theory. Both results are correct and are not contradictory; they differ because different expansions of the same quantity are involved. Possible modifications of the extended partially conserved axial-vector current relation, one of the starting points in the derivation of the LET, are discussed. An alternate, more transparent form of the LET is presented.

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The low-energy theorem (LET) for photoproduction of neutral pions from protons is the subject of an on-going discussion. The reason for this is that the recent calculations in the framework of chiral perturbation theory (CHPT) [1, 2] and heavy baryon chiral perturbation theory (HBCHPT) [3] are claimed to contradict the older result, i.e. the 'classical' LET [4]. This lead to the statement that this LET[1] is actually not a theorem [1, 2, 3, 5]. In this paper, we point out that the LET is a theorem in the sense that it is based on a few general principles, and once these are given the final result is model-independent. It is also shown that the commonly used extrapolation [4, 6] of the off-shell to the physical pion-nucleon coupling can be avoided by means of the exact, i.e. non-extrapolated Goldberger-Treiman relation. Furthermore, the validity of the particular form of chiral symmetry breaking used in deriving the LET is investigated. However, as will be discussed below, the apparent discrepancy between the CHPT[2] calculation to order q^3 and the LET is due to the fact that expansions in different parameters, pion mass versus energy, are made. Therefore, both results are correct within their respective frameworks and both can be called a LET. To avoid confusion, however, we will refer to the CHPT low-energy theorem as 'CHPT-LET'.

A model-independent result following from general principles should be regarded as a theorem. The LET under consideration is based on Lorentz invariance, gauge invariance, crossing symmetry, the partially conserved axial-vector current (PCAC) *hypothesis*, and its extension to include the electromagnetic interaction [7]. This hypothesis formulates the underlying chiral symmetry and its breaking. Note that, besides spontaneous, also explicit symmetry breaking is included. Recently, the LET has been carefully rederived [6] starting from the principles mentioned above. The consequences of isospin symmetry breaking [8] and the explicitly broken chiral symmetry for the LET [9] have also been addressed. The LET was shown not to be modified. Let us recall some details relevant for this work, which mainly concern the implementation of extended PCAC,

$$(i\partial^\mu + e_\pi A^\mu) J^{\pm,0}_{5,\mu} = i f_\pi M_\pi^2 \phi^{\pm,0} \,. \tag{1}$$

Here, A_μ is the photon field, $J_{5,\mu}$ is the axial-vector current, ϕ is the pion (interpolating) field, M_π is the pion mass, f_π is the pion decay constant, and e_π is the pion charge. Taking the appropriate matrix element of Eq. (1), one finds

$$\begin{aligned} &\frac{f_\pi M_\pi^2}{M_\pi^2 - q^2} \ \langle N(\mathbf{p}') \, | j^{\pm,0}_\pi | N(\mathbf{p}), \gamma(\mathbf{k})\rangle \\ &\quad = i q^\mu \ \langle N(\mathbf{p}') \, | J^{\pm,0}_{5,\mu} | N(\mathbf{p}), \gamma(\mathbf{k})\rangle \\ &\quad\ -e_\pi \ \langle N(\mathbf{p}') \, | J^{\pm,0}_{5,\mu} A^\mu | N(\mathbf{p}), \gamma(\mathbf{k})\rangle \,. \end{aligned} \tag{2}$$

The left-hand side of this equation contains the pion production amplitude expressed by means of the pion source, j_π. The second term on the right-hand side of Eq. (2) does not contribute to neutral pion production. The first term on the right-hand side of Eq. (2) requires a careful treatment because of possible nucleon pole contributions. Therefore, the amplitude is first divided into two general classes: class A diagrams, which contain the dressed nucleon propagator and half off-shell γNN and πNN vertices, and generalized non-pole contributions (class B diagrams). As the above mentioned principles provide enough constraints on these contributions to determine the LET, no (microscopic) model or theory for the hadron structure is needed. This does not imply, however, that the internal structure of the hadrons is ignored.

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[1] From now on, we will omit 'classical'.

[2] No distinction between CHPT and HBCHPT will be made in this paper.

Since Eq. (2) is a priori defined only for virtual pions [6], the production of off-shell pions with **q** = 0 (the pion three momentum in the cm frame) and $q_0 \to 0$ (the pion energy in the cm frame), has been considered. This does not mean that the pion mass is taken to be zero. As a technical tool, an artificial mass difference between the in- and outgoing nucleons is introduced [4] (this procedure is not unique and other methods, yielding the same final result, have been used [10]). In this way, two small parameters appear, $\omega = q_0/M$ and $\delta = (M' - M)/M$ (M being the nucleon mass). These mathematical tools are necessary to determine unknown amplitudes. In particular, the mass splitting[3] enables one to separate out possible pole terms, i.e. contributions of order $1/q_0$. After this separation, it is assumed that an expansion of the amplitude in ω and δ is valid, with δ taking on its physical value in the end, i.e. $\delta = 0$. The expansion in ω is supposed to hold for $q_0 < M_\pi$ (the validity of this assumption and the limit $q_0 \to M_\pi$ will be discussed below).

