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A Unified Point of View on the Theory of Generalized Bessel Functions

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Abstract—Bessel functions have been generalized in a number of ways and many of these generalizations have been proved to be important tools in applications. In this paper we present a unified treatment, thus proving that many of the seemingly different generalizations may be viewed as particular cases of a two-variable function of the type introduced by Miller during the sixties.

Keywords—Besel functions, Miller functions, Clifford functions, Hermite polynomials, Wright functions, Group theory.

1. INTRODUCTION

The theory of Bessel functions (BFs) is rich and wide, and certainly provides an inexhaustible field of research. A large number of functions are recognized as belonging to the BF family and it is not clear whether a unifying feature, characterizing the various generalizations, may be identified. The Miller [1] and Wright [2] functions are classified as BFs because they satisfy recurrence relations similar to those of ordinary BFs. This is also true for the Bourget-Giuliani [3,4] and Bessel-Clifford [5] functions, just to quote a few among the rich BF-type "zoology."

Many variable BFs were introduced at the beginning of this century [6,7], forgotten for many years and recently rediscovered within the context of various physical applications [8].

These kinds of function, usually called generalized (G) BFs, have opened a new chapter in the theory of special functions, displaying interesting connections with elliptic functions [9] and multivariable Hermite polynomials [10].

The functions belonging to the Bessel domain seem to proliferate. A unifying approach might be useful to have a deeper understanding of the theory and to provide general rules to construct "exotic" forms of BFs.

In this paper, we attempt such a generalization, introducing a two-variable BF from which we deduce, as particular cases, the functions quoted at the beginning of this introduction. We also show that they provide a natural basis for GBFs and discuss their importance within the context of the theory of generalized Hermite polynomials.

The functions we introduce may be viewed as two-variable Miller's and yield a powerful framework for investigating the theory of generalized BFs. In view of the importance played by these BFs in the paper, we recall in this introduction their elementary properties.

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Miller introduced a new class of BF, denoted by $J_n^{(p,q)}(x)$, with generating function

$$\sum_{n=-\infty}^{+\infty} t^n J_n^{(p,q)}(x) = \exp\left[\frac{ix}{p+q}\right](t^p + t^{-q}), \tag{1.1}$$

(p,q) being relatively prime positive integers. The recurrence relations are easily derived and read

$$\begin{aligned} p\frac{d}{dx}J_n^{(p,q)}(x) + \frac{n}{x}J_n^{(p,q)}(x) &= iJ_{n-q}^{(p,q)}(x),\\ q\frac{d}{dx}J_n^{(p,q)}(x) - \frac{n}{x}J_n^{(p,q)}(x) &= iJ_{n+p}^{(p,q)}(x). \end{aligned} \tag{1.2}$$

According to the above relations, the $J_n^{(p,q)}(x)$ can be shown to satisfy the $(q+p)^{\text{th}}$ order ordinary differential equation[1]

$$\prod_{s=1}^{s=p}\left(p\frac{d}{dx} + \frac{n+qs}{x}\right)\prod_{r=q-1}^{0}\left(q\frac{d}{dx} - \frac{n+pr}{x}\right)J_n^{(p,q)}(x) = i^{p+q}\,J_n^{(p,q)}(x), \quad q \geq 1 \tag{1.3}$$

and furthermore, it is easily understood that the properties of ordinary BFs can be derived from those of $J_n^{(p,q)}(x)$, being

$$J_n^{(1,1)}(x) = I_n(ix) = i^n J_n(x), \tag{1.4}$$

where $J_n(x)$ is the first kind cylinder function and $I_n(x)$ its modified version.

2. TWO-VARIABLE BF

Let us consider the following generating function:

$$F(x,y,t \mid 1,m) = e^{xt - y/t^m}, \qquad x,y \in R, \quad 0 < |t| < \infty, \quad m \text{ positive integer.} \tag{2.1}$$

Expanding in Laurent series the r.h.s. of (2.1) we find

$$\begin{aligned} e^{xt-y/t^m} &= \sum_{n=-\infty}^{+\infty} t^n\,\mathcal{D}_n^{(1,m)}(x,y),\\ \mathcal{D}_n^{(1,m)}(x,y) &= \sum_{r=0}^{\infty}\frac{(-1)^r x^{n+mr} y^r}{(n+mr)!r!}. \end{aligned} \tag{2.2}$$

The recurrence properties of the function $\mathcal{D}^{(1,m)}(x,y)$ are easily obtained. Deriving, indeed, with respect to t the first of (2.2) and rearranging the summation index we find

$$m\,\mathcal{D}_n^{(1,m)}(x,y) = x\,\mathcal{D}_{n-1}^{(1,m)}(x,y) + my\,\mathcal{D}_{n+m}^{(1,m)}(x,y). \tag{2.3a}$$

On the other hand, the derivation with respect to x and y leads to

$$\begin{aligned} \frac{\partial}{\partial x}\mathcal{D}_n^{(1,m)}(x,y) &= \mathcal{D}_{n-1}^{(1,m)}(x,y),\\ \frac{\partial}{\partial y}\mathcal{D}_n^{(1,m)}(x,y) &= -\mathcal{D}_{n+m}^{(1,m)}(x,y). \end{aligned} \tag{2.3b}$$

The above recurrences can be combined to get the further relations

$$n\mathcal{D}_n^{(1,m)}(x,y) = \left[x\frac{\partial}{\partial x} - my\frac{\partial}{\partial y}\right]\mathcal{D}_n^{(1,m)}(x,y), \tag{2.3c}$$

$$\frac{\partial^{m+1}}{\partial x^m \partial y}\mathcal{D}_n^{(1,m)}(x,y) = -\mathcal{D}_n^{(1,m)}(x,y), \tag{2.3d}$$

$$\left[x\frac{\partial}{\partial x} + \left(n\frac{\partial^n}{\partial x^n} - my\right)\frac{\partial}{\partial y}\right]\mathcal{D}_n^{(1,m)}(x,y) = 0. \tag{2.3e}$$

[1]The products are ordered according to $\prod_{m=a}^{m=b} R_m = R_a \ldots R_b$.

