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ABSTRACT

We discuss the structure of the divergences in the multiloop vacuum diagrams for the closed bosonic strings in the framework of the Polyakov covariant formalism. We find, by an explicit computation, that all the divergencesd in the theory may be interpreted as due to tadpole diagrams in which the dilation goes into the vacuum.

Some early attempts to understand the nature of string theories at the multiloop level were made in the context of the old dual models [1]. Later, with the formulation of Polyakov [2], which treats string amplitudes as sums over random surfaces, an important step towards the computation of multiloop amplitudes was given by Alvarez [3], indicating the principal mathematical ingredients. More recently, with the emergence of interest in superstring theories[4], this study has been further revived both in the light cone approach [5,6] and in the Polyakov covariant formalism[7,8]; in particular, the one loop amplitude [9] and the off-shell string propagator [10] are now available in the latter approach.

In this note, we discuss the case of multiloop diagrams for closed bosonic strings with no external legs. We aim to isolate the divergent factors of the vacuum amplitude $\mathcal{Z}_h$ for an underlying two dimensional manifold $\mathcal{M}$ of genus $h \geq 2$, h being the same as the number of loops. We shall confine ourselves to spacetime dimensions D = 26 in the Polyakov formalism.

The functional integral is computed to give, in general [3,8]

$$\mathcal{Z}_h = V \int [d\,\text{Teich}] \left(\det{}' P_1^{\dagger} P_1\right)^{1/2} \left(\frac{2\pi}{\int d^2\sigma \sqrt{\hat{g}}} \det{}' \Delta_{\hat{g}}\right)^{-D/2} . \tag{1}$$

In Eq.(1), the integrations over X^{μ}, (including the centre of mass coordinate X_0^{μ} giving $V = \int d^D X_0$) and over the continuous diffeomorphisms have been completed. The factor $(\det{}' \Delta_{\hat{g}})^{-D/2}$ is the determinant arising from the X^{μ} integration and the prime denotes that the zero mode (related to the translational invariance) has been removed. $(\det{}' P_1^{+} P_1)^{1/2}$ is the Faddeev-Popov determinant, where P_1 and P_1^{+} are the operators introduced by Alvarez [3] and the prime again denotes the deletion of zero modes of P_1, which are however absent for surfaces of genus $h \geqslant 2$. Both the determinants are evaluated with respect to a fixed metric $\hat{g}$ of constant negative curvature. Since we work in D = 26 we ignore the conformal anomaly and the corresponding integration over the conformal factor. It is well known [11] that for $h \geqslant 2$ we can represent the Riemann surface by a fundamental domain H/Γ in the upper half of the complex plane H, where Γ is a Fuchsian group with only hyperbolic elements. The metric $\hat{g}$ will be induced by the natural Poincaré metric on H $ds^2 = dz d\bar{z}/(\mathrm{Im}\, z)^2$, which has curvature -1, defining a hyperbolic geometry on the manifold. The factor d[Teich] is the measure over T - [Teichmueller] deformations of the Riemann surface. The 6h-6 dimensional space of zero modes of P_1^+ is indeed the space of such deformations, which provide a parametrization of the conformally inequivalent surfaces, modulo a discrete gauge group (the T-modular group).

Having chosen the metric $\hat{g}$, one can now evaluate the functional determinants using the techniques of the Minakshisundaram-Pleijel ζ-function and the Selberg trace formula [12, 13]. The integrand appearing in Eq.(1) can be expressed [6,7,8] in terms of the Selberg Z-function:

$$Z(s) = \prod_{\{p\}} \prod_{n=0}^{\infty} \left(1 - e^{-(s+n)\ell_p}\right) \quad . \qquad (2)$$

Here ℓ_p are the hyperbolic minimal lengths, corresponding to inconjugate primitive elements of Γ, of closed curves on the surface belonging to definite homotopy classes. There is an infinite number of ℓ_p's, which are in principle functions of the T-parameters.