Applying the above principles then leads to the final result, the LET for $\gamma p \to p\pi^0$

$$E_{0+} = -\frac{eg_A}{8\pi f_\pi \cos\theta_c}\left[\omega - \frac{\omega^2}{2}(3 + \kappa_p)\right] + \mathscr{O}(\omega^3) , \tag{3}$$

where E_{0+} is the s-wave multipole. It is evidently an expansion in the kinematical variable ω, and only contains observables like the pion decay constant, the proton charge, e, the proton anomalous magnetic moment, κ_p, the nucleon axial-vector coupling constant, g_A, and the Cabibbo angle, θ_c. The rather uncommon prefactor in Eq. (3) stems from the fact that the result of the derivation in Ref. [6] actually contains the off-shell pion-nucleon coupling at $q^2 = 0$, i.e. $g(0)$,

$$E_{0+} = -\frac{eg(0)}{8\pi M}\left[\omega - \frac{\omega^2}{2}(3 + \kappa_p)\right] + \mathscr{O}(\omega^3) . \tag{4}$$

Of course, given the order of ω, it is consistent to replace $g(0)$ by $g(q_0^2)$. Finally putting in $\omega = \mu$ yields the more familiar form of the LET in terms of the physical pion-nucleon constant, g [4, 6]. This procedure appears like the extrapolation usually made in the derivation of the Goldberger-Treiman relation, i.e., $g(0) \approx g(M_\pi^2)$. The exact, *non*-extrapolated Goldberger-Treiman relation, however, reads

$$g_A = \cos\theta_c \frac{f_\pi g(0)}{M} . \tag{5}$$

It follows from the PCAC hypothesis, one of the basic ingredients in the derivation, and can therefore consistently be used to eliminate the off-shell pion-nucleon coupling $g(0)$ in Eq. (4), which immediately leads to the final result, Eq. (3). In order to compare with experiment, the limit $\omega \to M_\pi/M \equiv \mu$ is normally taken in Eq. (3), and the LET is usually presented at the point $\omega = \mu$. However, to exploit the main power of the LET, i.e. a theoretical check on models, the above limit is not required [11].

In CHPT, one makes the plausible assumption [12] that the low-energy regime of QCD is described by an effective Lagrangian [13] incorporating the global symmetries of QCD. One then arrives at a systematic expansion scheme in terms of small momenta and meson masses, denoted by q^n. Up to order q^3, the CHPT result [1, 2, 3] is

$$E_{0+} = -\frac{eg}{8\pi M}\left[\mu - \frac{\mu^2}{2}(3 + \kappa_p + \frac{M^2}{8f_\pi^2})\right] + \mathscr{O}(\mu^3) . \tag{6}$$

It should be remarked that the CHPT-LET, Eq. (6), is valid at threshold, and that no distinction is made between the pion energy and the pion mass. In other words, the CHPT amplitude is expanded in the pion mass regardless of its origin. It is obvious that Eqs. (3) and (6) disagree at $\omega = \mu$; numerically, Eq. (3) gives -2.1 $\times$ $10^{-3} M_{\pi^+}^{-1}$, while Eq. (6) gives +0.9 in the same units. The current experimental value [10, 14, 15] is -2.0 $\pm$ 0.2, and new experiments are underway at SAL and Mainz.

On the theoretical side, we conclude that for models satisfying extended PCAC, the CHPT-LET *and* the LET *simultaneously* serve as important theoretical checks. Nevertheless, the apparent conflict between the CHPT-LET and the LET justifies some clarifications concerning the derivation of the latter. First, we stress that all loop contributions are implicitly taken into account in the derivation of the LET. The general vertices and propagators, as well as the general non-pole contributions, in principle contain loops, and no contribution is dropped or neglected.