It is worth stressing that

$$\mathcal{D}_n^{(1,m)}(x,0) = \frac{x^n}{n!},$$
$$\mathcal{D}_n^{(1,m)}(0,y) = \begin{cases} (-1)^{n/m}\frac{y^{-n/m}}{(-n/m)!} & n \le 0, \quad \sqrt{n}/m \text{ positive integer}. \\ 0 & n \ge 0. \end{cases} \tag{2.4}$$

The generating function

$$F(x,y;t \mid m,1) = e^{xt^m - y/t} \tag{2.5}$$

can be used to introduce the function $\mathcal{D}_n^{(m,1)}(x,y)$ which will be said to be complementary to $\mathcal{D}_n^{(1,m)}(x,y)$, according to the meaning of relation (1)[2]

$$\mathcal{D}_{-n}^{(1,m)}(x,y) = \mathcal{D}_n^{(m,1)}(-y,-x). \tag{2.6}$$

An idea of the behavior of $\mathcal{D}_n^{(m,1)}(x,y)$ functions is given in the concluding remarks.

The reason why the functions discussed so far may be usefully exploited to generalize and unify the various BF forms is justified by the relations

$$\mathcal{D}_n^{(1,1)}\left(\frac{x}{2},\frac{x}{2}\right) = J_n(x),$$
$$\mathcal{D}_n^{(1,1)}\left(\frac{x}{2},-\frac{x}{2}\right) = I_n(x). \tag{2.7}$$

Furthermore,

$$\mathcal{D}_n^{(1,1)}(1,y) = C_n(y),$$
$$\mathcal{D}_n^{(1,m)}(1,y) = W_n^m(y), \tag{2.8}$$

with C_n and W_n^m denoting the Bessel-Clifford and Wright functions, respectively. The $C_n(x)$ functions can be shown to be related to the ordinary BF $J_n(x), I_n(x)$ by the relations

$$C_n(y) = \sum_{\ell=0}^{\infty} \frac{(1-y)^\ell}{\ell!} J_{n-\ell}(2y),$$
$$C_n(-y) = \sum_{\ell=0}^{\infty} \frac{(1-y)^\ell}{\ell!} I_{n-\ell}(2y). \tag{2.9}$$

Moreover, the recurrence relations of $C_n(x)$ immediately derived from (2.3) yield for $C_n(x)$ the second-order differential equation

$$\left[x\frac{d^2}{dx^2} + (n+1)\frac{d}{dx} + 1\right] C_n(x) = 0, \tag{2.10}$$

which is also recognized as the Tricomi equation, and thus, $C_n(x)$ is also understood as the Tricomi function [11], namely,

$$C_n(x) = x^{-n/2} J_n(2\sqrt{x}). \tag{2.11}$$

[2]The recurrence relations obeyed by $\mathcal{D}_n^{(m,1)}$ are

$$\frac{\partial}{\partial x}\mathcal{D}_n^{(m,1)}(x,y) = \mathcal{D}_{n-m}^{(m,1)}(x,y),$$
$$\frac{\partial}{\partial y}\mathcal{D}_n^{(m,1)}(x,y) = -\mathcal{D}_{n+1}^{(m,1)}(x,y),$$
$$n\,\mathcal{D}_n^{(m,1)}(x,y) = mx\,\mathcal{D}_{n-m}^{(m,1)}(x,y) + y\,\mathcal{D}_{n+1}^{(m,1)}(x,y).$$

As already remarked, the Wright function is a direct generalization of $C_n(x)$ and its recursion relations write

$$\begin{aligned}\left(n + mx\frac{d}{dx}\right) W_n^{(m)}(x) &= +W_{n-1}^{(m)}(x), \\ \frac{d}{dx} W_n^{(m)}(x) &= -W_{n+m}^{(m)}(x),\end{aligned} \tag{2.12a}$$

and therefore $W_n^{(m)}(x)$ is a solution of the $(m+1)^{\text{th}}$ order ordinary differential equation

$$\left[\left(\frac{d}{dx}\right) \prod_{r=m-1}^{q\geq 1} \left(n - r + mx\frac{d}{dx}\right) + 1\right] W_n^{0(m)}(x) = 0. \tag{2.12b}$$

The extension of $\mathcal{D}_n^{(1,m)}(x,y)$ to nonintegers n and m indices is quite straightforward and reads

$$\mathcal{D}_\nu^{(1,\mu)}(x,y) = \sum_{r=0}^{\infty} \frac{(-1)^r\, x^{\nu+\mu r}\, y^r}{\Gamma(\nu+\mu r+1) r!}. \tag{2.13}$$

The BF, $\mathcal{D}_\nu^{(1,\mu)}(x,y)$ satisfies the same recurrences (2.3a,b) provided that n and m are replaced by ν and μ, respectively. Furthermore, using the properties of fractional-order derivatives [12]

$$\frac{d^\mu}{dx^\mu} x^\alpha = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\mu+1)} x^{\alpha-\mu}, \tag{2.14}$$

the validity of (2.3c–e) can be easily stated for ν and μ real.