For instance, one gets [7,8]

$$\left(\det \Delta_{\hat{g}}\right)^{-D/2} = V.\left(\frac{2\pi}{\int d^2\sigma\sqrt{\hat{g}}}\det{}'\Delta_{\hat{g}}\right) = \text{const.}\,V.\left(Z'(1)\right)^{-D/2}, \tag{3}$$

where Z' = dZ/ds and the volume V comes from the zero mode. Next, by a similar computation, using the technique of ch.IV of ref [13] we get

$$\left(\det(P_1^+P_1)\right)^{\frac{1}{2}} = \text{const.}\ Z(2). \tag{4}$$

Thus we find

$$\mathcal{Z}_h = K.V.\int [d\,\mathrm{Teich}]\, Z(2)\left(Z'(1)\right)^{-D/2}, \tag{5}$$

where K is a constant that depends on the genus h. Of course, we still need the knowledge of the explicit dependence of Z(2) and Z'(1) on the 6h-6 real coordinates of the T-space, the measure [dTeich] and the fundamental domain of the corresponding modular group. However, for our purpose it is enough to assume that there exists a domain in the T-space which covers all the physically distinct configurations and that its measure is finite. The integration in Eq.(5) is meant to be restricted to this domain.

Clearly, the divergences could arise from possible zeroes of Z(s) corresponding to particular configurations, and from Eq.(2) one can see that they arise whenever one of the lengths, say ℓ_γ, contracts to zero. The problem is now to find how many such lengths can be simultaneously shrunk. When a homotopically non trivial geodesic on the surface shrinks and pinches it, in the limit it produces a node, i.e. two identified punctures, either reducing by one the number of handles or splitting the surface into two parts with a puncture on each (the Euler characteristic for a surface of genus h with n punctures is $\chi = (2-2h-n)$. This cannot be done in an arbitrary way, since the Gauss-Bonnet theorem provides a powerful constraint: the hyperbolic area $A = -2\pi\chi$ of the manifold remains constant during this deformation. Take γ to lie along the imaginary axis in H, which can always be achieved by a conformal transformation, and consider an annular piece of the surface containing it, called a collar (see fig. 1a). It is a mathematical result that there always exists a collar with a stricly positive area [14]. Since the area of the collar (EFCD) is $2\ell_\gamma \cot\theta$, then $\theta \sim \ell_\gamma$ as $\ell_\gamma \to 0$. Therefore any geodesic on the manifold $\mathfrak{M}$ intersecting γ must have, in the limit $\ell_\gamma \to 0$, an infinite hyperbolic length, equal to or greater than the length of BC which is $\sim |\ln \ell_\gamma|$ (We also note that the length of the boundaries CD and EF goes to a non zero constant in the limit). Hence only non intersecting geodesics can be shrunk simultaneously, and there are at most 3h-3 of these, giving a partition of $\mathfrak{M}$ into 2h-2 disjoint pieces. Further these 3h-3 geodesics can be chosen in a finite number of ways [15]. They can also be used to provide coordinates for T-space, the coordinates being their lengths and the twists on them [11] . For some partitions, some of these geodesics can be dividing geodesics (a dividing geodesic splits the surface into two distinct pieces) and their number in a partition can be at most (2h-3). These counting rules indicate that the manifold swept out by a propagating closed string can be represented by the Feynmann diagrams of a ϕ^3 theory. Fig (2a) shows two possible partitions of a genus h=2 surface.

Let us first consider the shrinking of one of the dividing geodesics γ. We evaluate the behaviour of Z(2) for $\ell_\gamma \to 0$, by isolating the factors that depend on ℓ_γ.

$$Z(2) = \prod_{n=2}^{\infty} (1 - e^{-n\ell_\gamma})^2 R(2) \quad , \tag{6}$$

where R(2) contains all other factors which are nonzero and bounded. Using a version of the Jacobi identity

$$\prod_{n=1}^{\infty} (1 - e^{-n\ell}) = \left(\frac{2\pi}{\ell}\right)^{1/2} e^{\frac{1}{24}\left(\ell - \frac{(2\pi)^2}{\ell}\right)} \prod_{n=1}^{\infty} \left(1 - e^{-(2\pi)^2 n/\ell}\right) \tag{7}$$

we find

$$Z(2) \xrightarrow[\ell_\gamma \to 0]{} \text{const.} \frac{1}{\ell_\gamma^3} \cdot e^{-(\pi^2/3\ell_\gamma)} \prod_{n=1}^{\infty} \left(1 - e^{-4\pi^2 n/\ell_\gamma}\right)^2 \tag{8}$$