Secondly, the coefficients of the ω expansion do *not* need to be analytic in μ in order to arrive at the LET. In CHPT some coefficients are divergent in the chiral limit and this is nothing else than the Li-Pagels mechanism [16]. Already before the CHPT calculations, this potential problem was addressed in detail, and it was shown that the LET does not change due to this [9]. It was also anticipated that the coefficients of the μ expansion could be different than the coefficients of the ω expansion due to the Li-Pagels mechanism. Differentiating between pion mass and energy, and only expanding in the latter, avoids this mechanism and the LET is not affected, i.e. the LET is valid even if the coefficients are nonanalytic in μ. We emphasize once more that an expansion has been made in the variables ω and δ. After implementation of PCAC, the fictitious mass difference, δ, can be put to zero and one is left with an expansion in the energy, ω. Usually, however, the on-shell limit, $\omega = \mu$, is taken, and the result takes on the *form* of an expansion in the pion-nucleon mass ratio. As already recognized by Kroll and Ruderman [17] and later by Vainsthein and Zakharov [18], there is no a priori reason that this expression should coincide with the μ expansion of the amplitude. The coefficients may also depend on μ and this dependence is not constrained by the principles used in the derivation of the LET. We stress that the validity of the ω expansion for $\omega < \mu$ is not disproved in the CHPT calculation. In fact, it was argued [19] that the expansion converges for $0 < \omega \leq \mu$. Moreover, it was shown [19] that expansion of the CHPT amplitude [2] in terms of ω produces the LET. In summary, neither CHPT nor QCD forbids an energy expansion with finite pion mass, and, as in Compton scattering (also discussed below), the coefficients of this expansion may be nonanalytic in the pion mass.

The arguments raised above can be illustrated in the linear sigma model [11] or the CHPT amplitude [19]. In particular, one can test if the assumed power expansion in ω

[3] After emission of the pion, the nucleon has mass M'. In this way, no problems with gauge invariance occur.

is valid for $\omega \leq \mu$. It was found for both the linear sigma model amplitude [11] and the CHPT amplitude [19] that the expansion converges for $\omega \leq \mu$, and the coefficients of ω^n (for $n \geq 3$) are $\sim 1/\mu^{(n-2)}$, in agreement with the anticipation of Li and Pagels [16]. It should be emphasized that to obtain the LET, *only* an expansion in ω is needed. In other words, no additional expansion in μ is necessary. In order to illustrate the behavior of the ω expansion at $\omega = \mu$, the ω coefficients were expanded in μ [11], and it was demonstrated that an infinite *convergent* series had to be summed to obtain the CHPT result. The off-shell behavior of the CHPT amplitude could, of course, be different than the off-shell behavior of the linear sigma model amplitude. In an effective Lagrangian approach, such as CHPT, off-shell ambiguities may arise due to the presence of terms which vanish on-shell, see, e.g. [20]. We re-emphasize that starting with the above principles, the LET is valid in the region $\omega \leq \mu$ and it has no off-shell ambiguity, i.e. models that satisfy PCAC should agree on the off-shell value of the E_{0+} multipole up to and including order ω^2.

The current experimental result happens to agree with the numerical value of the LET, and, consequently, not with the CHPT-LET. This disagreement between experiment and the order q^3 CHPT result is believed to be due to the slow convergence of the expansion in μ, and it is concluded that this reaction is not the ideal place to test the standard model [5]. Although convergence issues are beyond the predictive power of LET's in general, one can look at them given a reasonable model. Indeed, the ω expansion of the linear sigma model amplitude [11] converges slowly as $\omega \to \mu$, and therefore the agreement of the LET with the data is somewhat surprising. Another problem in CHPT is that the isospin violation corrections are not fully understood, but expected to be large. In contrast, the LET for this reaction is only trivially modified [8] by isospin violation corrections which are numerically small. In the linear sigma model, the main source of difference between the exact model result and the LET value arises from the intermediate $n\pi^+$ state. In nature, this threshold is 6 MeV above the $p\pi^0$ threshold. Including isospin symmetry breaking (by hand) in the linear sigma model, it was found that the LET value is more accurate, but still 35% higher than the exact model result [11].

In the discussions above, we have re-established the LET based on extended PCAC. We note here that the extension of the PCAC relation was derived assuming minimal electromagnetic coupling [7]. In effective theories, however, non-minimal terms can be present, for example the well-known anomalous magnetic moment terms. Although the latter terms do not, the pertinent question is whether such terms could possibly change the extended PCAC relation. Consequently, the low-energy expansion would be modified, for instance, by a contribution of the form $q_0 M_\pi^2$. To our knowledge this point has not been addressed in the literature, and as an explicit example we consider CHPT. Although such a contribution is not present in the order q^3 calculation of [2], it does appear at order q^4. Explicitly, one has, in the notation of [21], the non-minimal term [22, 23]

$$\mathscr{L} = ia\, Tr\left[\bar{B}\gamma_5\sigma_{\mu\nu}(F^{+\mu\nu}\rho + \rho F^{+\mu\nu})B\right] , \tag{7}$$

where a is an unknown constant. Looking at the neutral pion-nucleon sector, we find

$$\mathscr{L} \sim eaM_u F^{\mu\nu}\left[2\bar{p}\gamma_5\sigma_{\mu\nu}p\pi^0 + \bar{n}\gamma_5\sigma_{\mu\nu}n\pi^0\right] , \tag{8}$$

where M_u is the light-quark mass. The contribution to the E_{0+} multipole at the tree level is $\sim q_0 M_u \sim q_0 M_\pi^2$. As anticipated, this term, Eq. (7), also gives a contribution to the divergence of the axial-vector current,

$$\Delta\partial^\mu J^0_{5,\mu} \sim eaM_u F^{\mu\nu}\left[2\bar{p}\gamma_5\sigma_{\mu\nu}p + \bar{n}\gamma_5\sigma_{\mu\nu}n\right] . \tag{9}$$