It is now convenient to introduce the generating function

$$F(x,y;t \mid p,m) = \exp\left[xt^p - \frac{y}{t^m}\right] = \sum_{n=-\infty}^{+\infty} t^n\, \mathcal{D}_n^{(p,m)}(x,y). \tag{2.15}$$

The function $\mathcal{D}_n^{(p,m)}(x,y)$ satisfies the following recurrences:

$$\begin{aligned}\frac{\partial}{\partial x} \mathcal{D}_n^{(p,m)}(x,y) &= \mathcal{D}_{n-p}^{(p,m)}(x,y), \\ \frac{\partial}{\partial y} \mathcal{D}_n^{(p,m)}(x,y) &= -\mathcal{D}_{n+m}^{(p,m)}(x,y), \\ n\, \mathcal{D}_n^{(p,m)}(x,y) &= px\, \mathcal{D}_{n-p}^{(p,m)}(x,y) + my\, \mathcal{D}_{n+m}^{p,m}(x,y),\end{aligned} \tag{2.16a}$$

and also

$$\begin{aligned}\frac{\partial^{m+p}}{\partial x^m \partial y^p} \mathcal{D}_n^{(p,m)}(x,y) &= (-1)^p\, \mathcal{D}_n^{(p,m)}(x,y), \\ n\, \mathcal{D}_n^{(p,m)}(x,y) &= \left\{px\frac{\partial}{\partial x} + my\frac{\partial}{\partial y}\right\} \mathcal{D}_n^{(p,m)}(x,y).\end{aligned} \tag{2.16b}$$

The $\mathcal{D}_n^{(p,m)}(x,y)$ is given by the following series:

$$\mathcal{D}_n^{(p,m)}(x,y) = \sum_{r=0}^{\infty} \frac{(-1)^r\, x^{(n+mr)/p}\, y^r}{((n+mr)/p)!\, r!}, \tag{2.17}$$

with the only restriction that $(n+mr)/p$ should keep integer values. This last function is recognized as a two-variable Miller function according to

$$\mathcal{D}_n^{(p,m)}\left(\frac{ix}{p+m}, -\frac{ix}{p+m}\right) = J_n^{(p,m)}(x). \tag{2.18}$$

In the case in which (n, p, m) are real, equation (2.17) can be easily generalized, namely,

$$\mathcal{D}_{\nu}^{(\pi,\mu)}(x,y) = \sum_{r=0}^{\infty} \frac{(-1)^r x^{((\nu+\mu r)/\pi)} y^r}{\Gamma((\nu+\mu r)/\pi + 1) r!}, \tag{2.19}$$

and the recurrences (2.16a) can be shown to be valid for the noninteger case, too.

The first of equations (2.16b) holds for n and m nonintegers and p integer.

The examples we have discussed so far have given perhaps an idea of the generality and usefulness of the $\mathcal{D}_n^{(p,m)}(x,y)$ BF. In the next section, we present the relevant addition theorems and show that this class of two-variable function provides an interesting tool to study the properties of GBFs.

3. ADDITION THEOREMS AND GBFs

The Neumann addition theorem, valid for ordinary BFs [13], can be extended to $\mathcal{D}_n^{(1,m)}(x,y)$ functions. From equation (2.1) we find

$$\begin{aligned} F(x \pm u, y \pm z; t \mid 1, m) &= \sum_{n=-\infty}^{+\infty} t^n \mathcal{D}_n^{(1,m)}(x \pm u, y \pm z), \\ \mathcal{D}_n^{(1,m)}(x \pm u, y \pm z) &= \sum_{\ell=-\infty}^{+\infty} \mathcal{D}_{n-\ell}^{(1,m)}(x,y)\, \mathcal{D}_\ell^{(1,m)}(\pm u, \pm z). \end{aligned} \tag{3.1}$$

Furthermore, recalling equation (2.6), we infer that

$$\mathcal{D}_n^{(1,m)}(x-u, y-z) = \sum_{\ell=-\infty}^{+\infty} \mathcal{D}_{n+\ell}^{(1,m)}(x,y)\, \mathcal{D}_\ell^{(m,1)}(z,u). \tag{3.2}$$

Within this framework, the GBF is defined by the series expansion

$$J_n^{(m)}(x,y) = \sum_{\ell=-\infty}^{+\infty} J_{n-m\ell}(x) J_\ell(y), \tag{3.3a}$$

and possesses the generating function

$$\sum_{n=-\infty}^{+\infty} t^n J_n^{(m)}(x,y) = \exp\left[\frac{x}{2}\left(t - \frac{1}{t}\right) + \frac{y}{2}\left(t^m - \frac{1}{t^m}\right)\right], \tag{3.3b}$$

which can be viewed as a particular case of the quoted addition theorem; in fact,

$$\begin{aligned} \exp\left[\frac{x}{2}\left(t - \frac{1}{t}\right) + \frac{y}{2}\left(t^m - \frac{1}{t^m}\right)\right] &= F\left(\frac{x}{2}, \frac{y}{2}; t \mid 1, m\right) F\left(\frac{-x}{2}, \frac{-y}{2}; \frac{1}{t} \mid 1, m\right) \\ &= \sum_{n=-\infty}^{+\infty} t^n \left\{ \sum_{r=-\infty}^{+\infty} \mathcal{D}_{n+r}^{(1,m)}\left(\frac{x}{2}, \frac{y}{2}\right) \mathcal{D}_r^{(1,m)}\left(-\frac{x}{2}, -\frac{y}{2}\right) \right\}, \end{aligned} \tag{3.4a}$$

thus finally finding

$$J_n^{(m)}(x,y) = \sum_{r=-\infty}^{+\infty} \mathcal{D}_{n-r}^{(1,m)}\left(\frac{x}{2}, \frac{y}{2}\right) \mathcal{D}_r^{(m,1)}\left(\frac{y}{2}, \frac{x}{2}\right). \tag{3.4b}$$

A Graf-type addition theorem [13] can also be stated. It is not difficult, in fact, to cast the series

$$\mathcal{D}_n^{(1,m)}(x,y,u,v,t) = \sum_{\ell=-\infty}^{+\infty} t^\ell \mathcal{D}_{n-\ell}^{(1,m)}(x,y)\, \mathcal{D}_\ell^{(1,m)}(u,v), \qquad 0 < |t| < \infty, \tag{3.5}$$

in closed form. Multiplying indeed both sides of (3.5) by ξ^n and then summing up over the index n, we find

$$\sum_{n=-\infty}^{+\infty} \xi^n \sum_{\ell=-\infty}^{+\infty} t^\ell \mathcal{D}_{n-\ell}^{(1,m)}(x,y)\, \mathcal{D}_\ell^{(1,m)}(u,v) = \exp\left[\xi(x+tu) - \xi^{-m}(y+t^{-m}v)\right], \tag{3.6a}$$

thus getting

$$\sum_{\ell=-\infty}^{+\infty} t^\ell \mathcal{D}_{n-\ell}^{(1,m)}(x,y)\, \mathcal{D}_\ell^{(1,m)}(u,v) = \mathcal{D}_n^{(1,m)}\left(x+tu, y+\frac{v}{t^m}\right). \tag{3.6b}$$