The behaviour of the other factor, i.e. $(Z'(1))^{-D/2}$ in Eq.(7), requires more care. We recall that Z(1) = 0 because of a zero mode, and even when this is eliminated by taking the derivative Z'(1), we can still get in the limit an additional zero. Indeed $\mathcal{M}$ will split into two infinitely separated halves, and each of them could acquire a zero mode. We find it useful therefore to follow the strategy of dividing $\mathcal{M}$ into three parts with a collar Ω sandwiched between manifolds $\mathcal{M}_{1,2}$ with boundaries $\partial\mathcal{M}_{1,2} = \sigma_{1,2}$. (See Fig. 3.) By taking a collar of finite hyperbolic area, as discussed earlier, the (hyperbolic) lengths of the boundaries $\sigma_{1,2}$ remain nonzero and finite when $\ell_\gamma \to 0$ and Area($\mathcal{M}_1$) + Area($\mathcal{M}_2$) < Area($\mathcal{M}$). This ensures that $\mathcal{M}_{1,2}$ has no punctures or non trivial closed curves of vanishing length. We may represent $(Z'(1))^{-D/2}$ as a functional integral over X^μ (μ = 1,2,..26) by fixing first $X(\sigma) = \bar{X}_{1,2}(\sigma)$ on $\sigma_{1,2}$ and performing the integrations over $\bar{X}_1(\sigma)$, $\bar{X}_2(\sigma)$ at the end:

$$V.\left(z'(1)\right)^{-D/2} = \int \mathcal{D}\bar{X}_1\, \mathcal{D}\bar{X}_2 \int_{\mathcal{M}_1} \mathcal{D}X\, e^{X\Delta_{\hat{g}}X} \cdot \int_{\Omega} \mathcal{D}X\, e^{X\Delta_{\hat{g}}X} \cdot \int_{\mathcal{M}_2} \mathcal{D}X\, e^{X\Delta_{\hat{g}}X} \tag{9}$$

The factor in the middle is a functional integral over the collar Ω. We may use a Weyl transformation from hyperbolic to flat metric and then a conformal transformation

$$z = x + iy = \exp(-i\ell_\gamma \omega) \; ; \; \omega = u + iv \tag{10}$$

so that the collar transforms to a straight cylinder fig. 1(b), where the periodic variable is v with range (0,1) and the range of u is $(0, \lambda = \pi/\ell_\gamma)$. Each of these transformations gives a conformal anomaly which we may evaluate using the formula for a manifold with boundaries in Eq. 4.42 of ref. [3](disregarding the power-like divergences since we consistently use ζ-function regularization). One can check that the conformal anomaly is a constant in both cases. For example, for $\hat{g} = e^{2\sigma}\delta$, $\sigma = -\ln y$, we get $\int_{\Omega} d^2z\, \partial_a\sigma\partial_a\sigma =$ hyperbolic area of Ω = constant.

We now perform the integration over the manifold Ω to give as in ref. [10]:

$$\int \mathcal{D}X\, e^{X\Delta X} = \text{const.} \left(\frac{1}{\lambda}\right)^{D/2} \exp\left(\frac{\pi\lambda D}{6}\right) \prod_n \left(1 - e^{-4\pi\lambda n}\right)^{-D} \cdot \exp -\frac{(\bar{X}_{01} - \bar{X}_{02})^2}{\lambda} \exp -S\left(\bar{X}_{n1}, \bar{X}_{n2}\right) \tag{11}$$

where $\bar{X}_{n_{1,2}} = \int d\sigma\ e^{2\pi i n\sigma} \bar{X}_{1,2}(\sigma)$. In the limit $\ell_\gamma = \pi/\lambda \to 0$, the last exponential factorises :

$$\exp -S(\bar{X}_{n1}, \bar{X}_{n2}) \longrightarrow \exp -2\pi n_1 |\bar{X}_{n1}|^2 \exp -2\pi n_2 |\bar{X}_{n2}|^2 \tag{12}$$