For non–zero constant a, the extended PCAC relation, Eq. (1), is therefore modified in CHPT, obviously leading to contributions $\sim \omega\mu^2$ in the low-energy expansion, Eq. (3). We emphasize that there is no *formal* conflict with, or modification of, the LET since it was derived using the principle of minimal coupling. However, as demonstrated here, the low-energy expansion may be modified in models with non-minimal terms.

It is also interesting to consider how the interaction term shown in Eq. (8) modifies the electromagnetic vector current. In the $p\pi^0$ sector we obtain

$$\begin{aligned}\Delta J_\mu \sim M_u \big[(\partial^\nu\bar{p})\gamma_5\sigma_{\mu\nu}p\pi^0 &+ \bar{p}\gamma_5\sigma_{\mu\nu}(\partial^\nu p)\pi^0 \\ &+ \bar{p}\gamma_5\sigma_{\mu\nu}p(\partial^\nu\pi^0)\big] .\end{aligned} \tag{10}$$

One sees that the last term on the right-hand side of Eq. (10) involves a derivative of the pion field, and, in particular, the spatial components of this current evidently contain the time derivative of the pion field. As pointed out in Ref. [10], this is what is needed in effective theories to obtain a non-zero "photoproduction sigma term" [24]. Indeed, the tree-level contribution of Eq. (8) to the photoproduction amplitude is of the same form as the sigma term contribution found in Ref. [25].

It is important to recall that in theories with minimal electromagnetic coupling the extension of PCAC(-like) relations is known [7]. For example, in QCD, the divergence of the axial-vector current is expressed in terms of quark fields. The resulting expression is an acceptable pion interpolating field [26], $\phi^{\pm,0} \sim \bar{q}\gamma_5\tau^{\pm 0}q$ [27]. This choice of the pion interpolating field immediately yields PCAC; since this cannot be proved, PCAC remains a hypothesis [26]. Electromagnetic coupling to the elementary quarks leads to the extended PCAC relation. In other words, given the PCAC hypothesis starting from QCD, corresponding to the above choice of the interpolating pion field, its extension holds. Note, however, that the choice of the pion interpolating field in terms of quarks fields is not unique. Other choices may modify the PCAC relation and it would be interesting to study the possibilities and consequences of such modifications. In CHPT, for instance, PCAC is most likely modified at some order.

Probably the best known example of a LET is Compton scattering [28, 29]. From the theoretical point of view, there is no discussion about this LET. On the other hand, before the connection with experiment can be made, a careful treatment of infrared divergences is needed [30]. In one of the two classical derivations, a discussion on infrared divergences is included. Low [28] explicitly states that the proof applies to all orders in the electromagnetic coupling, provided the virtual photons are given a fictitious mass λ.

He cannot guarantee its validity in the limit $\lambda \to 0$. Surprisingly, the situation for neutral pion photoproduction is under better control. First, infrared divergences concerning virtual photons are not present because the LET is only valid to first order in the electromagnetic coupling. Secondly, there is no need to give the virtual pions a fictitious mass because they are already massive. A few clarifying remarks are in order. The fictitious mass difference for the nucleons was introduced to deal with the nucleon pole in the matrix element of the axial-vector current. This is not connected with infrared problems; moreover, in the end we can take the equal mass limit. Finally, for pion photoproduction as well as Compton scattering, the energy expansion holds from zero up to the pion threshold. Beyond this threshold one cannot make definite statements.

In summary, the 'classical' LET derived *assuming* the extended PCAC relation has been verified. In particular, the LET does not break down due to neglect of loops, as they are implicitly taken into account, nor due to assumptions about the expansion in the pion energy. It has been proposed to present the LET in the form of an energy expansion, reflecting its real content. Moreover, it was demonstrated that the extrapolation of the pion-nucleon coupling can be avoided by using the exact Goldberger-Treiman relation. Furthermore, we pointed out that non-minimal contact couplings, often appearing in effective theories, may affect the extended PCAC relation. This was explicitly shown in CHPT, and the consequences for its low-energy expansion were exhibited. In addition, it was established that non-minimal terms can give rise to a non-zero "photoproduction sigma term". We also commented on the PCAC hypothesis in the context of QCD, where, given PCAC, its extension follows. Finally, a brief discussion of issues related to the expansion of the Compton amplitude and its similarities to the expansion used in deriving the LET was presented.

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