In general, it can be proved that

$$\sum_{\ell=-\infty}^{+\infty} t^\ell \mathcal{D}_{n-\ell}^{(p,m)}(x,y)\, \mathcal{D}_\ell^{(p,m)}(u,v) = \mathcal{D}_n^{(p,m)}\left(x+t^p u, y+\frac{v}{t^m}\right). \tag{3.6c}$$

The usual addition theorems for ordinary BFs can be deduced from equation (3.6b) as particular cases. In fact[3]

$$\begin{aligned}\sum_{\ell=-\infty}^{+\infty} t^\ell J_{-\ell}(x) J_\ell(x) &= \sum_{\ell=-\infty}^{+\infty} t^\ell \mathcal{D}_{-\ell}^{(1,1)}\left(\frac{x}{2},\frac{x}{2}\right) \mathcal{D}_\ell^{(1,1)}\left(\frac{x}{2},\frac{x}{2}\right)\\ &= \mathcal{D}_0^{(1,1)}\left(\frac{x}{2}(1+t), \frac{x}{2}\left(\frac{1+t}{t}\right)\right) = J_0\left(x\frac{1+t}{\sqrt{t}}\right).\end{aligned} \tag{3.6d}$$

It is worth mentioning that the above-quoted theorems cannot be easily extended to series of the type

$$\mathcal{D}_n^{(p,m,r,s)}(x,y,u,v,t) = \sum_{\ell=-\infty}^{+\infty} t^\ell \mathcal{D}_{n-\ell}^{(p,m)}(x,y)\, \mathcal{D}_\ell^{(r,s)}(u,v), \qquad (p,m) \neq (r,s), \tag{3.7}$$

which seem to provide new functions whose properties will be discussed elsewhere. The recurrences of

$$\mathcal{D}_n^{(p,m,r,s)}(x,y) = \sum_{\ell=-\infty}^{+\infty} \mathcal{D}_{n-\ell}^{(p,m)}(x,y)\, \mathcal{D}_\ell^{(r,s)}(x,y) \tag{3.8}$$

may perhaps give an idea of the nature of functions of the type (3.7). Equations (2.16a) yield, indeed,

$$\begin{aligned}\frac{\partial}{\partial x}\mathcal{D}_n^{(p,m,r,s)}(x,y) &= \mathcal{D}_{n-p}^{(p,m,r,s)}(x,y) + \mathcal{D}_{n-r}^{(p,m,r,s)}(x,y),\\ \frac{\partial}{\partial y}\mathcal{D}_n^{(p,m,r,s)} &= -\left[\mathcal{D}_{n+m}^{(p,m,r,s)}(x,y) + \mathcal{D}_{n+s}^{(p,m,r,s)}(x,y)\right],\\ n\,\mathcal{D}_n^{(p,m,r,s)}(x,y) &= x\left(p\,\mathcal{D}_{n-p}^{(p,m,r,s)}(x,y) + r\,\mathcal{D}_{n-r}^{(p,m,r,s)}(x,y)\right)\\ &\quad + y\left(m\,\mathcal{D}_{n+m}^{(p,m,r,s)}(x,y) + s\,\mathcal{D}_{n+s}^{(p,m,r,s)}(x,y)\right).\end{aligned} \tag{3.9}$$

Equations (3.8) and (3.9) indicate that functions of the type (3.7) may be ascribed to the class of GBF. The next section is devoted to the theory of generalized Hermite polynomials, treated within the context of the point of view so far developed.

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[3]Equation is a consequence of the identity $\mathcal{D}_n^{(1,1)}((x/2)a,(x/2)b) = (a/b)^{n/2} J_n\left(\sqrt{ab}x\right)$, which can be inferred from the series (2.2).

4. GENERALIZED HERMITE POLYNOMIALS

The theory of two-variable generalized Hermite polynomials (GHPs) was originally developed by Appél and Kampé de Fériet [14], successively reconsidered by Gould and Hopper [15], and more recently by the present authors, who proposed a multivariable generalization and discussed the link with the multivariable GBFs [10], as well as their orthogonality properties [16]. In this section, we sketch the theory of GHPs, using a formalism close to that so far exploited.

We consider, in fact, the generating function

$$F\left(x,y;t \mid p,m^{-1}\right) = \exp\left[xt^p - yt^m\right] = \sum_{n=0}^{\infty} \frac{t^n H_n^{(p,m)}(x,y)}{n!}, \tag{4.1a}$$

(p and m being relatively prime integers) where the two-variable GHP $H_n^{(p,m)}(x,y)$ is defined by the series

$$H_n^{(p,m)}(x,y) = n! \sum_{r=0}^{[n/m]} (-1)^r \frac{x^{((n-mr)/p)} y^r}{((n-mr)/p)!r!}, \tag{4.1b}$$

with the restriction that $((n-mr)/p)$ be an integer. The recurrence relations of $H_n^{(p,m)}(x,y)$ are

$$\begin{aligned} &\frac{\partial}{\partial x} H_n^{(p,m)}(x,y) = \frac{n!}{(n-p)!} H_{n-p}^{(p,m)}(x,y), \qquad n \geq p, \\ &\frac{\partial}{\partial y} H_n^{(p,m)}(x,y) = -\frac{n!}{(n-m)!} H_{n-m}^{(p,m)}, \qquad n \geq m, \\ &H_n^{(p,m)}(x,y) = (n-1)! \left\{ \frac{px}{(n-p)!} H_{n-p}^{(p,m)}(x,y) - \frac{my}{(n-m)!} H_{n-m}^{(p,m)}(x,y) \right\}, \end{aligned} \tag{4.2a}$$

and also

$$\left\{ \left(\frac{\partial}{\partial x}\right)^m - \left(-\frac{\partial}{\partial y}\right)^p \right\} H_n^{(p,m)}(x,y) = 0, \qquad n \geq mp. \tag{4.2b}$$