and we can absorb the factors in the r.h.s. in the integration over $\bar{X}_{n_{1,2}}$. Substituting Eq.(11) in Eq.(9), we obtain

$$V.\left(Z'(1)\right)^{-D/2} \xrightarrow[\ell_\gamma \to 0]{} \text{const.}\ M_1 . M_2 . \int d\bar{X}_{01}\ d\bar{X}_{02} \left(\frac{1}{\lambda}\right)^{D/2} \exp\left(\frac{\pi\lambda D}{6}\right) . \prod_{n=1}^{\infty} (1 - e^{-4\pi n\lambda})^{-D} . \exp - \frac{(\bar{X}_{01} - \bar{X}_{02})^2}{\lambda} \tag{13}$$

The factors $M_{1,2}$ are the result of the functional integration over the manifolds $\mathcal{M}_{1,2}$ at fixed $\bar{X}_{01,2}$. There are therefore no zero modes and, because of the translational invariance, they are independent of $\bar{X}_{01,2}$. Further, the integration over $\mathcal{M}_{1,2}$ will be finite and non zero as these have no non-trivial closed curves that can be arbitrarily small. We may point out that Eq.(13), apart from the factor $\exp - \frac{(\bar{X}_{01} - \bar{X}_{02})^2}{\lambda}$, displays the same behaviour in $\ell_\gamma = \pi/\lambda$, as could be naïvely obtained by analysing Z(1), (which is actually zero) by means of the Jacobi identity, as was done for Z(2) in Eqs.(7) and (8).

Since we are looking at the corner of the T-space where $\ell_\gamma \to 0$, let us take ℓ_γ and the related twist α_γ to be a pair among the T-coordinates (in fact in our limit the integrand is independent of α_γ). The relevant measure will factorise

$$[d\,\mathrm{Teich}] = d[\ell_\gamma \alpha_\gamma]\, dT' \tag{14}$$

The coordinates ℓ_γ and α_γ parametrise a cylinder, which in our case links $\mathcal{M}_{1,2}$, but it could also be considered, for instance, to close and form a torus. In the case of the torus the measure $[\,\mathrm{dTeich}\,] = d[\ell_\gamma \alpha_\gamma]$ appearing in Eq.(1) is known [8,9] to be (with our definition, the factor coming from the zero mode of $(P_1^+ P_1)$ has to be inserted in the measure)

$$d[\ell_\gamma \alpha_\gamma] = \ell_\gamma\, d\ell_\gamma\, d\alpha_\gamma \tag{15}$$

We take Eq.(15) for our case too, since we expect the relevant part of the measure to be independent of what is attached to the cylinder. Putting together Eqs.(8),(13),(14),(15) into Eq.(5), we finally obtain for this corner of T-space:

$$\mathcal{Z}_h = \mathrm{const.} \int d\bar{X}_{01}\, d\bar{X}_{02} \int d\lambda \left(\frac{1}{\lambda}\right)^{13} \exp - (\bar{X}_{01} - \bar{X}_{02})^2/\lambda \;. \\ \exp 4\pi\lambda \,.\prod_{n=1}^{\infty} \left(1 - e^{-4\pi\lambda n}\right)^{-24} \tag{16}$$

The remaining integration over $\bar{X}_{01}$ and $\bar{X}_{02}$ will give, apart from the usual volume factor, the Euclidean propagator in momentum space [10]

$$\mathcal{Z}_h\,(\mathrm{corner}) \longrightarrow \mathrm{const.}\; V. \sum d(N) \frac{1}{p^2 + m^2(N)} \tag{17}$$

evaluated at $p^2 = 0$. Here d(N) is the degeneracy of the N^{th} level and we read from Eq.(16) that $m^2(0) = -16\pi$ representing the tachyon (in units $4\pi\alpha' = 1$), $m^2(1) = 0$ is the dilaton, all the other $(\text{mass})^2$ being positive. Our result displays explicitly that the configuration we are analysing gives rise to a divergence which can be interpreted as the dilaton tadpole (see fig. 2b). Such a divergence and its interpretation was, in fact, foreseen [16], and emphasized more recently [17] .