The ordinary Hermite polynomials are a particular case of $H_n^{(p,m)}(x,y)$, accordingly,

$$H_n(x) = H_n^{(1,2)}(2x,+1). \tag{4.3}$$

Multivariable GHPs have been introduced in [16] and are defined according to

$$\exp\left\{\sum_{s=1}^{M} x_s t^s\right\} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n^{(\{s\}_M)}(\{x_s\}_M), \qquad (\{s\}_M = (1,2,\ldots,M), \{x_s\}_M = (x_1,x_2,\ldots,x_M)). \tag{4.4}$$

It is easy to realize that $H_n^{(\{s\}_M)}(\{x_s\}_M)$ can be constructed using the HP $H_n^{(p,m)}(x,y)$. In the case of $M = 4$ we find, e.g.,

$$H_n^{(\{s\}_M)}\left(\{x_s\}_M\right) = n! \sum_{n=0}^{\infty} \frac{H_{n-r}^{(1,2)}\left(x_1,x_2\right) H_r^{(3,4)}\left(x_3,x_4\right)}{(n-r)!r!}. \tag{4.5}$$

A relation linking the $\mathcal{D}_n^{(p,m)}(x,y)$ functions and the $H_n^{(p,m)}(x,y)$ polynomials can be easily stated. Using the generating function, we get, indeed,

$$\sum_{q=-\infty}^{+\infty} \mathcal{D}_{n-q}^{(p,m)}(x,y)\, \mathcal{D}_q^{(\nu,r)}(z,u) = \sum_{s=0}^{\infty} \frac{H_{n+s}^{(p,\nu)}(x,-z) H_s^{(m,r)}(-y,u)}{(n+s)!s!}, \tag{4.6}$$

which can be specialized to the case

$$J_n(x,y) = \sum_{s=0}^{\infty} \frac{H_{n+s}^{(1,2)}(x,y)\, H_s^{(1,2)}(-x,-y)}{(n+s)!s!}, \tag{4.7}$$

where $H_n^{(1,1)}(x,y)$ are Appél-Kampé de Fériet polynomials (see [14]).

We will not discuss any further results relevant to $H_n^{(p,m)}(x,y)$ polynomials, which will be analyzed more carefully in a forthcoming note.

5. CONCLUDING REMARKS

In this section we will discuss two points:

(a) the behavior of the previously introduced functions;

(b) the underlying Lie algebraic structure.

An idea of the behavior of the $\mathcal{D}_n^{(1,1)}(1,y)$ function (namely, the Bessel-Clifford-Tricomi function) is offered by Figure 1, which clearly displays the double nature of this function. For negative values, $\mathcal{D}_n^{(1,1)}(1,y)$ is unbounded for increasing $|y|$, and for positive values the function behaves like a dampend oscillator. The reasons for this trend are easily understood from equation (2.11) and indeed for negative values of the argument we find[4]

$$\mathcal{D}_n^{(1,1)}(1,-y) = y^{-n/2} I_n(2\sqrt{y}). \tag{5.1}$$

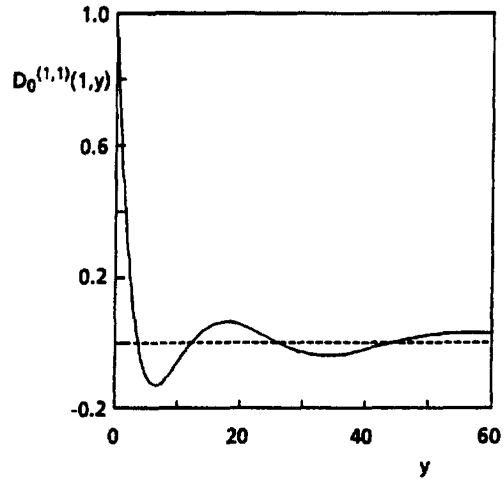

Figure 1. $\mathcal{D}_0^{(1,1)}(1,y)$ vs. y.

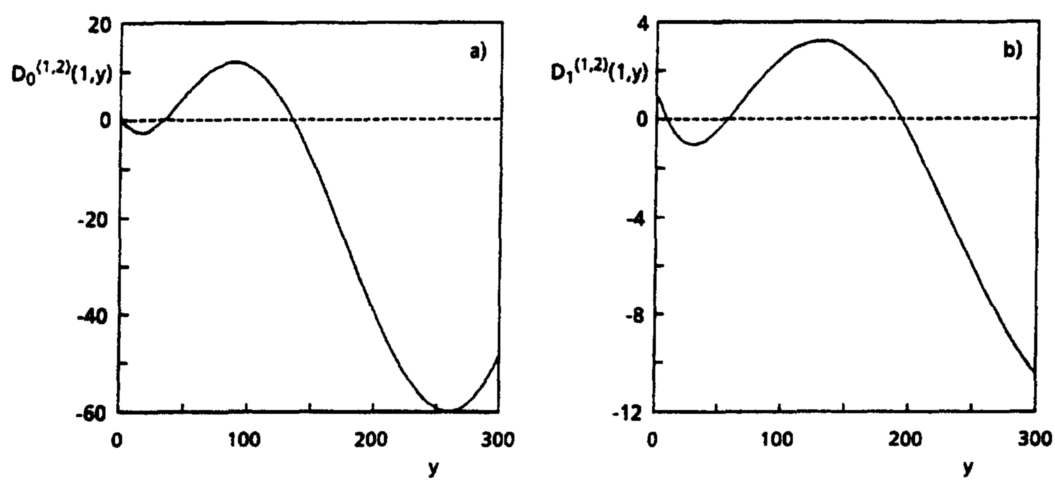

Figure 2. $\mathcal{D}_0^{(1,2)}(1,y)$ (a) and $\mathcal{D}_1^{(1,2)}(1,y)$ (b) vs. y.

In Figure 2, the Wright function, namely, $\mathcal{D}_n^{(1,m)}(1,y)$, is shown for $m=2$ and $n=0,1$. For negative values of the argument the Wright function has a somewhat similar behavior to that of $\mathcal{D}_n^{(1,1)}(1,y)$, while for $y>0$ the function oscillates with increasingly larger amplitudes and

[4]Let us recall that $J_m(ix) = i^n I_n(x)$.

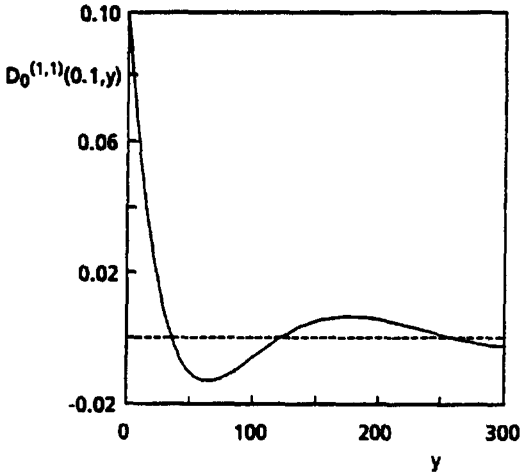

Figure 3. $\mathcal{D}_0^{(1,1)}(0.1, y)$ vs. y.

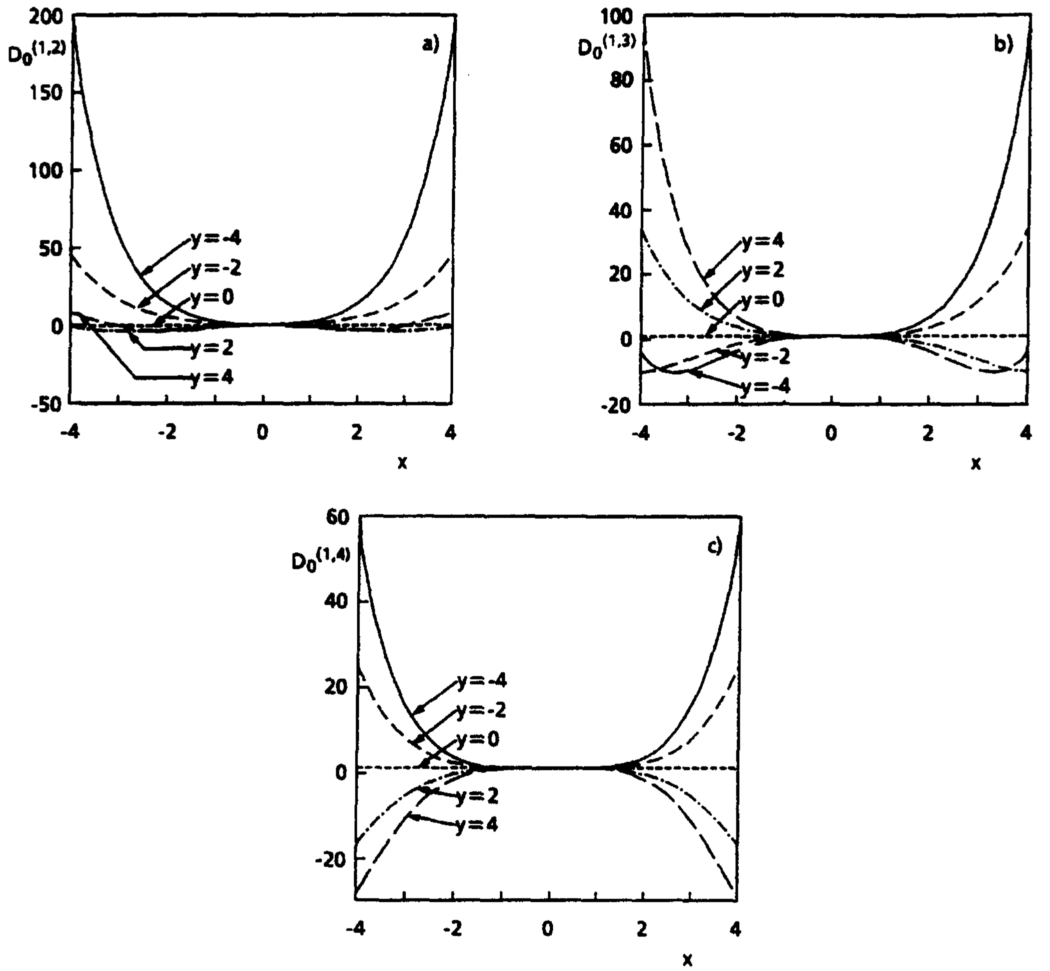

Figure 4. $\mathcal{D}_0^{(1,2)}(x, y)$ (a), $\mathcal{D}_0^{(1,3)}(x, y)$ (b) and $\mathcal{D}_0^{(1,4)}(x, y)$ (c) vs. x for different values of y.

periods. A similar behavior is shown in Figure 3 relevant to $\mathcal{D}_0^{(1,1)}(0.1, y)$, which for positive values of the argument oscillates with amplitudes and periods increasing with increasing y. The trend of the function $\mathcal{D}_n^{(1,m)}(x, y)$ can be inferred from Figures 4–6. In Figure 4a the function $\mathcal{D}_0^{(1,2)}(x, y)$ is plotted vs. x for different values of y. The various curves are all tangent at the origin

$(0,0)$, which is a source point providing a center of symmetry. Figures 4b–c report $\mathcal{D}_0^{(1,3)}$ and $\mathcal{D}_0^{(1,4)}$ vs. x for different values of y. The behavior of $\mathcal{D}_{1,2}^{(1,2-3)}(x,y)$ is finally shown in Figures 5 and 6. The general trend is clear and does not require further comments.