Since a surface of genus h can be drawn with at most (2h-3) dividing geodesics and therefore (2h-3) tadpoles, we expect the maximal divergerce of $\mathcal{Z}_h$ to be of this order. When primitive lengths other than the dividing geodesics are shrunk, we do not expect divergences other than the exponential factor exp $4\pi^2/\ell_\gamma$, signalling the tachyon. The tachyon divergences can be regularized in the same manner, as the Laplace transform of a diverging exponential is defined by analytic continuation. This would amount to some prescription on how to go around the tachyon pole in the integration over a propagator like the one in Eq.(17). We do not wish to insist on this procedure, since the tachyon is anyhow an unphysical feature of the bosonic string. The dilaton, on the other hand is not expected to give rise to any divergence other than the tadpole. This can be understood in terms of the suppression of the infrared divergences in Feynmann loop integrals coming from the high dimension of the phase space. This fact is also indicated by our discussion on the possible new zero modes, which can arise only for dividing geodesics.

As a final comment, we recall that in the (closed) superstring case there is no tachyon, and the residue of the dilaton tadpole is expected to be zero [18] , allowing the theory to be finite. To show this explicitly one would need the covariant formulation of the superstring, which has not yet been developed at a quantised level, or equivalently the formulation for the higher genus of the light cone gauge version of the theory [5,6]. Of course, for the case of external legs one should also understand possible external-line divergences [19].

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Note added in proof: After completing this work we have found in the paper "On the Weil-Petersson geometry of the moduli space of curves" by S. Wolpert (American Journal of Mathematics, Vol.107, 969 (1985) that our formula (15) agrees with the expression found by this author with very different techniques.

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FIGURE CAPTIONS

Fig. 1 - (a) The collar Ω

(b) Ω as a cylinder

Fig. 2 - (a) Partitions of a genus 2 surface

(b) A tadpole diagram

Fig. 3 - Division of the manifold $\mathcal{M}$

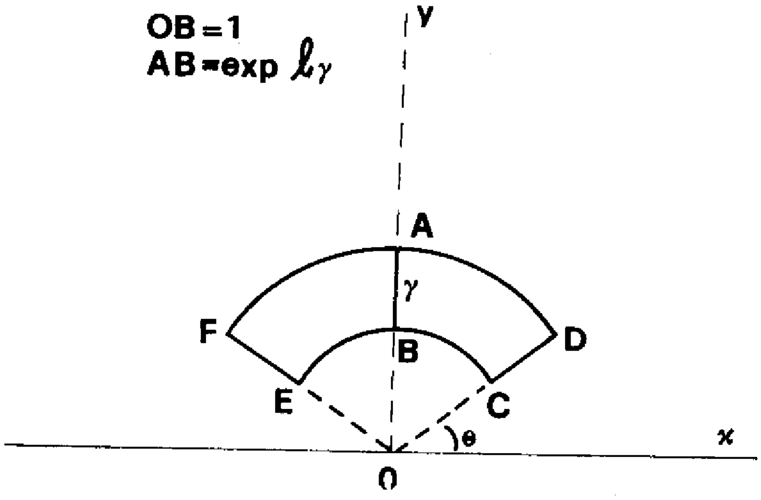

FIG. 1 (a)

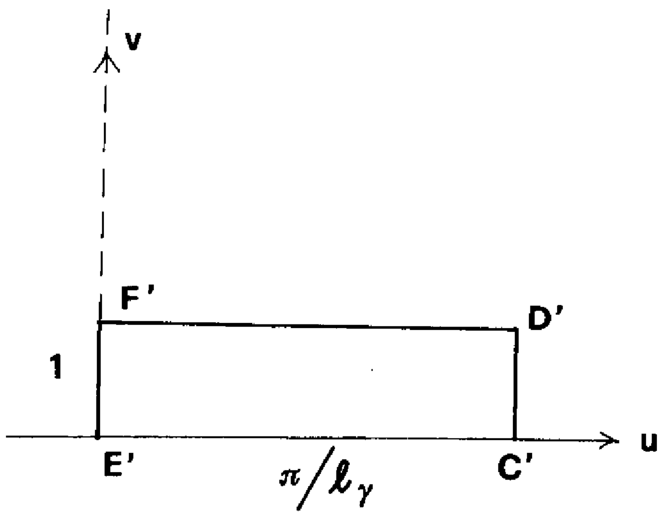

FIG. 1 (b)