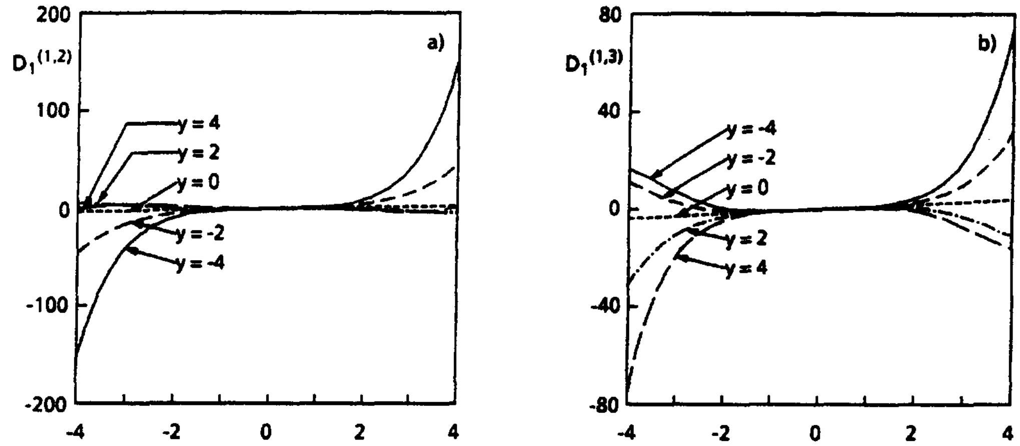

Figure 5. $\mathcal{D}_1^{(1,2)}(x,y)$ (a) and $\mathcal{D}_1^{(1,3)}(x,y)$ (b) vs. x for different values of y.

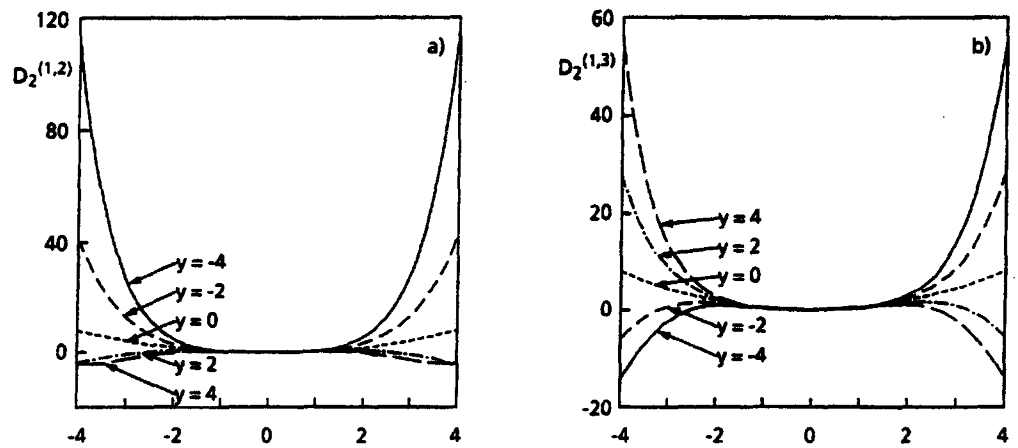

Figure 6. $\mathcal{D}_1^{(1,2)}(x,y)$ (a) and $\mathcal{D}_2^{(1,3)}(x,y)$ (b) vs. x for different values of y.

We will not develop any considerations on the asymptotic properties of the function, since they require an accurate analysis of the relevant differential equations. Such a study is beyond the scope of this paper and hence, will be presented elsewhere.

In the following we will discuss a few points relevant to the Lie algebraic structure of the functions introduced so far. To complete the unifying view proposed in the paper, we apply the formalism of [17–19].

We introduce the group $G(p,m)$ with elements

$$g \equiv (\xi, \eta, \alpha), \qquad \xi, \eta \in C, \quad \alpha \in [0, 2\pi], \tag{5.2}$$

and multiplication law

$$g_1 \circ g_2 \equiv \left(\xi_1 + \xi_2\, e^{ip\alpha_1}, \eta_1 + \eta_2\, e^{-im\alpha_1}, \alpha_1 + \alpha_2\right). \tag{5.3}$$

The integers (p,m) are assumed relative primes. The group $G(p,m)$ has a representation on the Hilbert space $\mathcal{L}_2(0,2\pi)$ of the square-summable functions $f(\psi)$ defined on the unit circle

$(0 \le \psi \le 2\pi)$. For this purpose we introduce the operator $\hat{\mathcal{J}}_\rho(g)$ acting on the function $f(\psi)$ as

$$\hat{\mathcal{J}}_\rho(g) f(\psi) = \exp\left\{ i\rho \left[\xi \, e^{ip\psi} + \eta \, e^{-im\psi} \right] \right\} f(\psi - \alpha). \tag{5.4}$$

Using as basis the set of harmonic functions: $\{e^{in\psi}, n \text{ integer}\}$, we can evaluate the matrix elements of $\hat{\mathcal{J}}_\rho(g)$ which read

$$\tau_{r,n}^{(\rho)}(g) = \frac{1}{2\pi} e^{-in\alpha} \int_0^{2\pi} d\psi \, e^{-i(r-n)\psi} \, e^{i\rho(\xi \, e^{ip\psi} + \eta \, e^{-im\psi})}, \tag{5.5}$$

which, for $p = m = 1$ and $\xi = \eta$, yields the well-known integral representation of the ordinary BF, apart from the factor $e^{-in\alpha}$. In the present more general framework we find

$$\tau_{r,n}^{(\rho)}(g) = e^{-in\alpha} \, (i)^{(r-n)/p} \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{((r-n+q\ell)/p)!} \, \frac{(\rho\xi)^{(r-n+m\ell)/p}}{\ell!} \left(i^{(m-p)/p} \rho\eta \right)^\ell, \tag{5.6}$$

which, once compared with (2.17), yields

$$\tau_{r,n}^{(\rho)}(g) = e^{-in\alpha} \, (i)^{(r-n)/p} \, \mathcal{D}_{r-n}^{(p,m)} \left(\rho\xi, i^{(m-p)/p} \rho\eta \right). \tag{5.7}$$

It is therefore clear that the relations (5.5–7) can be exploited to construct the general theory of $\mathcal{D}_n^{(p,m)}(x,y)$ functions. Furthermore, the above algebraic framework along with the composition law (5.3) is the most natural framework to study the addition theorems discussed in Section 3.

A quasi regular representation of the group $G(p,m)$ can be straightforwardly obtained on the Hilbert space of two complex variable functions $f(x,y)$, [17–19]. We defined indeed the operator $\hat{\mathcal{L}}(g)$ acting on $f(x,y)$ according to

$$\hat{\mathcal{L}}(g) f(\vec{x}) = f\left(g^{-1}\vec{x}\right), \tag{5.8}$$

where $\vec{x}$ is the complex vector (x,y). According to [19] each element g in $G(p,m)$ can be associated with a transformation in the vector space $C \times C$, namely

$$g \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} e^{ip\alpha} & 0 & \xi \\ 0 & e^{im\alpha} & \eta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}, \tag{5.9}$$

and the vector $g^{-1}\vec{x}$ is obtained acting on $\vec{x}$ with the inverse transformation g^{-1}. Within the above framework, it is easy to recognize that the $G(p,m)$ group is generated by

$$g_1 = \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}, \quad g_3 = \begin{pmatrix} e^{ipt} & 0 & 0 \\ 0 & e^{-imt} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.10}$$

In conclusion, the action of the $\hat{\mathcal{L}}_i \equiv \hat{\mathcal{L}}(g_i)$, $i = 1,2,3$ operators on $f(x,y)$ is specified by

$$\hat{\mathcal{L}}_1 f(x,y) = f(x-t,y), \quad \hat{\mathcal{L}}_2 f(x,y) = f(x,y-t), \quad \hat{\mathcal{L}}_3 f(x,y) = f(x\,e^{-ipt}, y\,e^{-imt}). \tag{5.11}$$

According to the above relation, the explicit expressions of the infinitesimal operators [19]

$$\hat{\mathcal{A}}_i f(x,y) = \frac{d}{dt} \hat{\mathcal{L}}_i f(x,y) \bigg|_{t=0}, \qquad i = 1,2,3 \tag{5.12}$$

can be easily obtained in the form

$$\hat{\mathcal{A}}_1 = -\frac{\partial}{\partial x}, \quad \hat{\mathcal{A}}_2 = -\frac{\partial}{\partial y}, \quad \hat{\mathcal{A}}_3 = -ipx \frac{\partial}{\partial x} + imy \frac{\partial}{\partial y}. \tag{5.13}$$

Introducing the operators

$$\hat{E}_{-p} = -\hat{\mathcal{A}}_1 = \frac{\partial}{\partial x}, \quad \hat{E}_{+m} = -\hat{\mathcal{A}}_2 = \frac{\partial}{\partial y}, \quad \hat{n} = i\hat{\mathcal{A}}_3 = px\,\frac{\partial}{\partial x} - my\,\frac{\partial}{\partial y}, \tag{5.14}$$

and using the integral representation of $\mathcal{D}_n^{(p,m)}(x,y)$, it is immediately recognized that

$$\begin{aligned} \hat{E}_{-p}\,\mathcal{D}_n^{(p,m)}(x,y) &= \mathcal{D}_{n-p}^{(p,m)}(x,y), \\ \hat{E}_{+m}\,\mathcal{D}_n^{(p,m)}(x,y) &= -\mathcal{D}_{n+m}^{(p,m)}(x,y), \\ \hat{n}\,\mathcal{D}_n^{(p,m)}(x,y) &= n\,\mathcal{D}_n^{(p,m)}(x,y). \end{aligned} \tag{5.15}$$

Therefore $\mathcal{D}_n^{(p,m)}(x,y)$ are basis functions of the representation $\hat{\mathcal{L}}(g)$.

Before closing this section, let us note that in the first section we mentioned the Bourget-Giuliani function but did not add any comment about its nature. Such a function recurrent in astronomical problems is specified by the integral representation

$$J_{n,k}(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - x\sin\theta)[2\cos\theta]^k\,d\theta, \qquad (k,n) \text{ integers}, \tag{5.16}$$

and it is example of a two-index function.

It is not difficult to understand that $J_{n,k}(x)$ can be cast in the form of a truncated sum, as follows:

$$J_{n,k}(x) = \sum_{m=0}^{[k/2]} \binom{k}{k-m} \left\{J_{n-(k-2m)}(x) + J_{n+(k-2m)}(x)\right\}, \tag{5.17}$$

and thus, all its properties can be derived from those of the ordinary BFs. A Bourget-Giuliani-type function can be associated to the $\mathcal{D}_n^{(p,m)}(x,y)$ BF. Noting indeed that the integral representation of $\mathcal{D}_n^{(p,q)}(x,y)$ is

$$\mathcal{D}_n^{(p,m)}(x,y) = \frac{1}{\pi}\int_0^{\pi} e^{(x\cos p\theta - y\cos m\theta)} \cos\left[x\sin(p\theta) + y\sin(m\theta) - n\theta\right]\,d\theta, \tag{5.18}$$

according to (5.16) a possible Bourget-Giuliani form associated to $\mathcal{D}_n^{(p,q)}(x,y)$ is

$$\mathcal{D}_{n,k}^{(p,m)}(x,y) = \frac{1}{\pi}\int_0^{\pi} (\cos\theta)^k\, e^{(x\cos p\theta - y\cos m\theta)} \cos\left[x\sin(p\theta) + y\sin(m\theta) - n\theta\right]\,d\theta. \tag{5.19}$$

In conclusion, in this paper we have discussed a number of problems and we have shown that a theory of two-variable BFs can be constructed on rather general grounds. We just touched on some problems, as, e.g., those associated with the theory of generalized two-variable Hermite polynomials and their link to BFs. Furthermore, we just derived the differential equations satisfied by the $\mathcal{D}_n^{(p,m)}(x,y)$ functions and did not mention further developments in that direction. All these topics will be the subject of forthcoming investigations.

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