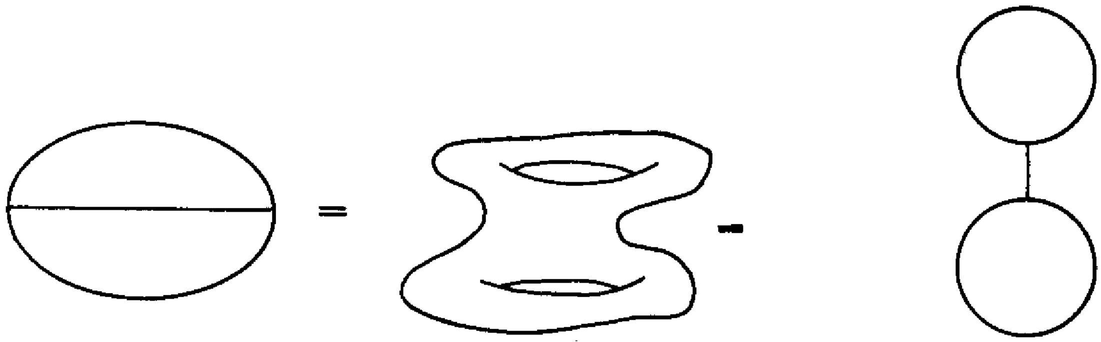

FIG. 2 (a)

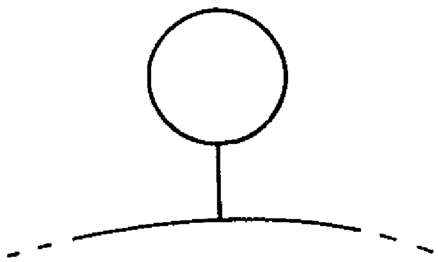

FIG. 2 (b)

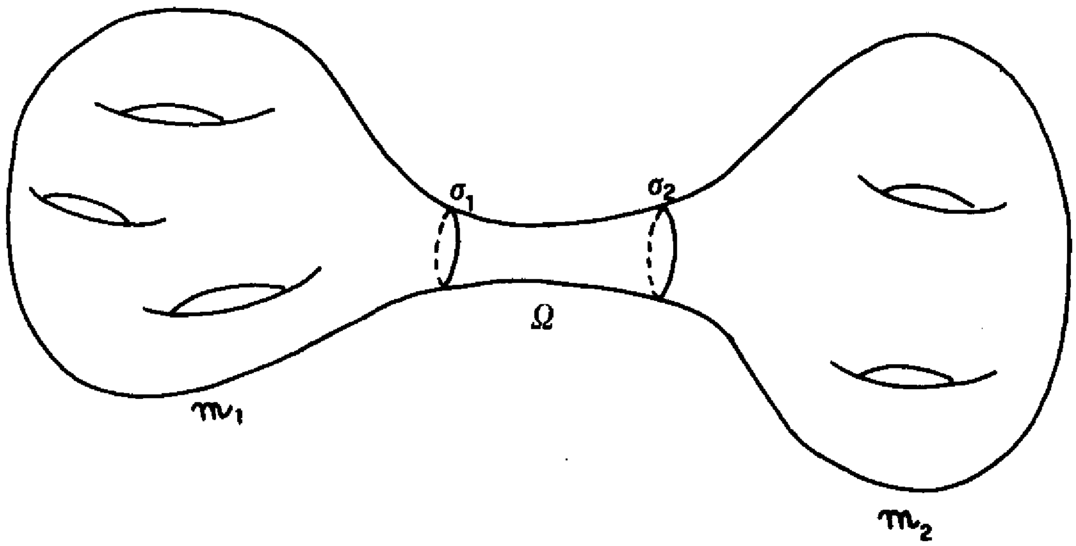

FIG. 3